Homework 3

Quantum computation

Task 1.

a) Calculate the probability to measure the first qubit in state 1 when we have the following 2-qubit system:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{-i}{2} \\ \frac{i}{2} \\ \frac{-1}{2} \end{pmatrix}$$

Solution: Probability for state 10 is $\left|\frac{i}{2}\right|^2 = \frac{1}{4}$ + probability for state 11 is $\left|\frac{-1}{2}\right|^2 = \frac{1}{4}$, so total probability is $\frac{1}{2}$.

b) Calculate the probability to measure the second qubit in state 0 when we have the following 2-qubit system:

$$\begin{pmatrix} \frac{1}{3} \\ \frac{i\sqrt{2}}{3} \\ \frac{2-i}{3} \\ \frac{-1}{3} \end{pmatrix}$$

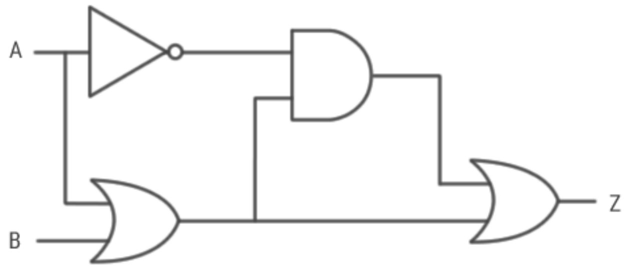
Solution: Probability for state 00 is $\left|\frac{1}{3}\right|^2 = \frac{1}{9}$ + probability for state 10 is $\left|\frac{2-i}{9}\right|^2 = \frac{5}{9}$, so total probability is $\frac{2}{3}$.

c) Calculate the probability to measure the state 11 when we have the following 2-qubit system:

$$\begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{-i}{3} \\ \frac{2-i\sqrt{3}}{3} \end{pmatrix}$$

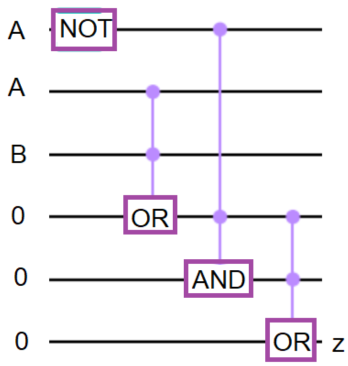
Solution: Probability for state 11 is $\left|\frac{2-i\sqrt{3}}{3}\right|^2 = \frac{7}{9}$.

Task 2. a) Implement the following classical circuit reversibly:



This circuit can be expressed as ((NOT A) AND (A OR B)) OR (A OR B)

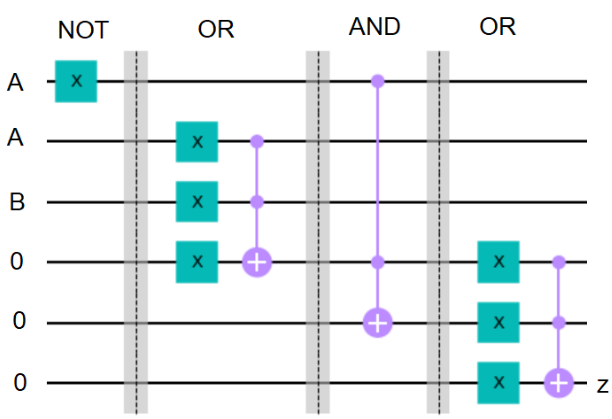
Solution:



b) Implement this circuit as a quantum circuit by using additional qubits, Toffoli gates and NOT gates.

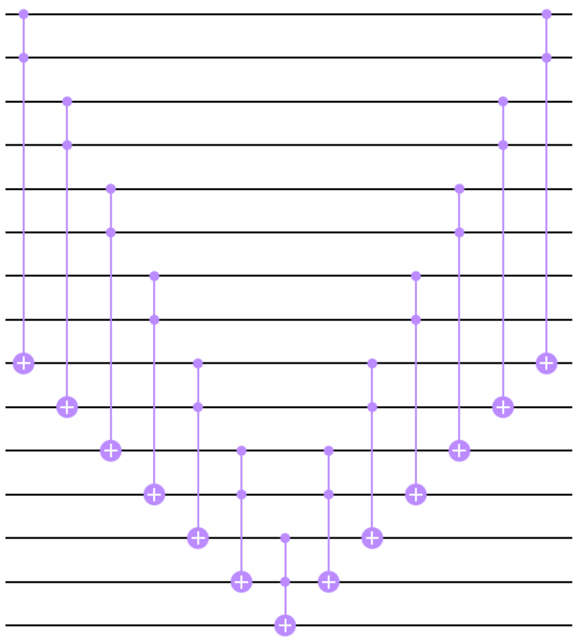
Remark: in task b) you can consider transforming OR operator into combination of AND and NOT operators by using de-Morgan laws.

Solution:



Task 3. Implement the quantum circuit that takes values of 8 qubits as input and puts the result of their multiplication in output qubit (which means output is equal to 1 only if all 8 qubits are in state 1; otherwise output is equal to 0). For this task use only Toffoli gates. For this implementation you will need ancilla qubits. Please try to make a circuit that has 15 qubits (8 for input, 6 ancillas and 1 for output).

Solution:



Task 4. Analyze behavior of Grover's Search when we have 4 elements and 2 of them are marked. What will be the outcome if we do the measurement after

a) 1 iteration of Grover's Search for such setting?

b) 2 iterations of Grover's Search for such setting?

c) 3 iterations of Grover's Search for such setting?

d) 4 iterations of Grover's Search for such setting?

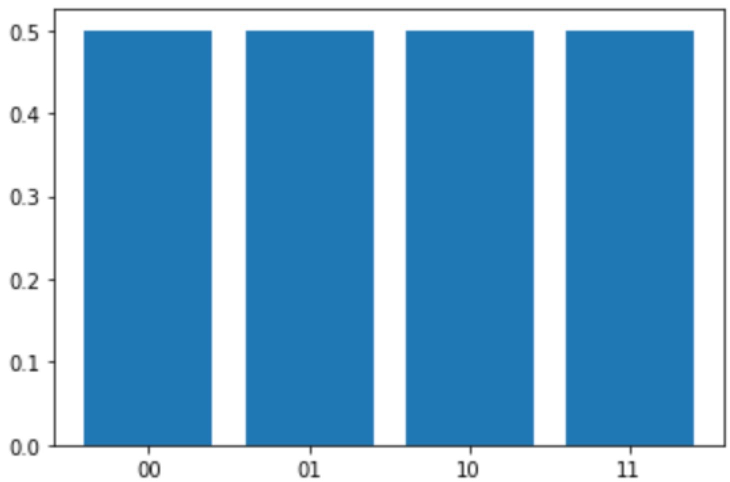
Bonus points if you explain behavior of Grover's Search for cases where exactly half of elements in search space is marked.

Solution:

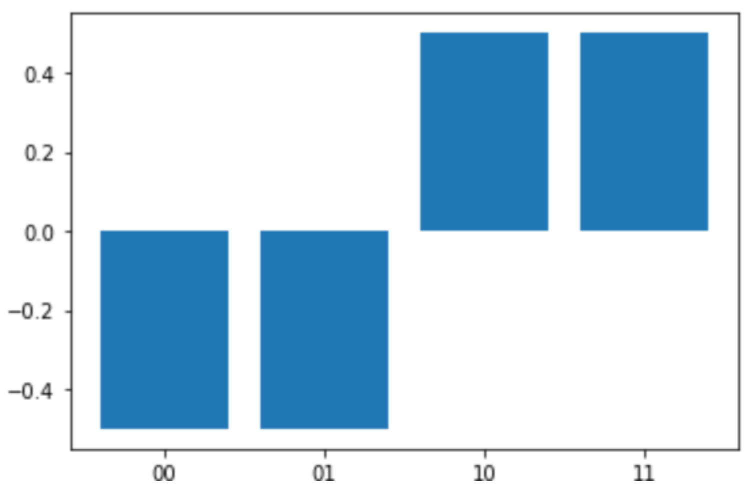
Initially we have an equal superposition for states that denote each of 4 elements. If 2 of elements are marked, after query phase we have 2 elements with positive amplitudes and 2 elements with negative amplitudes. When we apply operator V (diffusion), our state will change only by flipping signs of each amplitude (because average value of amplitudes is equal to 0). If we would do a measurement, we would get each element with equal probability. If we repeat iteration, we get all elements like in the beginning – we will have equal superposition with positive amplitudes. Iteration 3 and 4 will be same as 1 and 2.

Visual explanation with 4 elements, where ‘00’ and ‘01’ are marked. We show amplitude values of the states.

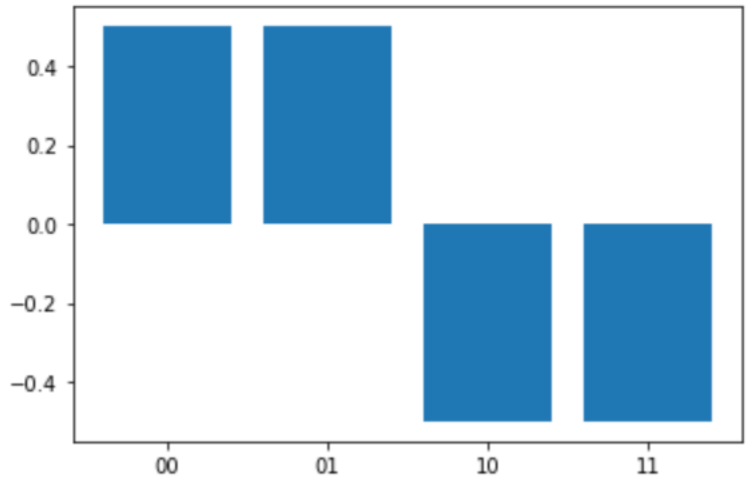
Initial amplitudes:



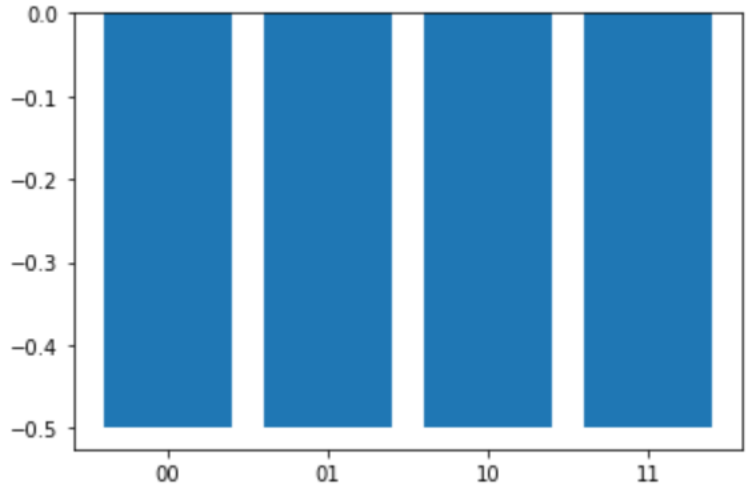
After first query:



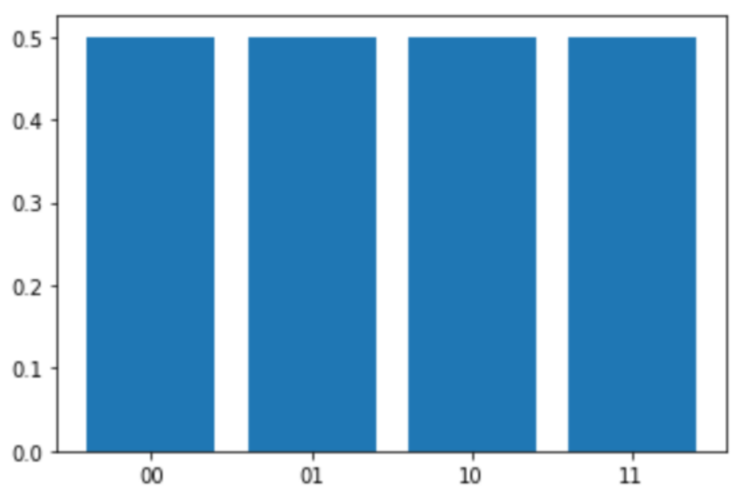
After operator V (end of iteration 1):



After second query:



After operator V (end of iteration 2):



After that situation repeats.

In general, if half of elements is marked, Grover’s Search is inefficient, because evolution of quantum state will not progress towards a solution and will remain in equal superposition.