Stochastic Modelling and Random Processes
Problem Sheet 3

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Contents

1 Geometric Brownian Motion

Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x),\ \mu \in \mathbb{R},\ \sigma > 0,$$

and initial condition $X_0 = 0$. **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}.$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator. Write down the associated SDE and Fokker-Planck equation.

Sol.

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(Y_t)] = \mathbb{E}[\mathcal{L}f(Y_t)],$$

to derive ODEs for the meane $m(t) := \mathbb{E}[Y_t]$ and the second moment $m_2(t) := \mathbb{E}[Y_t^2]$. (No need to solve the ODEs.)

Sol.

(c) Under which conditions on μ and σ^2 is $(Y_t : t \geq 0)$ a martingale?

What is the asymptotic behaviour of the variance $v(t) = m_2(t) - m(t)^2$ in that case?

Sol.

(d) Show that δ_0 is the unique stationary distribution of the process on the state space $[0, \infty)$. Under which conditions on μ and σ^2 does the process with $Y_0 = 1$ converge to the stationary distribution?

Under which conditions on μ and σ^2 is the process ergodic? Justify your answer.

Sol.

(e) For $\sigma^2 = 1$ choose $\mu = -1/2$ and two other values $\mu < -1/2$ and $\mu > 1/2$. Simulate and plot a sample path of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

Sol.