4. Fourier transform.

The Fourier transform is currently widely used in radio electronics for modeling signal conversion and for creating a neuro-computer interface. Neuro-computer interface is a set of software for controlling various devices using thought.

Function $y = f(x)$

is called periodic with period T>0 if the equality

$$f(x+T) = f(x)$$

Fourier series expansion is used only for periodic functions.

Definition. Functional series of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x +$$

$$+ b_3 \sin 3x + a_3 \cos 3x + \ldots + a_n \cos nx + b_n \sin nx + \ldots =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad (4.1)$$

called trigonometric series or Fourier series.

Constant numbers $a_0, a_1, b_1, a_2, b_2, \ldots$

are called coefficients of the trigonometric series.

Let the function be given $f(x)$ with a period of 2π.
It is represented by a trignometric series converging in a given function in the interval $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (4.2)$$

Let us determine the coefficients of the series.
Let's integrate the left and right sides of expression (4.1)

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx dx + \int_{-\pi}^{\pi} b_n \sin nx dx\right).$$

Let us separately calculate the integrals appearing on the right side.

$$\int_{-\pi}^{\pi} \frac{a_0}{2}dx = \frac{a_0}{2}x\Big|_{-\pi}^{\pi} = \frac{a_0}{2}\cdot(\pi-(-\pi)) = \pi a_0,$$

$$\int_{-\pi}^{\pi} a_n \cos nx dx = \frac{a_n}{n}\sin nx\Big|_{-\pi}^{\pi} = \frac{a_n}{n}(\sin n\pi + \sin n\pi) = 0,$$

$$\int_{-\pi}^{\pi} b_n \sin nx dx = -\frac{b_n}{n}\cos nx\Big|_{-\pi}^{\pi} = -\frac{b_n}{n}(\cos n\pi - \cos(-n\pi)) = 0.$$

Consequently

$$\int_{-\pi}^{\pi} f(x)dx = \pi a_0,$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx. \qquad (4.3)}$$

The remaining coefficients can be calculated using the formulas

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad (4.4)}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \qquad (4.5)}$$

The coefficients determined by formulas (4.3), (4.4) and (4.5) are called Fourier coefficients, and the trigonometric series (4.2) is called the Fourier series.

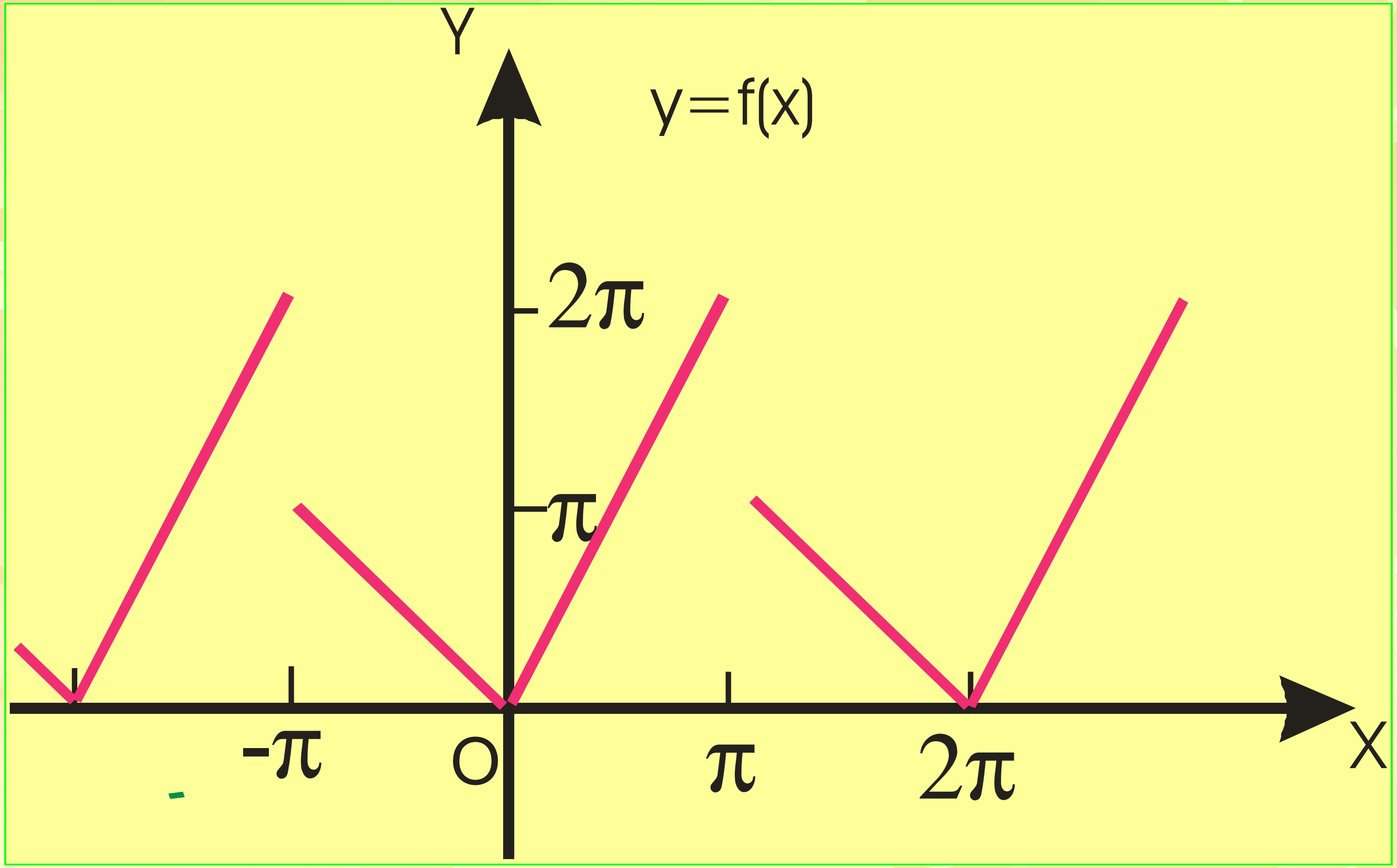
$$f(x)$$

Example.

Expand the $f(x)$ function into a Fourier series with a period of 2π specified on the interval $[-\pi,\pi]$ by the formula

$$f(x) = \begin{cases} 2x, & 0 \le x \le \pi, \\ -x, & -\pi \le x < 0. \end{cases}$$

Solution. Let's draw a graph of the function $f(x)$.



Finding the coefficients of the series

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)dx + \int_{0}^{\pi} 2xdx\right) =$$

$$= \frac{1}{\pi}\left(-\frac{x^2}{2}\Bigg|_{-\pi}^{0} + x^2\Bigg|_{0}^{\pi}\right) = \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \frac{3}{2}\pi,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx =$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}-x\cos nx dx+\int_{0}^{\pi}2x\cos nx dx\right)=\begin{bmatrix} x=u, dx=du, \\ d\upsilon=\cos nx dx, \\ \upsilon=\frac{1}{n}\sin nx \end{bmatrix}=$$

$$=\frac{1}{\pi}\left(-\frac{1}{n}x\sin nx\Big|_{-\pi}^{0}+\frac{1}{n}\int_{-\pi}^{0}\sin nx dx+\frac{2}{n}x\sin nx\Big|_{0}^{\pi}-\frac{2}{n}\int_{0}^{\pi}\sin nx dx\right)=$$

$$=\frac{1}{\pi}\left(-\frac{1}{n^2}\cos nx\Big|_{-\pi}^{0}+\frac{2}{n^2}\cos nx\Big|_{0}^{\pi}\right)=$$

$$=\frac{1}{\pi}\left(-\frac{1}{n^2}+\frac{1}{n^2}\cos n\pi+\frac{2}{n^2}\cos n\pi-\frac{2}{n^2}\right)=$$

$$=\frac{1}{\pi}\left(-\frac{3}{n^2}+\frac{3}{n^2}\cos n\pi\right)=\frac{3}{\pi n^2}(-1+\cos n\pi)=$$

$$=\frac{3}{\pi n^2}(-1+(-1)^n),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}-x\sin nx dx+\int_{0}^{\pi}2x\sin nx dx\right)=\left[\begin{array}{l}x=u, dx=du,\\ d\upsilon=\sin nx dx,\\ \upsilon=-\frac{1}{n}\cos nx\end{array}\right]=$$

$$=\frac{1}{\pi}\left(\frac{1}{n}x\cos nx\Big|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\cos nx dx-\frac{2}{n}x\cos nx\Big|_{0}^{\pi}+\frac{2}{n}\int_{0}^{\pi}\cos nx dx\right)=$$

$$=\frac{1}{\pi}\left(\frac{\pi}{n}\cos n\pi-\frac{1}{n^2}\sin nx\Big|_{-\pi}^{0}-\frac{2\pi}{n}\cos n\pi+\frac{2}{n^2}\sin nx\Big|_{0}^{\pi}\right)=$$

$$=\frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right)=-\frac{1}{n}(-1)^n=\frac{(-1)^{n+1}}{n}.$$

Original function $f(x)$ corresponds to the series

$$f(x)\sim S(x)=\frac{3}{4}\pi+\sum_{n=1}^{\infty}\left(-\frac{3}{\pi n^2}(1-(-1)^n)\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right).$$

$$f(x)=S(x)=\frac{3}{4}\pi-\frac{6}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)+$$

$$+\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots$$

Let's consider computer simulation of the Fourier series

4.1 Fourier series expansion of even and odd functions

If the function $f(x)$ expanded on the interval [-π,π] in the Fourier series is even or odd, then

this will be reflected in the formulas of the Fourier coefficients (their calculation is simplified) and the series itself becomes incomplete.

If the function $f(x)$ is even, then its Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \quad (4.6)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx. \quad (4.7)$$

This is due to the fact that the product of an even function and an odd one gives an odd function, and the integral on the symmetric limits of an odd function is equal to zero. Thus, the expansion of an even function into a Fourier series will contain "only cosines".

If the function $f(x)$ odd, then the product

$f(x)\cos nx$ there is an odd function.

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0, \qquad (4.8)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \qquad (4.9)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx, \quad (4.10)$$

Thus, the Fourier series of an odd function contains “only sines”

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \quad (11.62)$$