

Stochastic Modelling and Random Processes

Problem Sheet 2

Yiming MA

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1 Notations

- All row vectors are represented in a bold font, such as $\boldsymbol{\pi}$, and sometimes, it is also written as $\langle \boldsymbol{\pi} |$ according to the [bra-ket notation](#).
- All column vectors are represented with an arrow above, such as $\vec{0}$ which is the vector whose elements are all 0.
- Uppercase letters usually represent a matrix, such as G .
- A particular notation, $\boldsymbol{\pi}_t(i)$, means the i -th component of the row vector $\boldsymbol{\pi}_t$.
- $\Delta \in \mathbb{R}^n$ is the region $\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \vec{1} = 1, \boldsymbol{x}_i \geq 0, i = 1, \dots, n\}$.
- \boldsymbol{e}_i is the row vector whose i -th element is 1, and all other elements are 0. For example,

$$\boldsymbol{e}_1 = [1, 0, \dots, 0].$$

(The dimension depends on the context.)

2 Kingman's Coalescent

Consider a system of L well mixed, coalescing particles. Each of the $\binom{L}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of L individuals in a population model, tracing back to a single common ancestor.

- (a) Let N_t be the number of particles at time t with $N_0 = L$. Give the transition rates of the process $(N_t : t \geq 0)$ on the state space $\{1, \dots, L\}$. Write down the generator $(\mathcal{L}f)(n)$ for $n \in \{1, \dots, L\}$ and the master equation. Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

Sol. It's easy to see $G_{i,i-1} = \binom{i}{2} \times 1 = \binom{i}{2}$, for $i \in \{2, \dots, L\}$. Since $\sum_{j=1}^L G_{i,j} = 0$ and $G_{i,j} = 0$ for $j \notin \{i-1, i\}$, we know $G_{i,i} = -\binom{i}{2}$. To summarize,

$$G_{i,j} = \begin{cases} \binom{i}{2} & j = i-1, i \in \{2, \dots, L\} \\ -\binom{i}{2} & j = i, i \in \{2, \dots, L\} \\ 0 & \text{Otherwise} \end{cases} \quad (1)$$

By $\mathcal{L}(f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (1), we know

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = 1 \\ \binom{n}{2}[f(n-1) - f(n)] & n \in \{2, \dots, L\} \end{cases} \quad (2)$$

The master equation is $\frac{d}{dt}\pi_t(x) = \sum_{y \neq x} \pi_t(y)G_{y,x} - \sum_{y \neq x} \pi_t(x)G_{x,y}$. Use (1) again, and we get

$$\begin{cases} \frac{d\pi_t(1)}{dt} = \pi_t(2) \\ \frac{d\pi_t(i)}{dt} = \binom{i+1}{2}\pi_t(i+1) - \binom{i}{2}\pi_t(i), i = 2, \dots, L-1 \\ \frac{d\pi_t(L)}{dt} = -\binom{L}{2}\pi_t(L) \end{cases} \quad (3)$$

Obviously, state 1 is absorbing, and furthermore, it forms an absorbing component $\{1\}$. Thus, the process is SP-ergodic.

To find all stationary distributions, we need to solve $\pi G = \vec{0}$ with $\pi_t \in \Delta$, which is equivalent to

$$\begin{cases} 0\pi_t(1) + \binom{2}{2}\pi_t(2) = 0 \\ -\binom{i}{2}\pi_t(i) + \binom{i+1}{2}\pi_t(i+1) = 0, i = 2, \dots, L-1 \\ -\binom{L}{2}\pi_t(L) = 0 \end{cases} \quad (4)$$

and

$$\sum_{i=1}^L \pi_t(i) = 1. \quad (5)$$

Using backward substitution performed on (4) results in $\pi_t(i) = 0$, for $i = 2, \dots, L$. And using (5), we have $\pi_t(1) = 1$. So the only stationary distribution is

$$\pi_t = [1, \underbrace{0, \dots, 0}_{(L-1)'s 0}].$$

- (b) Show that the mean time to absorption is given by $\mathbb{E}(T) = 2(1 - \frac{1}{L})$.

Sol. Let W_i be the holding time for the process to leave state i , i.e. $W_i = \inf\{t \in \mathbb{R}_+ | N_t \neq i, N_0 = i\}$, for $i = L, L-1, \dots, 2$. Notice that

$$T = \sum_{i=2}^L W_i \quad (6)$$

since the process can only go from state i to $i - 1$ at a time.

From lecture, we know $W_i \sim \text{Exponential}(|G_{i,i}|)$. So

$$\mathbb{E}[W_i] = \frac{1}{|G_{i,i}|} = \frac{1}{\binom{i}{2}} = \frac{(i-2)! \times 2!}{i!} = \frac{2}{i(i-1)}. \quad (7)$$

Thus, using (6) together with the linearity of expectation and (7), we get

$$\begin{aligned} \mathbb{E}(T) &= \mathbb{E} \left(\sum_{i=2}^L W_i \right) \\ &= \sum_{i=2}^L \mathbb{E}(W_i) \\ &= \sum_{i=2}^L \left(\frac{2}{i(i-1)} \right) \\ &= \sum_{i=2}^L 2 \left(\frac{1}{i-1} - \frac{1}{i} \right) \\ &= 2 \left(1 - \frac{1}{L} \right). \end{aligned}$$

- (c) Write the generator of the rescaled process $(N_t/L : t \geq 0)$ and Taylor expand it up to the second order. Show that the slowed-down, rescaled process $(X_t^L : t \geq 0)$ where

$$X_t^L := \frac{1}{L} N_{\frac{t}{L}},$$

converges to the process $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}}f(x) = -\frac{x^2}{2}f'(x)$$

and state space $(0, 1]$ with $X_0 = 1$.

Convince yourself that this process is “deterministic”, i.e. $X_t = \mathbb{E}(X_t)$ for all $t \geq 0$, and compute X_t explicitly. How is your result compatible with the result from (b)?

Sol. For the rescaled process $(N_t/L : t \geq 0)$, the rates of transitions are not changed, but the state space is replaced with $\{\frac{1}{L}, \dots, \frac{L}{L}\}$. Thus, the rate matrix is

$$G_{i,j} = \begin{cases} \binom{iL}{2} & j = i - \frac{1}{L}, i \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \\ -\binom{iL}{2} & j = i, i \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \\ 0 & \text{Otherwise} \end{cases} \quad (8)$$

By $(\mathcal{L}f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (8), we have

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \binom{nL}{2}[f(n - \frac{1}{L}) - f(n)] & n \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \end{cases}. \quad (9)$$

Now Taylor expand (9) to the second order, which results in

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \binom{nL}{2}[-\frac{1}{L}f'(n) + \frac{1}{2L^2}f''(n) + o(\frac{1}{L^2})] & n \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \end{cases}.$$

We need to derive the generator of the process $(X_t^L : t \geq 0)$ first. The rate matrix of $(X_t^L : t \geq 0)$ is

$$G_{i,j} = \begin{cases} \frac{1}{L} \binom{iL}{2} & j = i - \frac{1}{L}, i \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \\ -\frac{1}{L} \binom{iL}{2} & j = i, i \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \\ 0 & \text{Otherwise} \end{cases}. \quad (10)$$

By $(\mathcal{L}f)(n) = \sum_{k \neq n} G_{n,k}[f(k) - f(n)]$ and (10), we have

$$\mathcal{L}(f)(n) = \begin{cases} 0 & n = \frac{1}{L} \\ \frac{1}{L} \binom{nL}{2} [f(n - \frac{1}{L}) - f(n)] & n \in \{\frac{2}{L}, \dots, \frac{L}{L}\} \end{cases}. \quad (11)$$

Notice that

$$\begin{aligned} & \frac{1}{L} \binom{nL}{2} \left[f(n - \frac{1}{L}) - f(n) \right] \\ &= \frac{1}{L} \cdot \frac{(nL)!}{(nL-2)!2!} \cdot \left[f(n) - \frac{1}{L} f'(n) + \frac{1}{2L^2} f''(n) + o(\frac{1}{L^2}) - f(n) \right] \\ &= \frac{1}{L} \cdot \frac{(nL)(nL-1)}{2} \cdot \left[-\frac{1}{L} f'(n) + \frac{1}{2L^2} f''(n) + o(\frac{1}{L^2}) \right] \\ &= \frac{n(nL-1)}{2} \cdot \left[-\frac{1}{L} f'(n) + \frac{1}{2L^2} f''(n) + o(\frac{1}{L^2}) \right] \\ &= \frac{n}{2} \left[-\frac{nL-1}{L} f'(n) + \frac{nL-1}{2L^2} f''(n) + o(\frac{nL-1}{L^2}) \right], \end{aligned}$$

so as $L \rightarrow \infty$, we have

$$\lim_{L \rightarrow \infty} \frac{1}{L} \binom{nL}{2} \left[f(n - \frac{1}{L}) - f(n) \right] = -\frac{n^2}{2} f'(n).$$

In conclusion, we have, as $L \rightarrow \infty$, the process $(X_t^L : t \geq 0)$ converges to $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}}(f)(x) = -\frac{x^2}{2} f'(x) \quad (12)$$

and state space $(0, 1]$ with $X_0 = 1$.

- (d) Generate sample paths of the process $(X_t^L : t \geq 0)$ for $L = 10, L = 100, L = 1000$ and compare to the solution X_t from (c) in a single plot.

Sol.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 Ls = [10, 100, 1000]
5 colors = ["lightcoral", 'orange', 'cyan']
6 for i in range(0, len(Ls)):
7     L = Ls[i]
8     color = colors[i]
9     time = 0
10    WT = 0
11
12    for n in range(L, 1, -1):
13        waitTime = np.random.exponential(scale=2/(n*(n-1)))
```

```

14     plt.plot([time/L, (time+waitTime)/L], [n/L, n/L], color=color, lw=2)
15     time += waitTime
16     WT = waitTime
17
18     plt.plot([time/L, (time+2*WT)/L], [1/L, 1/L], color=color, label=r"$L = " +
19     str(L) + "$")
20 plt.title("Kingman's Coalescent")
21 plt.legend()
22 plt.xlabel('$t$')
23 plt.ylabel('$N_t$')
24 plt.yscale('linear')
25 plt.xscale('log')
26 plt.savefig("Kingman_Coalescent.png")
27 plt.show()

```

The output image is Figure 1.

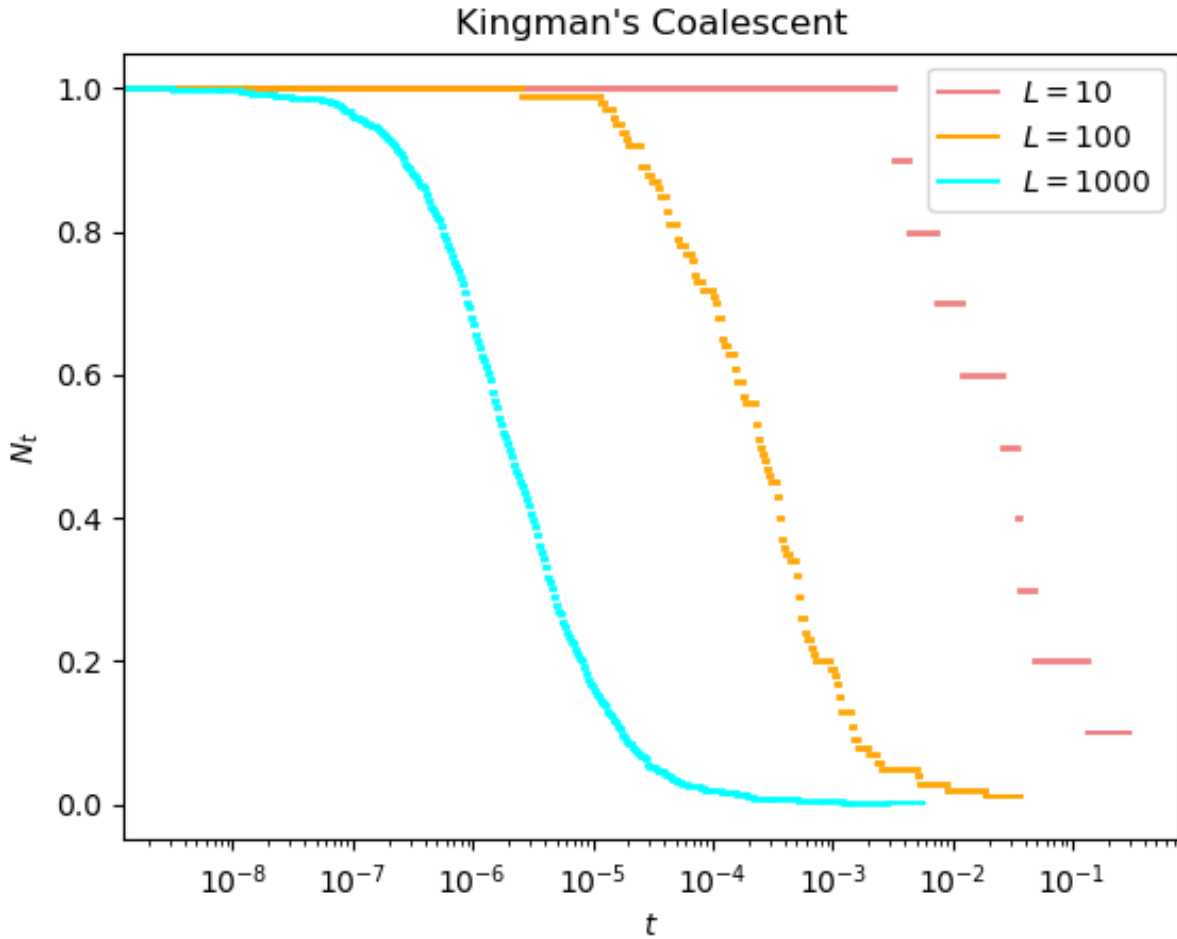


Figure 1: Kingman's Coalescent with $L = 10$, $L = 100$, $L = 1000$

3 Ornstein-Uhlenbeck Processes

The Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ is a diffusion process on \mathbb{R} with generator

$$(\mathcal{L}f)(x) = -\alpha x f'(x) + \frac{1}{2} \sigma^2 f''(x)$$

with $\alpha, \sigma^2 > 0$, and we consider a fixed initial condition $X_0 = x_0$.

(a) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[\mathcal{L}f(X_t)], \quad (13)$$

to derive ODEs for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - m(t)^2$, and solve them.

Sol. Set $f(x) = x$ in the evolution equation (13), then we have

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[\mathcal{L}f(X_t)] = \mathbb{E}[-\alpha X_t] = -\alpha\mathbb{E}[X_t],$$

which is

$$\frac{dm(t)}{dt} = -\alpha m(t) \quad (14)$$

The general solution to (14) is

$$m(t) = C \cdot e^{-\alpha t},$$

where $C \in \mathbb{R}$ is a constant. By $m(0) = \mathbb{E}[X_0] = \mathbb{E}[x_0] = x_0$, we know $C = x_0$. Thus, the solution to (14) is

$$m(t) = x_0 \cdot e^{-\alpha t}. \quad (15)$$

Setting $f(x) = x^2$ in (13) gives

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[X_t^2] &= \mathbb{E}[-\alpha X_t \cdot 2X_t + \frac{1}{2}\sigma^2 \cdot 2] \\ &= \mathbb{E}[-2\alpha X_t^2 + \sigma^2] \\ &= -2\alpha\mathbb{E}[X_t^2] + \sigma^2 \end{aligned} \quad (16)$$

To solve (16), we need to find the general solution $h(t)$ of its homogeneous version and a particular solution $p(t)$ of it separately. So $h(t)$ satisfies

$$\frac{dh(t)}{dt} = -2\alpha h(t).$$

Using the method of separation of variables again, we know $h(t) = C_1 \cdot e^{-2\alpha t}$, where $C_1 \in \mathbb{R}$ is a constant.

Now suppose $p(t) = e^{-2\alpha t} + C_2$ with $C_2 \in \mathbb{R}$ being a constant. Then we have

$$\begin{aligned} \frac{dp(t)}{dt} &= -2\alpha p(t) + \sigma^2 \\ -2\alpha e^{-2\alpha t} &= -2\alpha(e^{-2\alpha t} + C_2) + \sigma^2 \\ 2\alpha C_2 &= \sigma^2 \\ C_2 &= \frac{\sigma^2}{2\alpha}. \end{aligned}$$

So a particular solution $p(t)$ is $p(t) = e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$.

Thus, the general solution of (16) is $\mathbb{E}[X_t^2] = C_1 \cdot e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$. By $\mathbb{E}[X_0^2] = \mathbb{E}[x_0^2] = x_0^2$, we have $C_1 = x_0^2 - \frac{\sigma^2}{2\alpha}$. So the second central moment of X_t is

$$\mathbb{E}[X_t^2] = \left(x_0^2 - \frac{\sigma^2}{2\alpha}\right) e^{-2\alpha t} + \frac{\sigma^2}{2\alpha},$$

and the variance of X_t is

$$\begin{aligned} v(t) &= \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 \\ &= \left(x_0^2 - \frac{\sigma^2}{2\alpha} \right) e^{-2\alpha t} + \frac{\sigma^2}{2\alpha} - (x_0 \cdot e^{-\alpha t})^2 \\ &= \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-2\alpha t}). \end{aligned}$$

- (b) Using the fact that $(X_t : t \geq 0)$ is a Gaussian process, give the distribution of X_t for all $t \geq 0$. What is the stationary distribution of the process?

Sol. In (a), we have solved $\mathbb{E}[X_t]$ and $\mathbb{V}\mathbb{A}\mathbb{R}[X_t]$:

$$\mathbb{E}[X_t] = m(t) = x_0 \cdot e^{-\alpha t} \quad \text{and} \quad \mathbb{V}\mathbb{A}\mathbb{R}[X_t] = v(t) = \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-2\alpha t}).$$

Since the process $(X_t : t \geq 0)$ is a Gaussian process, we know X_t follows a Gaussian distribution. Thus,

$$X_t \sim \mathcal{N} \left(x_0 \cdot e^{-\alpha t}, \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-2\alpha t}) \right).$$

From lecture, we know the stationary density of the diffusion process with time-independent $a(y) \in \mathbb{R}$ and $\sigma^2(y) > 0$ has the unnormalized stationary density

$$p(x) = \exp \left(\int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)} dy \right).$$

Since the Ornstein-Uhlenbeck process is a special case of the diffusion process with $a(y) = -\alpha y$ and $\sigma^2(y) = \sigma^2$, we know the unnormalized stationary density of the Ornstein-Uhlenbeck process is

$$\begin{aligned} p(x) &= \exp \left(\int_0^x \frac{-2\alpha y - 0}{\sigma^2} dy \right) \\ &= \exp \left(-\frac{\alpha x^2}{\sigma^2} \right) \\ &= \exp \left(-\frac{(x - 0)^2}{2 \cdot \frac{\sigma^2}{2\alpha}} \right). \end{aligned}$$

So the stationary distribution of the process is $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$.

- (c) For $\alpha = 1$, $\sigma^2 = 1$ and $x_0 = 5$, simulate and plot a sample path of the process up to time $t = 10$, by numerically integrating the SDE with time steps $\Delta t = 0.1$ and $\Delta t = 0.01$.

Sol.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3 import sdeint
4
5 np.random.seed(1234)
6
7 alpha = 1
```

```

8 sigma = 1
9 x_0 = 5
10 t_max = 10
11
12 dts = [0.1, 0.01]
13 colors = ["skyblue", "violet"]
14
15
16 def f(x, t):
17     return -alpha*x
18
19
20 def g(x, t):
21     return sigma*np.sin(t)
22
23
24 plt.figure(figsize=(20, 8))
25
26 for i in range(0, len(dts)):
27     dt = dts[i]
28     color = colors[i]
29
30     times = np.arange(0, t_max, dt)
31     result = sdeint.itoint(f, g, x_0, times)
32
33     label = r"$\Delta t = " + str(dt) + "$"
34     plt.plot(times, result, color=color, label=label)
35
36
37 plt.legend()
38 plt.xlabel(r'$t$')
39 plt.ylabel(r'$X_t$')
40 plt.title(r'Ornstein-Uhlenbeck process with $\alpha$ = {}, $\sigma$ = {}'.format(alpha, sigma))
41 plt.savefig("Ornstein_Uhlenbeck.png")
42 plt.show()

```

The output image is [Figure 2](#).

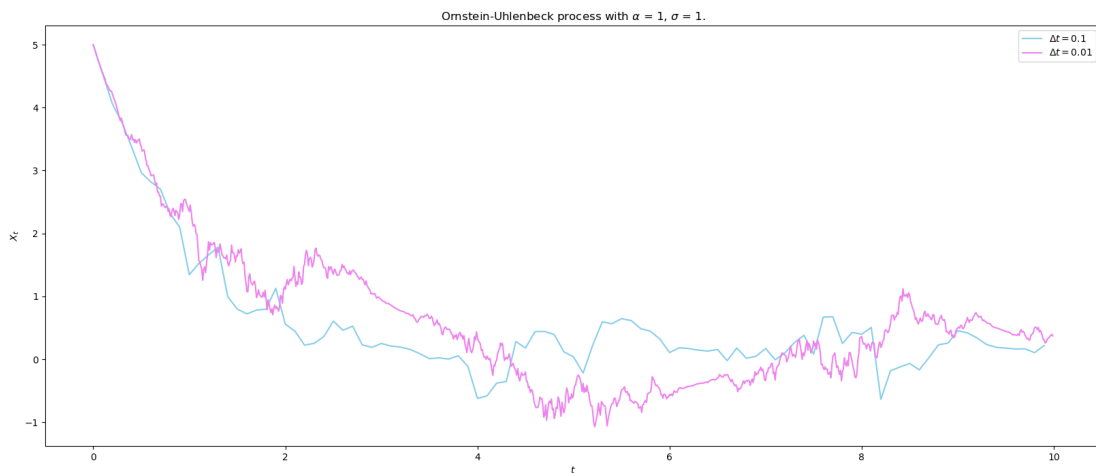


Figure 2: Ornstein-Uhlenbeck Process with $\alpha = 1$, $\sigma^2 = 1$, $x_0 = 5$

4 Moran Model and Wright-Fisher Diffusion

Consider a fixed population of L individuals. At time $t = 0$, each individual i has a different type $X_0(i)$, and for simplicity, we simply put $X_0(i) = i$. In continuous time, each individual indepently, with rate 1, imposes its type on another randomly chosen individual (or equivalently, kills it and puts its own offspring in its place).

- (a) Give the state space of the Markov chain $(X_t : t \geq 0)$. Is it irreducible? What are the stationary distributions?

Sol. Since $X_t(i)$ ($i \in \{1, \dots, L\}$) is the type of the i -th individual at time t and there are L types in total, the state space is $\{1, \dots, L\}$.

The process is not irreducible. Suppose at time t_0 , individual 1 with type 1 dies and individual L with type L reproduces to substitute, i.e.

$$t_0 = \inf\{t > 0 : \sum_{i=1}^L \delta_{X_t(i), 1} \neq 1\},$$

such that $X_{t_0}(1) = X_{t_0}(L) = L$. Since there are no individuals with type 1 any more, $P_t(1, y) = 0$ for any $t \geq t_0$ and $y \in \{2, \dots, L\}$.

Stationary distributions mean once these distributions are entered, the process will stay in them forever. So the stationary distributions of the process are \mathbf{e}_i , with $i = 1, \dots, L$.

- (b) Let $N_t = \sum_{i=1}^L \delta_{X_t(i), k}$ be the number of individuals of a given type $k \in \{1, \dots, L\}$ at time t , with $N_0 = 1$.

- Is $(N_t : t \geq 0)$ a Markov process? Given the state space and the generator.

Sol. The process is obviously Markov, as the distribution of $N_{t_{n+1}}$ given $N_{t_n}, N_{t_{n-1}}, \dots, N_{t_0}$ only depends on N_{t_n} .

The state space of $(N_t : t \geq 0)$ is $\{0, 1, \dots, L\}$.

Suppose $N_t = i$, where $i \in \{1, \dots, L-1\}$. Since there are i individuals of type k and each of them reproduces at rate 1, the total rate of reproduction of individuals of type k is just i . Also, we need to select 1 out of the $L-i$ individuals with other types to be replaced, and this gives the probability $\frac{L-i}{L}$. So

$$G_{i,i+1} = \frac{i(L-i)}{L}.$$

Similarly, we also have

$$G_{i,i-1} = \frac{i(L-i)}{L}.$$

Since $\sum_{j=1}^{L+1} G_{i,j} = 1$, we know

$$G_{i,i} = -\frac{2i(L-i)}{L}.$$

Since state 0 and state L are absorbing, $G_{0,i} = G_{L,i} = 0$, for any $i \in \{0, \dots, L\}$.

Hence, the rate matrix G (indices start from 0 and end at L), whose (i, j) element represents the transition rate from state $i-1$ into state $j-1$, is given by

$$G_{i,j} = \begin{cases} \frac{i(L-i)}{L} & j = i-1, i \in \{1, \dots, L-1\} \\ -\frac{2i(L-i)}{L} & j = i, i \in \{1, \dots, L-1\} \\ \frac{i(L-i)}{L} & j = i+1, i \in \{1, \dots, L-1\} \\ 0 & \text{Otherwise} \end{cases}.$$

- Is the process irreducible? What are the stationary distributions?

Sol. The process is not irreducible, since state 0 and state L are absorbing while others are not. Following the same argument in (a), the stationary distributions are \mathbf{e}_0 and \mathbf{e}_L .

- What is the limiting distribution as $t \rightarrow \infty$ for the initial condition $N_0 = 1$?

Sol. As $t \rightarrow \infty$, all types have the equal possibility to become the only type of the population. So

$$\lim_{t \rightarrow \infty} \mathbb{P}[N_t = L] = \frac{1}{L}$$

and

$$\lim_{t \rightarrow \infty} \mathbb{P}[N_t = 0] = \frac{L-1}{L}.$$

(c) From now consider general initial conditions $N_0 = n \in \{0, \dots, L\}$.

- Compute $m_1(t) = \mathbb{E}[N_t]$ for all $t \geq 0$.
- Compute $m_2(t) = \mathbb{E}[N_t^2]$. What happens in the limit $t \rightarrow \infty$?
- Compute the absorption probabilities as a function of the initial condition n .

Sol. To solve $m_1(t)$ and $m_2(t)$, we solve $\mathbb{E}[f(N_t)]$ where $f : S := \{0, \dots, L\} \mapsto \mathbb{R}$ first.

Obviously, the following two statements hold.

$$\mathbb{E}[f(N_t)] = f(0), \quad \text{when } N_0 = 0. \quad (17)$$

$$\mathbb{E}[f(N_t)] = f(L), \quad \text{when } N_0 = L. \quad (18)$$

Now suppose $N_0 = n \in \{2, \dots, L-1\}$. By

$$\mathbb{E}[f(N_t)] = \sum_{x \in S} \pi_t(x) f(x) = \langle \pi_t | f \rangle$$

and

$$\frac{d}{dt} \langle \pi_t | = \langle \pi_t | G,$$

we have

$$\frac{d}{dt} \mathbb{E}[f(N_t)] = \frac{d}{dt} \langle \pi_t | f \rangle = \langle \pi_t | G f \rangle = \mathbb{E}[(Gf)(N_t)].$$

So

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(N_t)] &= \sum_{k \in S} P_t(n, k) \left(\sum_{j \neq k} G_{k,j} [f(j) - f(k)] \right) \\ &= \sum_{k \in S} P_t(n, k) (G_{k,k-1} [f(k-1) - f(k)] + G_{k,k+1} [f(k+1) - f(k)]) \\ &= \sum_{k \in S} P_t(n, k) \cdot \frac{k(L-k)}{L} \cdot [f(k-1) + f(k+1) - 2f(k)]. \end{aligned} \quad (19)$$

Set $f(x) = x$ in (17), then we have

$$\frac{d}{dt} \mathbb{E}[N_t] = \sum_{k \in S} P_t(n, k) \cdot \frac{k(L-k)}{L} \cdot [(k-1) + (k+1) - 2k] = 0,$$

which means $m_1(t) = m_1(0) = n$, for $n \in \{2, \dots, L-1\}$. Along with (17) and (18), we know $m_1(t) = n$, for all $n \in S$.

Set $f(x) = x^2$ in (17), and we get

$$\begin{aligned}
\frac{d}{dt}m_2(t) &= \frac{d}{dt}\mathbb{E}[N_t^2] \\
&= \sum_{k \in S} P_t(n, k) \cdot \frac{k(L-k)}{L} \cdot [(k-1)^2 + (k+1)^2 - 2k^2] \\
&= \sum_{k \in S} P_t(n, k) \cdot \frac{k(L-k)}{L} \cdot 2 \\
&= \frac{2}{L} \left(L \cdot \sum_{k \in S} P_t(n, k) \cdot k - \sum_{k \in S} P_t(n, k) \cdot k^2 \right) \\
&= \frac{2}{L} \cdot (L \cdot \mathbb{E}[X_t] - \mathbb{E}[X_t^2]) \\
&= 2\mathbb{E}[X_t] - \frac{2}{L}\mathbb{E}[X_t^2] \\
&= 2n - \frac{2}{L}m_2(t)
\end{aligned} \tag{20}$$

The general solution to the homogeneous version of (20) is

$$h(t) = C_1 e^{-\frac{2}{L}t},$$

where $C_1 \in \mathbb{R}$ is a constant. Now suppose $p(t) = e^{-\frac{2}{L}t} + C_2$ is a particular solution to (20), then

$$-\frac{2}{L}e^{-\frac{2}{L}t} = \frac{dp(t)}{dt} = 2n - \frac{2}{L}p(t) = 2n - \frac{2}{L}(e^{-\frac{2}{L}t} + C_2),$$

which gives $C_2 = nL$. So the general solution to (20) is

$$m_2(t) = C_1 e^{-\frac{2}{L}t} + nL.$$

By $m_2(0) = \mathbb{E}[X_0^2] = \mathbb{E}[n^2] = n^2$, we know $C_1 = n^2 - nL = n(n-L)$. Together with (17) and (18), we have

$$m_2(t) = n(n-L)e^{-\frac{2}{L}t} + nL. \tag{21}$$

Let $t \rightarrow \infty$ in (21), and we have

$$\lim_{t \rightarrow \infty} m_2(t) = nL.$$

Let $\tau = \inf\{t \geq 0 : N_t \in \{0, L\}\}$, which is the time when the process enters the either one of two absorption states 0 and L . By $\mathbb{E}[N_t] = m_2(t) = n$, we have

$$n = \mathbb{E}[N_\tau] = 0 \cdot \mathbb{P}[N_\tau = 0] + L \cdot \mathbb{P}[N_\tau = L] = L \cdot \mathbb{P}[N_\tau = L],$$

so $\mathbb{P}[N_\tau = L] = \frac{n}{L}$, which is the probability that the process eventually falls in the state L . Thus, the probability that the process eventually fixed in the state 0 is $\frac{L-n}{L}$.

(d) Consider the rescaled process $(M_t^L : t \geq 0)$ where

$$M_t^L = \frac{1}{L}N_{tL^\alpha}$$

on the state space $[0, 1]$. For which value of $\alpha > 0$ does $(M_t^L : t \geq 0)$ have a (non-trivial) scaling limit $(M_t : t \geq 0)$?

Compute the generator of this process and write down the Fokker-Planck equation. (The scaling limit is called **Wright-Fisher diffusion**.)

Sol. The rate matrix of the rescaled process is

$$G_{i,j} = \begin{cases} i(L-i)L^{\alpha-1} & j = i - \frac{1}{L}, i \in \{\frac{1}{L}, \dots, \frac{L-1}{L}\} \\ -2i(L-i)L^{\alpha-1} & j = i, i \in \{\frac{1}{L}, \dots, \frac{L-1}{L}\} \\ i(L-i)L^{\alpha-1} & j = i + \frac{1}{L}, i \in \{\frac{1}{L}, \dots, \frac{L-1}{L}\} \\ 0 & \text{Otherwise} \end{cases}.$$

By $(\mathcal{L}f)(X_t) = (Gf)(X_t)$, we have

$$(\mathcal{L}f)(x) = \sum_{y \neq x} G_{x,y} [f(y) - f(x)].$$

For $x = 0$ and $x = 1$, $(\mathcal{L}f)(0) = (\mathcal{L}f)(1) = 0$. So now suppose $x \in \{\frac{1}{L}, \dots, \frac{L-1}{L}\}$, then we have

$$\begin{aligned} (\mathcal{L}f)(x) &= x(L-x)L^{\alpha-1} \left[f\left(x + \frac{1}{L}\right) + f\left(x - \frac{1}{L}\right) - 2f(x) \right] \\ &= xL^{\alpha} \left[f\left(x + \frac{1}{L}\right) + f\left(x - \frac{1}{L}\right) - 2f(x) \right] \\ &\quad - x^2L^{\alpha-1} \left[f\left(x + \frac{1}{L}\right) + f\left(x - \frac{1}{L}\right) - 2f(x) \right] \end{aligned} \quad (22)$$

Suppose f is smooth enough to be Taylor expanded into the second order, and perform Taylor expansion of terms involving f in (22):

$$\begin{aligned} f\left(x + \frac{1}{L}\right) + f\left(x - \frac{1}{L}\right) - 2f(x) &= f(x) + \frac{1}{L}f'(x) + \frac{1}{2L^2}f''(x) + o\left(\frac{1}{L^2}\right) \\ &\quad + f(x) - \frac{1}{L}f'(x) + \frac{1}{2L^2}f''(x) + o\left(\frac{1}{L^2}\right) \\ &\quad - 2f(x) \\ &= \frac{1}{L^2}f''(x) + o\left(\frac{1}{L^2}\right) \end{aligned} \quad (23)$$

Plug (23) expansion into (22), then we have

$$(\mathcal{L}f)(x) = xL^{\alpha-2}f''(x) - x^2L^{\alpha-3}f''(x) + o(L^{\alpha-2}).$$

So in order that the process has a scaling limit, $\alpha > 0$ should satisfy $\alpha - 2 \leq 0$ and $\alpha - 3 \leq 0$ at the same time, which gives $\alpha \in (0, 2]$. α has to be 2 such that the limiting process is non-trivial, and the corresponding limiting process has the generator

$$(\bar{\mathcal{L}}f)(x) = xf''(x).$$

Now, we are going to derive the Fokker-Planck equation of $(M_t : t \geq 0)$.

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(M_t)] &= \mathbb{E}[(\bar{\mathcal{L}}f)(M_t)] \\ \frac{d}{dt} \int_0^1 P_t(x, y) f(y) dy &= \mathbb{E}[X_t f''(M_t)] \\ \int_0^1 \frac{\partial}{\partial t} P_t(x, y) f(y) dy &= \int_0^1 P_t(x, y) y f''(y) dy. \end{aligned} \quad (24)$$

Doing integration by parts on the right-hand side of (24) gives

$$\begin{aligned}
\int_0^1 P_t(x, y) y f''(y) dy &= \int_0^1 P_t(x, y) y df'(y) \\
&= P_t(x, y) y f'(y) \Big|_{y=0}^{y=1} - \int_0^1 f'(y) \frac{\partial}{\partial y} (P_t(x, y) y) dy \\
&= P_t(x, y) y f'(y) \Big|_{y=0}^{y=1} - \int_0^1 f'(y) \left(\frac{\partial P_t(x, y)}{\partial y} y + P_t(x, y) \right) dy \quad (25)
\end{aligned}$$

Assuming $\lim_{t \rightarrow \infty} P_t(x, y) = 0$ and t is large enough, the right-hand side of (25) becomes

$$\begin{aligned}
&P_t(x, y) y f'(y) \Big|_{y=0}^{y=1} - \int_0^1 f'(y) \left(\frac{\partial P_t(x, y)}{\partial y} y + P_t(x, y) \right) dy \\
&\approx - \int_0^1 f'(y) \left(\frac{\partial P_t(x, y)}{\partial y} y + P_t(x, y) \right) dy \\
&= - \int_0^1 \frac{\partial P_t(x, y)}{\partial y} y f'(y) dy - \int_0^1 P_t(x, y) f'(y) dy \\
&= - \int_0^1 \frac{\partial P_t(x, y)}{\partial y} y df(y) - \int_0^1 P_t(x, y) df(y) \\
&= - \frac{\partial P_t(x, y)}{\partial y} y f(y) \Big|_{y=0}^{y=1} + \int_0^1 f(y) \left(\frac{\partial^2 P_t(x, y)}{\partial y^2} y - \frac{\partial P_t(x, y)}{\partial y} \right) dy \\
&\quad - P_t(x, y) f(y) \Big|_{y=0}^{y=1} + \int_0^1 \frac{\partial P_t(x, y)}{\partial y} f(y) dy \\
&\approx - \frac{\partial P_t(x, y)}{\partial y} y f(y) \Big|_{y=0}^{y=1} + \int_0^1 f(y) \left(\frac{\partial^2 P_t(x, y)}{\partial y^2} y - \frac{\partial P_t(x, y)}{\partial y} \right) dy \\
&\quad + \int_0^1 \frac{\partial P_t(x, y)}{\partial y} f(y) dy \\
&= - \frac{\partial P_t(x, y)}{\partial y} y f(y) \Big|_{y=0}^{y=1} + \int_0^1 f(y) \cdot \frac{\partial^2 P_t(x, y)}{\partial y^2} y dy \quad (26)
\end{aligned}$$

Also, assume $\lim_{t \rightarrow \infty} \frac{\partial P_t(x, y)}{\partial y} = 0$, then in (26) we have

$$P_t(x, y) y f'(y) \Big|_{y=0}^{y=1} - \int_0^1 f'(y) \left(\frac{\partial P_t(x, y)}{\partial y} y + P_t(x, y) \right) dy \approx \int_0^1 f(y) \cdot \frac{\partial^2 P_t(x, y)}{\partial y^2} y dy$$

Combined with the (24), we have

$$\int_0^1 \frac{\partial}{\partial t} P_t(x, y) f(y) dy \approx \int_0^1 f(y) \cdot \frac{\partial^2 P_t(x, y)}{\partial y^2} y dy.$$

So the Fokker-Planck equation is

$$\frac{\partial}{\partial t} P_t(x, y) = \frac{\partial^2}{\partial y^2} P_t(x, y). \quad (27)$$

- (e) For the limit process $(M_t : t \geq 0)$ in (d) compute $m(t) = \mathbb{E}[M_t]$ and $v(t) = \mathbb{E}[M_t^2] - m(t)^2$. Is it a Gaussian process?

Sol. Recall $\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[\bar{\mathcal{L}}f(X_t)] = \mathbb{E}[X_t f''(X_t)]$. Let $f(x) = x$, then we have

$$\frac{d}{dt}\mathbb{E}[X_t] = 0.$$

So $m(t) = \mathbb{E}[M_t] = \mathbb{E}[M_0] = m(0)$.

Let $f(x) = x^2$, then we have

$$\frac{d}{dt}\mathbb{E}[M_t^2] = \mathbb{E}[2M_t] = 2\mathbb{E}[M_t] = 2m(t) = 2m(0).$$

So $\mathbb{E}[M_t^2] = 2m(0)t + \mathbb{E}[M_0^2]$, and as a result, $v(t) = \mathbb{E}[M_t^2] - (\mathbb{E}[M_t])^2 = 2m(0)t + \mathbb{E}[M_0^2] - (\mathbb{E}[M_0])^2 = 2m(0)t + \text{VAR}[M_0]$.

5 Birth-Death Process

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \dots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \text{ for all } x \in S, \quad x \xrightarrow{\beta_x} x-1 \text{ for all } x \geq 1.$$

According to the article [KM57] of Samuel Karlin and James McGregor, a sufficient and necessary condition for the states of a birth-death process being recurrent is

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \infty, \quad (28)$$

sufficient and necessary conditions for them being null recurrent is

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} = \infty, \quad (29)$$

and sufficient and necessary conditions for them being ergodic is

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} < \infty, \quad (30)$$

(a) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$. Consider different cases depending on the choice of α and β where necessary:

- Is $(X_t : t \geq 0)$ irreducible? Give all communicating classes in \mathbb{N}_0 and state whether they are transient or null/positive recurrent.

Sol. Since all states are accessible from each other, with positive rates α and β , all states in S are communicating states. Thus, the process is irreducible.

$\forall i \geq 1, \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \prod_{n=1}^i \frac{\beta}{\alpha} = \left(\frac{\beta}{\alpha}\right)^i$, so

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i.$$

According to (28), these states are recurrent if and only if

$$\sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i = \infty,$$

which is equivalent to

$$\frac{\beta}{\alpha} \geq 1.$$

Thus, the states are transient if and only if

$$0 < \frac{\beta}{\alpha} < 1.$$

And similarly, by (29), the states are null recurrent if and only if

$$\frac{\beta}{\alpha} = 1,$$

and they are positive recurrent if and only if

$$\frac{\beta}{\alpha} > 1.$$

- Give all stationary distributions and state whether they are reversible.

Sol. We can use the $\langle \boldsymbol{\pi} | G = \langle \mathbf{0} |$ to solve stationary distributions. Notice that

$$G = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & 0 & \cdots \\ \beta & -(\alpha + \beta) & \alpha & 0 & 0 & \cdots \\ 0 & \beta & -(\alpha + \beta) & \alpha & 0 & \cdots \\ 0 & 0 & \beta & -(\alpha + \beta) & \alpha & \cdots \\ 0 & 0 & 0 & \beta & -(\alpha + \beta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so

$$-\alpha\boldsymbol{\pi}_0 + \beta\boldsymbol{\pi}_1 = 0 \tag{31}$$

$$\alpha\boldsymbol{\pi}_0 - (\alpha + \beta)\boldsymbol{\pi}_1 + \beta\boldsymbol{\pi}_2 = 0 \tag{32}$$

$$\alpha\boldsymbol{\pi}_1 - (\alpha + \beta)\boldsymbol{\pi}_2 + \beta\boldsymbol{\pi}_3 = 0 \tag{33}$$

\vdots

To solve these equations, notice that (31) gives

$$\beta\boldsymbol{\pi}_1 = \alpha\boldsymbol{\pi}_0. \tag{34}$$

Plug (34) into (32), and we get

$$\beta\boldsymbol{\pi}_2 = \alpha\boldsymbol{\pi}_1. \tag{35}$$

And plugging (35) into (33) results in

$$\beta\boldsymbol{\pi}_3 = \alpha\boldsymbol{\pi}_2. \tag{36}$$

So we can keep doing this recursively and get

$$\beta\boldsymbol{\pi}_{n+1} = \alpha\boldsymbol{\pi}_n, \quad \forall n \in \mathbb{N}_0,$$

or equivalently,

$$\boldsymbol{\pi}_n = \left(\frac{\alpha}{\beta}\right)^n \boldsymbol{\pi}_0.$$

If $\frac{\alpha}{\beta} \geq 1$, there is no stationary distribution, because

$$\sum_{n=0}^{\infty} \pi_n = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^n \pi_0 \neq 1.$$

If $\frac{\alpha}{\beta} < 1$, then solving $\sum_{n=0}^{\infty} \pi_n = 1$ gives

$$\pi_0 = \frac{\beta - \alpha}{\beta},$$

so

$$\pi_n = \left(\frac{\alpha}{\beta}\right)^n \cdot \frac{\beta - \alpha}{\beta}.$$

Since $G_{x,y}\pi_x = G_{y,x}\pi_y$ for all $x, y \in S$, the process is time reversible.

- Is the process ergodic?

Sol. To check the ergodicity of the process, we only need to verify (30).

$$\begin{aligned} \infty &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta}{\alpha} \\ &= \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i, \end{aligned}$$

which holds only when

$$\frac{\beta}{\alpha} \geq 1. \tag{37}$$

$$\begin{aligned} \infty &> \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha}{\beta} \\ &= \sum_{i=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^i \end{aligned}$$

which can be true only when

$$\frac{\alpha}{\beta} < 1$$

Based on (37) and (5), the process is ergodic if and only if

$$\frac{\beta}{\alpha} > 1.$$

- (b) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$. Consider different cases depending on the choice of α and β where necessary:

- Is $(X_t : t \geq 0)$ irreducible? Give all communicating classes in N_0 and state whether they are transient or null/positive recurrent.

Sol. Notice that $\alpha_0 = 0$ and $\beta_0 = \beta > 0$, so state 0 is absorbing, and thus positive recurrent. For other states, they are communicating. So state space can be decomposed into $\{0\}$ and N_+ , which means the process is not irreducible.

Since

$$\begin{aligned} \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta}{\alpha n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta}{\alpha} \\ &= \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha} \right)^i, \end{aligned}$$

by (28), states $1, 2, \dots$ are transient if and only if $\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} < \infty$, which is equivalent to $\frac{\beta}{\alpha} < 1$.

Now suppose $\frac{\beta}{\alpha} \geq 1$ so that states $1, 2, \dots$ are recurrent. Since

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} = \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha(n-1)}{\beta n} = 0 < \infty,$$

by (29), states $1, 2, \dots$ positive recurrent.

- Give all stationary distributions and state whether they are reversible.

Sol. The transition matrix G is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \beta & -(\alpha + \beta) & \alpha & 0 & 0 & \dots \\ 0 & 2\beta & -2(\alpha + \beta) & 2\alpha & 0 & \dots \\ 0 & 0 & 3\beta & -3(\alpha + \beta) & 3\alpha & \dots \\ 0 & 0 & 0 & 4\beta & -4(\alpha + \beta) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$\langle \pi | G = \langle 0 |$ gives

$$\beta \pi_1 = 0 \tag{38}$$

$$-(\alpha + \beta) \pi_1 + 2\beta \pi_2 = 0 \tag{39}$$

$$\alpha \pi_1 - 2(\alpha + \beta) \pi_2 + 3\beta \pi_3 = 0 \tag{40}$$

$$2\alpha \pi_2 - 3(\alpha + \beta) \pi_3 + 4\beta \pi_4 = 0 \tag{41}$$

\vdots

Solve (38) first, and this gives $\pi_1 = 0$. Using “forward-substitution” in (39), (40), (41), ... gives $\pi_2 = \pi_3 = \pi_4 = \dots = 0$. Thus, the only stationary distribution is \mathbf{e}_1 , which is

$$[1, 0, 0, 0, \dots].$$

Once being absorbed into state 0, the process cannot go back into other states, so the process is not time reversible.

- Derive an equation for $\mu_t = \mathbb{E}[X_t]$ and solve it for initial condition $\mu_0 = 1$.

Sol. For any function $f : S \mapsto \mathbb{R}$,

$$\begin{aligned} (\mathcal{L}f)(x) &= G[f](x) \\ &= \sum_{y \neq x} G_{x,y} [f(y) - f(x)] \\ &= x\beta[f(x-1) - f(x)] + x\alpha[f(x+1) - f(x)] \end{aligned}$$

So

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(X_t)] &= \mathbb{E}[(\mathcal{L}f)(X_t)] \\ &= \mathbb{E}[X_t\beta[f(X_t-1) - f(X_t)] + X_t\alpha[f(X_t+1) - f(X_t)]] \end{aligned}$$

Let $f(x) = x$, then we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[X_t] &= \mathbb{E}[X_t\beta(X_t-1-X_t) + X_t\alpha(X_t+1-X_t)] \\ &= \mathbb{E}[-\beta X_t + \alpha X_t] \\ &= (\alpha - \beta) \mathbb{E}[X_t], \end{aligned}$$

whose general solution is give by

$$\mathbb{E}[X_t] = C e^{(\alpha-\beta)t}. \quad (42)$$

By the intial condition $\mathbb{E}[X_0] = \mu_0 = 1$, we know $C = 1$ in (42). So the solution is

$$\mu_t = e^{(\alpha-\beta)t}.$$

- Set up a recursion for the “extinction probability” $h_x = \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.

Sol. Since we consider the limit of X_t as $t \rightarrow \infty$, it is reasonable to assume t is large enough so that jumps can happen. Obviously, we have $h_0 = 1$, so now assume $x > 0$.

Let $J_1 = \inf\{t > 0 : X_t \neq X_0\}$ be the time of the first jump, so

$$\mathbb{P}[X_{J_1} = x+1 | X_0 = x] = \frac{G_{x,x+1}}{|G_{x,x}|} = \frac{x\alpha}{x(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta},$$

and

$$\mathbb{P}[X_{J_1} = x-1 | X_0 = x] = \frac{G_{x,x-1}}{|G_{x,x}|} = \frac{x\beta}{x(\alpha+\beta)} = \frac{\beta}{\alpha+\beta}.$$

Now condition on X_{J_1} , we have

$$\begin{aligned} h_x &= \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x] \\ &= \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x, X_{J_1} = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\ &\quad + \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x, X_{J_1} = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \\ &= \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_{J_1} = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\ &\quad + \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_{J_1} = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \quad (\text{by the Markov property}) \\ &= \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x+1] \cdot \mathbb{P}[X_{J_1} = x+1 | X_0 = x] \\ &\quad + \mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x-1] \cdot \mathbb{P}[X_{J_1} = x-1 | X_0 = x] \quad (\text{by homogeneity}) \\ &= h_{x+1} \cdot \frac{\alpha}{\alpha+\beta} + h_{x-1} \cdot \frac{\beta}{\alpha+\beta} \end{aligned} \quad (43)$$

The characteristic equation of (43) is

$$\begin{aligned} t^2 \cdot \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} &= t \\ \alpha t^2 - (\alpha + \beta)t + \beta &= 0 \\ (\alpha t - \beta)(t - 1) &= 0, \end{aligned}$$

whose roots are

$$t_1 = \frac{\beta}{\alpha} \quad \& \quad t_2 = 1.$$

If $\alpha = \beta$, then $t_1 = t_2 = 1$, and the general solution to (43) is

$$\begin{aligned} h_n &= a \cdot 1^n + b \cdot n \cdot 1^2 \\ &= a + bn, \end{aligned}$$

where $a, b \in \mathbb{R}$ are two constants. By $h_0 = 1$, we have $a = 1$, so $h_x = 1 + bx$. But h_x is a probability, which means it should range between 0 and 1, so we have $b = 0$. So in the case of $\alpha = \beta$, $h_x \equiv 1$.

Now suppose $\alpha \neq \beta$, so $t_1 \neq t_2$. Then the general solution to (43) is

$$\begin{aligned} h_n &= a \cdot 1^n + b \cdot \left(\frac{\beta}{\alpha}\right)^n \\ &= a + b \left(\frac{\beta}{\alpha}\right)^n, \end{aligned}$$

where $a, b \in \mathbb{R}$ are two constants. Again, use the boundary condition $h_0 = 1$, and we have $a = 1 - b$. Thus,

$$h_x = 1 - b + b \left(\frac{\beta}{\alpha}\right)^x.$$

If $\beta > \alpha$, then $\frac{\beta}{\alpha} > 1$, and as a result, $\lim_{x \rightarrow \infty} (\beta/\alpha)^x = \infty$. In order that h_x is a probability, b has to be 0. So in this case $h_x \equiv 1$.

If $\beta < \alpha$, then $\frac{\beta}{\alpha} < 1$, and thus, $(\beta/\alpha)^x \leq 1$ for all $x \geq 0$. To ensure $h_x \leq 1$, let

$$\begin{aligned} 1 - b + b \left(\frac{\beta}{\alpha}\right)^x &\leq 1 \\ \left(\left(\frac{\beta}{\alpha}\right)^x - 1\right) b &\leq 0 \\ b &\geq 0. \end{aligned}$$

In order to make h_x nonnegative, we need

$$\begin{aligned} 0 &\leq 1 - b + b \left(\frac{\beta}{\alpha}\right)^x \\ 1 &\geq \left(1 - \left(\frac{\beta}{\alpha}\right)^x\right) b \\ b &\leq \frac{1}{1 - \left(\frac{\beta}{\alpha}\right)^x} \\ b &\leq 1. \end{aligned}$$

So the range of b is $[0, 1]$.

To conclude, the solution to (43) is

$$h_x = \begin{cases} 1 & \beta \geq \alpha > 0 \\ 1 - b + b \left(\frac{\beta}{\alpha}\right)^x & 0 < \beta < \alpha \end{cases},$$

where $b \in [0, 1]$ is a constant.

Obviously, we have made $1 - b + b \left(\frac{\beta}{\alpha}\right)^x \leq 1$ when we determined the range of b , so the smaller solution is

$$1 - b + b \left(\frac{\beta}{\alpha}\right)^x,$$

with $0 < \beta < \alpha$ and $b \in [0, 1]$. Let $H(b) = 1 - b + b \left(\frac{\beta}{\alpha}\right)^x$, then

$$H'(b) = -1 + \left(\frac{\beta}{\alpha}\right)^x \leq 0,$$

which means $H(b)$ is monotonically decreasing in $[0, 1]$. So the smallest solution is reached when $b = 1$, and in this circumstance,

$$h_x = \left(\frac{\beta}{\alpha}\right)^x.$$

- Is the process ergodic?

Sol. By (30), the process is ergodic if and only if

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} < \infty.$$

The first condition is

$$\begin{aligned} \infty &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta_n}{\alpha_n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta n}{\alpha n} \\ &= \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\beta}{\alpha} \\ &= \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i, \end{aligned}$$

which is equivalent to $\beta \geq \alpha$.

As for the second condition, we have

$$\sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha_{n-1}}{\beta_n} = \sum_{i=1}^{\infty} \prod_{n=1}^i \frac{\alpha(n-1)}{\beta n} = 0 < \infty.$$

So the process is ergodic if and only if $\beta \geq \alpha$.

Reference

- [KM57] Samuel Karlin and James McGregor. “The classification of birth and death processes”. In: *Transactions of the American Mathematical Society* 86.2 (1957), pp. 366–400.