MLE Proof for P(dev) projection over 8 years

Salim Damerdji

July 2021

Let A be the event that an inventory site is permitted in the RHNA cycle, which is r years long. Random variable B is the year a permit is approved for a site.

Assume:

$$A \sim Bernoulli(p) \tag{1}$$

$$B|A \sim Unif(1, r) \tag{2}$$

Let r and w be known constants such that $r > 0$, $w > 0$, and $w < r$. Then:

$$P(A) = p \tag{3}$$

$$P(B \leq w|A) = w/r \tag{4}$$

$$P(A, B \leq w) = P(A)P(B \leq w|A) = pw/r \tag{5}$$

Let x_1 be the count of observations where A, $B \leq w$ occurs. Let x_2 be the count of all other observations. Then, $n = x_1 + x_2$.

$$\begin{aligned}
&f(x_1, x_2; p) = \frac{n!}{x_1 x_2}(\frac{pw}{r})^{x_1}(1 - \frac{pw}{r})^{x_2} \\
&l(p) = \ln n! - \ln x_1 - \ln x_2 + x_1 \ln \frac{pw}{r} + x_2 \ln(1 - \frac{pw}{r}) \\
&l'(p) = \frac{x_1 w}{r}(\frac{r}{pw}) - \frac{w x_2}{r - pw} \\
&0 = \frac{x_1}{p} - \frac{w x_2}{r - pw} \\
&0 = x_1 r - x_1 pw - w x_2 p \\
&x_1 r = (x_1 w + x_2 w)p \\
&p = \frac{x_1 r}{(x_1 + x_2)w} \\
&\widehat{p}_{MLE} = \frac{x_1 r}{nw}
\end{aligned} \tag{6}$$