DFA : regular language

Union $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Concatenation $A \circ B = \{xy : x \in A \text{ and } y \in B\}$

Star $A^* = \{w_1 \cdots w_k : w_i \in A,\ k \geq 0\}$

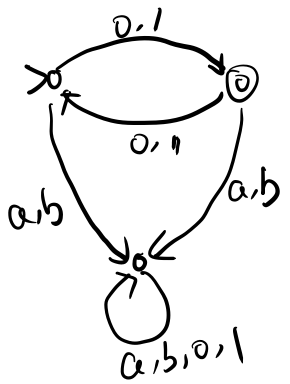
Theorem

If A and B are regular, so is $A \cup B$

A over $\{0,1\}$

B over $\{a,b\}$

over $\{0,1,a,b\}$



M_1

M_2

M_3

$\Sigma = \{a,b,0,1\}$

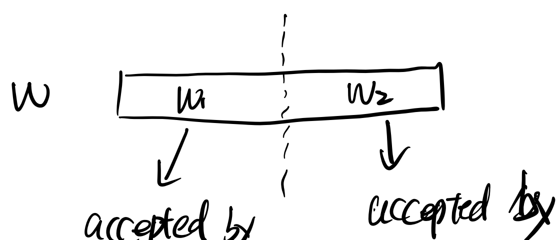
Theorem

If A and B are regular, so is $A \circ B$.

Idea:

$\exists M_1$ accepts A

$\exists M_2$ accepts B $\longrightarrow$ M_3 accepts $A \circ B$



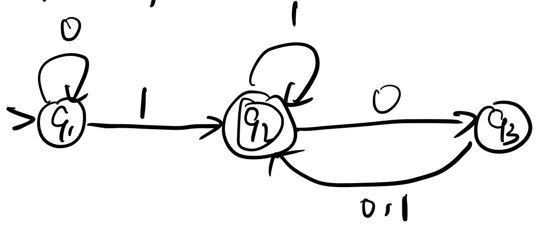
$w \in A \circ B$ M_3 accept w.

M_1 M_2

Non-determinism

DFA

δ: transition function $\delta : K \times \Sigma \to K$

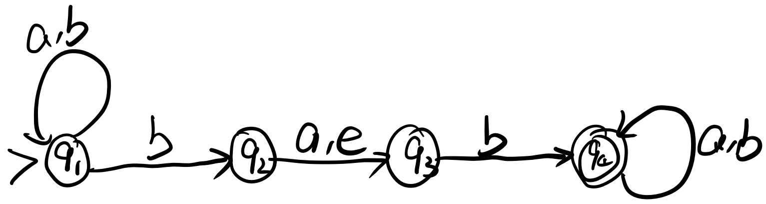


input: 1001

$(q_1, 1001) \vdash_M (q_2, 001) \vdash_M (q_3, 01) \vdash_M (q_2, 1) \vdash_M (q_2, e)$ path

Non-deterministic finite automata (NFA)

1. several choices for the next state
2. may switch states without reading any input symbols.



$K = \{q_1, \ldots, q_4\}$

$\Sigma = \{a, b\}$

$s = q_1$

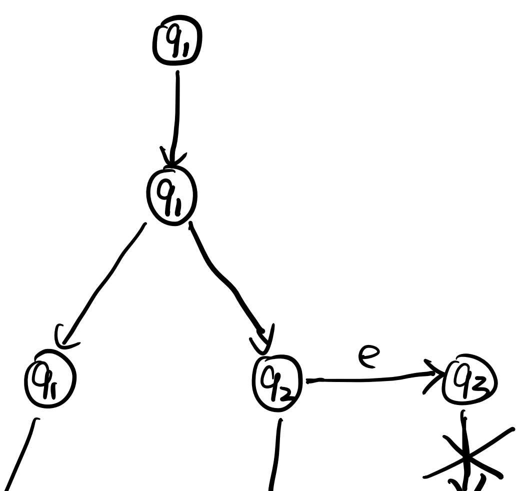
$F = \{q_4\}$

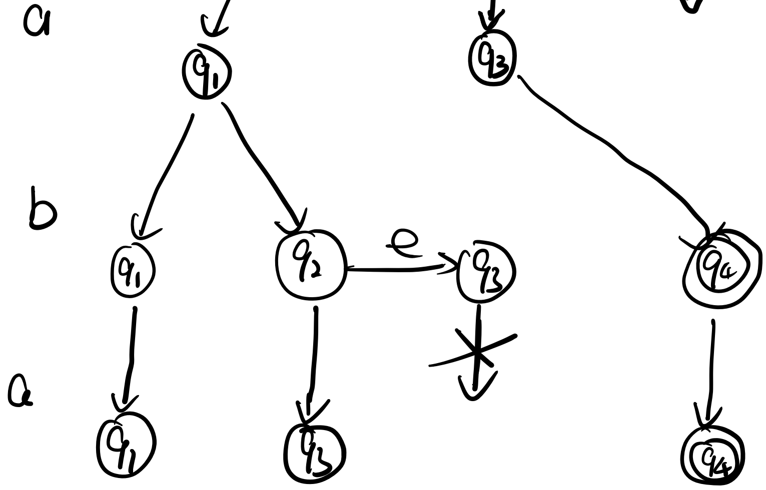
|a|b|a|b|a|

$\Delta = \{ (q_1, a, q_1),$
$(q_1, b, q_1),$
$(q_1, b, q_2),$
$(q_2, a, q_3),$
$(q_2, e, q_3),$ ϕ
$\}$

a

b





Definition

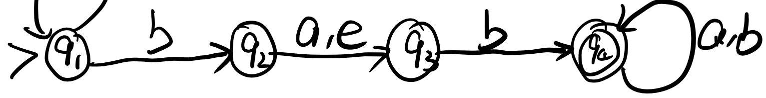
A NFA is 5-tuple $(K, \Sigma, \Delta, s, F)$

- K : a set of states
- Σ : " " of input symbols
- $s \in K$: initial state
- $F \subseteq K$: a set of final state.
- Δ transition relation : a subset of $\underbrace{K}_{\text{current state}} \times \underbrace{(\Sigma \cup \{e\})}_{\text{input symbol}} \times \underbrace{K}_{\text{next state.}}$

$(q, a, p) \in \Delta$

A configuration for a NFA $M = (K, \Sigma, \Delta, s, F)$ is an element of $\underbrace{K}_{\text{current state}} \times \underbrace{\Sigma^*}_{\text{unread input}}$.

$(q, w) \vdash_M (q', w')$ if $\exists u \in \Sigma \cup \{e\}$ s.t $w = uw'$ and $(q, u, q') \in \Delta$



a	b	a	b	a

$\vdash_M (q_1, aba)$

$(q_1, ababa) \vdash_M (q_1, baba) \vdash_M (q_2, aba)$ . . . . . .

$\vdash_M$

(q_3, aba)

$(q, w) \vdash_M^* (q', w')$ if $(q, w) = (q', w')$ or

$\exists (q_0, w_0), \ldots\ldots, (q_i, w_i)$ for some $i \geq 1$

s.t $(q, w) = (q_0, w_0) \vdash_M \cdots\cdots \vdash_M (q_i, w_i) = (q', w')$

NFA M accepts w if $(s, w) \vdash_M^* (q, e)$ for some $q \in F$

$L(M) = \{ w \in \Sigma^* : M \text{ accepts } w \}$

M accept $L(M)$

(recoganizes)

observation

DFA is a special type of NFA

function

$q, a \rightarrow \delta(q, a)$

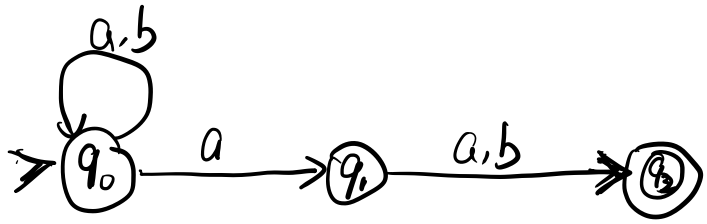
relation

$(q, a, \delta(q, a))$

Exercise

Design NFA to accept

$L = \{ w \in \{a,b\}^* : \text{the second symbol from the end of } w \text{ is } a \}$



Input: aba

bab

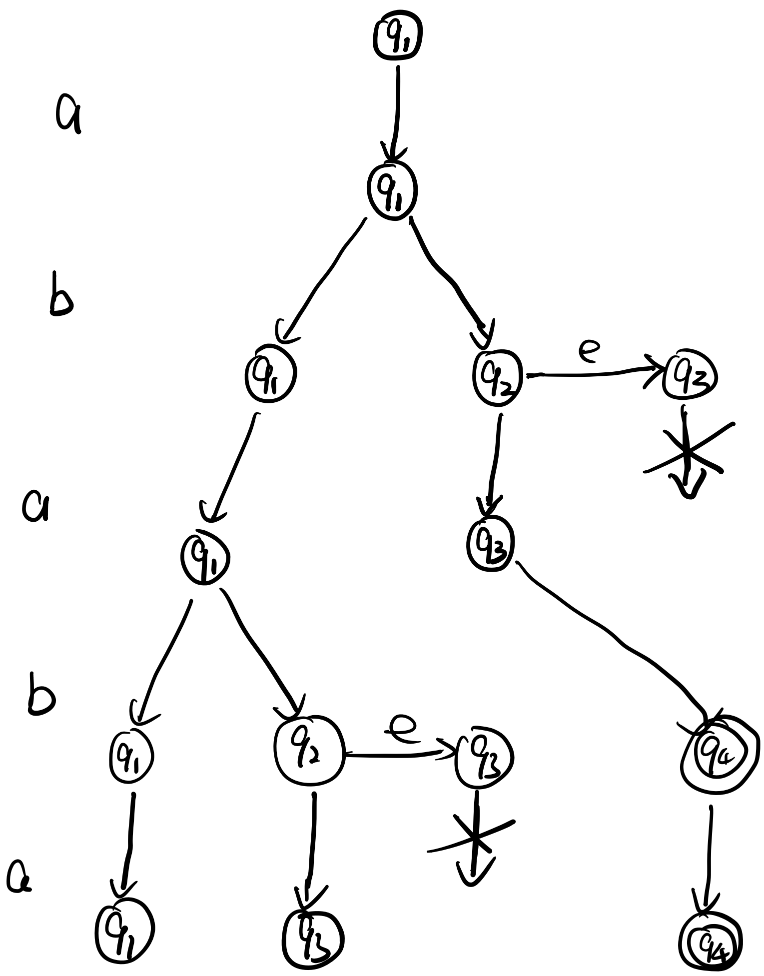
Theorem

For any NFA M, there alway exists DFA M'

such that $L(M) = L(M')$

Idea

M' simulate the tree-like computation of M



$$\{q_1\} \downarrow \{q_1\} \downarrow \{q_1, q_2, q_3\} \downarrow \{q_1, q_3\} \downarrow \{q_1, q_2, q_3, q_4\} \downarrow \{q_1, q_3, q_4\}$$

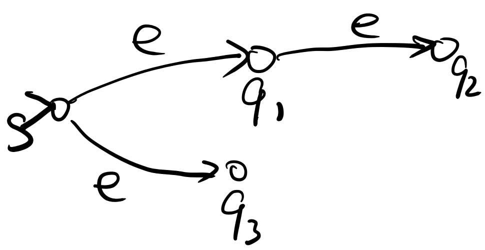
Proof.

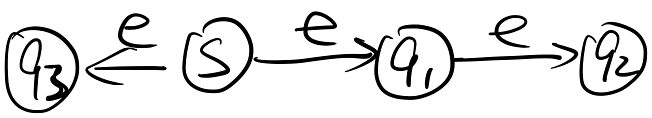
Given NFA $M = (K, \Sigma, \Delta, s, F)$,

Construct a DFA $M' = (K', \Sigma, \delta, s', F')$ s.t $L(M') = L(M)$

$$K' = 2^K = \{Q : Q \subseteq K\}$$

$$s' = \cancel{\{s\}}\ E(s)$$





$\{s, q_1, q_2, q_3\}$

$$\forall q \in K,\ E(q) = \{p \in K : (q, e) \vdash_M^* (p, e)\}$$

$$E(q_0) = \{q_0\}$$
$$E(q_1) = \{q_1, q_2, q_3\}$$
$$E(q_2) = \{q_2, q_3\}$$
$$E(q_3) = \{q_3\}$$

$$\forall Q \subseteq K,\ E(Q) = \bigcup_{q \in Q} E(q)$$

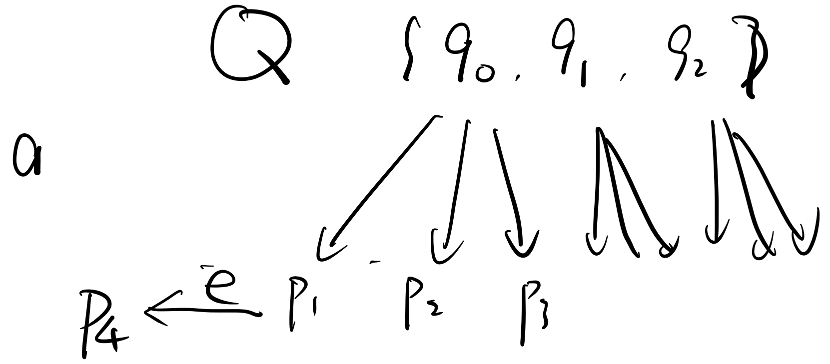
$$F' = \{Q \subseteq K : Q \cap F \neq \emptyset\}$$

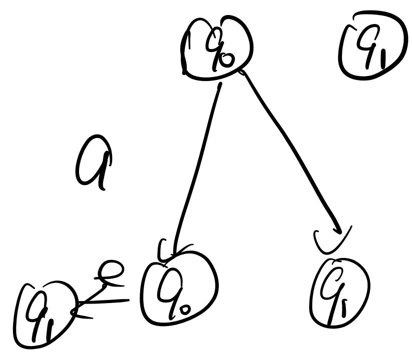
$\delta : 2^K \times \Sigma \longrightarrow 2^K$ transition function

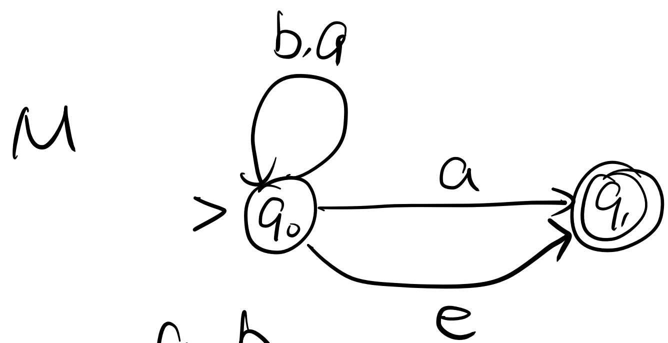
$\forall Q \subseteq K, \ \forall a \in \Sigma$

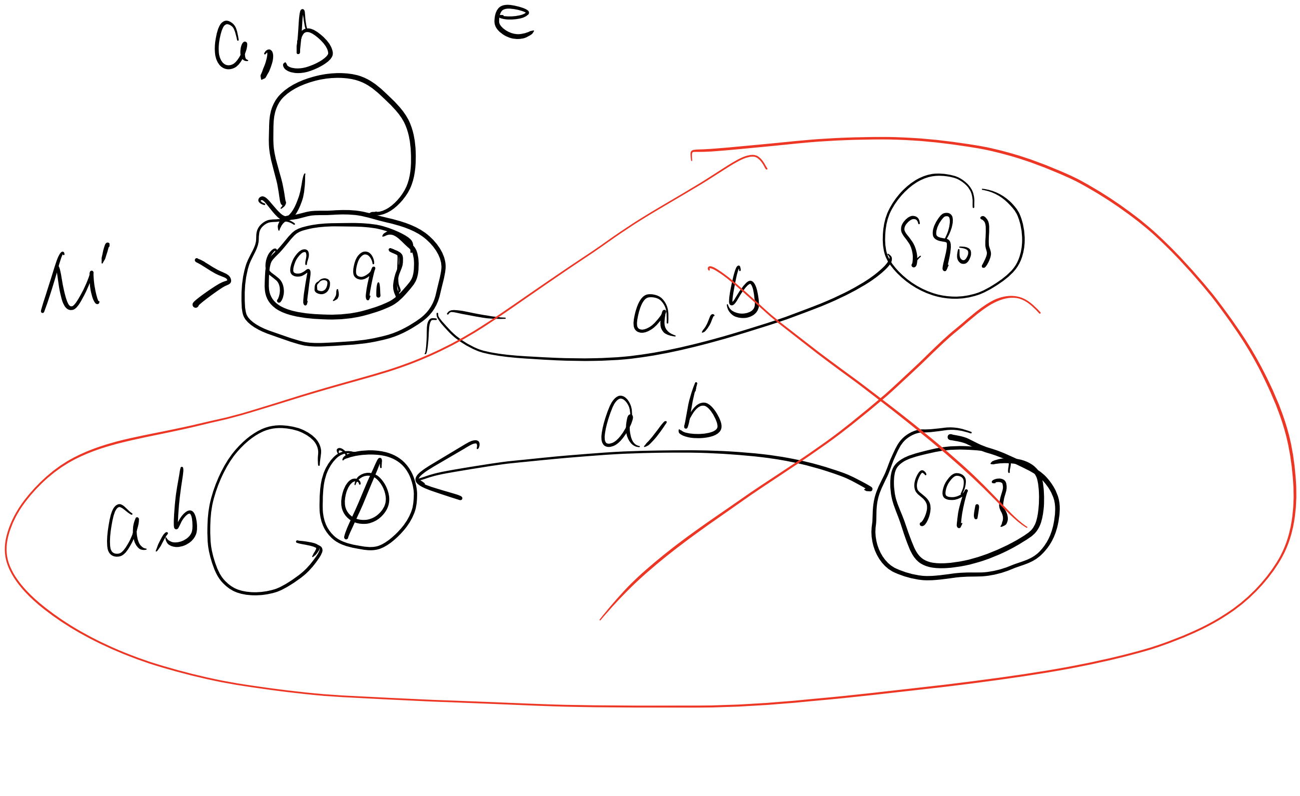
$$\delta(Q, a) = \bigcup_{q \in Q} E\left(\{p \in K : (q, a, p) \in \Delta\}\right)$$

NFA









$L(M') = L(M)$

Claim

for any $p, q \in K$ and $w \in \Sigma^*$

$(p, w) \vdash_M^* (q, e)$ iff $(E(p), w) \vdash_{M'}^* (Q, e)$

for some Q containing q.

by induction on $|w|$

M accepts w

$\Leftrightarrow (s, w) \vdash_M^* (q, e)$ for some $q \in F$

$\Leftrightarrow (E(s), w) \vdash_M^* (Q, e)$ with $q \in Q$

$Q \cap F \neq \emptyset$

$Q \in F'$

$\Leftrightarrow M'$ accept w

Corollary

A language is regular iff it is accepted by some NFA.

Proof:

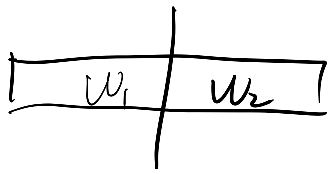
$\Rightarrow$ regular $\Rightarrow \exists$ DFA $\Rightarrow \exists$ NFA

$\Leftarrow$ $\exists$ NFA $\Rightarrow \exists$ DFA $\Rightarrow$ regular

Theorem

If A and B are regular, so is $A \circ B$

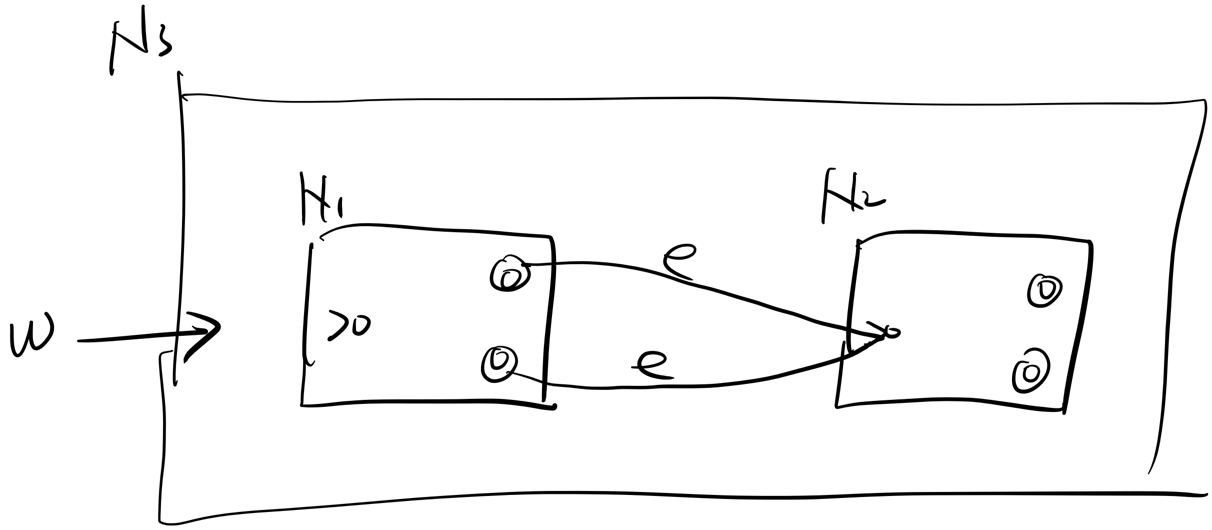
Idea

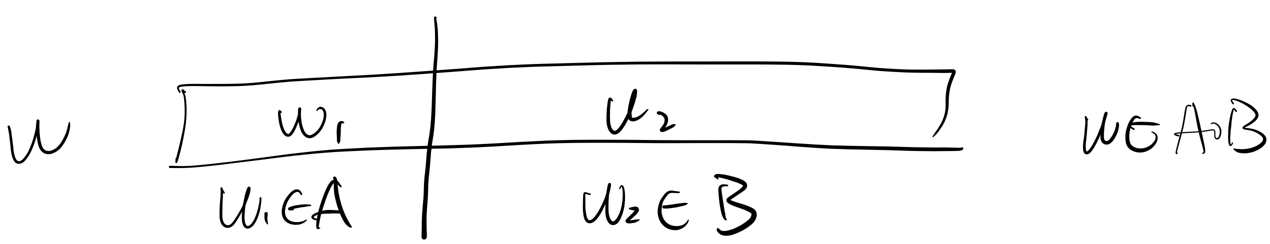


$\exists\, N_1$ accept A

$\exists\, N_2$ accept B

$\longrightarrow$ N_3 accept $A \circ B$





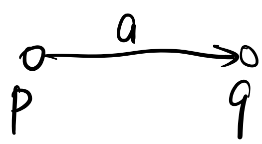
Proof:

$N_1 = (K_1, \Sigma, \Delta_1, s_1, F_1)$

$N_2 = (K_2, \Sigma, \Delta_2, s_2, F_2)$

$N_3 = (K_3, \Sigma, \Delta_3, s_3, F_3)$

- $K_3 = K_1 \cup K_2$
- $s_3 = s_1$
- $F_3 = F_2$
- $\Delta_3 = \Delta_1 \cup \Delta_2 \cup \{(q, e, s_2) : q \in F_1\}$



$(p, e) \vdash_M^* (q, e)$