Stochastic Modelling and Random Processes
Problem Sheet 3

Yiming MA

December 15, 2020

Contents

1 Geometric Brownian Motion

Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x),\ \mu \in \mathbb{R},\ \sigma > 0,$$

and initial condition $X_0 = 0$. **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}.$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator. Write down the associated SDE and Fokker-Planck equation.

Sol. Notice that

$$\begin{aligned}\mathbb{E}[(\mathcal{L}_Y f)(Y_t)] =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(Y_t)] \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(e^{X_t})]. \end{aligned} \tag{1}$$

Let $F = f \circ \exp$, then (1) becomes

$$\begin{aligned}\mathbb{E}[(\mathcal{L}_Y f)(Y_t)] =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(e^{X_t})] \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[F(X_t)] \\ =& \mathbb{E}[(\mathcal{L}_X F)(X_t)]. \end{aligned} \tag{2}$$

Since (2) holds for all $f \in C^1\mathbb{R}$, we know

$$
\begin{aligned}
(\mathcal{L}_Y f)(Y_t) =& (\mathcal{L}_X F)(X_t) \\
=& \mu \frac{\mathrm{d}}{\mathrm{d}x} f(e^{X_t}) + \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(e^{X_t}) \\
=& \mu f'(e^{X_t})e^{X_t} + \frac{1}{2}\sigma^2 \frac{\mathrm{d}}{\mathrm{d}x}(f'(e^{X_t})e^{X_t}) \\
=& \mu f'(e^{X_t})e^{X_t} + \frac{1}{2}\sigma^2 (f''(e^{X_t})e^{2X_t} + f'(e^{X_t})e^{X_t}) \\
=& \mu f'(Y_t)Y_t + \frac{1}{2}\sigma^2 (f''(Y_t)Y_t^2 + f'(Y_t)Y_t) \\
=& (\mu + \frac{1}{2}\sigma^2)Y_t f'(Y_t) + \frac{1}{2}(\sigma Y_t)^2 f''(Y_t),
\end{aligned}
$$

which shows $(Y_t : t \geq 0)$ is a diffusion process with the drift $(\mu + \frac{1}{2}\sigma^2)y$ and the diffusion σy.

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{\partial}{\partial t} p_t(x,y) f(y)\,\mathrm{d}y \\
=& \quad \frac{\partial}{\partial t} \int_{\mathbb{R}} p_t(x,y) f(y)\,\mathrm{d}y \\
=& \quad \frac{\partial}{\partial t} \mathbb{E}[f(Y_t)] \\
=& \quad \mathbb{E}[(\mathcal{L}_Y f)(Y_t)] \\
=& \quad \mathbb{E}\left[(\mu + \frac{1}{2}\sigma^2)Y_t f'(Y_t) + \frac{1}{2}(\sigma Y_t)^2 f''(Y_t)\right] \\
=& \quad \int_{\mathbb{R}} \left[(\mu + \frac{1}{2}\sigma^2) y f'(y) + \frac{1}{2}(\sigma y)^2 f''(y)\right] p_t(x,y)\,\mathrm{d}y \\
=& \quad (\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} y f'(y) p_t(x,y)\,\mathrm{d}y + \frac{1}{2}\sigma^2 \int_{\mathbb{R}} y^2 f''(y) p_t(x,y)\,\mathrm{d}y \\
=& \quad (\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} y p_t(x,y)\,\mathrm{d}f(y) + \frac{1}{2}\sigma^2 \int_{\mathbb{R}} y^2 p_t(x,y)\,\mathrm{d}f'(y) \\
=& \quad (\mu + \frac{1}{2}\sigma^2) \left\{ y p_t(x,y) f(y)\Big|_{y=-\infty}^{y=\infty} - \int_{\mathbb{R}} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \right\} \\
& \quad + \frac{1}{2}\sigma^2 \left\{ y^2 p_t(x,y) f'(y)\Big|_{y=-\infty}^{y=\infty} - \int_{\mathbb{R}} f'(y) \left[2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \right\} \\
=& \quad -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \int_{\mathbb{R}} f'(y) \left[2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \\
=& \quad -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \int_{\mathbb{R}} 2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\,\mathrm{d}f(y) \\
=& \quad -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \left\{ [2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)] f(y) \Big|_{y=-\infty}^{y=\infty} \right. \\
& \quad \left. - \int_{\mathbb{R}} f(y)[2p_t(x,y) + 4y \frac{\partial}{\partial y} p_t(x,y) + y^2 \frac{\partial^2}{\partial y^2} p_t(x,y)]\,\mathrm{d}y \right\} \\
=& \quad \int_{\mathbb{R}} f(y) \left[(\frac{1}{2}\sigma^2 - \mu) p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu) y \frac{\partial}{\partial y} p_t(x,y) + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} p_t(x,y)\right] \mathrm{d}y,
\end{aligned}
$$

so the Fokker-Planck equation is

$$\frac{\partial}{\partial t}p_t(x,y) = (\frac{1}{2}\sigma^2 - \mu)p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu)y\frac{\partial}{\partial y}p_t(x,y) + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2}p_t(x,y).$$

An easier way is to use the conclusion for a diffusion process directly:

$$\begin{aligned}\frac{\partial}{\partial t}p_t(x,y) =& -\frac{\partial}{\partial y}\left[(\mu + \frac{1}{2}\sigma^2)yp_t\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2 y^2 p_t\right) \\ =& -(\mu + \frac{1}{2}\sigma^2)(p_t + y\frac{\partial}{\partial y}p_t(x,y)) + \frac{1}{2}\sigma^2\frac{\partial}{\partial y}\left(2yp_t + y^2\frac{\partial}{\partial y}p_t(x,y)\right) \\ =&(\frac{1}{2}\sigma^2 - \mu)p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu)y\frac{\partial}{\partial y}p_t(x,y) + \frac{1}{2}\sigma^2 y^2\frac{\partial^2}{\partial y^2}p_t(x,y).\end{aligned}$$

The associated SDE is

$$dY_t = (\mu + \frac{1}{2}\sigma^2)Y_t\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}B_t.$$

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(Y_t)] = \mathbb{E}[\mathcal{L}f(Y_t)],$$

to derive ODEs for the meane $m(t) := \mathbb{E}[Y_t]$ and the second moment $m_2(t) := \mathbb{E}[Y_t^2]$. (No need to solve the ODEs.)

Sol.

(c) Under which conditions on μ and σ^2 is $(Y_t : t \geq 0)$ a martingale?

What is the asymptotic behaviour of the variance $v(t) = m_2(t) - m(t)^2$ in that case?

Sol.

(d) Show that δ_0 is the unique stationary distribution of the process on the state space $[0, \infty)$. Under which conditions on μ and σ^2 does the process with $Y_0 = 1$ converge to the stationary distribution?

Under which conditions on μ and σ^2 is the process ergodic? Justify your answer.

Sol.

(e) For $\sigma^2 = 1$ choose $\mu = -1/2$ and two other values $\mu < -1/2$ and $\mu > 1/2$. Simulate and plot a sample path of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

Sol.