Stochastic Modelling and Random Processes

Yiming MA

Contents

Chapter 1

Introduction

1.1 Motivation

Suppose we are modelling COVID. Let

- S be he number of the susceptible;
- I be the number of the infected;
- R be the number of the removed (those who have either recovered or died).

1.1.1 A Deterministic Model

A deterministic model might be

$$\begin{aligned}\dot{S} =& -\beta IS,\\ \dot{I} =& \beta IS - \gamma I,\\ \dot{R} =& \gamma I.\end{aligned}$$

But there are some problems in this model:

- S, I and R are integers, so it does not make sense to talk about $\dot{S}$, $\dot{I}$ and $\dot{R}$.
- There is variability in when contacts are made and lead to infection.

1.1.2 A Stochastic Model

A better model might be stochastic

$$\begin{aligned}\mathbb{P}(S \to S-1 \;\&\; I \to I-1 \text{ in } \Delta t) =& \beta IS\Delta t + o(\Delta t)\\ \mathbb{P}(I \to I-1 \;\&\; R \to R+1 \text{ in } \Delta t) =& \gamma I\Delta t + o(\Delta t).\end{aligned}$$

The problem of this model is that contacts are usually not made uniformally in the whole population.

1.1.3 A Network Model

We can use a network model, in which nodes represent individuals and edge weights represent contact rates, to avoid uniform contacts. But tis is unrealistic: the network is too big to represent 60 million people in the UK.

1.1.4 A Random Network Model

Based on the network model, we can make probability distributions on networks and derive probabilistic conclusions over the combination of stochastic dynamics and randomness of networks.

Chapter 2

Probability and Random Variables

2.1 Probability Theory

Suppose we are doing an experiment which have different random outcomes.

Definition 2.1.1 (Sample Spaces)**.** *The* ***sample space*** *of the experiment is the set of all possible outcomes, denoted as* Ω*.*

Definition 2.1.2 (Sigma Algebra)**.** *The* $\boldsymbol{\sigma}$***-algebra*** *of subsets of* Ω*, denoted as* $\mathcal{F}$*, is a set of subsets of* Ω *which satisfies:*

- $\Omega \in \mathcal{F}$;
- $A \in \mathcal{F} \implies A^c \in \mathcal{F}$;
- $\{A_i | i \in \mathcal{I}\} \subset \mathcal{F}$ *with* $\mathcal{I}$ *being countable* $\implies \bigcup\limits_{i \in \mathcal{I}} A_i \in \mathcal{F}$.

Remark. *We say* $\mathcal{I}$ *is countable if there exists a one-to-one map from* $\mathcal{I}$ *into* $\mathbb{Z}$*, so "countable" includes "finite".*

Example 2.1.1. If Ω is countable, we usually take $\mathcal{F} = 2^\Omega$, which is the power set of Ω.

Example 2.1.2. When Ω is not countable, e.g. $[0, 1]$, if you allow Axiom of Choice, then there exist unmeasurable subsets, and