Stochastic Modelling and Random Processes
Problem Sheet 2

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Contents

1 Notations

- All row vectors are represented in a bold font, such as $\boldsymbol{\pi}$, and somtimes, it is also written as $\langle\boldsymbol{\pi}|$ according to the bra–ket notation.
- All column vectors are represented with an arrow above, such as $\vec{0}$ which is the vector whose elements are all 0.
- Uppercase letters usually represent a matrix, such as G.
- A particular notation, $\boldsymbol{\pi}_t(i)$, means the i-th component of the row vector $\boldsymbol{\pi}_t$.
- $\Delta \in \mathbb{R}^n$ is the region $\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \vec{1} = 1,\, \boldsymbol{x}_i \geq 0,\, i = 1, \cdots, n\}$.
- $\boldsymbol{e}_i$ is the row vector whose i-th element is 1, and all other elements are 0. For example,

$$\boldsymbol{e}_1 = [1, 0, \cdots, 0].$$

  (The dimension depends on the context.)

2 Kingman's Coalescent

Consider a system of L well mixed, coalescing particles. Each of the $\binom{L}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of L individuals in a population model, tracing back to a single common ancestor.

(a) Let N_t be the number of particles at time t with $N_0 = L$. Give the transition rates of the process $(N_t : t \geq 0)$ on the state space $\{1, \cdots, L\}$. Write down the generator $(\mathcal{L}f)(n)$ for $n \in \{1, \cdots, L\}$ and the master equation. Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

Sol. It's easy to see $G_{i,i-1} = \binom{i}{2} \times 1 = \binom{i}{2}$, for $i \in \{2, \cdots, L\}$. Since $\sum_{j=1}^{L} G_{i,j} = 0$ and $G_{i,j} = 0$ for $j \notin \{i-1, i\}$, we know $G_{i,i} = -\binom{i}{2}$. To summarize,

$$G_{i,j} = \begin{cases} \binom{i}{2} & j = i-1,\ i \in \{2, \cdots, L\} \\ -\binom{i}{2} & j = i,\ i \in \{2, \cdots, L\} \\ 0 & \text{Otherwise} \end{cases}. \tag{1}$$

By $\mathcal{L}(f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (1), we know

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = 1 \\ \binom{n}{2}[f(n-1) - f(n)] & n \in \{2, \cdots, L\} \end{cases}. \tag{2}$$

The master equation is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\pi}_t(x) = \sum_{y \neq x} \boldsymbol{\pi}_t(y) G_{y,x} - \sum_{y \neq x} \boldsymbol{\pi}_t(x) G_{x,y}$. Use (1) again, and we get

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\pi}_t(1)}{\mathrm{d}t} = \boldsymbol{\pi}_t(2) \\ \frac{\mathrm{d}\boldsymbol{\pi}_t(i)}{\mathrm{d}t} = \binom{i+1}{2}\boldsymbol{\pi}_t(i+1) - \binom{i}{2}\boldsymbol{\pi}_t(i),\ i = 2, \cdots, L-1 \\ \frac{\mathrm{d}\boldsymbol{\pi}_t(L)}{\mathrm{d}t} = -\binom{L}{2}\boldsymbol{\pi}_L \end{cases}. \tag{3}$$

Obviously, state 1 is absorbing, and furthermore, it forms an absorbing component $\{1\}$. Thus, the process is SP-ergodic.

To find all stationary distributions, we need to solve $\boldsymbol{\pi} G = \vec{0}$ with $\boldsymbol{\pi}_t \in \Delta$, which is equivalent to

$$\begin{cases} 0\boldsymbol{\pi}_t(1) + \binom{2}{2}\boldsymbol{\pi}_t(2) = 0 \\ -\binom{i}{2}\boldsymbol{\pi}_t(i) + \binom{i+1}{2}\boldsymbol{\pi}_t(i+1) = 0,\ i = 2, \cdots, L-1 \\ -\binom{L}{2}\boldsymbol{\pi}_t(L) = 0 \end{cases}. \tag{4}$$

and

$$\sum_{i=1}^{L} \boldsymbol{\pi}_t(i) = 1. \tag{5}$$

Using backward substitution performed on (4) results in $\boldsymbol{\pi}_t(i) = 0$, for $i = 2, \cdots, L$. And using (5), we have $\boldsymbol{\pi}_t(1) = 1$. So the only stationary distribution is

$$\boldsymbol{\pi}_t = [\ 1,\ \underbrace{0, \cdots, 0}_{(L-1)'s\,0}\ ].$$

(b) Show that the mean time to absorption is given by $\mathbb{E}(T) = 2\left(1 - \frac{1}{L}\right)$.

Sol. Let W_i be the holding time for the process to leave state i, i.e. $W_i = \inf\{t \in \mathbb{R}_+ | N_t \neq i,\ N_0 = i\}$, for $i = L, L-1, \cdots, 2$. Notice that

$$T = \sum_{i=2}^{L} W_i \tag{6}$$

since the process can only go from state i to $i-1$ at a time.

From lecture, we know $W_i \sim \text{Exponential}(|G_{i,i}|)$. So

$$\mathbb{E}[W_i] = \frac{1}{|G_{i,i}|} = \frac{1}{\binom{i}{2}} = \frac{(i-2)! \times 2!}{i!} = \frac{2}{i(i-1)}. \tag{7}$$

Thus, using (6) together with the linearity of expectation and (7), we get

$$\begin{aligned}
\mathbb{E}(T) =& \mathbb{E}\left(\sum_{i=2}^{L} W_i\right) \\
=& \sum_{i=2}^{L} \mathbb{E}(W_i) \\
=& \sum_{i=2}^{L} \left(\frac{2}{i(i-1)}\right) \\
=& \sum_{i=2}^{L} 2\left(\frac{1}{i-1} - \frac{1}{i}\right) \\
=& 2\left(1 - \frac{1}{L}\right).
\end{aligned}$$

(c) Write the generator of the rescaled process $(N_t/L : t \geq 0)$ and Taylor expand it up to the second order. Show that the slowed-down, rescaled process $(X_t^L : t \geq 0)$ where

$$X_t^L := \frac{1}{L} N_{\frac{t}{L}},$$

converges to the process $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}} f(x) = -\frac{x^2}{2} f'(x)$$

and state space $(0, 1]$ with $X_0 = 1$.

Convince yourself that this process is "deterministic", i.e. $X_t = \mathbb{E}(X_t)$ for all $t \geq 0$, and compute X_t explicitly. How is your result compatible with the result from (b)?

Sol. For the rescaled process $(N_t/L : t \geq 0)$, the rates of transitions are not changed, but the state space is replaced with $\{\frac{1}{L}, \cdots, \frac{L}{L}\}$. Thus, the rate matrix is

$$G_{i,j} = \begin{cases} \binom{iL}{2} & j = i - \frac{1}{L},\ i \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \\ -\binom{iL}{2} & j = i,\ i \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \\ 0 & \text{Otherwise} \end{cases} \tag{8}$$

By $(\mathcal{L}f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (8), we have

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \binom{nL}{2}[f(n - \frac{1}{L}) - f(n)] & n \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \end{cases}. \tag{9}$$

Now Taylor expand (9) to the sencond order, which results in

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \binom{nL}{2}[-\frac{1}{L} f'(n) + \frac{1}{2L^2} f''(n) + o(\frac{1}{L^2})] & n \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \end{cases}.$$

We need to derive the generator of the process $(X_t^L : t \geq 0)$ first. The rate matrix of $(X_t^L : t \geq 0)$ is

$$G_{i,j} = \begin{cases} \frac{1}{L}\binom{iL}{2} & j = i - \frac{1}{L},\, i \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \\ -\frac{1}{L}\binom{iL}{2} & j = i,\, i \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \\ 0 & \text{Otherwise} \end{cases}. \tag{10}$$

By $(\mathcal{L}f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (10), we have

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \frac{1}{L}\binom{nL}{2}[f(n - \frac{1}{L}) - f(n)] & n \in \{\frac{2}{L}, \cdots, \frac{L}{L}\} \end{cases}. \tag{11}$$

Notice that

$$\begin{aligned} & \frac{1}{L}\binom{nL}{2}\left[f(n - \frac{1}{L}) - f(n)\right] \\ = \; & \frac{1}{L} \cdot \frac{(nL)!}{(nL-2)!2!} \cdot \left[f(n) - \frac{1}{L}f'(n) + \frac{1}{2L^2}f''(n) + o(\frac{1}{L^2}) - f(n)\right] \\ = \; & \frac{1}{L} \cdot \frac{(nL)(nL-1)}{2} \cdot \left[-\frac{1}{L}f'(n) + \frac{1}{2L^2}f''(n) + o(\frac{1}{L^2})\right] \\ = \; & \frac{n(nL-1)}{2} \cdot \left[-\frac{1}{L}f'(n) + \frac{1}{2L^2}f''(n) + o(\frac{1}{L^2})\right] \\ = \; & \frac{n}{2}\left[-\frac{nL-1}{L}f'(n) + \frac{nL-1}{2L^2}f''(n) + o(\frac{nL-1}{L^2})\right], \end{aligned}$$

so as $L \to \infty$, we have

$$\lim_{L \to \infty} \frac{1}{L}\binom{nL}{2}\left[f(n - \frac{1}{L}) - f(n)\right] = -\frac{n^2}{2}f'(n).$$

In conclusion, we have, as $L \to \infty$, the process $(X_t^L : t \geq 0)$ converges to $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}}(f)(x) = -\frac{x^2}{2}f'(x) \tag{12}$$

and state space $(0, 1]$ with $X_0 = 1$.

(d) Generate sample paths of the process $(X_t^L : t \geq 0)$ for $L = 10$, $L = 100$, $L = 1000$ and compare to the solution X_t from (c) in a single plot.

Sol.

```python
import numpy as np
import matplotlib.pyplot as plt

Ls = [10, 100, 1000]
colors = ["lightcoral", 'orange', 'cyan']
for i in range(0, len(Ls)):
    L = Ls[i]
    color = colors[i]
    time = 0
    WT = 0

    for n in range(L, 1, -1):
        waitTime = np.random.exponential(scale=2/(n*(n-1)))
```

```
        plt.plot([time/L, (time+waitTime)/L], [n/L, n/L], color=color, lw=2)
        time += waitTime
        WT = waitTime

    plt.plot([time/L, (time+2*WT)/L], [1/L, 1/L], color=color, label=r"$L = " +
    str(L) + "$")

plt.title("Kingman's Coalescent")
plt.legend()
plt.xlabel('$t$')
plt.ylabel('$N_t$')
plt.yscale('linear')
plt.xscale('log')
plt.savefig("Kingman_Coalescent.png")
plt.show()
```

The output image is Figure 1.

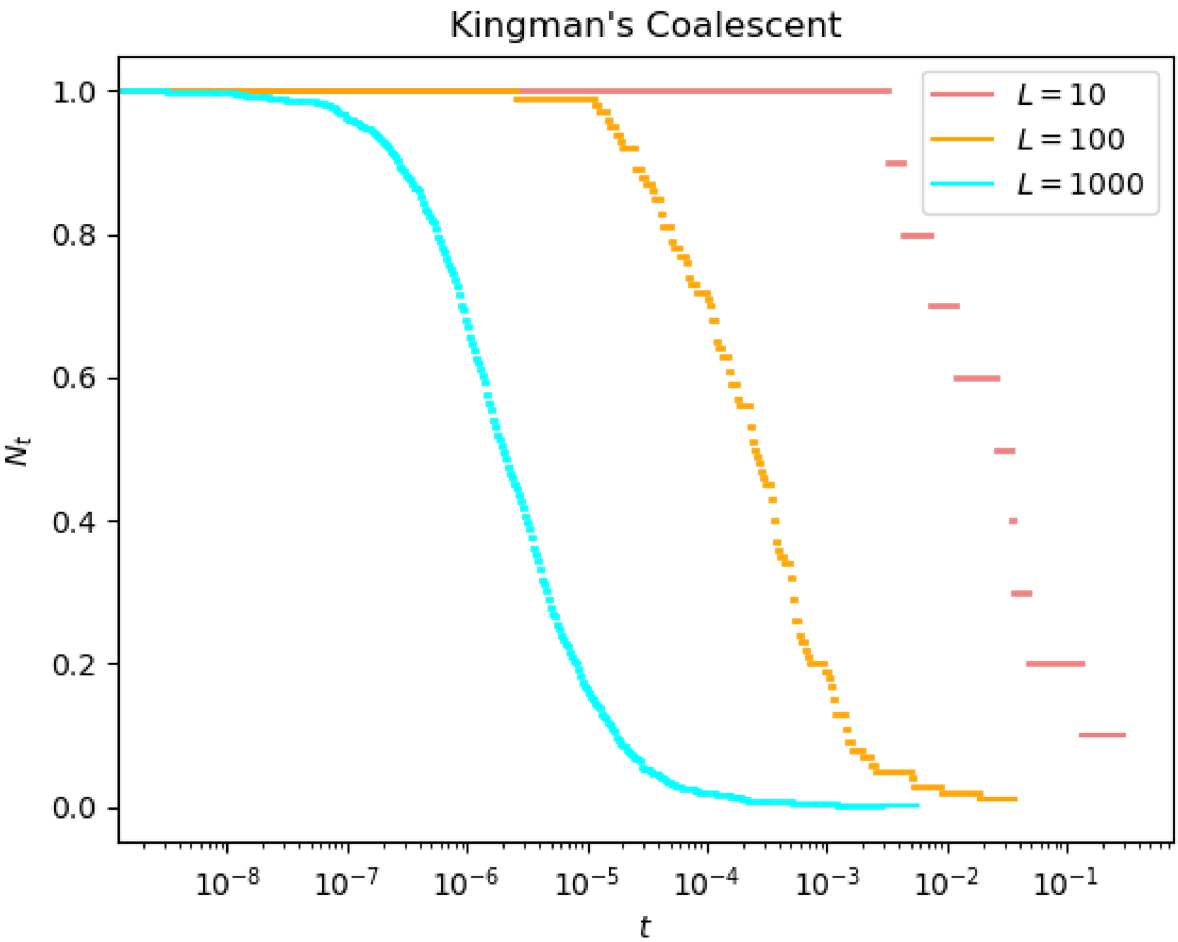

Figure 1: Kingman's Coalescent with $L = 10$, $L = 100$, $L = 1000$

3 Ornstein-Uhlenbeck Processes

The Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ is a diffusion process on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = -\alpha x f'(x) + \frac{1}{2}\sigma^2 f''(x)$$

with α, $\sigma^2 > 0$, and we consider a fixed initial condition $X_0 = x_0$.

(a) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t)] = \mathbb{E}[\mathcal{L}f(X_t)], \tag{13}$$

to derive ODEs for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - m(t)^2$, and solve them.

Sol. Set $f(x) = x$ in the evolution equation (13), then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X_t] = \mathbb{E}[\mathcal{L}f(X_t)] = \mathbb{E}[-\alpha X_t] = -\alpha\mathbb{E}[X_t],$$

which is

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = -\alpha m(t) \tag{14}$$

The general solution to (14) is

$$m(t) = C \cdot e^{-\alpha t},$$

where $C \in \mathbb{R}$ is a constant. By $m(0) = \mathbb{E}[X_0] = \mathbb{E}[x_0] = x_0$, we know $C = x_0$. Thus, the solution to (14) is

$$m(t) = x_0 \cdot e^{-\alpha t}. \tag{15}$$

Setting $f(x) = x^2$ in (13) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X_t^2] =& \mathbb{E}[-\alpha X_t \cdot 2X_t + \frac{1}{2}\sigma^2 \cdot 2] \\ =& \mathbb{E}[-2\alpha X_t^2 + \sigma^2] \\ =& -2\alpha\mathbb{E}[X_t^2] + \sigma^2\end{aligned} \tag{16}$$

To solve (16), we need to find the general solution $h(t)$ of its homogeneous version and a pariticular solution $p(t)$ of it separatively. So $h(t)$ satisfies

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = -2\alpha h(t).$$

Using the method of separation of variables again, we know $h(t) = C_1 \cdot e^{-2\alpha t}$, where $C_1 \in \mathbb{R}$ is a constant.

Now suppose $p(t) = e^{-2\alpha t} + C_2$ with $C_2 \in \mathbb{R}$ being a constant. Then we have

$$\begin{aligned}\frac{\mathrm{d}p(t)}{\mathrm{d}t} =& -2\alpha p(t) + \sigma^2 \\ -2\alpha e^{-2\alpha t} =& -2\alpha(e^{-2\alpha t} + C_2) + \sigma^2 \\ 2\alpha C_2 =& \sigma^2 \\ C_2 =& \frac{\sigma^2}{2\alpha}.\end{aligned}$$

So a particular solution $p(t)$ is $p(t) = e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$.

Thus, the general solution of (16) is $\mathbb{E}[X_t^2] = C_1 \cdot e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$. By $\mathbb{E}[X_0^2] = \mathbb{E}[x_0^2] = x_0^2$, we have $C_1 = x_0^2 - \frac{\sigma^2}{2\alpha}$. So the second central moment of X_t is

$$\mathbb{E}[X_t^2] = \left(x_0^2 - \frac{\sigma^2}{2\alpha}\right) e^{-2\alpha t} + \frac{\sigma^2}{2\alpha},$$

and the variance of X_t is

$$\begin{aligned} v(t) =& \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 \\ =& \left(x_0^2 - \frac{\sigma^2}{2\alpha}\right) e^{-2\alpha t} + \frac{\sigma^2}{2\alpha} - \left(x_0 \cdot e^{-\alpha t}\right)^2 \\ =& \frac{\sigma^2}{2\alpha} \cdot \left(1 - e^{-2\alpha t}\right). \end{aligned}$$

(b) Using the fact that $(X_t : t \geq 0)$ is a Gaussian process, give the distribution of X_t for all $t \geq 0$. What is the stationary distribution of the process?

Sol. In (a), we have solved $\mathbb{E}[X_t]$ and $\mathbb{VAR}[X_t]$:

$$\mathbb{E}[X_t] = m(t) = x_0 \cdot e^{-\alpha t} \quad \text{and} \quad \mathbb{VAR}[X_t] = v(t) = \frac{\sigma^2}{2\alpha} \cdot \left(1 - e^{-2\alpha t}\right).$$

Since the process $(X_t : t \geq 0)$ is a Gaussian process, we know X_t follows a Gaussian distribution. Thus,

$$X_t \sim \mathcal{N}\left(x_0 \cdot e^{-\alpha t}, \frac{\sigma^2}{2\alpha} \cdot \left(1 - e^{-2\alpha t}\right)\right).$$

From lecture, we know the stationary density of the diffusion process with time-independent $a(y) \in \mathbb{R}$ and $\sigma^2(y) > 0$ has the unnormalized stationary density

$$p(x) = \exp\left(\int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)} \,\mathrm{d}y\right).$$

Since the Ornstein-Uhlenbeck process is a special case of the diffusion process with $a(y) = -\alpha y$ and $\sigma^2(y) = \sigma^2$, we know the unnormalized stationary density of the Ornstein-Uhlenbeck process is

$$\begin{aligned} p(x) =& \exp\left(\int_0^x \frac{-2\alpha y - 0}{\sigma^2} \,\mathrm{d}y\right) \\ =& \exp\left(-\frac{\alpha x^2}{\sigma^2}\right) \\ =& \exp\left(-\frac{(x-0)^2}{2 \cdot \frac{\sigma^2}{2\alpha}}\right). \end{aligned}$$

So the stationary distribution of the process is $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$.

(c) For $\alpha = 1$, $\sigma^2 = 1$ and $x_0 = 5$, simulate and plot a sample path of the process up to time $t = 10$, by numerically integrating the SDE with time steps $\Delta t = 0.1$ and $\Delta t = 0.01$.

Sol.

```python
import numpy as np
import matplotlib.pyplot as plt
import sdeint

np.random.seed(1234)

alpha = 1
```

```
sigma = 1
x_0 = 5
t_max = 10

dts = [0.1, 0.01]
colors = ["skyblue", "violet"]


def f(x, t):
    return -alpha*x


def g(x, t):
    return sigma*np.sin(t)


plt.figure(figsize=(20, 8))

for i in range(0, len(dts)):
    dt = dts[i]
    color = colors[i]

    times = np.arange(0, t_max, dt)
    result = sdeint.itoint(f, g, x_0, times)

    label = r"$\Delta t = " + str(dt) + "$"
    plt.plot(times, result, color=color, label=label)


plt.legend()
plt.xlabel(r'$t$')
plt.ylabel(r'$X_t$')
plt.title(r'Ornstein-Uhlenbeck process with $\alpha$ = {}, $\sigma$ = {}.'.
    format(alpha, sigma))
plt.savefig("Ornstein_Uhlenbeck.png")
plt.show()
```

The output image is Figure 2.

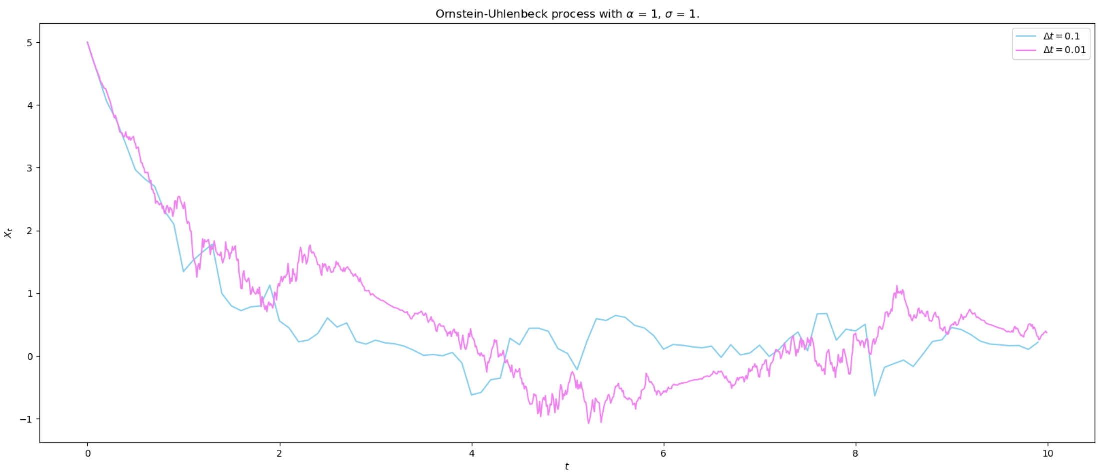

Figure 2: Ornstein-Uhlenbeck Process with $\alpha = 1$, $\sigma^2 = 1$, $x_0 = 5$

4 Moran Model and Wright-Fisher Diffusion

Consider a fixed population of L individuals. At time $t = 0$, each individuum i has a different type $X_0(i)$, and for simplicity, we simply put $X_0(i) = i$. In continuous time, each individuum indepently, with rate 1, imposes its type on another randomly chosen individuum (or equivalently, kills it and puts its own offspring in its place).

(a) Give the state space of the Markov chain $(X_t : t \geq 0)$. Is it irreducible? What are the stationary distributions?

Sol. Since $X_t(i)$ $(i \in \{1, \cdots, L\})$ is the type of the i-th individual at time t and there are L types in total, the state space is $\{1, \cdots, L\}$.

The process is not irreducible. Suppose at time t_0, individual 1 with type 1 dies and individual L with type L reproduces to substitute, i.e.

$$t_0 = \inf\{t > 0 : \sum_{i=1}^{L} \delta_{X_t(i),1} \neq 1\},$$

such that $X_{t_0}(1) = X_{t_0}(L) = L$. Since there are no individuals with tpye 1 any more, $P_t(1, y) = 0$ for any $t \geq t_0$ and $y \in \{2, \cdots, L\}$.

Stationary distributions mean once these distributions are entered, the process will stay in them forever. So the stationary distributions of the process are $\boldsymbol{e}_i$, with $i = 1, \cdots, L$.

(b) Let $N_t = \sum_{i=1}^{L} \delta_{X_t(i),k}$ be the number of individuals of a given type $k \in \{1, \cdots, L\}$ at time t, with $N_0 = 1$.

- Is $(N_t : t \geq 0)$ a Markov process? Given the state space and the generator.

  Sol. The process is obviously Markov, as the distribution of $N_{t_{n+1}}$ given N_{t_n}, $N_{t_{n-1}}$, $\cdots$, N_{t_0} only depends on N_{t_n}.

  The state space of $(N_t : t \geq 0)$ is $\{0, 1, \cdots, L\}$.

  Suppose $N_t = i$, where $i \in \{1, \cdots, L-1\}$. Since there are i individuals of type k and each of them reproduces at rate 1, the total rate of reproduction of individuals of type k is just i. Also, we need to select 1 out of the $L - i$ individuals with other types to be replaced, and this gives the probability $\frac{L-i}{L}$. So

  $$G_{i,i+1} = \frac{i(L-i)}{L}.$$

  Similarly, we also have

  $$G_{i,i-1} = \frac{i(L-i)}{L}.$$

  Since $\sum_{j=1}^{L+1} G_{i,j} = 1$, we know

  $$G_{i,i} = -\frac{2i(L-i)}{L}.$$

  Since state 0 and state L are absorbing, $G_{0,i} = G_{L,i} = 0$, for any $i \in \{0, \cdots, L\}$.

  Hence, the rate matrix G (indices start from 0 and end at L), whose (i, j) element represents the transition rate from state $i - 1$ into state $j - 1$, is given by

  $$G_{i,j} = \begin{cases} \frac{i(L-i)}{L} & j = i-1,\ i \in \{1, \cdots, L-1\} \\ -\frac{2i(L-i)}{L} & j = i,\ i \in \{1, \cdots, L-1\} \\ \frac{i(L-i)}{L} & j = i+1,\ i \in \{1, \cdots, L-1\} \\ 0 & \text{Otherwise} \end{cases}.$$

- Is the process irreducible? What are the stationary distributions?

  Sol. The process is not irreducible, since state 0 and state L are absorbing while others are not. Following the same argument in (a), the stationary distributions are $\boldsymbol{e}_0$ and $\boldsymbol{e}_L$.

- What is the limiting distribution as $t \to \infty$ for the initial condition $N_0 = 1$?

  Sol. As $t \to \infty$, all types have the equal possibility to become the only type of the population. So

$$\lim_{t\to\infty} \mathbb{P}[N_t = L] = \frac{1}{L}$$

  and

$$\lim_{t\to\infty} \mathbb{P}[N_t = 0] = \frac{L-1}{L}.$$

(c) From now consider general initial conditions $N_0 = n \in \{0, \cdots, L\}$.

- Compute $m_1(t) = \mathbb{E}[N_t]$ for all $t \geq 0$.
- Compute $m_2(t) = \mathbb{E}[N_t^2]$. What happens in the limit $t \to \infty$?
- Compute the absorption probabilities as a function of the initial condition n.

Sol. To solve $m_1(t)$ and $m_2(t)$, we solve $\mathbb{E}[f(N_t)]$ where $f : S := \{0, \cdots, L\} \mapsto \mathbb{R}$ first. Obviously, the following two statements hold.

$$\mathbb{E}[f(N_t)] = f(0), \quad \text{when } N_0 = 0. \tag{17}$$

$$\mathbb{E}[f(N_t)] = f(L), \quad \text{when } N_0 = L. \tag{18}$$

Now suppose $N_0 = n \in \{2, \cdots, L-1\}$. By

$$\mathbb{E}[f(N_t)] = \sum_{x\in S} \boldsymbol{\pi}_t(x) f(x) = \langle \boldsymbol{\pi}_t | f \rangle$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{\pi}_t | = \langle \boldsymbol{\pi}_t | G,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[f(N_t)] = \frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{\pi}_t | f \rangle = \langle \boldsymbol{\pi}_t | G | f \rangle = \mathbb{E}[(Gf)(N_t)].$$

So

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[f(N_t)] &= \sum_{k\in S} P_t(n,k) \left(\sum_{j\neq k} G_{k,j} [f(j) - f(k)] \right) \\
&= \sum_{k\in S} P_t(n,k) \left(G_{k,k-1}[f(k-1) - f(k)] + G_{k,k+1}[f(k+1) - f(k)] \right) \\
&= \sum_{k\in S} P_t(n,k) \cdot \frac{k(L-k)}{L} \cdot [f(k-1) + f(k+1) - 2f(k)].
\end{aligned} \tag{19}$$

Set $f(x) = x$ in (17), then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[N_t] = \sum_{k\in S} P_t(n,k) \cdot \frac{k(L-k)}{L} \cdot [(k-1) + (k+1) - 2k] = 0,$$

which means $m_1(t) = m_1(0) = n$, for $n \in \{2, \cdots, L-1\}$. Along with (17) and (18), we know $m_1(t) = n$, for all $n \in S$.

Set $f(x) = x^2$ in (17), and we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_2(t) =&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[N_t^2] \\ =&\sum_{k\in S} P_t(n,k)\cdot\frac{k(L-k)}{L}\cdot[(k-1)^2+(k+1)^2-2k^2] \\ =&\sum_{k\in S} P_t(n,k)\cdot\frac{k(L-k)}{L}\cdot 2 \\ =&\frac{2}{L}\left(L\cdot\sum_{k\in S}P_t(n,k)\cdot k-\sum_{k\in S}P_t(n,k)\cdot k^2\right) \\ =&\frac{2}{L}\cdot\left(L\cdot\mathbb{E}[X_t]-\mathbb{E}[X_t^2]\right) \\ =&2\mathbb{E}[X_t]-\frac{2}{L}\mathbb{E}[X_t^2] \\ =&2n-\frac{2}{L}m_2(t) \end{aligned} \tag{20}$$

The general solution to the homogeneous version of (20) is

$$h(t) = C_1 e^{-\frac{2}{L}t},$$

where $C_1 \in \mathbb{R}$ is a constant. Now suppose $p(t) = e^{-\frac{2}{L}t} + C_2$ is a particular solution to (20), then

$$-\frac{2}{L}e^{-\frac{2}{L}t} = \frac{\mathrm{d}p(t)}{\mathrm{d}t} = 2n - \frac{2}{L}p(t) = 2n - \frac{2}{L}\left(e^{-\frac{2}{L}t} + C_2\right),$$

which gives $C_2 = nL$. So the general solution to (20) is

$$m_2(t) = C_1 e^{-\frac{2}{L}t} + nL.$$

By $m_2(0) = \mathbb{E}[X_0^2] = \mathbb{E}[n^2] = n^2$, we know $C_1 = n^2 - nL = n(n-L)$. Together with (17) and (18), we have

$$m_2(t) = n(n-L)e^{-\frac{2}{L}t} + nL. \tag{21}$$

Let $t \to \infty$ in (21), and we have

$$\lim_{t\to\infty} m_2(t) = nL.$$

Let $\tau = \inf\{t \geq 0 : N_t \in \{0, L\}\}$, which is the time when the process enters the either one of two absorption states 0 and L. By $\mathbb{E}[N_t] = m_2(t) = n$, we have

$$n = \mathbb{E}[N_\tau] = 0\cdot\mathbb{P}[N_\tau = 0] + L\cdot\mathbb{P}[N_\tau = L] = L\cdot\mathbb{P}[N_\tau = L],$$

so $\mathbb{P}[N_\tau = L] = \frac{n}{L}$, which is the probability that the process eventually falls in the state L. Thus, the probability that the process eventually fixed in the state 0 is $\frac{L-n}{L}$.

(d) Consider the rescaled process $(M_t^L : t \geq 0)$ where

$$M_t^L = \frac{1}{L}N_{tL^\alpha}$$

on the state space $[0, 1]$. For which value of $\alpha > 0$ does $(M_t^L : t \geq 0)$ have a (non-trivial) scaling limit $(M_t : t \geq 0)$?

Compute the generator of this process and write down the Fokker-Planck equation. (The scaling limit is called **Wright-Fisher diffusion**.)

Sol. The rate matrix of the rescaled process is

$$G_{i,j} = \begin{cases} i(L-i)L^{\alpha-1} & j = i - \frac{1}{L},\ i \in \left\{\frac{1}{L}, \cdots, \frac{L-1}{L}\right\} \\ -2i(L-i)L^{\alpha-1} & j = i,\ i \in \left\{\frac{1}{L}, \cdots, \frac{L-1}{L}\right\} \\ i(L-i)L^{\alpha-1} & j = i + \frac{1}{L},\ i \in \left\{\frac{1}{L}, \cdots, \frac{L-1}{L}\right\} \\ 0 & \text{Otherwise} \end{cases}.$$

By $(\mathcal{L}f)(X_t) = (Gf)(X_t)$, we have

$$(\mathcal{L}f)(x) = \sum_{y \neq x} G_{x,y}[f(y) - f(x)].$$

For $x = 0$ and $x = 1$, $(\mathcal{L}f)(0) = (\mathcal{L}f)(1) = 0$. So now suppose $x \in \left\{\frac{1}{L}, \cdots, \frac{L-1}{L}\right\}$, then we have

$$\begin{aligned}(\mathcal{L}f)(x) =& x(L-x)L^{\alpha-1}\left[f\left(x+\frac{1}{L}\right) + f\left(x-\frac{1}{L}\right) - 2f(x)\right] \\ =& xL^{\alpha}\left[f\left(x+\frac{1}{L}\right) + f\left(x-\frac{1}{L}\right) - 2f(x)\right] \\ & - x^2L^{\alpha-1}\left[f\left(x+\frac{1}{L}\right) + f\left(x-\frac{1}{L}\right) - 2f(x)\right] \end{aligned} \tag{22}$$

Suppose f is smooth enough to be Taylor expanded into the second order, and perform Taylor expansion of terms involving f in (22):

$$\begin{aligned} f\left(x+\frac{1}{L}\right) + f\left(x-\frac{1}{L}\right) - 2f(x) =& f(x) + \frac{1}{L}f'(x) + \frac{1}{2L^2}f''(x) + o(\frac{1}{L^2}) \\ & + f(x) - \frac{1}{L}f'(x) + \frac{1}{2L^2}f''(x) + o(\frac{1}{L^2}) \\ & - 2f(x) \\ =& \frac{1}{L^2}f''(x) + o(\frac{1}{L^2}) \end{aligned} \tag{23}$$

Plug (23) expansion into (22), then we have

$$(\mathcal{L}f)(x) = xL^{\alpha-2}f''(x) - x^2L^{\alpha-3}f''(x) + o(L^{\alpha-2}).$$

So in order that the process has a scaling limit, $\alpha > 0$ should satisfy $\alpha - 2 \leq 0$ and $\alpha - 3 \leq 0$ at the same time, which gives $\alpha \in (0, 2]$. α has to be 2 such that the limiting process is non-trivial, and the corresponding limiting process has the generator

$$(\bar{\mathcal{L}}f)(x) = xf''(x).$$

Now, we are going to derive the Fokker-Planck equation of $(M_t : t \geq 0)$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(M_t)] =& \mathbb{E}[(\bar{\mathcal{L}}f)(M_t)] \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 P_t(x,y)f(y)\,\mathrm{d}y =& \mathbb{E}[X_tf''(M_t)] \\ \int_0^1 \frac{\partial}{\partial t}P_t(x,y)f(y)\,\mathrm{d}y =& \int_0^1 P_t(x,y)yf''(y)\,\mathrm{d}y. \end{aligned} \tag{24}$$

Doing integration by parts on the right-hand side of (24) gives

$$\begin{aligned}\int_0^1 P_t(x,y)yf''(y)\,\mathrm{d}y =& \int_0^1 P_t(x,y)y\,\mathrm{d}f'(y)\\ =& P_t(x,y)yf'(y)|_{y=0}^{y=1} - \int_0^1 f'(y)\frac{\partial}{\partial y}(P_t(x,y)y)\,\mathrm{d}y\\ =& P_t(x,y)yf'(y)|_{y=0}^{y=1} - \int_0^1 f'(y)\left(\frac{\partial P_t(x,y)}{\partial y}y + P_t(x,y)\right)\mathrm{d}y \end{aligned} \tag{25}$$

Assuming $\lim_{t\to\infty} P_t(x,y) = 0$ and t is large enough, the right-hand side of (25) becomes

$$\begin{aligned} & P_t(x,y)yf'(y)|_{y=0}^{y=1} - \int_0^1 f'(y)\left(\frac{\partial P_t(x,y)}{\partial y}y + P_t(x,y)\right)\mathrm{d}y\\ \approx & -\int_0^1 f'(y)\left(\frac{\partial P_t(x,y)}{\partial y}y + P_t(x,y)\right)\mathrm{d}y\\ = & -\int_0^1 \frac{\partial P_t(x,y)}{\partial y}yf'(y)\,\mathrm{d}y - \int_0^1 P_t(x,y)f'(y)\,\mathrm{d}y\\ = & -\int_0^1 \frac{\partial P_t(x,y)}{\partial y}y\,\mathrm{d}f(y) - \int_0^1 P_t(x,y)\,\mathrm{d}f(y)\\ = & -\frac{\partial P_t(x,y)}{\partial y}yf(y)\Big|_{y=0}^{y=1} + \int_0^1 f(y)\left(\frac{\partial^2 P_t(x,y)}{\partial y^2}y - \frac{\partial P_t(x,y)}{\partial y}\right)\mathrm{d}y\\ & -P_t(x,y)f(y)|_{y=0}^{y=1} + \int_0^1 \frac{\partial P_t(x,y)}{\partial y}f(y)\,\mathrm{d}y\\ \approx & -\frac{\partial P_t(x,y)}{\partial y}yf(y)\Big|_{y=0}^{y=1} + \int_0^1 f(y)\left(\frac{\partial^2 P_t(x,y)}{\partial y^2}y - \frac{\partial P_t(x,y)}{\partial y}\right)\mathrm{d}y\\ & + \int_0^1 \frac{\partial P_t(x,y)}{\partial y}f(y)\,\mathrm{d}y\\ = & -\frac{\partial P_t(x,y)}{\partial y}yf(y)\Big|_{y=0}^{y=1} + \int_0^1 f(y)\cdot\frac{\partial^2 P_t(x,y)}{\partial y^2}y\,\mathrm{d}y \end{aligned} \tag{26}$$

Also, assume $\lim_{t\to\infty}\frac{\partial P_t(x,y)}{\partial y} = 0$, then in (26) we have

$$P_t(x,y)yf'(y)|_{y=0}^{y=1} - \int_0^1 f'(y)\left(\frac{\partial P_t(x,y)}{\partial y}y + P_t(x,y)\right)\mathrm{d}y \approx \int_0^1 f(y)\cdot\frac{\partial^2 P_t(x,y)}{\partial y^2}y\,\mathrm{d}y$$

Combined with the (24), we have

$$\int_0^1 \frac{\partial}{\partial t}P_t(x,y)f(y)\,\mathrm{d}y \approx \int_0^1 f(y)\cdot\frac{\partial^2 P_t(x,y)}{\partial y^2}y\,\mathrm{d}y.$$

So the Fokker-Planck equation is

$$\frac{\partial}{\partial t}P_t(x,y) = \frac{\partial^2}{\partial t^2}P_t(x,y). \tag{27}$$

(e) For the limit process $(M_t : t \geq 0)$ in (d) compute $m(t) = \mathbb{E}[M_t]$ and $v(t) = \mathbb{E}[M_t^2] - m(t)^2$. Is it a Gaussian process?

Sol. Recall $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t)] = \mathbb{E}[\bar{\mathcal{L}}f(X_t)] = \mathbb{E}[X_t f''(X_t)]$. Let $f(x) = x$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X_t] = 0.$$

So $m(t) = \mathbb{E}[M_t] = \mathbb{E}[M_0] = m(0)$.

Let $f(x) = x^2$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[M_t^2] = \mathbb{E}[2M_t] = 2\mathbb{E}[M_t] = 2m(t) = 2m(0).$$

So $\mathbb{E}[M_t^2] = 2m(0)t + \mathbb{E}[M_0^2]$, and as a result, $v(t) = \mathbb{E}[M_t^2] - (\mathbb{E}[M_t])^2 = 2m(0)t + \mathbb{E}[M_0^2] - (\mathbb{E}[M_0])^2 = 2m(0)t + \mathbb{VAR}[M_0]$.

5 Birth-Death Process

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \cdots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \text{ for all } x \in S,\ x \xrightarrow{\beta_x} x-1 \text{ for all } x \geq 1.$$

According to the article [KM57] of Samuel Karlin and James McGregor, a sufficient and necessary condition for the states of a birth-death process being recurrent is

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \infty, \tag{28}$$

sufficient and necessary conditions for them being null recurrent is

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha_{n-1}}{\beta_n} = \infty, \tag{29}$$

and sufficient and necessary conditions for them being ergodic is

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha_{n-1}}{\beta_n} < \infty, \tag{30}$$

(a) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$. Consider different cases depending on the choice of α and β where necessary:

- Is $(X_t : t \geq 0)$ irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

  Sol. Since all states are accessible from each other, with positive rates α and β, all states in S are communicating states. Thus, the process is irreducible.

  $\forall i \geq 1$, $\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \prod_{n=1}^{i}\frac{\beta}{\alpha} = \left(\frac{\beta}{\alpha}\right)^i$, so

  $$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \sum_{i=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^i.$$

  According to (28), these states are recurrent if and only if

  $$\sum_{i=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^i = \infty,$$

which is equivalent to

$$\frac{\beta}{\alpha} \geq 1.$$

Thus, the states are transient if and only if

$$0 < \frac{\beta}{\alpha} < 1.$$

And similarly, by (29), the states are null recurrent if and only if

$$\frac{\beta}{\alpha} = 1,$$

and they are positive recurrent if and only if

$$\frac{\beta}{\alpha} > 1.$$

- Give all stationary distributions and state whether they are reversible.

  Sol. We can use the $\langle \boldsymbol{\pi} | G = \langle \mathbf{0} |$ to solve stationary distributions. Notice that

$$G = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & 0 & \cdots \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 & \cdots \\ 0 & \beta & -(\alpha+\beta) & \alpha & 0 & \cdots \\ 0 & 0 & \beta & -(\alpha+\beta) & \alpha & \cdots \\ 0 & 0 & 0 & \beta & -(\alpha+\beta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so

$$-\alpha \boldsymbol{\pi}_0 + \beta \boldsymbol{\pi}_1 = 0 \tag{31}$$
$$\alpha \boldsymbol{\pi}_0 - (\alpha+\beta)\boldsymbol{\pi}_1 + \beta \boldsymbol{\pi}_2 = 0 \tag{32}$$
$$\alpha \boldsymbol{\pi}_1 - (\alpha+\beta)\boldsymbol{\pi}_2 + \beta \boldsymbol{\pi}_3 = 0 \tag{33}$$
$$\vdots$$

To solve these equations, notice that (31) gives

$$\beta \boldsymbol{\pi}_1 = \alpha \boldsymbol{\pi}_0. \tag{34}$$

Plug (34) into (32), and we get

$$\beta \boldsymbol{\pi}_2 = \alpha \boldsymbol{\pi}_1. \tag{35}$$

And pluging (35) into (33) results in

$$\beta \boldsymbol{\pi}_3 = \alpha \boldsymbol{\pi}_2. \tag{36}$$

So we can keep doing this recursively and get

$$\beta \boldsymbol{\pi}_{n+1} = \alpha \boldsymbol{\pi}_n, \quad \forall n \in \mathbb{N}_0,$$

or equivalently,

$$\boldsymbol{\pi}_n = \left(\frac{\alpha}{\beta}\right)^n \boldsymbol{\pi}_0.$$

If $\frac{\alpha}{\beta} \geq 1$, there is no stationary distribution, because

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_n = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^n \boldsymbol{\pi}_0 \neq 1.$$

If $\frac{\alpha}{\beta} < 1$, then solving $\sum_{n=0}^{\infty} \boldsymbol{\pi}_n = 1$ gives

$$\boldsymbol{\pi}_0 = \frac{\beta - \alpha}{\beta},$$

so

$$\boldsymbol{\pi}_n = \left(\frac{\alpha}{\beta}\right)^n \cdot \frac{\beta - \alpha}{\beta}.$$

Since $G_{x,y}\boldsymbol{\pi}_x = G_{y,x}\boldsymbol{\pi}_y$ for all $x, y \in S$, the process is time reversible.

- Is the process ergodic?

  Sol. To check the ergodicity of the process, we only need to verify (30).

$$\begin{aligned} \infty &= \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\beta_n}{\alpha_n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\beta}{\alpha} \\ &= \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i, \end{aligned}$$

which holds only when

$$\frac{\beta}{\alpha} \geq 1. \tag{37}$$

$$\begin{aligned} \infty &> \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\alpha_{n-1}}{\beta_n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\alpha}{\beta} \\ &= \sum_{i=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^i \end{aligned}$$

which can be true only when

$$\frac{\alpha}{\beta} < 1$$

Based on (37) and (5), the process is ergodic if and only if

$$\frac{\beta}{\alpha} > 1.$$

(b) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$. Consider different cases depending on the choice of α and β where necessary:

- Is $(X_t : t \geq 0)$ irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

  Sol. Notice that $\alpha_0 = 0$ and $\beta_0 = \beta > 0$, so state 0 is absorbing, and thus positive recurrent. For other states, they are communicating. So state space can be decomposed into $\{0\}$ and $\mathbb{N}_+$, which means the process is not irreducible.

  Since

$$\begin{aligned}\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} &= \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta n}{\alpha n}\\ &= \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta}{\alpha}\\ &= \sum_{i=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^i,\end{aligned}$$

  by (28), states $1, 2, \cdots$ are transient if and only if $\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} < \infty$, which is equivalent to $\frac{\beta}{\alpha} < 1$.

  Now suppoes $\frac{\beta}{\alpha} \geq 1$ so that states $1, 2, \cdots$ are recurrent. Since

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha_{n-1}}{\beta_n} = \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha(n-1)}{\beta n} = 0 < \infty,$$

  by (29), states $1, 2, \cdots$ positive recurrent.

- Give all stationary distributions and state whether they are reversible.

  Sol. The transition matrix G is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 & \cdots \\ 0 & 2\beta & -2(\alpha+\beta) & 2\alpha & 0 & \cdots \\ 0 & 0 & 3\beta & -3(\alpha+\beta) & 3\alpha & \cdots \\ 0 & 0 & 0 & 4\beta & -4(\alpha+\beta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  $\langle \boldsymbol{\pi} | G = \langle \mathbf{0} |$ gives

$$\begin{aligned} \beta\boldsymbol{\pi}_1 &= 0 && (38)\\ -(\alpha+\beta)\boldsymbol{\pi}_1 + 2\beta\boldsymbol{\pi}_2 &= 0 && (39)\\ \alpha\boldsymbol{\pi}_1 - 2(\alpha+\beta)\boldsymbol{\pi}_2 + 3\beta\boldsymbol{\pi}_3 &= 0 && (40)\\ 2\alpha\boldsymbol{\pi}_2 - 3(\alpha+\beta)\boldsymbol{\pi}_3 + 4\beta\boldsymbol{\pi}_4 &= 0 && (41)\\ &\vdots \end{aligned}$$

  Solve (38) first, and this gives $\boldsymbol{\pi}_1 = 0$. Using "forward-substitution" in (39), (40), (41), ... gives $\boldsymbol{\pi}_2 = \boldsymbol{\pi}_3 = \boldsymbol{\pi}_4 = \cdots = 0$. Thus, the only stationary distribution is $\boldsymbol{e}_1$, which is

$$[1,\ 0,\ 0,\ 0,\ \cdots].$$

  Once being absorbed into state 0, the process cannot go back into other states, so the process is not time reversible.

- Derive an equation for $\mu_t = \mathbb{E}[X_t]$ and solve it for initial condition $\mu_0 = 1$.

  Sol. For any function $f : S \mapsto \mathbb{R}$,

$$
\begin{aligned}
(\mathcal{L}f)(x) =& G|f\rangle(x) \\
=& \sum_{y \neq x} G_{x,y}[f(y) - f(x)] \\
=& x\beta[f(x-1) - f(x)] + x\alpha[f(x+1) - f(x)]
\end{aligned}
$$

  So

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t)] =& \mathbb{E}[(\mathcal{L}f)(X_t)] \\
=& \mathbb{E}[X_t\beta[f(X_t - 1) - f(X_t)] + X_t\alpha[f(X_t + 1) - f(X_t)]].
\end{aligned}
$$

  Let $f(x) = x$, then we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X_t] =& \mathbb{E}[X_t\beta(X_t - 1 - X_t) + X_t\alpha(X_t + 1 - X_t)] \\
=& \mathbb{E}[-\beta X_t + \alpha X_t] \\
=& (\alpha - \beta)\mathbb{E}[X_t],
\end{aligned}
$$

  whose general solution is give by

$$
\mathbb{E}[X_t] = Ce^{(\alpha-\beta)t}. \tag{42}
$$

  By the intial condition $\mathbb{E}[X_0] = \mu_0 = 1$, we know $C = 1$ in (42). So the solution is

$$
\mu_t = e^{(\alpha-\beta)t}.
$$

- Set up a recursion for the "extinction probability" $h_x = \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.

  Sol. Since we consider the limit of X_t as $t \to \infty$, it is reasonable to assume t is large enough so that jumps can happen. Obviously, we have $h_0 = 0$, so now assume $x > 0$. Let $J_1 = \inf\{t > 0 : X_t \neq X_0\}$ be the time of the first jump, so

$$
\mathbb{P}[X_{J_1} = x + 1 | X_0 = x] = \frac{G_{x,x+1}}{|G_{x,x}|} = \frac{x\alpha}{x(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta},
$$

  and

$$
\mathbb{P}[X_{J_1} = x - 1 | X_0 = x] = \frac{G_{x,x-1}}{|G_{x,x}|} = \frac{x\beta}{x(\alpha+\beta)} = \frac{\beta}{\alpha+\beta}.
$$

  Now condition on X_{J_1}, we have

$$
\begin{aligned}
h_x =& \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x] \\
=& \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x, X_{J_1} = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\
&+ \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x, X_{J_1} = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \\
=& \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_{J_1} = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\
&+ \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_{J_1} = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \quad \text{(by the Markov property)} \\
=& \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\
&+ \mathbb{P}[\lim_{t\to\infty} X_t = 0 | X_0 = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \quad \text{(by homogeneity)} \\
=& h_{x+1} \cdot \frac{\alpha}{\alpha+\beta} + h_{x-1} \cdot \frac{\beta}{\alpha+\beta}
\end{aligned} \tag{43}
$$

The characteristic equation of (43) is

$$\begin{aligned}t^2 \cdot \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} &= t \\ \alpha t^2 - (\alpha+\beta)t + \beta = 0 \\ (\alpha t - \beta)(t-1) = 0,\end{aligned}$$

whose roots are

$$t_1 = \frac{\beta}{\alpha} \quad \& \quad t_2 = 1.$$

If $\alpha = \beta$, then $t_1 = t_2 = 1$, and the general solution to (43) is

$$\begin{aligned}h_n =& a \cdot 1^n + b \cdot n \cdot 1^2 \\ =& a + bn,\end{aligned}$$

where $a, b \in \mathbb{R}$ are two constants. By $h_0 = 1$, we have $a = 1$, so $h_x = 1 + bx$. But h_x is a probability, which means it should range between 0 and 1, so we have $b = 0$. So in the case of $\alpha = \beta$, $h_x \equiv 1$.

Now suppose $\alpha \neq \beta$, so $t_1 \neq t_2$. Then the general solution to (43) is

$$\begin{aligned}h_n =& a \cdot 1^n + b \cdot \left(\frac{\beta}{\alpha}\right)^n \\ =& a + b\left(\frac{\beta}{\alpha}\right)^n,\end{aligned}$$

where $a, b \in \mathbb{R}$ are two constants. Again, use the boundary condition $h_0 = 1$, and we have $a = 1 - b$. Thus,

$$h_x = 1 - b + b\left(\frac{\beta}{\alpha}\right)^x.$$

If $\beta > \alpha$, then $\frac{\beta}{\alpha} > 1$, and as a result, $\lim_{x\to\infty}(\beta/\alpha)^n = \infty$. In order that h_x is a probability, b has to be 0. So in this case $h_x \equiv 1$.

If $\beta < \alpha$, then $\frac{\beta}{\alpha} < 1$, and thus, $(\beta/\alpha)^x le 1$ for all $x \geq 0$. To ensure $h_x \leq 1$, let

$$\begin{aligned}1 - b + b\left(\frac{\beta}{\alpha}\right)^x \leq& 1 \\ \left(\left(\frac{\beta}{\alpha}\right)^x - 1\right) b \leq& 0 \\ b \geq& 0.\end{aligned}$$

In order to make h_x nonnegative, we need

$$\begin{aligned}0 \leq& 1 - b + b\left(\frac{\beta}{\alpha}\right)^x \\ 1 \geq& \left(1 - \left(\frac{\beta}{\alpha}\right)^x\right) b \\ b \leq& \frac{1}{1 - \left(\frac{\beta}{\alpha}\right)^x} \\ b \leq& 1.\end{aligned}$$

So the range of b is $[0,1]$.
To conclude, the solution to (43) is

$$h_x = \begin{cases} 1 & \beta \geq \alpha > 0 \\ 1 - b + b\left(\frac{\beta}{\alpha}\right)^x & 0 < \beta < \alpha \end{cases},$$

where $b \in [0,1]$ is a constant.
Obviously, we have made $1 - b + b\left(\frac{\beta}{\alpha}\right)^x \leq 1$ when we determined the range of b, so the smaller solution is

$$1 - b + b\left(\frac{\beta}{\alpha}\right)^x,$$

with $0 < \beta < \alpha$ and $b \in [0,1]$. Let $H(b) = 1 - b + b\left(\frac{\beta}{\alpha}\right)^x$, then

$$H'(b) = -1 + \left(\frac{\beta}{\alpha}\right)^x \leq 0,$$

which means $H(b)$ is monotonically decreasing in $[0,1]$. So the smallest solution is reached when $b = 1$, and in this circumstance,

$$h_x = \left(\frac{\beta}{\alpha}\right)^x.$$

- Is the process ergodic?
  Sol. By (30), the process is ergodic if and only if

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha_{n-1}}{\beta_n} < \infty.$$

The first condition is

$$\begin{aligned} \infty &= \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta_n}{\alpha_n} \\ &= \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta n}{\alpha n} \\ &= \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\beta}{\alpha} \\ &= \sum_{i=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^i, \end{aligned}$$

which is equivalent to $\beta \geq \alpha$.
As for the second condition, we have

$$\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha_{n-1}}{\beta_n} = \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\alpha(n-1)}{\beta n} = 0 < \infty.$$

So the process is ergodic if and only if $\beta \geq \alpha$.

Reference

[KM57] Samuel Karlin and James McGregor. "The classification of birth and death processes". In: *Transactions of the American Mathematical Society* 86.2 (1957), pp. 366–400.