Stochastic Modelling and Random Processes

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Contents

Chapter 1

Introduction

1.1 Motivation

Suppose we are modelling COVID. Let

- S be he number of the susceptible;
- I be the number of the infected;
- R be the number of the removed (those who have either recovered or died).

1.1.1 A Deterministic Model

A deterministic model might be

$$\begin{aligned}\dot{S} =& -\beta I S,\\ \dot{I} =& \beta I S - \gamma I,\\ \dot{R} =& \gamma I.\end{aligned}$$

But there are some problems in this model:

- S, I and R are integers, so it does not make sense to talk about $\dot{S}$, $\dot{I}$ and $\dot{R}$.
- There is variability in when contacts are made and lead to infection.

1.1.2 A Stochastic Model

A better model might be stochastic

$$\begin{aligned}\mathbb{P} S \to S-1 \,\&\, I \to I-1 \text{ in } \Delta t =& \beta I S \Delta t + o(\Delta t)\\ \mathbb{P} I \to I-1 \,\&\, R \to R+1 \text{ in } \Delta t =& \gamma I \Delta t + o(\Delta t).\end{aligned}$$

The problem of this model is that contacts are usually not made uniformally in the whole population.

1.1.3 A Network Model

We can use a network model, in which nodes represent individuals and edge weights represent contact rates, to avoid uniform contacts. But tis is unrealistic: the network is too big to represent 60 million people in the UK.

1.1.4 A Random Network Model

Based on the network model, we can make probability distributions on networks and derive probabilistic conclusions over the combination of stochastic dynamics and randomness of networks.

Chapter 2

Probability and Random Variables

2.1 Probability Theory

Suppose we are doing an experiment which have different random outcomes.

Definition 2.1.1 (Sample Spaces)**.** *The* ***sample space*** *of the experiment is the set of all possible outcomes, denoted as* Ω.

Definition 2.1.2 (Sigma Algebra)**.** *The* $\boldsymbol{\sigma}$***-algebra*** *of subsets of* Ω*, denoted as* $\mathcal{F}$*, is a set of subsets of* Ω *which satisfies:*

- $\Omega \in \mathcal{F}$;
- $A \in \mathcal{F} \implies A^c \in \mathcal{F}$;
- $\{A_i | i \in \mathcal{I}\} \subset \mathcal{F}$ *with* $\mathcal{I}$ *being countable* $\implies \bigcup_{i\in\mathcal{I}} A_i \in \mathcal{F}$.

Remark. *We say* $\mathcal{I}$ *is countable if there exists a one-to-one map from* $\mathcal{I}$ *into* $\mathbb{Z}$*, so "countable" includes "finite".*

Example 2.1.1. If Ω is countable, we usually take $\mathcal{F} = 2^{\Omega}$, which is the power set of Ω.

Example 2.1.2. When Ω is not countable, e.g. $[0, 1]$, if you allow Axiom of Choice [1], then there exist unmeasurable subsets, and we exclude them from $\mathcal{F}$, i.e. $\mathcal{F}$ is the set of all Lebesgue-measurable subsets on $[0, 1]$.

Definition 2.1.3 (Events)**.** *The members of* $\mathcal{F}$ *are called* ***events***.

Definition 2.1.4 (Probability)**.** $\mathbb{P}[\cdot] : \mathcal{F} \mapsto \mathbb{R}$ *is called a probability if*

- $\mathbb{P}[A] \in [0, 1], \forall A \in \mathcal{F}$;
- $\mathbb{P}[\Omega] = 1$;
- $\mathbb{P}[\cdot]$ *satisfies the* ***countable additivity***: $\forall \{A_i | i \in \mathcal{I}\} \subset \mathcal{F}$*, where* $\mathcal{I}$ *is a countable set, if* A_i*'s are disjoint, then*

$$\mathbb{P}\left[\bigcup_{i\in\mathcal{I}} A_i\right] = \sum_{i\in\mathcal{I}} \mathbb{P}\left[A_i\right].$$

[1] A Cartesian product of a collection of nonempty sets is nonempty.

Definition 2.1.5 (Independence)**.** *Say $A, B \in \mathcal{F}$ are **independent** if*

$$\mathbb{P}\left[A \cap B\right] = \mathbb{P}[A]\mathbb{P}[B].$$

Definition 2.1.6 (Conditional Probabilities)**.** *If $\mathbb{P}[B] > 0$, then the **conditional probability** $\mathbb{P}\left[A|B\right]$ is defined by*

$$\mathbb{P}\left[A|B\right] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}, \quad \forall A \in \mathcal{F}.$$

Definition 2.1.7 (Partitions)**.** *$\{B_i | i \in \mathcal{I}\}$ is called a **partition** of the sample space Ω if:*

- *B_i's are **pairwise disjoint**: $B_i \cap B_j = \emptyset$, $\forall i, j \in \mathcal{I}$, $i \neq j$;*
- *$B_i \neq \emptyset$, $\forall i \in \mathcal{I}$;*
- *$\{B_i | i \in \mathcal{I}\}$ **covers** Ω: $\bigcup_{i \in \mathcal{I}} B_i = \Omega$.*

Theorem 2.1.1 (The Law of Total Probability)**.** *Let $\{B_i | i \in \mathcal{I}\}$ be a countable partition of Ω, with $B_i \in \mathcal{F}$ and $\mathbb{P}\left[B_i\right] > 0$, $\forall i \in \mathcal{I}$. Then $\forall A \subset \mathcal{F}$, we have*

$$\mathbb{P}\left[A\right] = \sum_{i \in \mathcal{I}} \mathbb{P}\left[A|B_i\right] \mathbb{P}\left[B_i\right].$$

Chapter 3

Discrete-Time Markov Chain

Chapter 4

Continuous-Time Markov Chain

Chapter 5

Continuous State Space Markov Processes

Chapter 6

Stochastic Particle Systems

Chapter 7

Networks

Chapter 8

Random Networks