Stochastic Modelling and Random Processes

Yiming MA

December 13, 2020

Contents

1 Discrete-Time Markov Chains **2**
1.1 Countable Discrete-Time Markov Chains 2

2 Continuous-Time Markov Chains **4**
2.1 Continuous-Time Markov Chains 4

Chapter 1

Discrete-Time Markov Chains

1.1 Countable Discrete-Time Markov Chains

One can extend much of what we have done for finite discrete-time Markov chains to the countably infinite case, e.g. the **simple random walk** on $\mathbb{Z}$, but some results become more subtle. For example, the simple random walk is *not SP-ergodic*, despite being *irreducible*. Actually, it even *fails to have a stationary probability*; also it is *not aperiodic*, and it has a *period* 2.

Example 1.1.1. Using definition of the simple random walk:

$$Y_n = \sum_{i=0}^{n-1} X_i,$$

where X_i's are independent and identically distributed, with

$$X_i = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases},$$

Compute the $\mathbb{E}[Y_n]$ and $\mathrm{Var}[Y_n]$.

One has to refine various concepts.

Definition 1.1.1 (The First Return Time)**.** The **first return time** to state x is defined as

$$T_x = \inf\{n \geq 1 : X_n = x | X_0 = x\}.$$

Remark. Notice that when the state space is finite and x is recurrent, T_x is finite. Since the state space here is countably infinite, T_x is allowed to be infinte.

Definition 1.1.2 (Transience)**.** Say $x \in S$ is **transient** if

$$\mathbb{P}[T_x = \infty] > 0.$$

Remark. If $x \in S$ is transient, then with probability 1 X_n comes back to x only finitely many times.

Definition 1.1.3 (Null Recurrence)**.** Say $x \in S$ is **null recurrent** if

$$\mathbb{P}[T_x < \infty] = 1 \quad \text{and} \quad \mathbb{E}[T_x] = \infty.$$

Definition 1.1.4 (Positive Recurrence)**.** Say $x \in S$ is **positive recurrent** if

$$\mathbb{P}[T_x < \infty] = 1 \quad \text{and} \quad \mathbb{E}[T_x] < \infty.$$

Remark. A communicating class is either **null recurrent**, which means every member is null recurrent, or **positive recurrent** which means every member is positive recurrent.

Theorem 1.1.1 (Stationarity $\iff$ Positive Recurrence)**.** *An absorbing class has a stationary probability if and only if it is positive recurrent. Furthermore, if the class has one stationary probability, then it is uniquely determined by*

$$\boldsymbol{\pi}_x = \frac{1}{\mathbb{E}[T_x]}.$$

Chapter 2

Continuous-Time Markov Chains

2.1 Continuous-Time Markov Chains

We are now considering a continuous-time markov chain with a countable state space S and the domain $T \in \mathbb{R}$ (or $T \in \mathbb{R}_+$), and we restrict $X : \mathbb{R} \mapsto S$ to those which are *piecewise constant* and *right-continuous*, meaning

$$X(t) = \begin{cases} \vdots & \vdots \\ s & t \in [J_s, J_{s'}) \\ s' & t \in [J_{s'}, J_{s''}) \\ \vdots & \vdots \end{cases}$$

Definition 2.1.1 (Continuous-Time Markov Chains). $X(t) : \mathbb{R} \mapsto S$ is a **continuous-time Markov chain**, if it satisfies the **Markov property**

$$\mathbb{P}[X(t_{n+1}) \in A | X(t_n) = s_n, \cdots, X(t_1) = s_1] = \mathbb{P}[X(t_{n+1}) \in A | X(t_n) = s_n],$$

where $A \subset S$ and $t_1 < \cdots t_n < t_{n+1}$.

Definition 2.1.2 (Homogeneity). A continuous-time Markov chain is **homogeneous** if

$$\mathbb{P}[X(t+u) \in A | X(u) = s] = \mathbb{P}[X(t) \in A | X(0) = s].$$

Remark. Homogeneity means time translation invariance.

Definition 2.1.3 (Transition Matrices). Let $(P_t)_{i,j} := \mathbb{P}[X(t) = j | X(0) = i]$, then P_t is the transition matrix with time step t.

Remark. The (i, j) element of the transition matrix P_t can also be expressed as $P_t(i, j)$.

Theorem 2.1.1 (Chapman-Kolmogorov Equation). *The transition matrix P of a homogeneous Markov chain satisfies*

$$P_{t+u} = P_t P_u,\ P_0 = I.$$

Proof. Notice that

$$
\begin{aligned}
(P_{t+u})_{i,j} =& \mathbb{P}[X(t+u)=j|X(0)=i] \\
=& \sum_{k\in S} \mathbb{P}[X(t+u)=j|X(t)=k,\ X(0)=i]\mathbb{P}[X(t)=k|X(0)=i] \\
=& \sum_{k\in S} \mathbb{P}[X(t+u)=j|X(t)=k]\mathbb{P}[X(t)=k|X(0)=i] \\
=& \sum_{k\in S} \mathbb{P}[X(u)=j|X(0)=k]\mathbb{P}[X(t)=k|X(0)=i] \\
=& \sum_{k\in S} (P_u)_{k,j}(P_t)_{i,k} \\
=& (P_t)_{i,:}\ (P_u)_{:,j},
\end{aligned}
$$

where $(P_t)_{i,:}$ is the i-th row of P_t and $(P_u)_{:,j}$ is the j-th column of P_u. Thus, $P_{t+u} = P_t P_u$. And by definition, $(P_0)_{i,j} = \mathbb{P}[X_0 = j|X_0 = i] = \delta_{i,j}$, so $P_0 = I$. $\square$

Definition 2.1.4 (Generator / Rate Matrix)**.** Suppose P_t is differentiable with respect to t at $t = 0$, then

$$
G := \left.\frac{\mathrm{d}P_t}{\mathrm{d}t}\right|_{t=0}
$$

is called the **generator** or the **rate matrix** of the process.

Proposition 2.1.1. *$P_t = \exp(tG)$ in the sense of power series.*

Proof. By the Chapman-Kolmogorov equation, we have

$$
\begin{aligned}
P_{t+u} =& P_t P_u \\
P_{t+u} - P_t =& P_t(P_u - I) \\
\frac{P_{t+u} - P_t}{u} =& P_t \cdot \frac{P_u - I}{u} \\
\lim_{u\to 0} \frac{P_{t+u} - P_t}{u} =& \lim_{u\to 0} P_t \cdot \frac{P_u - I}{u} \\
\lim_{u\to 0} \frac{P_{t+u} - P_t}{u} =& P_t \cdot \lim_{u\to 0} \frac{P_u - I}{u} \\
\frac{\mathrm{d}P_t}{\mathrm{d}t} =& P_t G,
\end{aligned}
$$

So $P_t = C \cdot \exp(tG)$, where C is a constant diagonal matrix with diagonal elements being equal. By $P_0 = I$, we konw $C = I$. $\square$

Proposition 2.1.2. *The generator G also satisfies*

$$
G\vec{1} = \vec{0}.
$$

Proof. For any probability distribution $\boldsymbol{\pi}_t = \boldsymbol{\pi}_0 P_t$ with initial distribution $\boldsymbol{\pi}_0$, evolves by

$$
\frac{\mathrm{d}\boldsymbol{\pi}_t}{\mathrm{d}t} = \boldsymbol{\pi}_0 \frac{\mathrm{d}P_t}{\mathrm{d}t} = \boldsymbol{\pi}_0 P_t G = \boldsymbol{\pi}_t G.
$$

And by conservation of probability, we have $\boldsymbol{\pi}_t\vec{1} = \vec{1}$, which implies $\boldsymbol{\pi}_t G\vec{1} = \frac{\mathrm{d}\boldsymbol{\pi}_t\vec{1}}{\mathrm{d}t} = 0$. Since $\boldsymbol{\pi}_t$ is arbitrary, we have $G\vec{1} = 0$. $\square$

Theorem 2.1.2 (The Master Equation). *The equation*

$$\frac{\mathrm{d}\boldsymbol{\pi}_t}{\mathrm{d}t} = \boldsymbol{\pi}_t G$$

can be written into

$$\frac{\mathrm{d}(\boldsymbol{\pi}_t)_i}{\mathrm{d}t} = \underbrace{\sum_{j \neq i} (\boldsymbol{\pi}_t)_j G_{j,i}}_{\text{"gain"}} - \underbrace{\sum_{j \neq i} (\boldsymbol{\pi}_t)_i G_{i,j}}_{\text{"loss"}},$$

which is called the ***master equation****.*

Proof. For $i \neq j$, since $G_{i,j}$ is the rate at which the process goes from state i to j, we have $G_{i,j} \geq 0$. By $G\vec{1} = \vec{0}$, we have

$$G_{i,i} = -\sum_{j \neq i} G_{i,j}.$$

So

$$\begin{aligned} \frac{\mathrm{d}(\boldsymbol{\pi}_t)_i}{\mathrm{d}t} &= \boldsymbol{\pi}_t G_{:,i} \\ &= \sum_{j \in S} (\boldsymbol{\pi}_t)_j G_{j,i} \\ &= \sum_{j \neq i} (\boldsymbol{\pi}_t)_j G_{j,i} - \sum_{j \neq i} (\boldsymbol{\pi}_t)_i G_{i,j}. \end{aligned}$$

□

Remark. The name "master equation" is exaggerated; it does not tell everything about the process, such as the correlations between states at different times.

Definition 2.1.5 (Stationarity). Say $\boldsymbol{\pi} \in \Delta$ is **stationary** if $\boldsymbol{\pi} G = 0$.

Definition 2.1.6 (Reversibility). Say $\boldsymbol{\pi} \in \Delta$ is **reversible** if

$$\boldsymbol{\pi}_i G_{i,j} = \boldsymbol{\pi}_j G_{j,i}, \ \forall i, j \in S.$$

Proposition 2.1.3 (Reversibility $\implies$ Stationarity). *If $\boldsymbol{\pi} \in \Delta$ is reversibile, then it is also stationary.*

Proposition 2.1.4. *S is fintie $\implies$ $\exists$ stationary $\boldsymbol{\pi}$.*

There is an analogous decomposition of the state space S into transient and recurrent states, and of the set of recurrent states into communicating components. And we have the same definition of an absorbing component.

Proposition 2.1.5. *If S is finite, then each absorbing component has a unique stationary probability $\boldsymbol{\pi}$, and the space of starionary $\boldsymbol{\pi}$ for the whole continuous-time Markov chain (up to normalisation) is the span of those for its absorbing components. Furthermore, 0 is a semisimple eigenvalue of G.*

Theorem 2.1.3. *Suppose S is finite and G has a unique absorbing component, then the process is SP-ergodic, which means*

$$\lim_{t \to \infty} \boldsymbol{\pi}_t = \boldsymbol{\pi}_A,$$

where $\boldsymbol{\pi}_A$ is the stationary distribution of the absorbing component.

Remark. Aperiodicity is automatic in continuous time.