2. Nonlinear equations

2.1. Equations with one unknown

Nonlinear equations can be divided into two classes - algebraic and transcendental.

Algebraic equations are equations that contain only algebraic functions (integer, rational, irrational).

In particular, a polynomial is an entire algebraic function.

Equations containing other functions (trigonometric, exponential, logarithmic, etc.) are called transcendental.

Methods for solving nonlinear equations are divided into direct and iterative.

Direct methods allow you to write roots in the form of some finite relation (formula).

The algorithm for finding the root using the iterative method consists of two stages:

a) finding an approximate value of the root or a segment containing it;

b) refinement of the approximate value to a certain specified degree of accuracy.

The iterative process consists of sequential refinement of the initial approximation x_0.

Each such step is called an iteration.

2.2 Method of dividing a segment in half (bisection method)

Let's write the equation in the form F(x) = 0

Let's say we found a segment $[a,b]$,

in which the desired value of the root is located

$x = c$, those $a < c < b.$

As an initial approximation of the root c_0

we take the middle of this segment, thus $c_0 = (a+b)/2.$

Next we examine the values of the function $F(x)$

at the ends of the segments $[a, c_0]$ **and** $[c_0, b]$

thus at points $a, c_0, b.$

That segment at the ends of which $F(x)$

takes values of different signs, contains the desired root, so it is accepted as a new segment.

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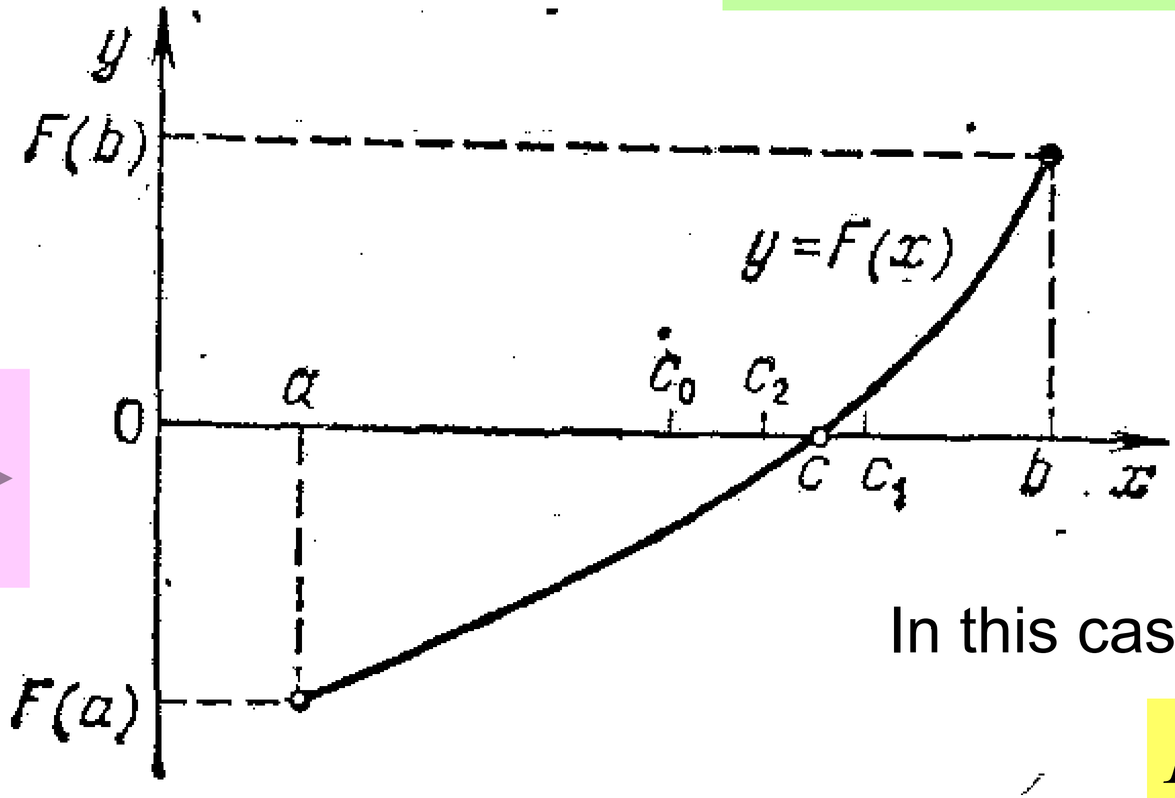
The other half of the segment $[a,b]$ on which the sign $F(x)$ does not change, discard it.

We take the middle of the new segment as the first iteration of the root, and so on.

Thus, after each iteration, the segment on which the root is located is halved.

Let for certainty $F(a) < 0, F(b) > 0$



As the initial approximation of the root we take

$$c_0 = (a+b)/2$$

In this case

$$F(c_0) < 0,$$

therefore we consider only the segment $[c_0, b].$

Next approximation: $c_1 = (c_0 + b)/2$

We similarly find other approximations:

$$c_2 = (c_0 + c_1)/2$$

and so on

We continue the iterative process until the value of the function $F(x)$

after the nth iteration will not be less than the absolute value of some given small number ε,

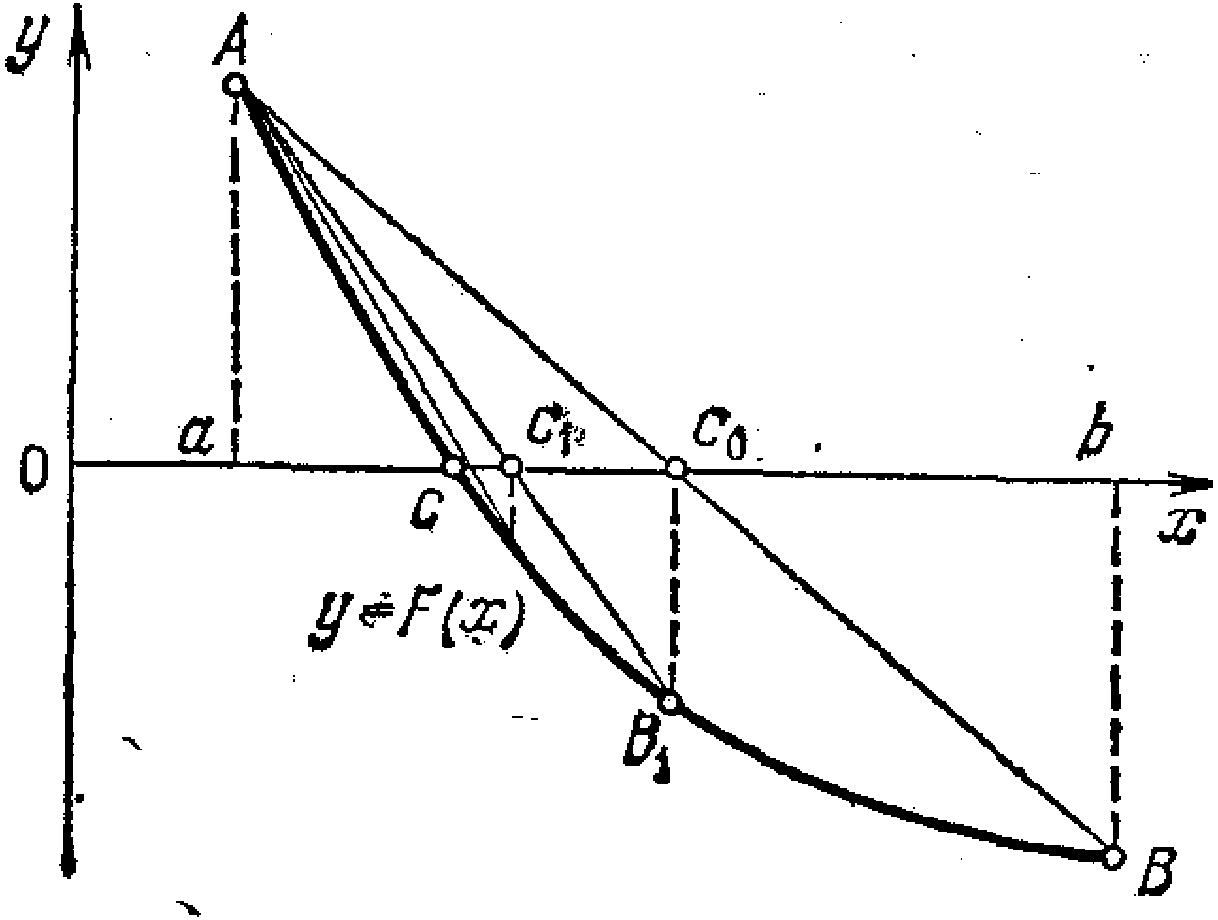
Thus $|F(c_n)| < \varepsilon.$

The method of dividing a segment in half is quite slow, but it always converges, i.e. when using it, a solution is always obtained, and with a given accuracy.

2.3 Chord method.

Let us find a segment $[a,b]$, on which the function $F(x)$ changes sign.

Let's say $F(a) > 0, F(b) < 0.$



In this method, the iteration process is that as approximations to the root of the equation

$$F(x)=0 \quad (3.1)$$

values accepted $c_0, c_1, \ldots$

points of intersection of the chord with the abscissa axis.

First we find the equation of the chord AB:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$ (equation of a line passing through two points)

$$\frac{y-F(a)}{F(b)-F(a)}=\frac{x-a}{b-a}.$$

For the point of intersection of the chord with the abscissa axis

$$(x=c_0,\, y=0)$$

we get the equation

$$\frac{0-F(a)}{F(b)-F(a)}=\frac{c_0-a}{b-a}.$$

$$c_0-a=-\frac{F(a)}{F(b)-F(a)}(b-a)$$

$$c_0=a-\frac{b-a}{F(b)-F(a)}F(a) \qquad (5.2)$$

Comparing the signs of functions $F(a)$ $F(b)$ **and** $F(c_0)$

we come to the conclusion that the root is in the segment $[a, c_0]$.

The next iteration is to define a new approximation

c_1, as the intersection points of chord AB1

and x-axis, and so on

The iterative process continues until the value

$$F(c_n)$$

will be modulo less than a given number ε.

The algorithms for the method of dividing a segment in half and the method of chords are similar, but the second of them in some cases gives faster convergence of the iterative process.

At the same time, the success of its application, as in the method of dividing a segment in half, is guaranteed.

2.4 Newton's method

(tangent method).

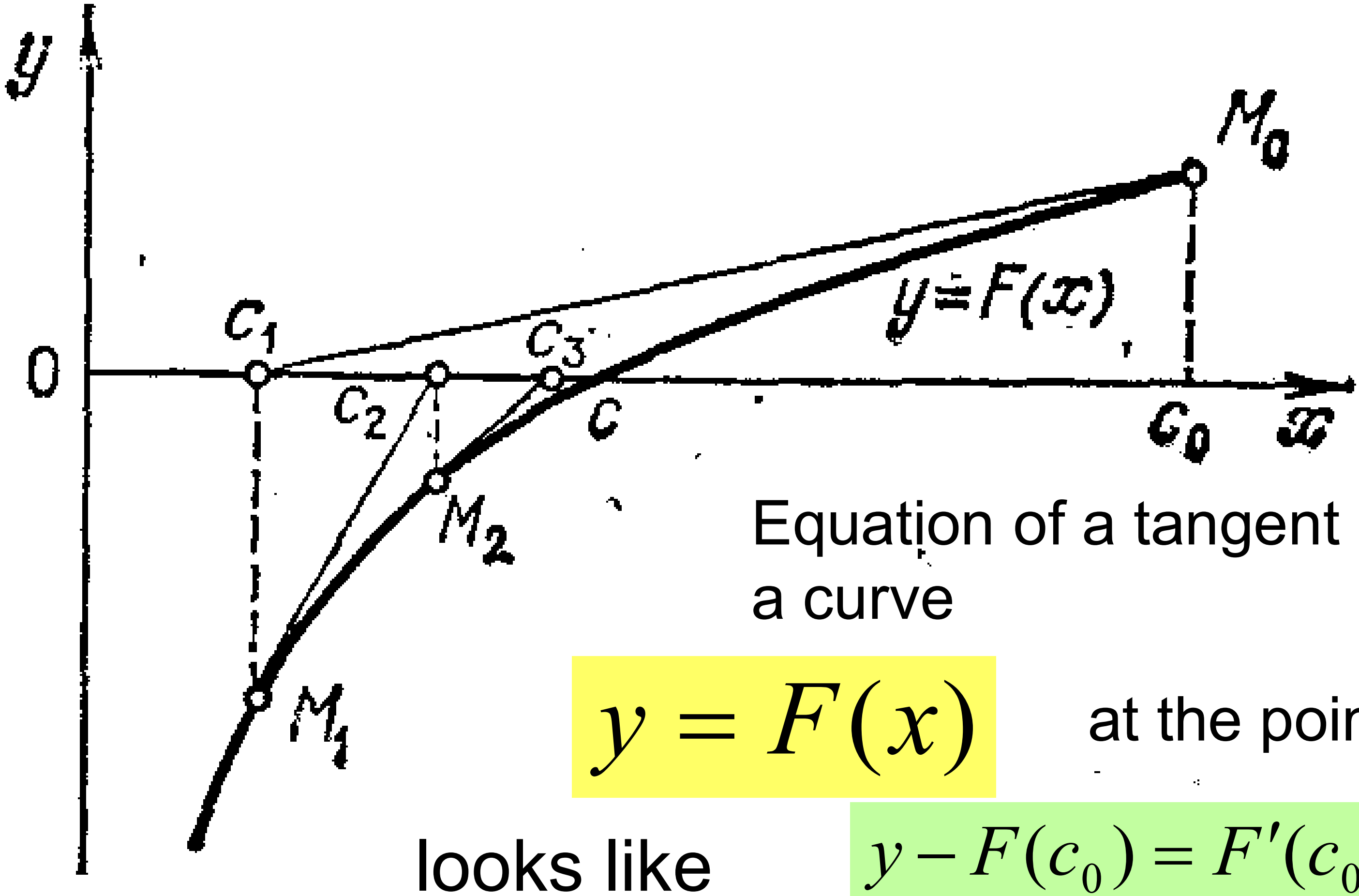
Its difference is that at the kth iteration, instead of a chord, a tangent to the curve is drawn

$y = F(x)$ at $x = c_k$

this is the point of intersection of the tangent with the x-axis.

In this case, it is not necessary to specify a segment $[a,b]$,

containing the root of the equation, and it is enough just to find some initial approximation of the root $x = c_0$.



Equation of a tangent drawn to a curve

$y = F(x)$ at the point $M_0(c_0, F(c_0))$

looks like $y - F(c_0) = F'(c_0)(x - c_0).$

From here we find the next approximation of the root c_1

as the abscissa of the point of intersection of the tangent with the axis $x \quad (y = 0)$

$$0 - F(c_0) = F'(c_0)(c_1 - c_0).$$

$$c_1 - c_0 = -F(c_0) / F'(c_0)$$

$$c_1 = c_0 - F(c_0) / F'(c_0)$$

Similarly, the following approximations can be found as the points of intersection with the abscissa axis of the tangents drawn at the points $M_1, M_2, \ldots$

Formula for $n+1$ th approximation has the form

$$\boxed{c_{n+1} = c_n - F(c_n) / F'(c_n) \quad (4.3)}$$

In this case, it is necessary that the derivative

was not equal to zero $F'(c_n) \neq 0$

To end the iterative process, the condition can be used $|F(c_n)| < \varepsilon,$

The amount of calculations in Newton's method is greater than in the previously discussed methods, since it is necessary to find the value of not only the functions $F(x),$ but also its derivative.

However, the convergence rate here is much higher than in other methods.

The difficulty in applying Newton's method is the choice of initial approximation. Therefore, it is sometimes convenient to use a mixed algorithm.

First, the always convergent method is used (for example, the method of dividing a segment in half), and after a certain number of iterations, Newton’s quickly convergent method is used.