Homework 4

Quantum computation

Task 1.

Suppose that we have the following (mixed) quantum state: $\frac{1}{3}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{2}{3}\binom{0}{1}$.

a) Write density matrix expression for the mentioned quantum system.

Solution:

$$\rho = \frac{1}{3}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{2}{3}\binom{0}{1}(0 \quad 1) = \frac{1}{3}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

b) Write expression with trace and projection and estimate the probabilities for state 0 and state 1.

Solution:

$$P(\rho, |0\rangle\langle 0|) = Tr(\rho|0\rangle\langle 0|) = Tr\left(\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \frac{1}{6}$$

$$P(\rho, |1\rangle\langle 1|) = Tr(\rho|1\rangle\langle 1|) = Tr\left(\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \frac{5}{6}$$

Task 2.

Consider the decoherence operator D that we discussed in lecture 19. Apply D to the following quantum state represented by density matrix:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Please write each step of D affecting the quantum system.

Solution:

$$\rho = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

First step: $\rho \to \rho \otimes |0\rangle\langle 0|$, we get

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Second step: "copy" the original qubit into the ancilla, using $\Lambda(\sigma^x)$: $|a, b\rangle \to |a, a \oplus b\rangle$

We get $\begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$

Last step: we take the partial trace over the ancilla, which yields the diagonal matrix

$$Tr\left(\begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Task 3.

Consider the topic of the lecture 21. Apply measuring operator $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to the state $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Remember that we start by joining the system with $|0\rangle\langle 0|$. What is the outcome?

Solution:

We apply $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

We should get:

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}+$$

$$+\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2} & 0\\0 & 0\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

At the end, we make the instrument (second part of the system) classical by applying the decoherence transformation (remember, it removes off-diagonal elements):

$$\begin{pmatrix}\frac{1}{2} & 0\\0 & 0\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

You can write it as a bipartite mixed state (like in slides and in the book), or state the following:

With probability $\frac{1}{2}$ quantum part of the system is in state $|0\rangle\langle 0|$, and in such case classical part is in state 0 with probability $\frac{1}{2}$ and in state 1 with probability $\frac{1}{2}$. With probability $\frac{1}{2}$ quantum part of the system is in state $|1\rangle\langle 1|$, and then classical system is in state 1.

Task 4.

Write down step-by-step application of Shor's algorithm to the number 15. When you describe steps, use $a = 7$ when random number is picked in step 3.

Solution:

We have input $y = 15$.

Step 1. Check y for parity. It is not even, so not divisible by 2.

Step 2. Check whether y is the k-th power of an integer for $k = 1,2,3,4$. We can conclude y cannot be expressed as m^k for some integer m.

Step 3. Choose an integer a randomly and uniformly between 1 and $y - 1$. Compute $b = gcd(a, y)$. We have given $a = 7$. $\gcd(7,15) = 1$.

Step 4. Compute $r = per_y(a)$. $7^4 = 2401 = 1(mod\ 15)$. This means $r = 4$, it is not odd number, so we proceed to Step 5.

Step 5. Compute $d = \gcd\left(a^{\frac{r}{2}} - 1, y\right) = \gcd(49 - 1, 15) = \gcd(48, 15) = 3.$

We have found prime factor 3.