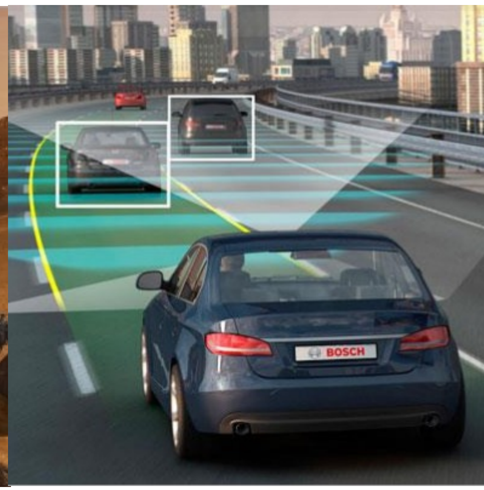
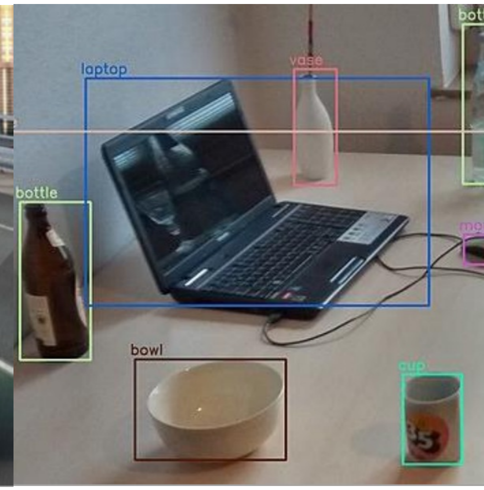
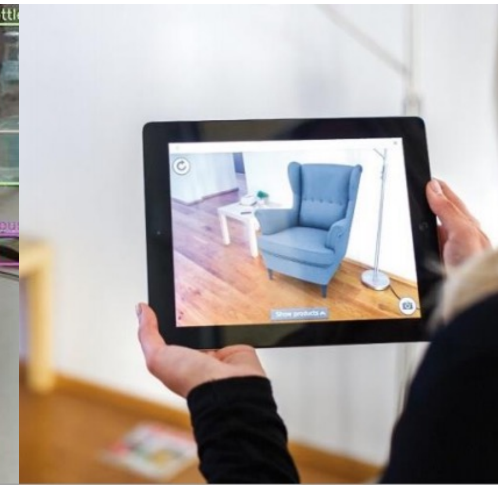
计算机视觉导论



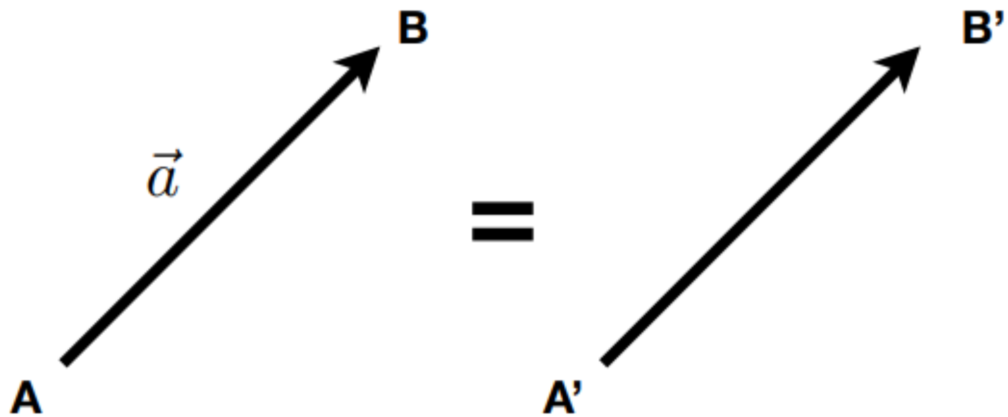


Review of Linear Algebra

2021.09.21

Slides adapted from GAMES101 by Lingqi Yan

Vectors

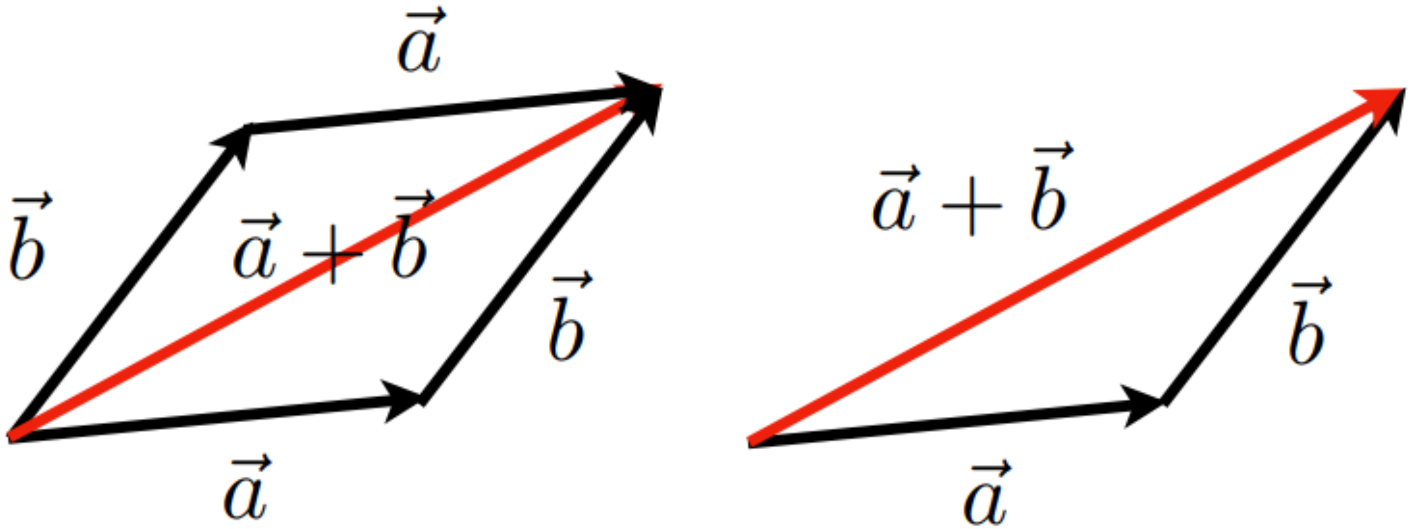


- Usually written as $\vec{a}$ or in bold $\boldsymbol{a}$
- Or using start and end points $\overrightarrow{AB} = B - A$
- Direction and length
- No absolute starting position

Vector Normalization

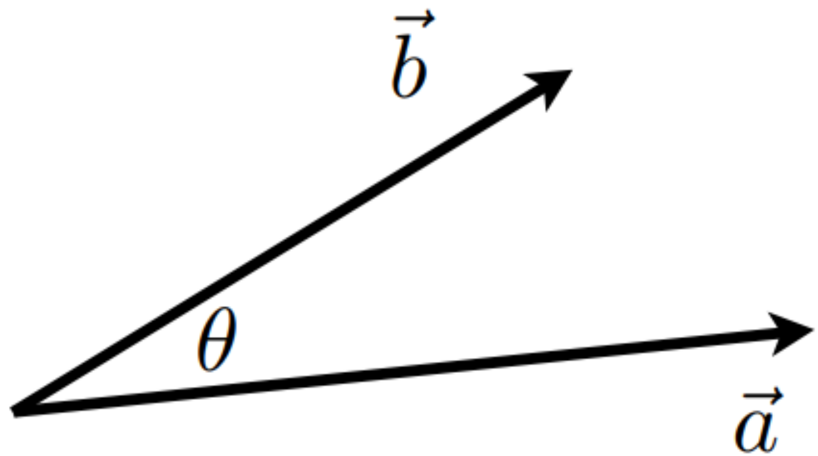
- Magnitude (length) of a vector written as $\|\vec{a}\|$
- Unit vector
  - A vector with magnitude of 1
  - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a}/\|\vec{a}\|$
  - Used to represent directions

Vector Addition



- Geometrically: Parallelogram law & Triangle law
- Algebraically: Simply add coordinates

Dot (scalar) Product



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

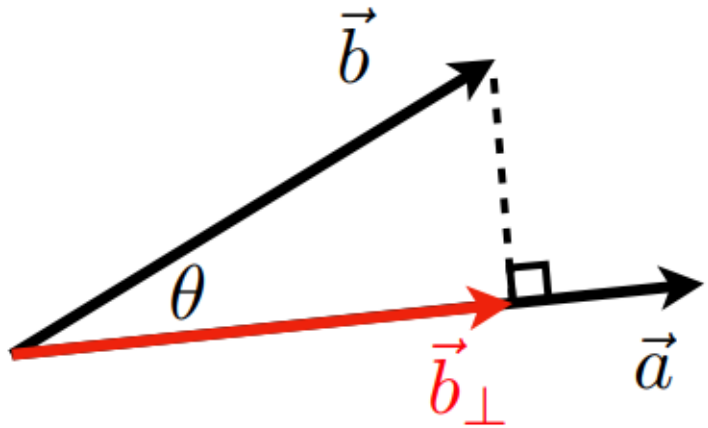
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

- For unit vectors

$$\cos\theta = \hat{a} \cdot \hat{b}$$

Dot Product for Projection

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $\vec{b}_\perp = k\hat{a}$
  - What's its magnitude k?
    - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$



What is a matrix

- Array of numbers ($m \times n$ = m rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition and multiplication by a scalar are trivial: element by element

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & ? & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & ? \end{pmatrix}$$

- Element (i, j) in the product is
  the dot product of row i from A and column j from B

Matrix-Matrix Multiplication

- Properties
  - **Non-commutative**
    (AB and BA are different in general)
  - Associative and distributive
    - $(AB)C=A(BC)$
    - $A(B+C) = AB + AC$
    - $(A+B)C = AC + BC$

Identity Matrix and Inverses

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

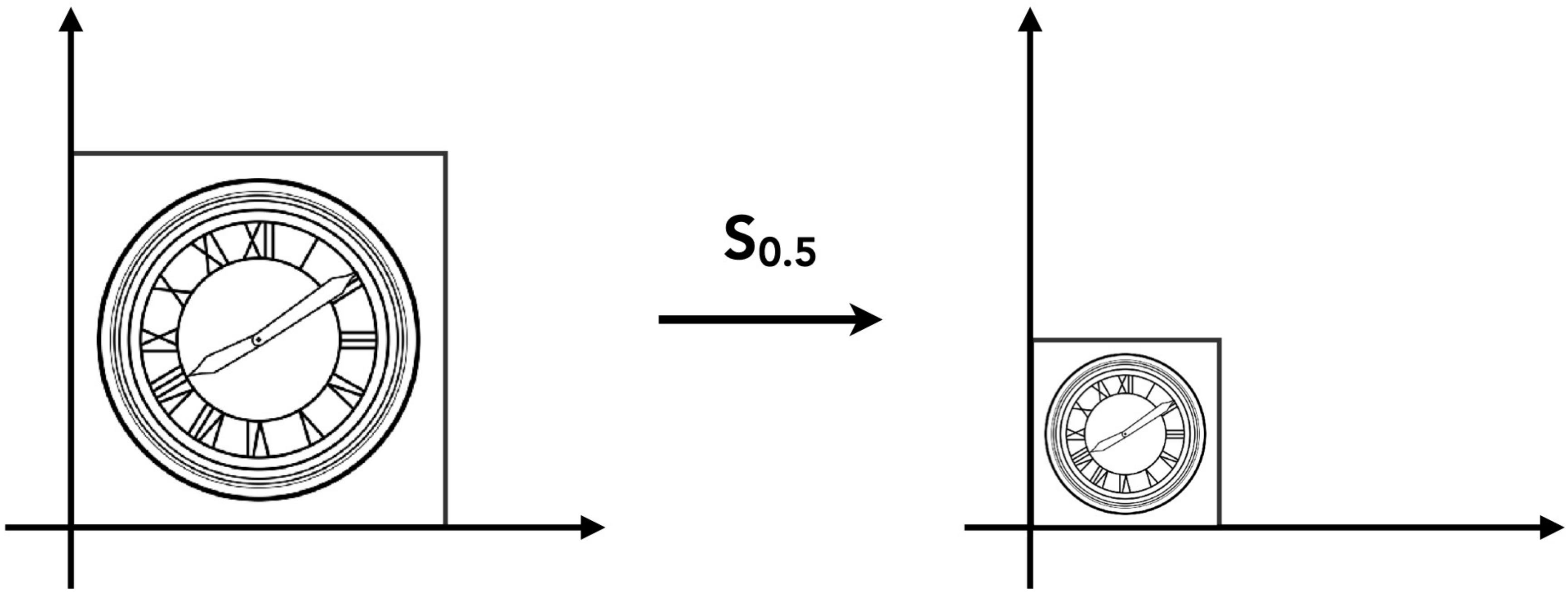
Matrix-Vector Multiplication

- The result of Matrix-Vector is a vector.

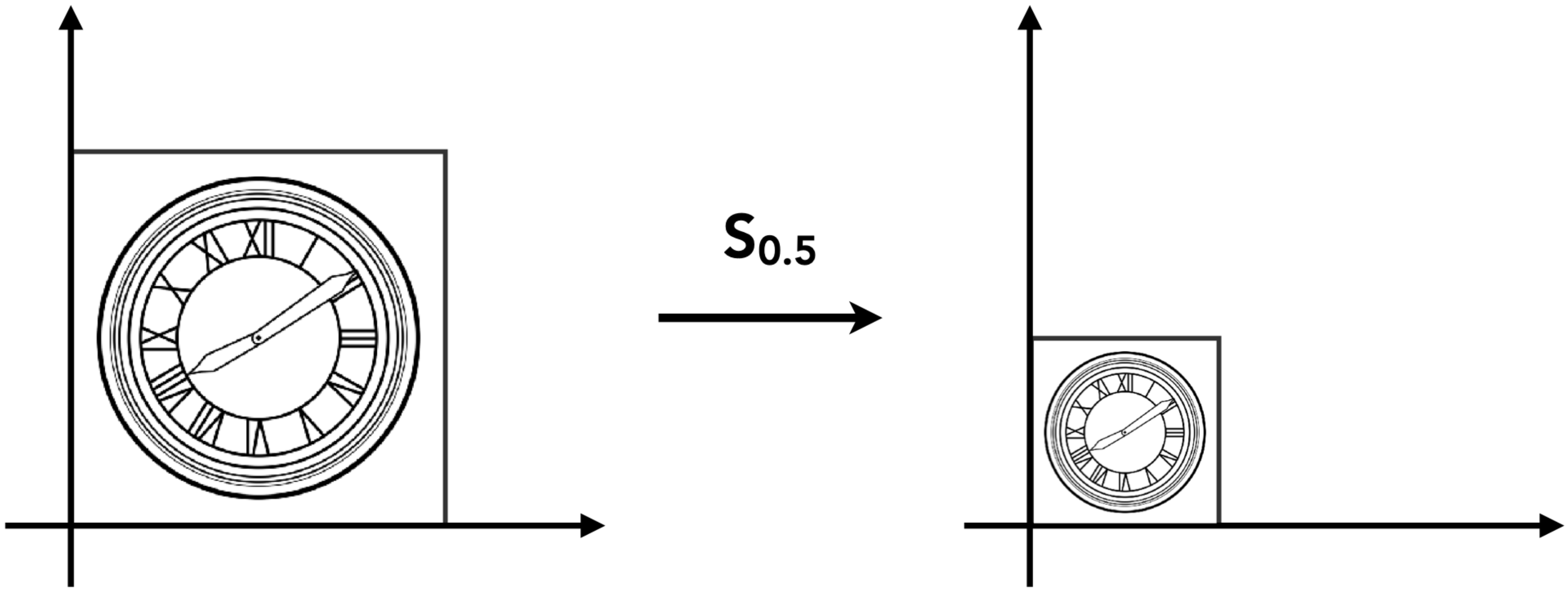
Example: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

- Each Matrix can be regarded as a geometric transformation.

Scale



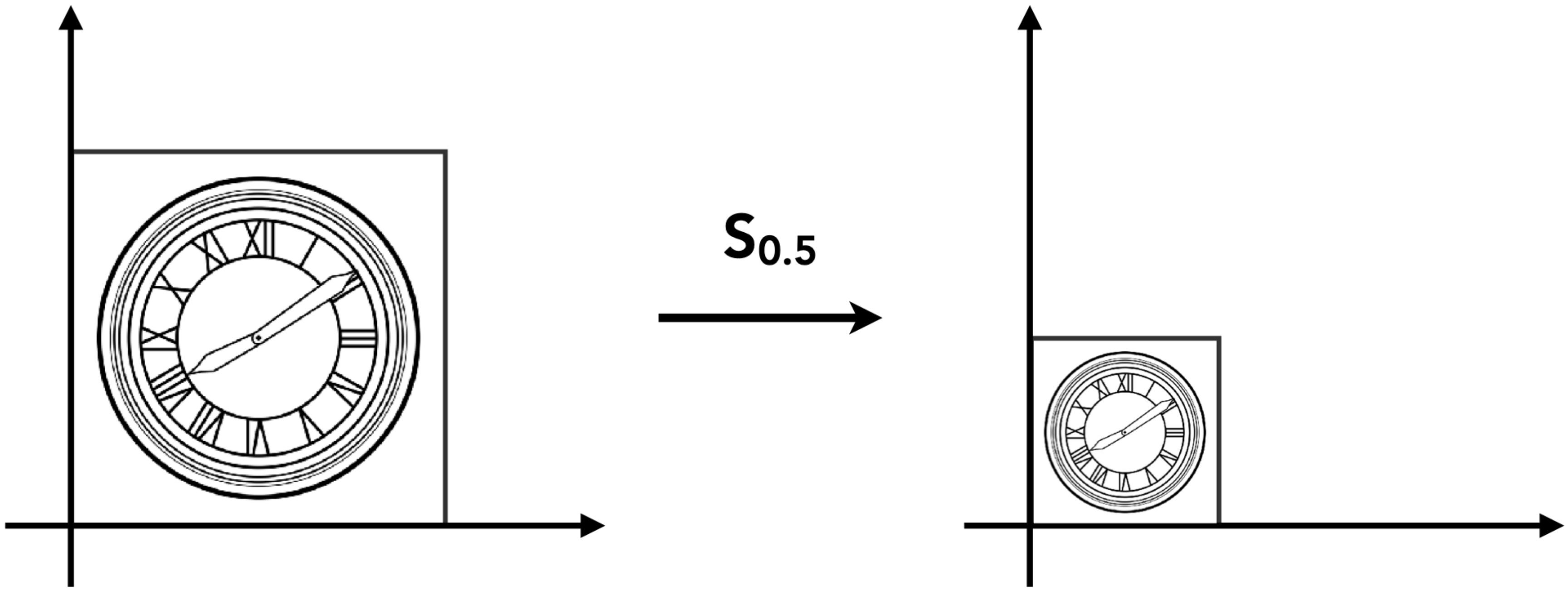
Scale Transform



$$x' = sx$$

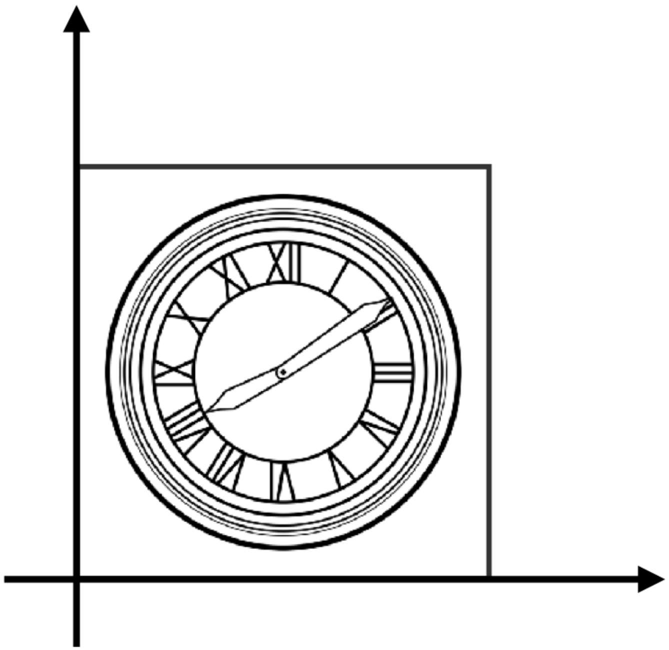
$$y' = sy$$

Scale Matrix

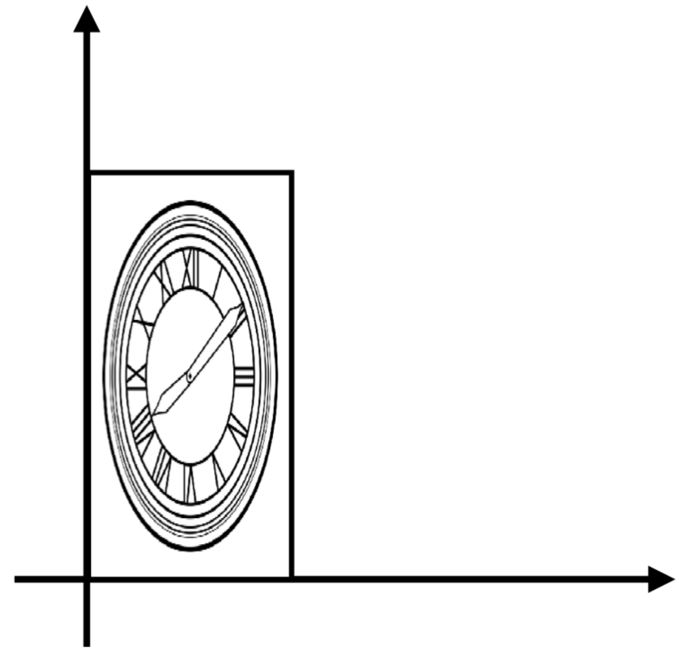


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale (Non-Uniform)

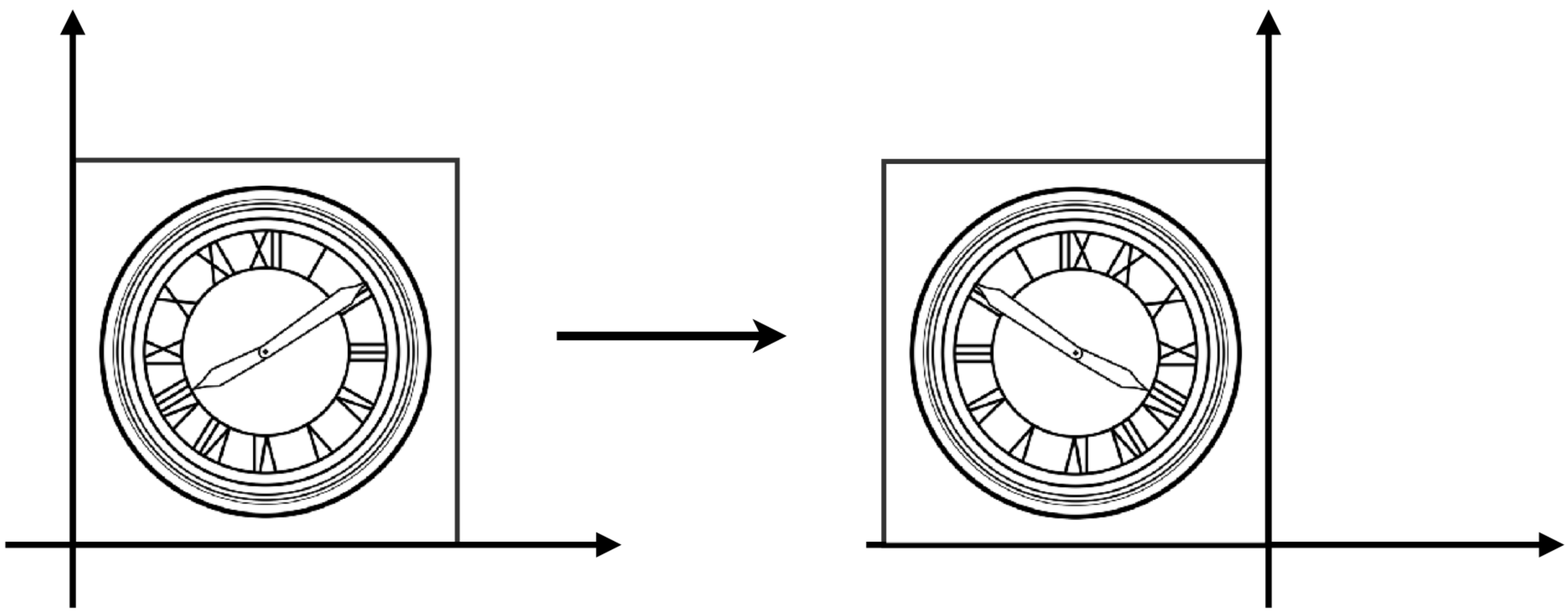


$\mathbf{S}_{0.5,1.0}$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection Matrix

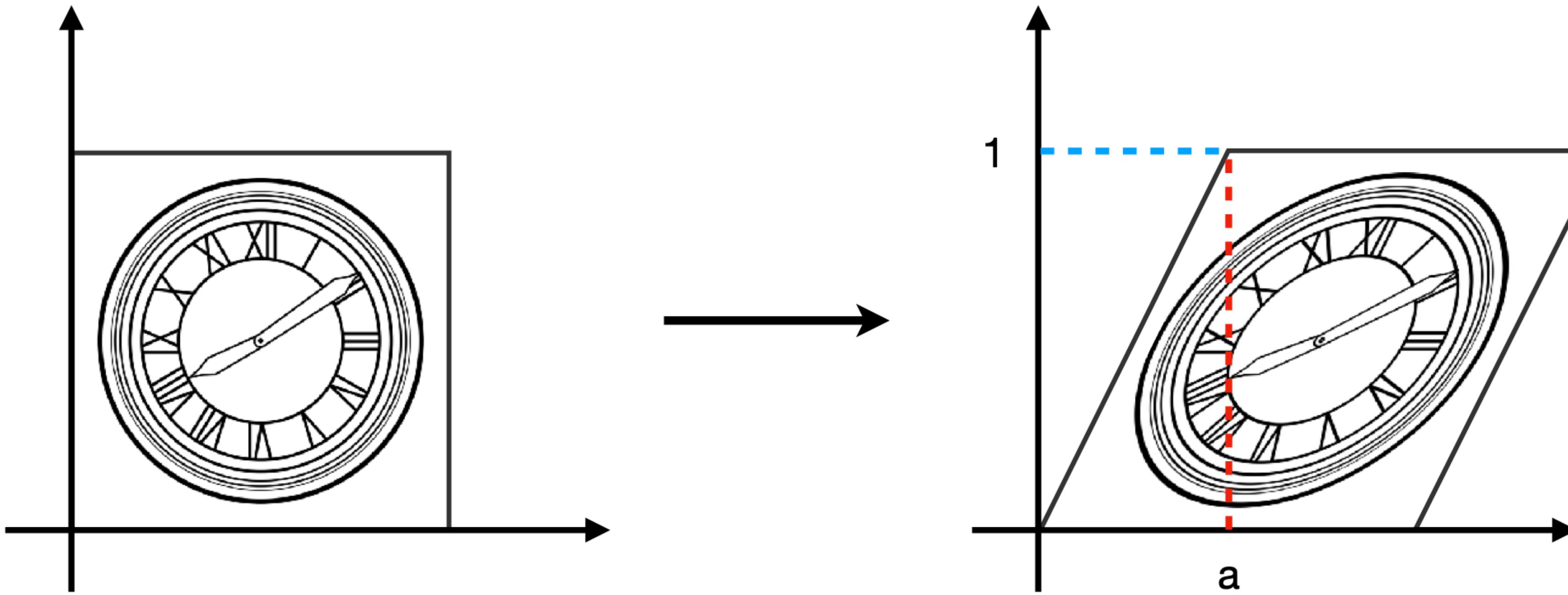


Horizontal reflection:

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear Matrix



Hints:

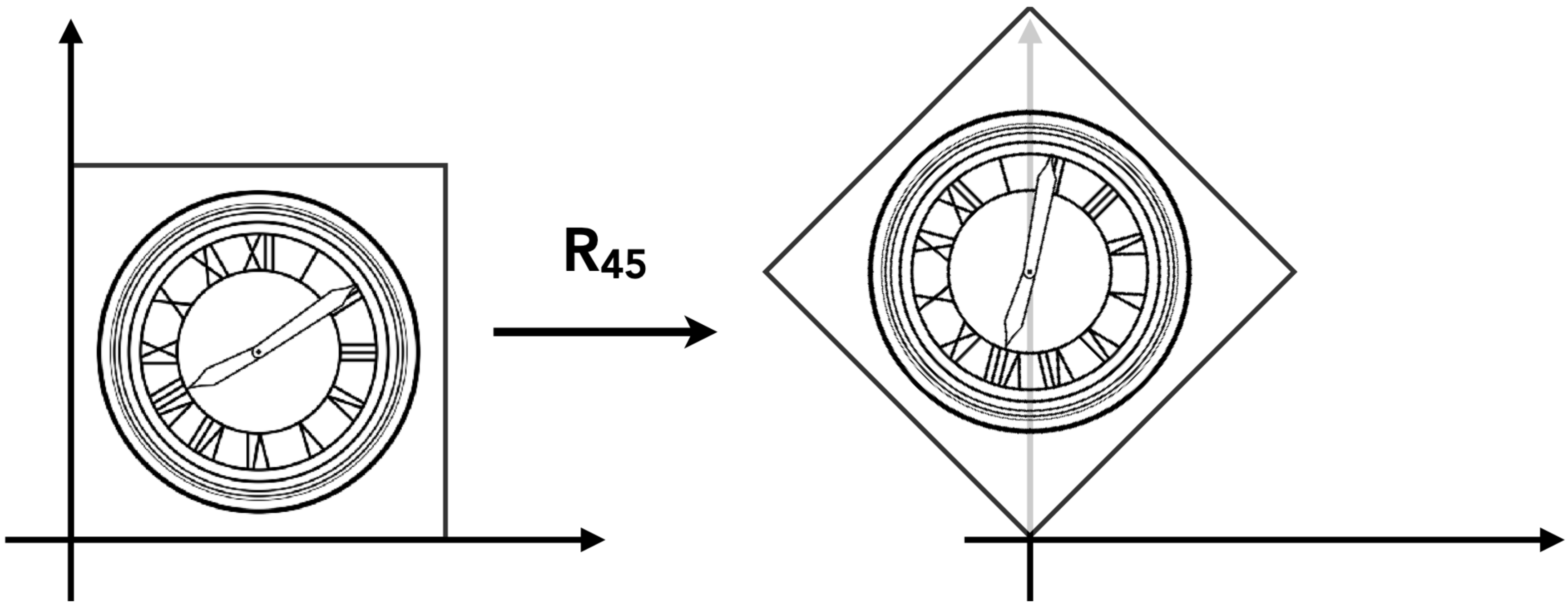
Horizontal shift is 0 at y=0

Horizontal shift is a at y=1

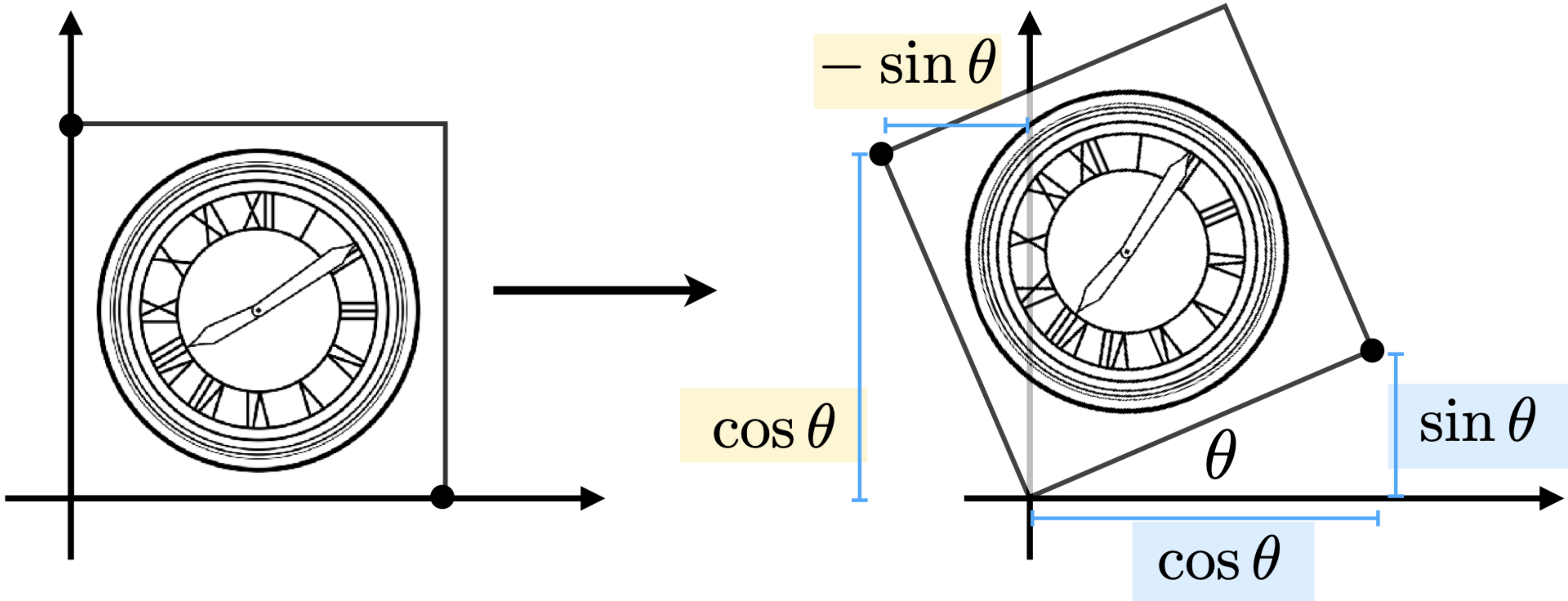
Vertical shift is always 0

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate (about the origin (0, 0), CCW by default)



Rotation Matrix



$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Linear Transforms = Matrices

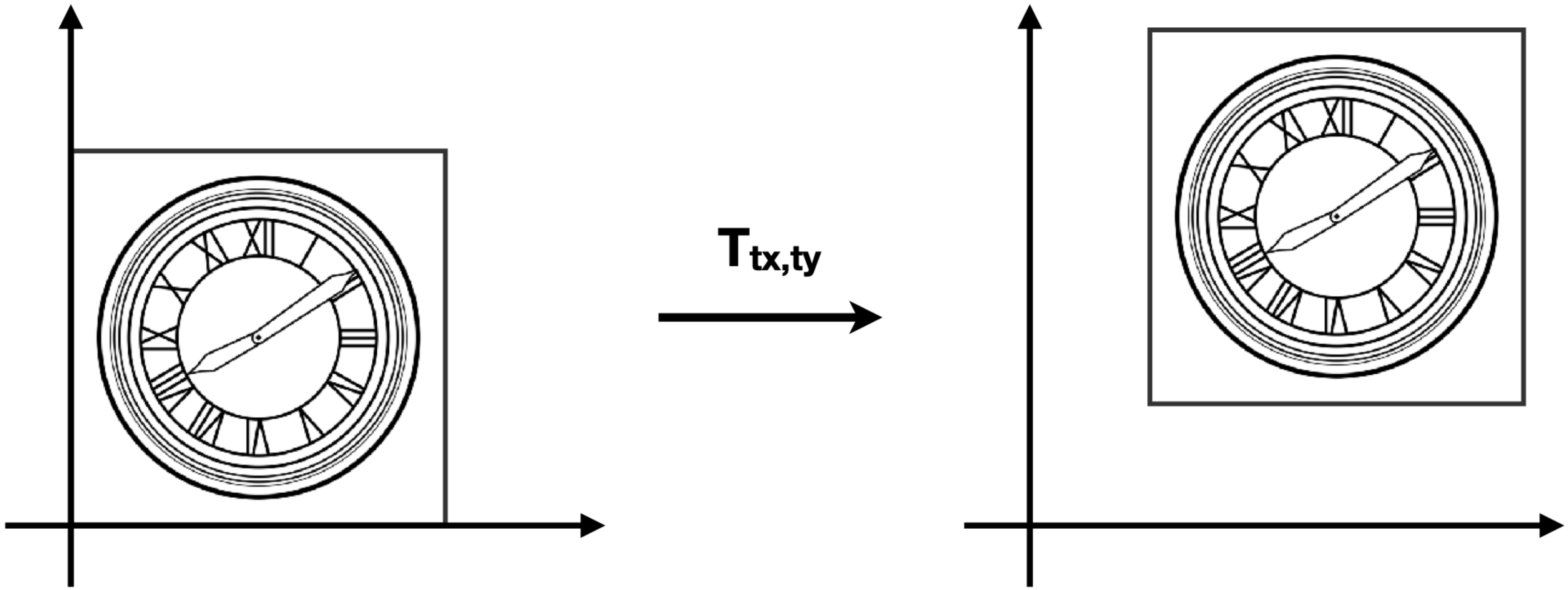
$$x' = a\,x + b\,y$$
$$y' = c\,x + d\,y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{M}\,\mathbf{x}$$

Columns of M are new coordinates of the standard basis after transformation

Translation??



$$x' = x + t_x$$
$$y' = y + t_y$$

Is translation linear transformation?

Affine Transformations

Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

How to write it as matrix-vector multiplication?

Affine Transformations

Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

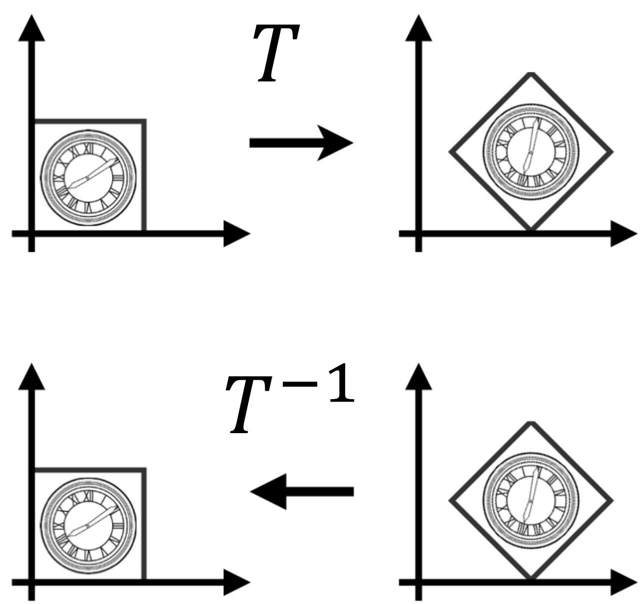
Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Inverse Transform

$$T^{-1}$$

T^{-1} **is the inverse of transform** T **in both a matrix and geometric sense**



Matrix determinant

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} \right)$$

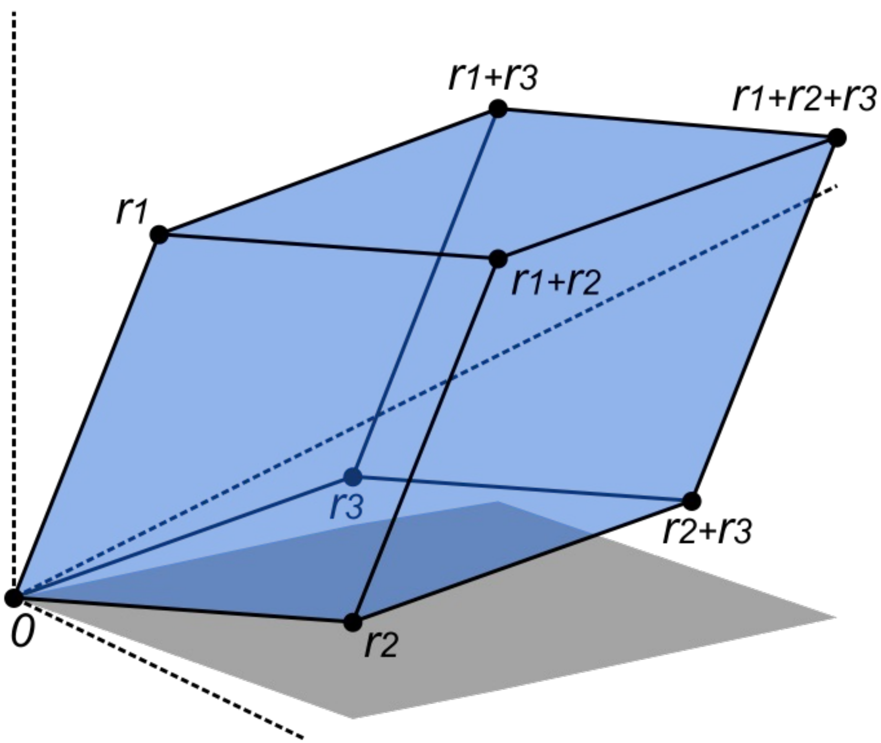
- S_n is a permutation of the set $\{1,2 \dots, n\}$

Permutations of $\{1, 2, 3\}$ and their contribution to the determinant

Permutation σ	$\operatorname{sgn}(\sigma)$	$\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$
1, 2, 3	$+1$	$+a_{1,1} a_{2,2} a_{3,3}$
1, 3, 2	-1	$-a_{1,1} a_{2,3} a_{3,2}$
3, 1, 2	$+1$	$+a_{1,3} a_{2,1} a_{3,2}$
3, 2, 1	-1	$-a_{1,3} a_{2,2} a_{3,1}$
2, 3, 1	$+1$	$+a_{1,2} a_{2,3} a_{3,1}$
2, 1, 3	-1	$-a_{1,2} a_{2,1} a_{3,3}$

Matrix determinant (geometric meaning)

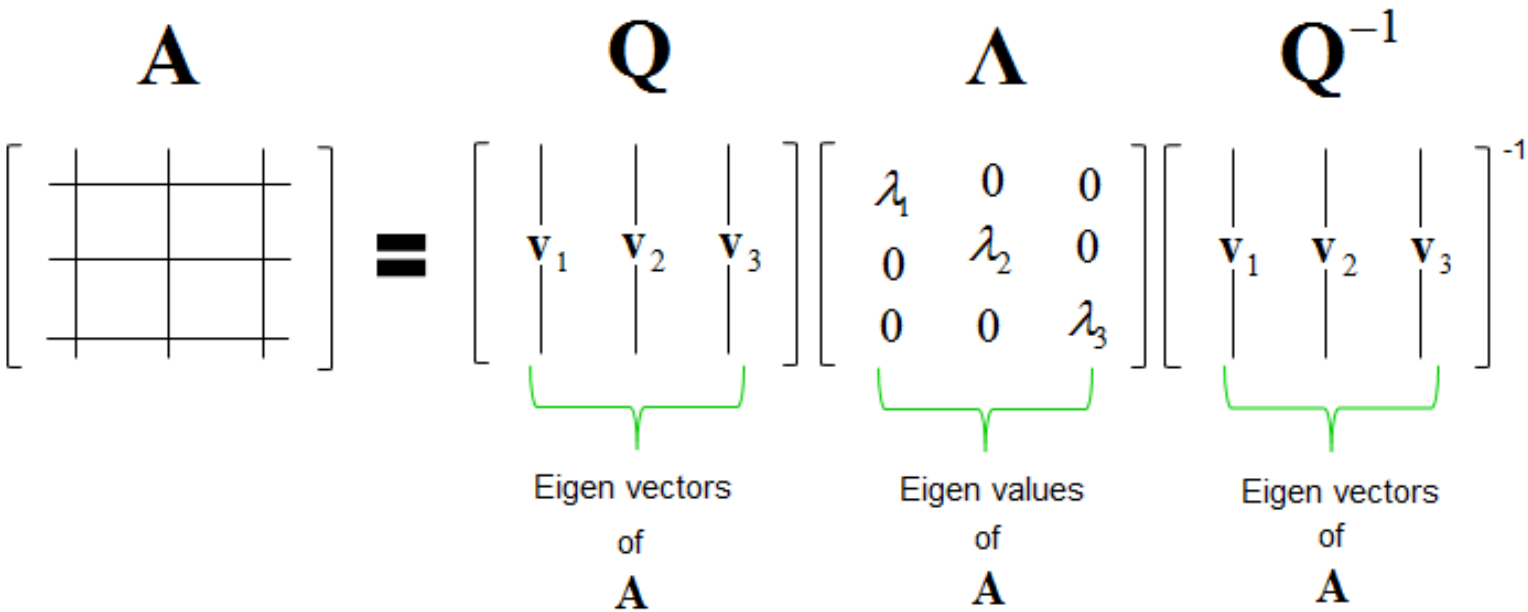
- The determinant is the volume of an n-dimensional parallel body.



Eigenvectors and eigenvalues

- $Ax = \lambda x$
- $x \neq 0$, x is the eigenvector of A with eigenvalue λ
- $A \in R^{n*n}$
- What is the geometric meaning of x and λ?

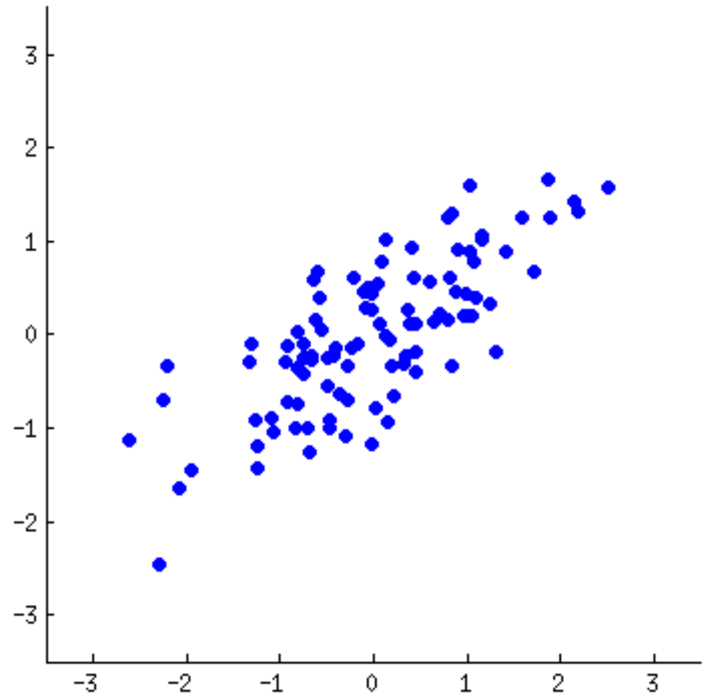
Eigen decomposition



https://www.sharetechnote.com/html/Handbook_EngMath_Matrix_EigenDecomposition.html

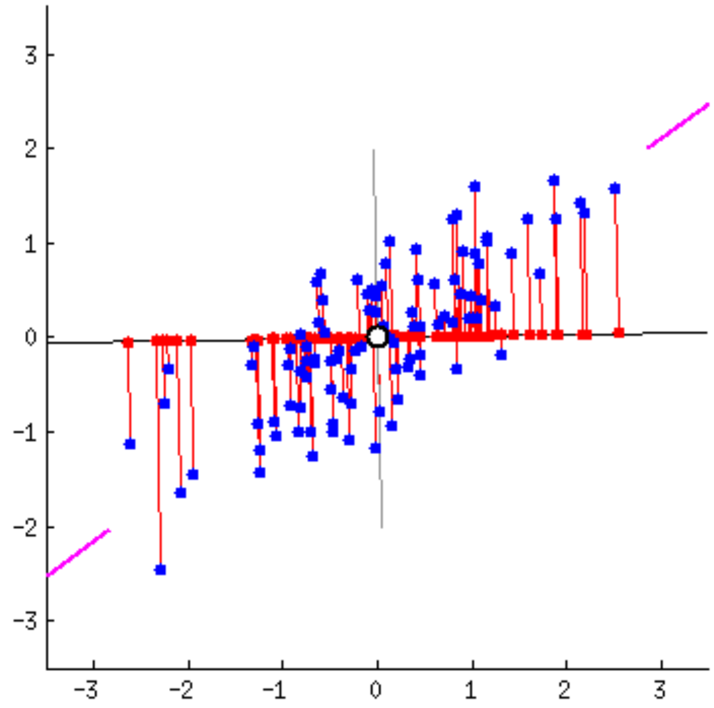
Application of Eigenvalues and eigenvectors.

- Principal component analysis: find the principal direction of the data.



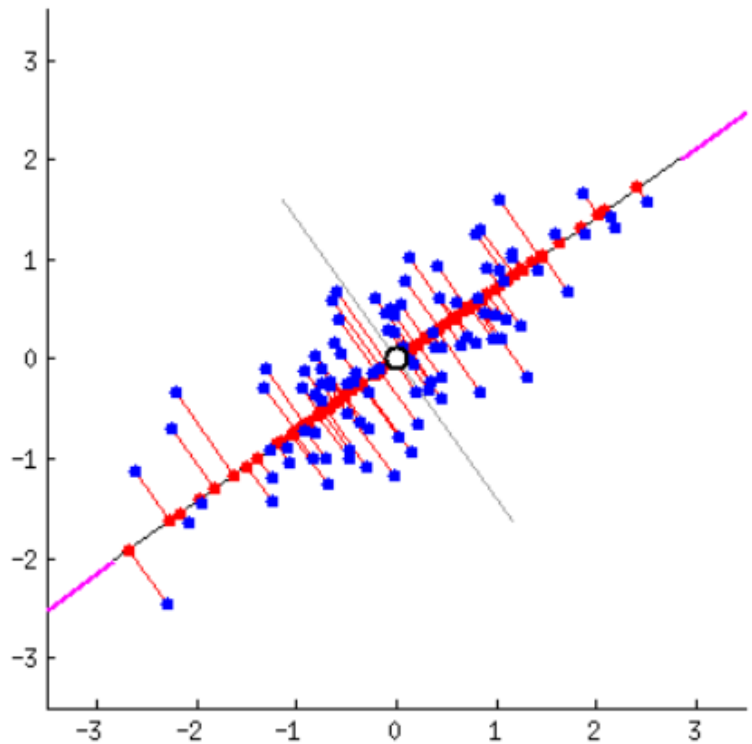
Application of Eigenvalues and eigenvectors.

- Principal component analysis: find the principal direction of the data.



Application of Eigenvalues and eigenvectors.

- Principal components = eigen vectors of $A^T A$



$$A = \begin{bmatrix} x_1, y_1 \\ x_2, y_2 \\ \vdots \\ x_n, y_n \end{bmatrix}$$

Questions?