Stochastic Modelling and Random Processes
Problem Sheet 3

Yiming MA

December 16, 2020

Contents

1 Geometric Brownian Motion

Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x),\ \mu \in \mathbb{R},\ \sigma > 0,$$

and initial condition $X_0 = 0$. **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}.$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator. Write down the associated SDE and Fokker-Planck equation.

Sol. Notice that

$$\begin{aligned}\mathbb{E}[(\mathcal{L}_Y f)(Y_t)] =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(Y_t)] \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(e^{X_t})]. \end{aligned} \tag{1}$$

Let $F = f \circ \exp$, then (1) becomes

$$\begin{aligned}\mathbb{E}[(\mathcal{L}_Y f)(Y_t)] =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(e^{X_t})] \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[F(X_t)] \\ =& \mathbb{E}[(\mathcal{L}_X F)(X_t)]. \end{aligned} \tag{2}$$

Since (2) holds for all $f \in C^1(\mathbb{R})$, we know

$$
\begin{aligned}
(\mathcal{L}_Y f)(Y_t) =& (\mathcal{L}_X F)(X_t) \\
=& \mu \frac{\mathrm{d}}{\mathrm{d}x} f(e^{X_t}) + \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(e^{X_t}) \\
=& \mu f'(e^{X_t}) e^{X_t} + \frac{1}{2}\sigma^2 \frac{\mathrm{d}}{\mathrm{d}x}(f'(e^{X_t})e^{X_t}) \\
=& \mu f'(e^{X_t}) e^{X_t} + \frac{1}{2}\sigma^2 (f''(e^{X_t})e^{2X_t} + f'(e^{X_t})e^{X_t}) \\
=& \mu f'(Y_t) Y_t + \frac{1}{2}\sigma^2 (f''(Y_t) Y_t^2 + f'(Y_t) Y_t) \\
=& (\mu + \frac{1}{2}\sigma^2) Y_t f'(Y_t) + \frac{1}{2}(\sigma Y_t)^2 f''(Y_t),
\end{aligned} \tag{3}
$$

which shows $(Y_t : t \geq 0)$ is a diffusion process with the drift $(\mu + \frac{1}{2}\sigma^2)y$ and the diffusion σy. To derive the Fokker-Planck equation, notice that

$$
\begin{aligned}
&\int_{\mathbb{R}_+} \frac{\partial}{\partial t} p_t(x,y) f(y)\,\mathrm{d}y \\
=& \frac{\partial}{\partial t} \int_{\mathbb{R}_+} p_t(x,y) f(y)\,\mathrm{d}y \\
=& \frac{\partial}{\partial t} \mathbb{E}[f(Y_t)] \\
=& \mathbb{E}[(\mathcal{L}_Y f)(Y_t)] \\
=& \mathbb{E}\left[(\mu + \frac{1}{2}\sigma^2) Y_t f'(Y_t) + \frac{1}{2}(\sigma Y_t)^2 f''(Y_t)\right] \\
=& \int_{\mathbb{R}_+} \left[(\mu + \frac{1}{2}\sigma^2) y f'(y) + \frac{1}{2}(\sigma y)^2 f''(y)\right] p_t(x,y)\,\mathrm{d}y \\
=& (\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}_+} y f'(y) p_t(x,y)\,\mathrm{d}y + \frac{1}{2}\sigma^2 \int_{\mathbb{R}_+} y^2 f''(y) p_t(x,y)\,\mathrm{d}y \\
=& (\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}_+} y p_t(x,y)\,\mathrm{d}f(y) + \frac{1}{2}\sigma^2 \int_{\mathbb{R}_+} y^2 p_t(x,y)\,\mathrm{d}f'(y) \\
=& (\mu + \frac{1}{2}\sigma^2) \left\{ y p_t(x,y) f(y)\Big|_{y=0}^{y=\infty} - \int_{\mathbb{R}_+} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \right\} \\
&+ \frac{1}{2}\sigma^2 \left\{ y^2 p_t(x,y) f'(y)\Big|_{y=0}^{y=\infty} - \int_{\mathbb{R}_+} f'(y) \left[2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \right\} \\
=& -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}_+} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \int_{\mathbb{R}_+} f'(y) \left[2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y \\
=& -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}_+} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \int_{\mathbb{R}_+} 2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)\,\mathrm{d}f(y) \\
=& -(\mu + \frac{1}{2}\sigma^2) \int_{\mathbb{R}_+} f(y) \left[p_t(x,y) + y \frac{\partial}{\partial y} p_t(x,y)\right] \mathrm{d}y - \frac{1}{2}\sigma^2 \left\{ [2y p_t(x,y) + y^2 \frac{\partial}{\partial y} p_t(x,y)] f(y) \Big|_{y=0}^{y=\infty} \right. \\
&\left. - \int_{\mathbb{R}_+} f(y)[2p_t(x,y) + 4y \frac{\partial}{\partial y} p_t(x,y) + y^2 \frac{\partial^2}{\partial y^2} p_t(x,y)]\,\mathrm{d}y \right\} \\
=& \int_{\mathbb{R}_+} f(y) \left[(\frac{1}{2}\sigma^2 - \mu) p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu) y \frac{\partial}{\partial y} p_t(x,y) + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} p_t(x,y)\right] \mathrm{d}y,
\end{aligned}
$$

so the Fokker-Planck equation is

$$\frac{\partial}{\partial t}p_t(x,y) = (\frac{1}{2}\sigma^2 - \mu)p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu)y\frac{\partial}{\partial y}p_t(x,y) + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2}p_t(x,y).$$

An easier way is to use the conclusion for a diffusion process directly:

$$\begin{aligned}\frac{\partial}{\partial t}p_t(x,y) =& -\frac{\partial}{\partial y}\left[(\mu + \frac{1}{2}\sigma^2)yp_t\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2 y^2 p_t\right)\\ =& -(\mu + \frac{1}{2}\sigma^2)(p_t + y\frac{\partial}{\partial y}p_t(x,y)) + \frac{1}{2}\sigma^2\frac{\partial}{\partial y}\left(2yp_t + y^2\frac{\partial}{\partial y}p_t(x,y)\right)\\ =&(\frac{1}{2}\sigma^2 - \mu)p_t(x,y) + (\frac{3}{2}\sigma^2 - \mu)y\frac{\partial}{\partial y}p_t(x,y) + \frac{1}{2}\sigma^2 y^2\frac{\partial^2}{\partial y^2}p_t(x,y).\end{aligned}$$

The associated SDE is

$$dY_t = (\mu + \frac{1}{2}\sigma^2)Y_t\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}B_t.$$

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(Y_t)] = \mathbb{E}[\mathcal{L}f(Y_t)],$$

to derive ODEs for the meane $m(t) := \mathbb{E}[Y_t]$ and the second moment $m_2(t) := \mathbb{E}[Y_t^2]$. (No need to solve the ODEs.)

Sol. We have calculated the generator of Y_t in (a), which is given by (3). Let f be the identity function, i.e. $f(Y_t) = Y_t$, and plug it and (3) in the evolution to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[Y_t] =&\mathbb{E}[\mathcal{L}f(Y_t)]\\ =&\mathbb{E}[(\mu + \frac{1}{2}\sigma^2)Y_t]\\ =&(\mu + \frac{1}{2}\sigma^2)\mathbb{E}[Y_t].\end{aligned}$$

Thus, $m(t)$ satisfies

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = (\mu + \frac{1}{2}\sigma^2)m(t). \tag{4}$$

Set $f(Y_t) = Y_t^2$, and use it and (3), to derive the ordinary differential equation for $m_2(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[Y_t^2] =&\mathbb{E}[\mathcal{L}f(Y_t)]\\ =&\mathbb{E}[2(\mu + \frac{1}{2}\sigma^2)Y_t^2 + (\sigma Y_t)^2]\\ =&\mathbb{E}[(2\mu + \sigma^2)Y_t^2 + \sigma^2 Y_t^2]\\ =&2(\mu + \sigma^2)\mathbb{E}[Y_t^2].\end{aligned}$$

Therefore, $m_2(t)$ satisfies

$$\frac{\mathrm{d}m_2(t)}{\mathrm{d}t} = 2(\mu + \sigma^2)m_2(t). \tag{5}$$

(c) Under which conditions on μ and σ^2 is $(Y_t : t \geq 0)$ a martingale?

What is the asymptotic behaviour of the variance $v(t) = m_2(t) - m(t)^2$ in that case?

Sol. To make the process $(Y_t : t \geq 0)$ with respect to the process $(X_t : t \geq 0)$, we need to ensure

- $\forall t \geq 0$,
$$m(t) = \mathbb{E}[Y_t] = \mathbb{E}[|Y_t|] < \infty; \tag{6}$$
- $\forall s \leq t$ and $s \geq 0$,
$$\mathbb{E}\left[Y_t \mid \{X_u : 0 \leq u \leq s\}\right] = Y_s. \tag{7}$$

The general solution to (4) is
$$m(t) = Ce^{(\mu+\frac{1}{2}\sigma^2)t},$$
for some constant $C \in \mathbb{R}$, which can be determined by the initial condition
$$C = m(0) = \mathbb{E}[Y_0] = \mathbb{E}[e^{X_0}] = \mathbb{E}[e^0] = 1.$$

Thus, the solution to (4) is
$$m(t) = e^{(\mu+\frac{1}{2}\sigma^2)t}.$$
To make sure condition (6) fullfilled, μ and σ^2 should satisfy
$$\mu + \frac{1}{2}\sigma^2 \leq 0. \tag{8}$$

Now, let us delve into (7).
$$\begin{aligned}\mathbb{E}[Y_t|\{X_u : 0 \leq u \leq s\}] =& \mathbb{E}[e^{X_t}|\{X_u : 0 \leq u \leq s\}]\\ =& \mathbb{E}[e^{X_s+X_t-X_s}|\{X_u : 0 \leq u \leq s\}]\\ =& e^{X_s}\mathbb{E}[e^{X_t-X_s}|\{X_u : 0 \leq u \leq s\}]\\ =& Y_s\mathbb{E}[e^{X_t-X_s}|\{X_u : 0 \leq u \leq s\}],\end{aligned} \tag{9}$$
where $(X_t : t \geq 0)$ is the general Brownian motion with drift μ and noise σ. Since it has stationary and independent increments, we know $X_t - X_s|\{X_u : 0 \leq u \leq s\}$ and $X_{t-s}|X_0$ has the same distribution, which is $\mathcal{N}(\mu(t-s), \sigma^2(t-s))$. Notice that the moment generation function $g_X(t)$ of a normal distribution $\mathcal{N}(\mu, \sigma^2)$ is
$$g_X(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2},$$
so
$$\mathbb{E}[e^{\mathcal{N}}] = g_X(1) = e^{\mu+\frac{1}{2}\sigma^2}.$$
Thus, in (9),
$$\mathbb{E}[e^{X_t-X_s}|\{X_u : 0 \leq u \leq s\}] = \mathbb{E}[e^{X_{t-s}}|X_0] = e^{\mu(t-s)+\frac{1}{2}\sigma^2(t-s)} = e^{(\mu+\frac{1}{2}\sigma^2)(t-s)},$$
which means to satisfy (7), we need $\mathbb{E}[e^{X_t-X_s}|\{X_u : 0 \leq u \leq s\}] = 1$, and therefore, $\mu+\frac{1}{2}\sigma^2 = 0$. Combined with (8), we konw, when
$$\mu + \frac{1}{2}\sigma^2 = 0,$$
the process $(Y_t : t \geq 0)$ is martingale with respect to the process $(X_t : t \geq 0)$.

Notice that $m_2(t)$ satisfies

$$\begin{cases} \frac{\mathrm{d}m_2(t)}{\mathrm{d}t} = 2(\mu+\sigma^2)m_2(t) \\ m_2(0) = \mathbb{E}[Y_0^2] = \mathbb{E}[(e^{X_0})^2] = \mathbb{E}[1] = 1 \end{cases},$$

so

$$m_2(t) = e^{2(\mu+\sigma^2)t}.$$

In this case,

$$\begin{aligned} v(t) =& m_2(t) - m(t)^2 \\ =& e^{2(\mu+\sigma^2)t} - \left(e^{(\mu+\frac{1}{2}\sigma^2)t}\right)^2 \\ =& e^{2(\mu+\sigma^2)t} - e^{2(\mu+\frac{1}{2}\sigma^2)t} \\ =& e^{\sigma^2 t} - e^0 \\ =& e^{\sigma^2 t} \to \infty, \qquad \text{as } t \to \infty. \end{aligned}$$

(d) Show that δ_0 is the unique stationary distribution of the process on the state space $[0,\infty)$. Under which conditions on μ and σ^2 does the process with $Y_0 = 1$ converge to the stationary distribution?

Under which conditions on μ and σ^2 is the process ergodic? Justify your answer.

Sol.

(e) For $\sigma^2 = 1$ choose $\mu = -1/2$ and two other values $\mu < -1/2$ and $\mu > 1/2$. Simulate and plot a sample path of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

Sol.

```python
import numpy as np
import matplotlib.pyplot as plt
np.random.seed(1234)
plt.rc('text', usetex=True)
plt.rc('font', family='serif')

sigma = 1

mu1 = -0.5
mu2 = -1
mu3 = 1
mus = [mu1, mu2, mu3]

dt1 = 0.1
dt2 = 0.01
dts = [dt1, dt2]


def drift(mu, y):
    return (mu + 0.5*sigma**2) * y


def noise(y):
    return sigma*y


```

```
plt.figure(figsize=(15, 15))
for j in range(len(mus)):
    plt.subplot(len(mus), 1, j+1)
    for dt in dts:
        ts = np.arange(0, 10+dt, step=dt)
        Ys = np.empty(ts.shape)
        Ys[0] = 1
        for i in range(len(Ys) - 1):
            Ys[i+1] = Ys[i] + drift(mus[j], Ys[i])*dt + noise(Ys[i])*np.random.normal(0, dt)
        plt.plot(ts, Ys, label=r"$\Delta t = %.2f$" % dt)
    plt.legend()
    plt.title(r"$Y_t$ when $\mu = $%.1f" % mus[j])
    plt.xlim(0, 10)
    plt.xlabel(r"$t$")
    plt.ylabel(r"$Y_t$")

plt.tight_layout()
plt.savefig("Q1-D.png")
```

The output image is Figure 1.

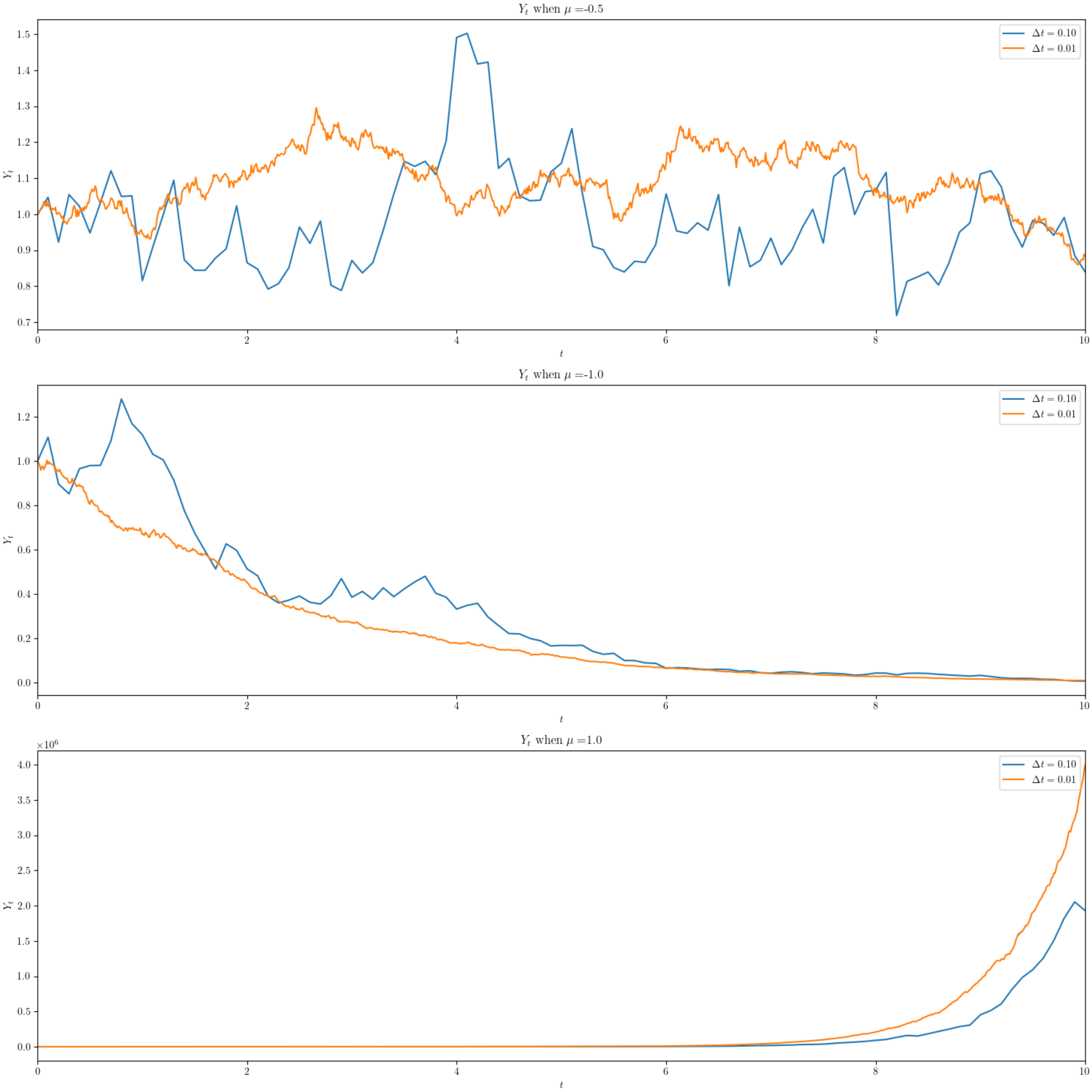

Figure 1: Geometric Brownian Motion with $\mu = -1/2$, -1 & 1 and $\Delta t = 0.1$ & 0.01

2 Barabási-Albert Model

Consider the Barabási-Albert model starting with $m_0 = 5$ connected nodes, adding in each timestep a node linked to $m = 5$ existing distinct nodes according to the preferential attachment rule. Simulate the model for $N = |V| = 1000$, with at least 20 independent realizations.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realization and for all 20, and compare to the power law with exponent -2 (all in a single plot).

Sol.

(b) Compute $k_{nn}(k) = \mathbb{E}\left[\sum_{i\in V} k_{nn,i}\delta_{k_i,k} / \sum_{i\in V}\delta_{k_i,k}\right]$ where $k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j$, and decide whether the graphs are typically uncorrelated or (dis-)assortative.

Sol.

(c) Plot the spectrum of the adjacency matrix $A = (a_{ij})$ using all realizations with a kernel density estimate, and compare it to the Wigner semi-circle law with $\sigma^2 = \mathrm{Var}[a_{ij}]$.

Sol.

3 Erdős Rényi Random Graphs

Consider the Erdős Rényi random graph model and simualte at least 20 realizations of $\mathcal{G}_{N,p}$ graphs with $p = p_N = z/N$, $z = 0.1, 0.2, \cdots, 3.0$ for $N = 100$ and $N = 1000$.

(a) Plot the average size of the two largest components in each realization divided by N, against z for both values of N in a single plot (4 date series in total, use different colours). Use all 20 (or more) realizations and include error bars indicating the standard deviation.

Sol.

(b) For $N = 1000$, plot the average local clustering coefficient $\langle C_i\rangle$ against z using all 20 realizations and $i = 1, \cdots, N$ for averaging, and including error bars indicating the standard deviation for all $20N$ data points.

Sol.

(c) For $N = 1000$ and your favourite value of $z \in [0.5, 2]$, plot the degree distribution $p(k)$ against $k = 0, 1, \cdots$ using 20 realizations, and compare it to the mass function of the $\mathrm{Poi}(z)$ Poisson distribution in a single plot.

Sol.

(d) Consider $z = 0.5, 1.5, 5$ and 10. Plot the spectrum of the adjacency matrix A using all 20 realizations with a kernel density estimate, and compare it to the Wigner semi-circle law.

Sol.