Quantum Algorithms
Lecture 21
Measuring operators

Zhejiang University

Introduction

More general operator

A measuring operator is a generalization of an operator with quantum control. Such operators are very useful in constructing quantum algorithms. After mastering this tool, we will be ready to tackle difficult computational problems.

Reminder: quantum control was considered when we discussed control-operators such as CNOT and Toffoli ($\Lambda(\sigma^x), \Lambda^2(\sigma^x)$).

Definition and examples

Measuring operator

$$W = \sum_j \Pi_{L_j} \otimes U_j$$

We have a state space $N \otimes K$, $N = \bigotimes_{j \in \{1,\ldots,r\}} L_j$ (pairwise orthogonal subspaces), Π_{L_j} is a projection on a subspace L_j, $U_j \in L(K)$.

We have projections in space N, if a system appears in subspace L_j, then U_j is applied to subspace K.

Measuring

We want to measure $\rho \in L(N)$.
First, add subsystem: joint state is $\rho \otimes |0\rangle\langle 0|$.
Then, apply $W = \sum_j \Pi_{L_j} \otimes U_j$.

$$W\left(\rho \otimes |0^m\rangle\langle 0^m|\right)W^\dagger = \sum_j \left(\Pi_{\mathcal{L}_j}\rho\Pi_{\mathcal{L}_j}\right) \otimes \left(U_j|0\rangle\langle 0|U_j^\dagger\right)$$

Properties of projection were considered:

- $\Pi^\dagger = \Pi$
- $\Pi^2 = \Pi$

Making measuring classical

$$W\Big(\rho \otimes |0^m\rangle\langle 0^m|\Big)W^\dagger = \sum_j \left(\Pi_{\mathcal{L}_j}\rho\Pi_{\mathcal{L}_j}\right) \otimes \left(U_j|0\rangle\langle 0|U_j^\dagger\right)$$

Making additional space classical by applying the decoherence transformation. Therefore, the matrix is diagonalized with respect to the second tensor factor. Here, arrow represents the effect of decoherence:

$$U_j|0\rangle\langle 0|U_j^\dagger \mapsto \sum_k \big|\langle k|U_j|0\rangle\big|^2 \, |k\rangle\langle k|$$

Bipartite mixed state

$$W\left(\rho\otimes|0^m\rangle\langle 0^m|\right)W^\dagger=\sum_j(\Pi_{\mathcal{L}_j}\rho\Pi_{\mathcal{L}_j})\otimes(U_j|0\rangle\langle 0|U_j^\dagger)$$

$$U_j|0\rangle\langle 0|U_j^\dagger\mapsto\sum_k\left|\langle k|U_j|0\rangle\right|^2|k\rangle\langle k|$$

We obtain:

$$\sum_j\sum_k\left(\Pi_{\mathcal{L}_j}\rho\Pi_{\mathcal{L}_j}\left|\langle k|U_j|0\rangle\right|^2,\,k\right)=\sum_j\sum_k\mathbf{P}(k|j)\cdot(\Pi_{\mathcal{L}_j}\rho\Pi_{\mathcal{L}_j},k)$$

Like if we did a measurement, that gives a classical outcome k.

One replacement occurred (conditional probability): $P(k|j)=\left|\langle k|U_j|0\rangle\right|^2$

Probabilistic projective measurement

Transformation summary:

$$T: \ \rho \mapsto \sum_{k,j} \mathbf{P}(k|j) \cdot \left(\Pi_{\mathcal{L}_j} \rho \Pi_{\mathcal{L}_j}, k\right)$$

It can also be viewed as a nondestructive version of the POVM measurement

$$\rho \mapsto \sum_{k} \mathrm{Tr}(\rho X_k) \cdot (k), \quad \text{where } X_k = \sum_{j} \mathbf{P}(k|j) \Pi_{\mathcal{L}_j}$$

Projective measurement is a special case:

$$P(k|j) = \delta_{kj}$$

Examples

The first example is just a unitary operation, controlled by one qubit: $\Lambda(U) = \Pi_0 \otimes I + \Pi_1 \otimes U$.

Here, we apply U to the second subsystem, if first qubit is in state 1.

Examples

In continuation of the example $\Lambda(U)$ was considered as a measuring with respect to the second subsystem.

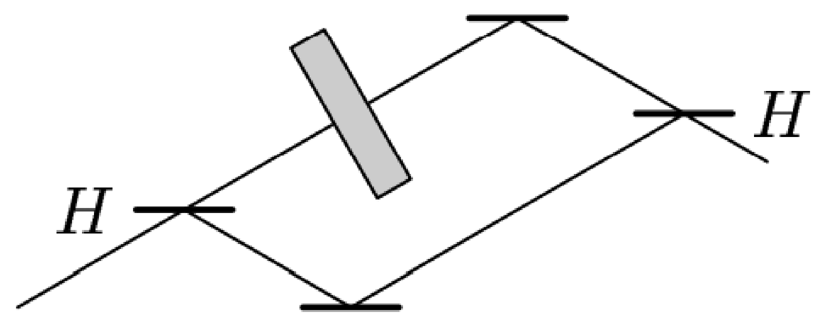
Unitary operator can be decomposed into the sum of projections onto the eigenspaces (related to the Spectral theorem): $U = \sum_j \lambda_j \Pi_{L_j}$

$$\Lambda(U) = \sum_j (\Pi_0 + \lambda_j \Pi_1) \otimes \Pi_{\mathcal{L}_j} = \sum_j \begin{pmatrix} 1 & 0 \\ 0 & \lambda_j \end{pmatrix} \otimes \Pi_{\mathcal{L}_j}$$

Conditional probabilities here are trivial: $P(0|j) = 1$, $P(1|j) = 0$.

Physicist's approach

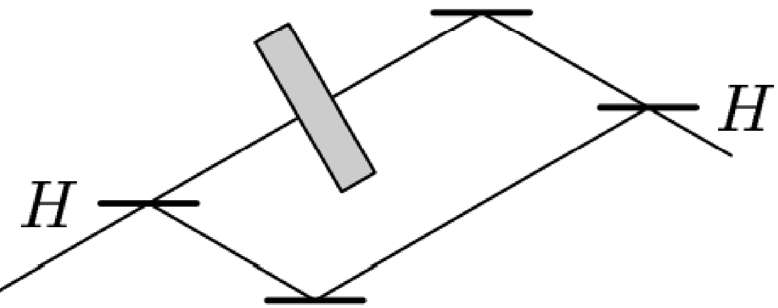
Let U be the operator of phase shift for light as it passes through a glass plate. We can split the light beam into two parts by having it pass through a semitransparent mirror. Then one of the two beams passes through the glass plate, after which the beams merge at another semitransparent mirror (see the diagram). The resulting interference will allow us to determine the phase shift.



Mathematical variant

$$\Xi(U) = (H \otimes I)\Lambda(U)(H \otimes I)\colon B^{\otimes N} \to B^{\otimes N}$$

$$\boldsymbol{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



If the initial vector has the form $|\psi\rangle = |\eta\rangle \otimes |\xi\rangle$ ($|\xi\rangle \in L_j$), then $\Xi(U)|\psi\rangle = |\eta'\rangle \otimes |\xi\rangle$, where

$$|\eta'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda_j \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}|\eta\rangle = \frac{1}{2}\begin{pmatrix} 1+\lambda_j & 1-\lambda_j \\ 1-\lambda_j & 1+\lambda_j \end{pmatrix}|\eta\rangle$$

Mathematical variant

$$|\eta'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \lambda_j\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}|\eta\rangle = \frac{1}{2}\begin{pmatrix}1+\lambda_j & 1-\lambda_j\\ 1-\lambda_j & 1+\lambda_j\end{pmatrix}|\eta\rangle$$

Here λ_j is a phase shift applied to the qubit $|\eta'\rangle$, so that amplitude of state $|1\rangle$ is multiplied by $\lambda_j = e^{\pi i \varphi_j}$.

$$\Xi(U) = \sum_j \overbrace{\frac{1}{2}\begin{pmatrix}1+\lambda_j & 1-\lambda_j\\ 1-\lambda_j & 1+\lambda_j\end{pmatrix}}^{R_j} \otimes \Pi_{\mathcal{L}_j}$$

We obtain the following conditional probabilities:

$$\mathbf{P}(0|j) = \left|\langle 0|R_j|0\rangle\right|^2 = \left|\frac{1+\lambda_j}{2}\right|^2 = \frac{1+\cos(2\pi\varphi)}{2}$$

About chapter 13

The measuring operator

$$\Xi(U) = (H \otimes I)\Lambda(U)(H \otimes I)\colon B^{\otimes N} \to B^{\otimes N}$$

will be used to estimate the eigenvalues of unitary operators. For this, we will need to apply the operator $\Xi(U)$ several times to the same “object” (the space N), but with different “instruments” (copies of the space B). But first we must make sure that this is correct, in the sense that the probabilities multiply as they should (next subchapter).

General properties

Property 1

Fixed orthogonal decomposition $N = \otimes_j L_j$

The product of measuring operators is a measuring operator:

$$W^{(2)}W^{(1)} = \sum_j R_j^{(2)} R_j^{(1)} \otimes \Pi_{\mathcal{L}_j}$$

This is true when projections are of orthogonal subspaces – this is what $\Pi_{L_j}\Pi_{L_k} = \delta_{jk}\Pi_{L_k}$ means, so $\delta_{jk} = 0$ when $j \neq k$.

Property 2

Fixed orthogonal decomposition $N = \otimes_j L_j$

The conditional probabilities for products of measuring operators with "different instruments" are multiplicative. This means, that if measuring operators use different additional subspaces, then

$$P(k1, k2|j) = P(k1|j)P(k2|j)$$

Here is how case with different additional subspaces may look like:

$$\left(\langle\xi_1| \otimes \langle\xi_2|\right)\left(U_1 \otimes U_2\right)\left(|\eta_1\rangle \otimes |\eta_2\rangle\right) = \langle\xi_1|U_1|\eta_1\rangle \langle\xi_2|U_2|\eta_2\rangle$$

Property 3

Fixed orthogonal decomposition $N = \otimes_j L_j$

Formula of total probability:

$$\mathbf{P}\Big(W\left(|0\rangle\langle 0| \otimes \rho\right)W^{\dagger}, \mathbb{C}\left(|k\rangle\right) \otimes \mathcal{N}\Big) = \sum_j \mathbf{P}(k|j)\,\mathbf{P}(\rho, \mathcal{L}_j)$$

The proof carefully uses properties of probability and trace. Not hard to follow if do it slowly, we will discuss during the lecture. Check Quantum probability property q4 from Chapter 10.

Problem 12.1.

The formulation of problem states that if our unitary U is approximated with precision δ, then corresponding measuring operator will be with the same approximation δ.

Garbage removal and composition of measurements

Useful result + garbage

Measurement operators are used to obtain some information about the value of the index j in the decomposition $N = \bigotimes_{j \in \{1,\ldots,r\}} L_j$.

The measurement operator can be written in the form:

$$W = \sum_{j \in \Omega} \Pi_{\mathcal{L}_j} \otimes R_j, \qquad R_j : \mathcal{B}^N \to \mathcal{B}^N, \qquad R_j|0\rangle = \sum_{y,z} c_{y,z}(j)|y, z\rangle$$

$y \in B^m$ is useful, $z \in B^{N-m}$ is garbage.

Useful result + garbage

$y \in B^m$ is useful, $z \in B^{N-m}$ is garbage.

$$\mathbf{P}(y|j) \stackrel{\text{def}}{=} \langle 0 | R_j^\dagger \Pi_{\mathcal{M}_y} R_j | 0 \rangle, \qquad \mathcal{M}_y = \mathbb{C}\big(|y\rangle\big) \otimes \mathcal{B}^{\otimes(N-m)}$$

Conditional probabilities without garbage are similar to ones that we discussed in previous lectures.

Working without garbage

Measuring operator without garbage has general solution where the result y is deterministic, namely, $P(y|j) = \delta_{y,f(j)}$ (equals to 1 if $y = f(j)$, that's why deterministic) for some function $f\colon \Omega \to B^m$ (so that we can say that W actually measures the value of $f(j)$). We can measure $f(j)$, copy the result, and "un-measure".

Three extensions

1. W measures f with some error probability ε -precision the above procedure corresponds to a garbage-free measurement is $\sqrt{2\varepsilon}$.
2. Formula for the conditional probabilities makes sense for an arbitrary orthogonal decomposition of $B^{\otimes N}$.
3. Instead of copying the result, we can apply any operator V that is measuring with respect to the indicated decomposition. The copying corresponds to $V: |y, z, v\rangle \to |y, z, y \oplus v\rangle$.

Problem 12.2.

Problem describes first extension on more details, when we measure a function with error probability. We see that in case of copy operator precision is increased by factor of $\sqrt{2}$. We can check closer the definition and proof of the formula during the lecture. Please try to check it before the beginning of the lecture, since it is a nice summary of the knowledge in the chapter.

Thank you for your attention!