Numerical Analysis 24 Fall

I. Multiple Choice Questions (16 pts)

1) **Evaluate the function** $f(x) = 2x^2 - 0.1x$ at $x = 5.21$ using 4-digit arithmetic with chopping. What is the result?

(A) 53.75

(B) 53.76

(C) 53.77

(D) 53.74

2) **Given a symmetric, positive real matrix A and initial eigenvalue guesses λ_1^*, λ_2^* such that $|\lambda_1^* - \lambda_1| > |\lambda_2^* - \lambda_2|$**, which iterative method will converge with the best rate?

(A) $x_n = (A - \lambda_1^* I)x_{n-1}$

(B) $x_n = (A - \lambda_2^* I)x_{n-1}$

(C) $(A - \lambda_1^* I)x_n = x_{n-1}$

(D) $(A - \lambda_2^* I)x_n = x_{n-1}$

3) **Which of the following iterative methods is unstable with respect to numerical error growth at x_0?**

(A) $x_{n+1} = 3x_n + 2$

(B) $x_{n+1} = \frac{1}{6}x_n + 100$

(C) $x_{n+1} = \frac{7}{8}x_n + 20$

(D) $x_{n+1} = 0.1x_n + 10$

4) **Given the points $x_0 = 1, x_1 = 2, x_2 = 3$, which of the following is not a Lagrange basis function?**

(A) $-(x-1)(x-3)$

(B) $\frac{(x-1)(x-2)}{2}$

(C) $\frac{(x-2)(x-3)}{2}$

(D) $\frac{(x-1)(x-3)}{2}$

II. Fill in the Blanks (30 pts)

1) **For the equation** $5x^2 + x - 6 = 0$, determine if the following fixed-point iterations starting with $x_0 = 0.9$ are convergent. Fill 'True' if convergent, 'False' if not. (2 pts each)

- $x = \sqrt{\frac{6-x}{5}}$
- $x = 6 - 5x^2$
- $x = \sqrt{\frac{-3x^2-x+6}{2}}$

2) **Given points** $x_0 = 1, x_1 = 2$, and the derivative at x_0, determine the three basis polynomials for Hermite interpolation. (2 pts each)

3) **Given the matrix** $\begin{bmatrix} 100 & 14 \\ 14 & 4 \end{bmatrix}$, find its eigenvalues and condition number under the spectral norm. (2 pts each)

4) **To minimize the local truncation error of the formula**

$$w_{l+1} = a_0 w_l + a_1 w_{l-1} + \beta h f_{l+1}$$

for solving the IVP $y' = f(t, y)$, find the values of a_0, a_1, and β. (2 pts each)

5) **Find the monic polynomials** $\varphi_k(x)$ (for $k = 0, 1, 2$) that are orthogonal on $[0, 4]$ with respect to the weight function $\rho(x) = 1$. (2 pts each)

III. Iterative Method Convergence (12 pts)

Given $A = \begin{bmatrix} 8 & 2 \\ 0 & 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the iterative method

$$\vec{x}^{(k)} = \vec{x}^{(k-1)} + \omega\left(A\vec{x}^{(k-1)} - \vec{b}\right)$$

answer the following:

1) **For which values of ω will the method converge?** (8 pts)

2) **For which values of ω will the method converge the fastest?** (4 pts)

IV. Vector Norm Proof (10 pts)

Prove that $\|X\|_1 = \sum_{i=1}^{n} |X_i|$ is a valid vector norm, where X_i is the i-th component of vector X.

V. Richardson Extrapolation (10 pts)

Given the formula for the second derivative approximation

$$f^*(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} - \frac{h^2}{12} f^{(4)}(x_0) - \frac{h^4}{360} f^{(6)}(\xi),$$

derive a better formula to approximate $f''(x_0)$ with error $O(h^4)$ using Richardson extrapolation.

VI. Least Squares Fit (12 pts)

Find the values of a and b such that $y = ax + bx^3$ fits the following data using least squares, weighted by the given weights:

X	1	2	3
Y	−4	24	6
Weights	1	1/4	1/9

VII. Region of Absolute Stability (10 pts)

For the following methods solving Initial-Value Problems for ODEs, calculate the region of absolute stability using the test equation $y' = \lambda y$ with $\mathrm{Re}\ (\lambda) < 0$. Which method is more stable (or are they the same)?

1) **Second-order Runge-Kutta implicit method**

$$W_{i+1} = w_i + hK_1, \quad K_1 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}K_1\right)$$

2) **Adams-Moulton one-step implicit method**

$$w_{i+1} = w_i + \frac{h}{2}(f_{i+1} + f_i)$$

Answer for Numerical Analysis 24 Fall

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