An Approximation Algorithm for Maximum Stable Matching with Ties and Constraints*

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Abstract

We present a polynomial-time $\frac{3}{2}$-approximation algorithm for the problem of finding a maximum-cardinality stable matching in a many-to-many matching model with ties and laminar constraints on both sides. We formulate our problem using a bipartite multigraph whose vertices are called workers and firms, and edges are called contracts. Our algorithm is described as the computation of a stable matching in an auxiliary instance, in which each contract is replaced with three of its copies and all agents have strict preferences on the copied contracts. The construction of this auxiliary instance is symmetric for the two sides, which facilitates a simple symmetric analysis. We use the notion of matroid-kernel for computation in the auxiliary instance and exploit the base-orderability of laminar matroids to show the approximation ratio.

In a special case in which each worker is assigned at most one contract and each firm has a strict preference, our algorithm defines a $\frac{3}{2}$-approximation mechanism that is strategy-proof for workers.

1 Introduction

The *college admission problem* (CA) is a many-to-one generalization of the well-known *stable marriage problem* [14, 27, 29], introduced by Gale and Shapley [12]. An instance of CA involves two disjoint agent sets called students and colleges. Each agent has a strict linear order of preference over agents on the opposite side, and each college has an upper quota for the number of assigned students. It is known that any instance of CA has a stable matching, we can find it efficiently, and all stable matchings have the same cardinality.

Recently, matching problems with constraints have been studied extensively [4, 7, 11, 22, 23]. Motivated by the matching scheme used in the higher education sector in Hungary, Biró et al. [2] studied CA *with common quotas*. In this problem, in addition to individual colleges, certain subsets of colleges, called *bounded sets*, have upper quotas. Such constraints are also called *regional caps* or *distributional constraints*, and they have been studied in [13, 24]. Meanwhile, motivated by academic hiring, Huang [17] introduced the *classified stable matching problem*. This is an extension of CA in which each individual college has quotas for subsets of students, called *classes*. Its many-to-many generalizations have been studied in [10, 36].[1] For these models, the laminar structure of constraints is commonly found to be the key to the existence of a stable matching. A family $\mathcal{L}$ of sets is called *laminar* if any $L, L' \in \mathcal{L}$ satisfy $L \subseteq L'$ or $L \supseteq L'$ or $L \cap L' = \emptyset$ (also called *nested* or *hierarchical*). In [2, 17], the authors showed that a stable

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[1] In [10, 13, 17, 36], not only upper quotas but also lower quotas are considered. With lower quotas, the existence of stable matching is not guaranteed. In this paper, we consider only upper quotas.

matching exists in their models if regions or classes form laminar families, whereas the existence is not guaranteed in the general case. Furthermore, in the laminar case, a stable matching can be found efficiently, and all stable matchings have the same cardinality. Applications with laminar constraints have been discussed in [24].

The purpose of this paper is to introduce ties to a matching model with laminar constraints. In the previous studies described above, the preferences of agents were assumed to be strictly ordered. However, ties naturally arise in real problems. The preference of an agent is said to contain a *tie* if she is indifferent between two or more agents on the opposite side. When ties are allowed, the existence of a stable matching is maintained; however, stable matchings vary in cardinalities. As it is desirable to produce a large matching in practical applications, we consider the problem of finding a maximum-cardinality stable matching.

Such a problem is known to be difficult even in the simple matching model without constraints. The problem of finding a maximum stable matching in the setting of *stable marriage with ties and incomplete lists*, called MAX-SMTI, is NP-hard [19, 30], as is obtaining an approximation ratio within $\frac{33}{29}$ [35]. For its approximability, several algorithms with improved approximation ratios have been proposed [20, 21, 25, 26, 31, 33]. The current best ratio is $\frac{3}{2}$ by a polynomial-time algorithm proposed by McDermid [31] as well as linear-time algorithms proposed by Paluch [33] and Király [26]. The $\frac{3}{2}$-approximability extends to the settings of CA with ties [26] and the student-project allocation problem with ties [6].

Our Contribution. We present a polynomial-time $\frac{3}{2}$-approximation algorithm for the problem of finding a maximum-cardinality stable matching in a many-to-many matching model with ties and laminar constraints on both sides. We call this problem MAX-SMTI-LC and formulate it using a bipartite multigraph, where we call the two vertex sets *workers* and *firms*, respectively, and each edge a *contract*. Each agent has upper quotas on a laminar family defined on incident contracts. Our formulation can deal with each agent's constraints, such as *classified stable matching*. Furthermore, distributional constraints such as CA *with common quotas* can be handled by considering a dummy agent that represents a consortium of the agents on one side (see the remark at the end of Section 2). Our algorithm runs in $O(k \cdot |E|^2)$ time, where E is the set of contracts and k is the maximum level of nesting of laminar constraints. The *level of nesting* of a laminar family $\mathcal{L}$ is the maximum length of a chain $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k$ of members of $\mathcal{L}$; hence, $k \leq |E|$.

Our algorithm is described as the computation of a stable matching in an auxiliary instance. Here, we explain the ideas underlying the construction of the auxiliary instance, which is inspired by the algorithms of Király [26] and Hamada, Miyazaki, and Yanagisawa [15].

First, we briefly explain Király's $\frac{3}{2}$-approximation algorithm for MAX-SMTI [26]. In this algorithm, each worker makes proposals from top to bottom in her list sequentially, as with the worker-oriented Gale–Shapley algorithm. A worker rejected by all firms is given a second chance for proposals. Each firm prioritizes a worker in the second cycle over a worker in the first cycle if they are tied in its preference list. This idea of *promotion* is used to handle ties in firms' preference lists. To handle ties in workers' lists, Király's algorithm lets each worker prioritize a currently unmatched firm over a currently matched firm if they are tied in her preference list. This priority rule depends on the states of firms at each moment, which makes the algorithm complicated when we introduce constraints on both sides.

Then, we introduce the idea of the algorithm of Hamada et al. [15], who proposed a worker-strategy-proof algorithm for MAX-SMTI that attains the $\frac{3}{2}$-approximation ratio when ties appear only in workers' lists. They modified Király's algorithm such that each worker's proposal order is predetermined and is not affected by the history of the algorithm. Their algorithm can be seen as a Gale–Shapley-type algorithm in which each worker makes proposals twice to each firm in a tie before proceeding to the next tie, and each firm prioritizes second proposals over first proposals regardless of its preference. By combining their algorithm with the promotion operation of Király's algorithm, we obtain a Gale–Shapley-type algorithm in which each worker makes at most three proposals to each firm.

Based on these observations, we propose a method for transforming a MAX-SMTI-LC instance I into an auxiliary instance I^*, which is also a MAX-SMTI-LC instance. Each contract e_i in I is replaced with three copies x_i, y_i, z_i in I^*. Each agent has a strict preference on the copied contracts, which reflects the priority rules in the algorithms of Király and Hamada et al. The instance I^* has an upper bound 1 for each triple $\{x_i, y_i, z_i\}$ and also has constraints corresponding to those in I. The construction of I^* is completely symmetric for workers and firms. We show that, for any stable matching M^* of I^*, its projection $M \coloneqq \{ e_i \mid \{x_i, y_i, z_i\} \cap M^* \neq \emptyset \}$ is a $\frac{3}{2}$-approximate solution for I. Both the stability and the approximation ratio of M are implied by the stability of M^* in I^*, and the process of computing M^* is irrelevant. Thus, our method enables us to conduct a symmetric and static analysis even with constraints.

Because the auxiliary instance I^* has no ties, we can find a stable matching of I^* efficiently by using the matroid framework of Fleiner [8, 9]. In the analysis of the approximation ratio, we exploit the fact that the family of feasible sets defined by laminar constraints forms a matroid with a property called base-orderability.

In the last section, we show that the result of Hamada et al. [15] mentioned above is generalized to a many-to-one matching setting with laminar constraints on the firm side. In other words, if we restrict MAX-SMTI-LC such that each worker is assigned at most one contract and each firm has a strict preference, then we can provide a worker-strategy-proof mechanism that returns a $\frac{3}{2}$-approximate solution. We obtain this conclusion using the strategy-proofness result of Hatfield and Milgrom [16].

Paper Organization. The remainder of this paper is organized as follows. Section 2 formulates our matching model, while Section 3 describes our algorithm. Section 4 presents a lemma on base-orderable matroids that is the key to our proof of the approximation ratio. Sections 5 and 6 are devoted to the proofs of correctness and time complexity, respectively. Section 7 presents a one-sided strategy-proof mechanism for a special case.

Throughout the paper, we denote the set of non-negative integers by $\mathbf{Z}_+$. For a subset $S \subseteq E$ and an element $e \in E$, we denote $S + e \coloneqq S \cup \{e\}$ and $S - e \coloneqq S \setminus \{e\}$.

2 Problem Formulation

An instance of the *stable matching with ties and laminar constraints*, which we call SMTI-LC, is a tuple $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$ defined as follows. Let W and F be disjoint finite sets called *workers* and *firms*, respectively. We call $a \in W \cup F$ an *agent* when we do not distinguish between workers and firms. We are provided a set E of *contracts*. Each contract $e \in E$ is associated with one worker and one firm, denoted by $\partial_W(e)$ and $\partial_F(e)$, respectively. Multiple contracts are allowed to exist between a worker–firm pair. Then, $(W, F; E)$ is represented as a bipartite multigraph in which W and F are vertex sets, and each $e \in E$ is an edge connecting $\partial_W(e)$ and $\partial_F(e)$. For each $a \in W \cup F$, we denote the set of associated contracts by E_a, i.e.,

$$E_w \coloneqq \{ e \in E \mid \partial_W(e) = w \} \quad (w \in W), \qquad E_f \coloneqq \{ e \in E \mid \partial_F(e) = f \} \quad (f \in F).$$

Then, the family $\{ E_w \mid w \in W \}$ forms a partition of E, as does $\{ E_f \mid f \in F \}$.

Each agent $a \in W \cup F$ has a laminar family $\mathcal{L}_a$ of subsets of E_a and a quota function $q_a \colon \mathcal{L}_a \to \mathbf{Z}_+$. For any subset $M \subseteq E$ of contracts and an agent $a \in W \cup F$, we denote by $M_a \coloneqq M \cap E_a$ the set of contracts assigned to a. We say that M is *feasible* for $a \in W \cup F$ if

$$\forall L \in \mathcal{L}_a : |M_a \cap L| \le q_a(L).$$

A set $M \subseteq E$ is called a *matching* if it is feasible for all agents in $W \cup F$.

Each agent $a \in W \cup F$ has a preference list P_a that consists of all elements in E_a and may contain ties. In this paper, a preference list is written in one row, from left to right according to preference, where two or more contracts with equal preference are included in the same parentheses. For example, if the preference list P_a of an agent $a \in W \cup F$ is represented as

$$P_a : \; e_2 \;\; (\; e_1 \;\; e_4 \;) \;\; e_3,$$

then e_2 is a's top choice, e_1 and e_4 are the second choices with equal preference, and e_3 is the last choice. For contracts $e, e' \in E_a$, we write $e \succ_a e'$ if a prefers e to e'. Furthermore, we write $e \succeq_a e'$ if $e \succ_a e'$ or a is indifferent between e and e' (including the case $e = e'$).

For a matching $M \subseteq E$, a contract $e \in E \setminus M$, and an associated agent $a \in \{\partial_W(e), \partial_F(e)\}$, we say that e is *free for* a in M if

- $M_a + e$ is feasible for a, or
- there is $e' \in M_a$ such that $e \succ_a e'$ and $M_a + e - e'$ is feasible for a.

In other words, a contract e is free for an agent a if a has an incentive to add e to the current assignment possibly at the expense of some less preferred contract e'. A contract $e \in E \setminus M$ *blocks* M if e is free for both $\partial_W(e)$ and $\partial_F(e)$. A matching M is *stable* if there is no contract in $E \setminus M$ that blocks M.

The goal of our problem MAX-SMTI-LC is to find a maximum-cardinality stable matching for a given SMTI-LC instance. Because MAX-SMTI-LC is a generalization of the NP-hard problem MAX-SMTI, we consider the approximability. Similarly to the case of MAX-SMTI, a 2-approximate solution can be obtained using an arbitrary tie-breaking method for MAX-SMTI-LC (see Appendix A). In the next section, we present a $\frac{3}{2}$-approximation algorithm.

Remark 1. We demonstrate that SMTI-LC includes several models investigated in previous works, which implies that our algorithm finds $\frac{3}{2}$-approximate solutions for the problems of finding maximum-cardinality stable matchings in those models with ties.

First, SMTI and the stable b-matching problem are special cases such that $E \subseteq W \times F$ and $\mathcal{L}_a = \{E_a\}$ for every $a \in W \cup F$. Furthermore, the two-sided laminar classified stable matching problem [10, 17], if lower quotas are absent, is a special case with $E \subseteq W \times F$.

To represent CA with laminar common quotas [2], let W be the set of students and let $F \coloneqq \{f\}$, where f is regarded as a consortium of all colleges in C. The set of contracts is defined by $E \coloneqq \{\, (w, f, c) \mid \text{a college } c \in C \text{ is acceptable for a student } w \in W \,\}$, where $\partial_W(e) = w$, $\partial_F(e) = f$ for any $e = (w, f, c)$. Note that $E = E_f$. A quota for a region $C' \subseteq C$ is then represented as a quota for the set $\{\, (w, f, c) \in E \mid c \in C' \,\} \subseteq E_f$. Thus, laminar common quotas can be represented as constraints on a laminar family on E_f.

For the student-project allocation problem [6], let W and F be the sets of students and lecturers, respectively, and $E \coloneqq \{\, (w, f, p) \mid p \text{ is a project offered by } f \in F \text{ and acceptable for } w \in W \,\}$. Let $E_{f,p} \subseteq E_f$ be the set of contracts associated with a project p offered by a lecturer f. Then, the lecturer's upper quota and projects' upper quotas define two-level laminar constraints on the family $\mathcal{L}_f = \{E_f\} \cup \{\, E_{f,p} \mid p \text{ is offered by } f \,\}$.

For the above-mentioned settings, we can appropriately set the preferences of agents such that the stability in the previous works coincides with the stability in SMTI-LC.

3 Algorithm

Our approximation algorithm for MAX-SMTI-LC consists of three steps: (i) construction of an auxiliary instance, (ii) computation of any stable matching of this auxiliary instance, and (iii) mapping the obtained matching to a matching of the original instance. In what follows, we describe how to construct an auxiliary instance I^* from a given instance I and how to map a matching of I^* to that of I.

Let $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$ be an instance of MAX-SMTI-LC, where the set E of contracts is represented as $E = \{\, e_i \mid i = 1, 2, \ldots, n \,\}$. We construct an auxiliary instance $I^* = (W, F, E^*, \{\mathcal{L}^*_a, q^*_a, P^*_a\}_{a \in W \cup F})$, which is also an SMTI-LC instance; however, each preference list P^*_a does not contain ties.

The sets of workers and firms in I^* are the same as those in I. The set E^* of contracts in I^* is given as $E^* = \{\, x_i, y_i, z_i \mid i = 1, 2, \ldots, n \,\}$, where x_i, y_i, and z_i are copies of e_i; hence, $\partial_W(x_i) = \partial_W(y_i) = \partial_W(z_i) = \partial_W(e_i)$ and $\partial_F(x_i) = \partial_F(y_i) = \partial_F(z_i) = \partial_F(e_i)$. We define a mapping $\pi : 2^{E^*} \to 2^E$ by $\pi(S^*) = \{\, e_i \mid \{x_i, y_i, z_i\} \cap S^* \neq \emptyset \,\}$ for any $S^* \subseteq E^*$.

For any agent $a \in W \cup F$, the laminar family $\mathcal{L}^*_a$ and the quota function $q^*_a : \mathcal{L}^*_a \to \mathbf{Z}_+$ are defined as follows. For each $e_i \in E_a$, we have $\{x_i, y_i, z_i\} \in \mathcal{L}^*_a$ and $q^*_a(\{x_i, y_i, z_i\}) = 1$. For each $L \in \mathcal{L}_a$, we have $L^* := \{\, x_i, y_i, z_i \mid e_i \in L \,\} \in \mathcal{L}^*_a$ and $q^*_a(L^*) = q_a(L)$. These are all that $\mathcal{L}^*_a$ contains. Then, for any set $M^* \subseteq E^*$ of contracts, we see that M^* is feasible for a in I^* if and only if M^* contains at most one copy of each $e_i \in E_a$ and the set $\pi(M^*)$ is feasible for a in I.

The preference list P^*_w of each worker $w \in W$ is defined as follows. Take a tie $(e_{i_1} e_{i_2} \cdots e_{i_\ell})$ in P_w. We replace it with a strict linear order of 2ℓ contracts $x_{i_1} x_{i_2} \cdots x_{i_\ell} y_{i_1} y_{i_2} \cdots y_{i_\ell}$. Apply this operation to all the ties in P_w, where we regard a contract not included in any tie as a tie of length one. Next, at the end of the resultant list, append the original list P_w with each e_i replaced with z_i and all the parentheses omitted. Here is a demonstration. If the preference list of a worker w is

$$P_w : \ (\ e_2 \ \ e_6\ ) \ \ e_1 \ \ (\ e_3 \ \ e_4\ ),$$

then her list in I^* is

$$P^*_w : \ \ x_2 \ x_6 \ y_2 \ y_6 \ x_1 \ y_1 \ x_3 \ x_4 \ y_3 \ y_4 \ z_2 \ z_6 \ z_1 \ z_3 \ z_4.$$

The preference list P^*_f of each firm $f \in F$ is defined in the same manner, where the roles of x_i and z_i are interchanged. For example, if the preference list of a firm f is

$$P_f : \ \ e_3 \ \ (\ e_2 \ \ e_4 \ \ e_7\ ) \ \ e_5,$$

then its list in I^* is

$$P^*_f : \ \ z_3 \ y_3 \ z_2 \ z_4 \ z_7 \ y_2 \ y_4 \ y_7 \ z_5 \ y_5 \ x_3 \ x_2 \ x_4 \ x_7 \ x_5.$$

Thus, we have defined the auxiliary instance I^*. As this is again an SMTI-LC instance, a stable matching of I^* is defined as before. The existence of a stable matching of I^* is guaranteed by the existing framework of Fleiner [8, 9], as will be explained in Section 6. Here is the main theorem of this paper, which states that any stable matching of I^* defines a $\frac{3}{2}$-approximate solution for I.

Theorem 1. *For a stable matching M^* of I^*, let $M := \pi(M^*)$. Then, M is a stable matching of I with $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$, where M_{OPT} is a maximum-cardinality stable matching of I.*

Algorithm 1 $\frac{3}{2}$-approximation algorithm for MAX-SMTI-LC

Require: An instance $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$.
Ensure: A stable matching M with $|M| \geq \frac{2}{3}|M_{\text{OPT}}|$, where M_{OPT} is an optimal solution.
1: Construct an auxiliary instance I^*.
2: Find any stable matching M^* of I^*.
3: Let $M = \pi(M^*)$ and return M.

We prove Theorem 1 in Section 5. This theorem guarantees the correctness of Algorithm 1.

Clearly, the first and third steps of Algorithm 1 can be performed efficiently. Furthermore, the second step can be executed in polynomial time by applying the generalized Gale–Shapley algorithm of Fleiner [8, 9]. In Section 6, we will explain this more precisely and present the time complexity represented in the following theorem.

Theorem 2. *One can find a stable matching M of I with $|M| \geq \frac{2}{3}|M_{\text{OPT}}|$ in $O(k \cdot |E|^2)$ time, where M_{OPT} is a maximum-cardinality stable matching and k is the maximum level of nesting of laminar families $\mathcal{L}_a$ $(a \in W \cup F)$.*

4 Base-orderable Matroids

For the proofs of Theorems 1 and 2, we introduce some concepts related to matroids (see, e.g., Oxley [32] for more information on matroids).

For a finite set E and a family $\mathcal{I} \subseteq 2^E$, a pair $(E, \mathcal{I})$ is called a *matroid* if the following three conditions hold: (I1) $\emptyset \in \mathcal{I}$, (I2) $S \subseteq T \in \mathcal{I}$ implies $S \in \mathcal{I}$, and (I3) for any $S, T \in \mathcal{I}$ with $|S| < |T|$, there exists $e \in T \setminus S$ such that $S + e \in \mathcal{I}$.

For a matroid $(E, \mathcal{I})$, each member of $\mathcal{I}$ is called an *independent set*. An independent set is called a *base* if it is inclusion-wise maximal in $\mathcal{I}$. We denote the family of all bases by $\mathcal{B}$. By the matroid axiom (I3), it follows that $|B_1| = |B_2|$ holds for any bases $B_1, B_2 \in \mathcal{B}$.

Definition 3 (Base-orderable Matroid)**.** A matroid $(E, \mathcal{I})$ is called *base-orderable* if for any two bases $B_1, B_2 \in \mathcal{B}$, there exists a bijection $\varphi \colon B_1 \to B_2$ with the property that, for every $e \in B_1$, both $B_1 - e + \varphi(e)$ and $B_2 + e - \varphi(e)$ are bases.

A class of base-orderable matroids includes *gammoids* [5, 18] and, in particular, includes laminar matroids defined below.

Example 4 (Laminar Matroid)**.** For a laminar family $\mathcal{L}$ on E and a function $q \colon \mathcal{L} \to \mathbf{Z}_+$, define $\mathcal{I} = \{\, S \subseteq E \mid \forall L \in \mathcal{L} : |S \cap L| \leq q(L) \,\}$. Then, $(E, \mathcal{I})$ is a base-orderable matroid.

A matroid is *laminar* if it can be defined in the above-mentioned manner for some $\mathcal{L}$ and q.

Base-orderability is known to be closed under the following operations (see, e.g., [3, 18]).

Contraction.[2] For a matroid $(E, \mathcal{I})$ and any $S \in \mathcal{I}$, define $\mathcal{I}_S \coloneqq \{\, T \subseteq E \setminus S \mid S \cup T \in \mathcal{I} \,\}$. Then, $(E \setminus S, \mathcal{I}_S)$ is a matroid. If $(E, \mathcal{I})$ is base-orderable, then so is $(E \setminus S, \mathcal{I}_S)$.

Truncation. For a matroid $(E, \mathcal{I})$ and any integer $p \in \mathbf{Z}_+$, define $\mathcal{I}_p \coloneqq \{\, S \in \mathcal{I} \mid |S| \leq p \,\}$. Then, $(E, \mathcal{I}_p)$ is a matroid. If $(E, \mathcal{I})$ is base-orderable, then so is $(E, \mathcal{I}_p)$.

Direct Sum. For matroids $(E_j, \mathcal{I}_j)$ $(j = 1, 2, \dots, \ell)$ such that E_j are all pairwise disjoint, let $E \coloneqq E_1 \cup E_2 \cup \cdots \cup E_\ell$ and $\mathcal{I} \coloneqq \{\, S_1 \cup S_2 \cup \cdots \cup S_\ell \mid S_j \in \mathcal{I}_j \ (j = 1, 2, \dots, \ell) \,\}$. Then, $(E, \mathcal{I})$ is a matroid. If all $(E_j, \mathcal{I}_j)$ are base-orderable, then so is $(E, \mathcal{I})$.

[2] Contraction is defined for any subset of E [32]; however this paper uses only contraction by independent sets.

On the intersection of two base-orderable matroids, we show the following property, which plays a key role in proving the $\frac{3}{2}$-approximation ratio of our algorithm. This generalizes the fact that, if (one-to-one) bipartite matchings M and N satisfy $|M| < \frac{2}{3}|N|$, then $M \triangle N$ contains a connected component that forms an alternating path of length at most three.

Lemma 5. *For base-orderable matroids $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$, suppose that $S, T \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|S| < \frac{2}{3}|T|$. If $S + e \notin \mathcal{I}_1 \cap \mathcal{I}_2$ for every $e \in T \setminus S$, then there exist distinct elements e_i, e_j, e_k such that $e_i, e_k \in T \setminus S$, $e_j \in S \setminus T$, and the following conditions hold:*

- $S + e_i \in \mathcal{I}_1$,
- *both* $S + e_i - e_j$ *and* $T - e_i + e_j$ *belong to* $\mathcal{I}_2$,
- *both* $S - e_j + e_k$ *and* $T + e_j - e_k$ *belong to* $\mathcal{I}_1$,
- $S + e_k \in \mathcal{I}_2$.

Proof. By the matroid axiom (I3), there is a subset $A_1 \subseteq T \setminus S$ such that $|A_1| = |T| - |S|$ and $S_1 \coloneqq S \cup A_1 \in \mathcal{I}_1$. Then, $|S_1| = |T|$; hence, $|S_1 \setminus T| = |T \setminus S_1|$. Let $(E', \mathcal{I}'_1)$ be a matroid obtained from $(E, \mathcal{I}_1)$ by contracting $S_1 \cap T$ and truncating with size $|S_1 \setminus T|$, i.e., $E' = E \setminus (S_1 \cap T)$ and $\mathcal{I}' \coloneqq \{\, R \subseteq E' \mid R \cup (S_1 \cap T) \in \mathcal{I}_1, |R| \le |S_1 \setminus T| \,\}$. Then, $S_1 \setminus T$ and $T \setminus S_1$ are bases of $(E', \mathcal{I}'_1)$. As $(E', \mathcal{I}'_1)$ is base-orderable, there is a bijection $\varphi_1 \colon S_1 \setminus T \to T \setminus S_1$ such that both $(S_1 \setminus T) - e + \varphi_1(e)$ and $(T \setminus S_1) + e - \varphi_1(e)$ are bases of $(E', \mathcal{I}'_1)$ for every $e \in S_1 \setminus T$. By the definition of $\mathcal{I}'_1$, this implies that both $S - e + \varphi_1(e)$ and $T + e - \varphi_1(e)$ belong to $\mathcal{I}_1$ for every $e \in S_1 \setminus T$. By the same argument, there exists $A_2 \subseteq T \setminus S$ such that $|A_2| = |T| - |S|$ and $S_2 \coloneqq S \cup A_2 \in \mathcal{I}_2$, and there exists a bijection $\varphi_2 \colon S_2 \setminus T \to T \setminus S_2$ such that both $S - e + \varphi_2(e)$ and $T + e - \varphi_2(e)$ belong to $\mathcal{I}_2$ for every $e \in S_2 \setminus T$.

We represent φ_1 and φ_2 using a bipartite graph as follows. Note that, for each $\ell \in \{1, 2\}$, we have $S_\ell \setminus T = S \setminus T$ and $T \setminus S_\ell = T \setminus (S \cup A_\ell) \subseteq T \setminus S$. Let $S \setminus T$ and $T \setminus S$ be two vertex sets and let $M_\ell \coloneqq \{\, (e, \varphi_\ell(e)) \mid e \in S \setminus T \,\}$ for $\ell = 1, 2$. Then, each M_ℓ is a one-to-one matching that covers $S \setminus T$ and $T \setminus (S \cup A_\ell)$. Note that the sets $A_1, A_2 \subseteq S \setminus T$ are mutually disjoint since, otherwise, some $e \in A_1 \cap A_2$ satisfies $S + e \in \mathcal{I}_1 \cap \mathcal{I}_2$, which contradicts the assumption. Then, $|T \setminus (S \cup A_1 \cup A_2)| = |T \setminus S| - |A_1| - |A_2| = |T \setminus S| - 2|T| + 2|S|$. Therefore, at most $2(|T \setminus S| - 2|T| + 2|S|)$ vertices in $S \setminus T$ are adjacent to $T \setminus (S \cup A_1 \cup A_2)$ via the edges in $M_1 \cup M_2$. Because $|S \setminus T| - 2(|T \setminus S| - 2|T| + 2|S|) = -3|S| + 2|T| + |S \cap T| > -3 \cdot \frac{2}{3}|T| + 2|T| + |S \cap T| \ge 0$, there exists $\tilde{e} \in S \setminus T$ that is not adjacent to $T \setminus (S \cup A_1 \cup A_2)$ via $M_1 \cup M_2$. This implies that $\varphi_2(\tilde{e}) \in A_1$ and $\varphi_1(\tilde{e}) \in A_2$; hence, $S + \varphi_2(\tilde{e}) \in \mathcal{I}_1$ and $S + \varphi_1(\tilde{e}) \in \mathcal{I}_2$. Let $e_i \coloneqq \varphi_2(\tilde{e})$, $e_j \coloneqq \tilde{e}$, and $e_k \coloneqq \varphi_1(\tilde{e})$. Then, these three elements satisfy all the required conditions. $\square$

5 Correctness

This section is devoted to showing Theorem 1, which establishes the correctness of Algorithm 1.

As in Section 3, let I be an SMTI-LC instance with $E = \{\, e_i \mid i = 1, 2, \ldots, n \,\}$ and let I^* be the auxiliary instance I^*, whose contract set is $E^* = \{\, x_i, y_i, z_i \mid i = 1, 2, \ldots, n \,\}$.

For any agent $a \in W \cup F$, let $E^*_a = \{\, x_i, y_i, z_i \mid e_i \in E_a \,\}$ and define families $\mathcal{I}_a$ and $\mathcal{I}^*_a$ by

$$\begin{aligned} \mathcal{I}_a &= \{\, S \subseteq E_a \mid \forall L \in \mathcal{L}_a : |S \cap L| \le q_a(L) \,\}, \\ \mathcal{I}^*_a &= \{\, S^* \subseteq E^*_a \mid \forall L^* \in \mathcal{L}^*_a : |S^* \cap L^*| \le q^*_a(L^*) \,\}, \end{aligned}$$

i.e., $\mathcal{I}_a$ and $\mathcal{I}^*_a$ are the families of feasible sets in I and I^*, respectively. Then, $(E_a, \mathcal{I}_a)$ and $(E^*_a, \mathcal{I}^*_a)$ are laminar matroids and base-orderable. The definitions of $\mathcal{L}^*_a$ and q^*_a imply the following fact. Recall that $\pi : 2^{E^*} \to 2^E$ is defined by $\pi(S^*) = \{\, e_i \mid \{x_i, y_i, z_i\} \cap S^* \ne \emptyset \,\}$.

Observation 6. *For a set $S^* \subseteq E^*_a$, we have $S^* \in \mathcal{I}^*_a$ if and only if $|\{x_i, y_i, z_i\} \cap S^*| \leq 1$ for every $e_i \in E_a$ and $\pi(S^*) \in \mathcal{I}_a$.*

Take any stable matching M^* of I^* and let $M := \pi(M^*)$. As M^* is feasible in I^*, it contains at most one copy of each contract e_i. For any $e_i \in M$, we denote by $\pi^{-1}(e_i)$ the unique element in $\{x_i, y_i, z_i\} \cap M^*$.

By the definitions of the preference lists $\{P^*_a\}_{a \in W \cup F}$ in I^*, we can observe the following properties. For any agent $a \in W \cup F$ and contracts $e, e' \in E^*_a$, we write $e \succ^*_a e'$ if a prefers e to e' with respect to P^*_a. Recall that P^*_a does not contain ties, while P_a may contain.

Observation 7. *For any $e_i \in E \setminus M$ and $e_j \in M$, the following conditions hold.*

- *For any agent $a \in W \cup F$, if $e_i, e_j \in E_a$ and $e_i \succ_a e_j$, then $y_i \succ^*_a \pi^{-1}(e_j)$ holds regardless of which of $\{x_i, y_i, z_i\}$ is $\pi^{-1}(e_i)$.*
- *For any worker $w \in W$, if $e_i, e_j \in E_w$ and $\pi^{-1}(e_j) \succ^*_w x_i$, then we have either $[\ \pi^{-1}(e_j) = x_j$ and $e_j \succeq_w e_i\ ]$ or $[\ \pi^{-1}(e_j) = y_j$ and $e_j \succ_w e_i\ ]$.*
- *For any firm $f \in F$, if $e_i, e_j \in E_f$ and $\pi^{-1}(e_j) \succ^*_f z_i$, then we have either $[\ \pi^{-1}(e_j) = z_j$ and $e_j \succeq_f e_i\ ]$ or $[\ \pi^{-1}(e_j) = y_j$ and $e_j \succ_f e_i\ ]$.*

First, we show the stability of M in I. For each agent $a \in W \cup F$, we write $M^*_a = M^* \cap E^*_a$, which implies that $\pi(M^*_a) = M_a$.

Lemma 8. *The set M is a stable matching of I.*

Proof. Since M^* is feasible for all agents in I^*, Observation 6 implies that $M = \pi(M^*)$ is feasible for all agents in I, i.e., M is a matching in I.

Suppose, to the contrary, that M is not stable. Then, some contract $e_i \in E \setminus M$ blocks M. Let $w = \partial_W(e_i)$ and $f = \partial_F(e_i)$. Then, e_i is free for both w and f in M. We now show that y_i is free for both w and f in M^*, which contradicts the stability of M^*.

As e_i is free for w in I, we have (i) $M_w + e_i \in \mathcal{I}_w$ or (ii) there exists $e_j \in M_a$ such that $e_i \succ_w e_j$ and $M_a + e_i - e_j \in \mathcal{I}_w$. Note that $e_i \in E \setminus M$ implies $\{x_i, y_i, z_i\} \cap M = \emptyset$. In case (i), we have $\pi(M^*_w + y_i) = M_w + e_i \in \mathcal{I}_a$, which implies $M^*_w + y_i \in \mathcal{I}^*_w$; hence, y_i is free for w in M^*. In case (ii), we have $\pi(M^*_w + y_i - \pi^{-1}(e_j)) = M_w + e_i - e_j \in \mathcal{I}_w$, which implies $M^*_w + y_i - \pi^{-1}(e_j) \in \mathcal{I}^*_w$. Furthermore, as $e_i \succ_w e_j$, the first statement of Observation 7 imply $y_i \succ^*_w \pi^{-1}(e_j)$. Thus, in each case, y_i is free for w in M^*.

Similarly, we can show that y_i is free for f in M^*. Thus, y_i blocks M^*, a contradiction. $\square$

Next, we show the approximation ratio using Lemma 5. Note that $\{\, E_w \mid w \in W \,\}$ is a partition of E, as is $\{\, E_f \mid f \in F \,\}$. Let $(E, \mathcal{I}_W)$ be the direct sum of base-orderable matroids $\{\, (E_w, \mathcal{I}_w) \mid w \in W \,\}$ and $(E, \mathcal{I}_F)$ be the direct sum of $\{\, (E_f, \mathcal{I}_f) \mid f \in F \,\}$. Then, they are both base-orderable matroids on E.

By definition, for any subset $N \subseteq E$, we have $N \in \mathcal{I}_W \cap \mathcal{I}_F$ if and only if $N_a := N \cap E_a$ is feasible for each $a \in W \cup F$, i.e., N is a matching. Furthermore, for any matching $N \in \mathcal{I}_W \cup \mathcal{I}_F$ and contract $e_i \in E \setminus N$, which is associated with a worker $w = \partial_W(e_i)$ (and a firm $f = \partial_F(e_i)$), the condition $N + e_i \in \mathcal{I}_W$ is equivalent to $N_w + e_i \in \mathcal{I}_w$. In addition, if $N + e_i \notin \mathcal{I}_W$, we have $N + e_i - e_j \in \mathcal{I}_W$ if and only if $e_j \in N_w$ and $N_w + e_i - e_j \in \mathcal{I}_w$. The same statements hold when w and W are replaced with f and F, respectively

Lemma 9. *The set M satisfies $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$, where M_{OPT} is a maximum-cardinality stable matching of I.*

Proof. Set $N := M_{\mathrm{OPT}}$ for notational simplicity. Since M and N are stable matchings, $M, N \in \mathcal{I}_W \cap \mathcal{I}_F$. In addition, $M + e_i \notin \mathcal{I}_W \cap \mathcal{I}_F$ for any $e_i \in N \setminus M$ since, otherwise, e_i blocks M. Suppose, to the contrary, that $|M| < \frac{2}{3}|N|$. Then, by Lemma 5 and the definitions of $\mathcal{I}_W$ and $\mathcal{I}_F$, there exist three contracts e_i, e_j, e_k such that $e_i, e_k \in N \setminus M$, $e_j \in M \setminus N$, and the following conditions hold:

- $M_w + e_i \in \mathcal{I}_w$,
- both $M_f + e_i - e_j$ and $N_f - e_i + e_j$ belong to $\mathcal{I}_f$,
- both $M_{w'} - e_j + e_k$ and $N_{w'} + e_j - e_k$ belong to $\mathcal{I}_{w'}$,
- $M_{f'} + e_k \in \mathcal{I}_{f'}$,

where $w = \partial_W(e_i)$, $f = \partial_F(e_i) = \partial_F(e_j)$, $w' = \partial_W(e_j) = \partial_W(e_k)$, $f' = \partial_F(e_k)$.

Since $e_i \notin M$ and $M_w + e_i \in \mathcal{I}_w$, we have $M^*_w + z_i \in \mathcal{I}^*_w$; hence, z_i is free for the worker $w = \partial_W(z_i)$ in M^*. Then, the stability of M^* implies that z_i is not free for the firm $f = \partial_F(z_i)$. Since $\pi(M^*_f + z_i - \pi^{-1}(e_j)) = M_f + e_i - e_j \in \mathcal{I}_f$ implies $M^*_f + z_i - \pi^{-1}(e_j) \in \mathcal{I}^*_f$, we should have $\pi^{-1}(e_j) \succ^*_f z_i$. Then, the third statement of Observation 7 implies that we have either $[\pi^{-1}(e_j) = z_j$ and $e_j \succeq_f e_i]$ or $[\pi^{-1}(e_j) = y_j$ and $e_j \succ_f e_i]$.

Meanwhile, since $e_k \notin M$ and $M_{f'} + e_k \in \mathcal{I}_{f'}$, we have $M^*_{f'} + x_i \in \mathcal{I}^*_{f'}$; hence, x_k is free for the firm $f' = \partial_W(x_k)$ in M^*. As M^* is stable, then x_k is not free for the worker $w' = \partial_W(x_k)$. Since $\pi(M^*_{w'} + x_i - \pi^{-1}(e_j)) = M_{w'} + e_i - e_j \in \mathcal{I}_{w'}$ implies $M^*_{w'} + x_i - \pi^{-1}(e_j) \in \mathcal{I}^*_{w'}$, we should have $\pi^{-1}(e_j) \succ^*_{w'} x_i$. Then, the second statement of Observation 7 implies that we have either $[\pi^{-1}(e_j) = x_j$ and $e_j \succeq_{w'} e_i]$ or $[\pi^{-1}(e_j) = y_j$ and $e_j \succ_{w'} e_i]$.

Because we cannot have $\pi^{-1}(e_j) = z_j$ and $\pi^{-1}(e_j) = x_j$ simultaneously, we must have $\pi^{-1}(e_j) = y_j$, $e_j \succ_f e_i$, and $e_j \succ_{w'} e_i$. As we have $N_{w'} + e_j - e_k \in \mathcal{I}_{w'}$ and $N_f - e_i + e_j \in \mathcal{I}_f$, these preference relations imply that e_j blocks N, which contradicts the stability of N. □

Proof of Theorem 1. Combining Lemmas 8 and 9, we obtain Theorem 1. □

6 Time Complexity

In this section, we explain how to implement the second step of Algorithm 1 and estimate its time complexity, which establishes Theorem 2. For this purpose, we introduce the notion of a matroid-kernel, which is a matroid generalization of a stable matching proposed by Fleiner [8, 9]. Note that it is defined not only for base-orderable matroids but for general matroids.

6.1 Matroid-kernels

A triple $\mathcal{M} = (E, \mathcal{I}, \succ)$ is called an *ordered matroid* if $(E, \mathcal{I})$ is a matroid and $\succ$ is a strict linear order on E. For an ordered matroid $\mathcal{M} = (E, \mathcal{I}, \succ)$ and an independent set $S \in \mathcal{I}$, an element $e \in E \setminus S$ is said to be *dominated* by S in $\mathcal{M}$ if $S + e \notin \mathcal{I}$ and there is no element $e' \in S$ such that $e \succ e'$ and $S + e - e' \in \mathcal{I}$.

Let $\mathcal{M}_1 = (E, \mathcal{I}_1, \succ_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2, \succ_2)$ be two ordered matroids on the same ground set E. Then, a set $S \subseteq E$ is called an $\mathcal{M}_1\mathcal{M}_2$-*kernel* if $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ and any element $e \in E \setminus S$ is dominated by S in $\mathcal{M}_1$ or $\mathcal{M}_2$.

In [8], an algorithm for finding a matroid-kernel has been described using choice functions defined as follows. For an ordered matroid $\mathcal{M} = (E, \mathcal{I}, \succ)$, give indices of elements in E such that $E = \{e^1, e^2, \ldots, e^n\}$ and $e^1 \succ e^2 \succ \cdots \succ e^n$. Define a function $C_{\mathcal{M}} : 2^E \to 2^E$ by letting

$C_{\mathcal{M}}$ be the output of the following greedy algorithm for every $S \subseteq E$. Let $T^0 := \emptyset$ and define T^ℓ for $\ell = 1, 2, \ldots, n$ by

$$T^\ell := \begin{cases} T^{\ell-1} + e^\ell & \text{if } \; e^\ell \in S \; \text{ and } \; T^{\ell-1} + e^\ell \in \mathcal{I}, \\ T^{\ell-1} & \text{otherwise;} \end{cases}$$

then, let $\mathcal{C}_{\mathcal{M}}(S) := T^n$.

Let $C_{\mathcal{M}_1}$, $C_{\mathcal{M}_2}$ be the choice functions defined from $\mathcal{M}_1 = (E, \mathcal{I}_1, \succ_1)$, $\mathcal{M}_2 = (E, \mathcal{I}_2, \succ_2)$, respectively. In [8, Theorem 2], Fleiner showed that an $\mathcal{M}_1\mathcal{M}_2$-kernel can be found using the following algorithm, which can be regarded as a generalization of the Gale–Shapley algorithm. First, set $R \leftarrow \emptyset$. Then, repeat the following three steps: (1) $S \leftarrow C_{\mathcal{M}_1}(E \setminus R)$, (2) $T \leftarrow C_{\mathcal{M}_2}(S \cup R)$, and (3) $R \leftarrow (S \cup R) \setminus T$. Stop the repetition if R is not changed at (3) and return T at that moment. In terms of the ordinary Gale–Shapley algorithm, R, S, and T correspond to the sets of contracts that are rejected by firms thus far, proposed by workers, and accepted by firms, respectively.

Theorem 10 (Fleiner [8, 9]). *For any pair of ordered matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set E, there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel. One can find an $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|E| \cdot \mathrm{EO})$ time, where* EO *is the time required to compute $C_{\mathcal{M}_1}(S)$ and $C_{\mathcal{M}_2}(S)$ for any $S \subseteq E$.*

6.2 Implementation of Our Algorithm

We show that the second step of Algorithm 1 is reduced to a computation of a matroid-kernel.

For an auxiliary instance I^* defined in Section 2, note that $\{\, E^*_w \mid w \in W \,\}$ is a partition of E^* and let $(E^*, \mathcal{I}^*_W)$ be the direct sum of $\{(E^*_w, \mathcal{I}^*_w)\}_{w \in W}$. Furthermore, let $\succ_W$ be a strict linear order on E^* that is consistent with the workers' preferences $\{P^*_w\}_{w \in W}$ in I^*. For example, obtain $\succ_W$ by concatenating the lists P^*_w of all workers in an arbitrary order. Then, $\mathcal{M}_W = (E^*, \mathcal{I}^*_W, \succ_W)$ is an ordered matroid on the contract set E^*. As $\{\, E^*_f \mid f \in F \,\}$ is also a partition of E^*, we can define an ordered matroid $\mathcal{M}_F = (E^*, \mathcal{I}^*_F, \succ_F)$ in the same manner from $\{(E^*_f, \mathcal{I}^*_f)\}_{f \in F}$ and $\{P^*_f\}_{f \in F}$.

We show that $\mathcal{M}_W\mathcal{M}_F$-kernels are equivalent to stable matchings of I. This has already been shown in several previous works [10, 36]. We present a proof for the completeness.

Lemma 11. *$M^* \subseteq E^*$ is a stable matching of I^* if and only if M^* is an $\mathcal{M}_W\mathcal{M}_F$-kernel.*

Proof. By the definitions of $(E^*, \mathcal{I}^*_W)$ and $(E^*, \mathcal{I}^*_f)$, a set $M^* \subseteq E^*$ is feasible for all agents in I^* if and only if $M^* \in \mathcal{I}^*_W \cap \mathcal{I}^*_F$. Recall that a contract $e \in E^* \setminus M^*$ is free for the associated worker $w := \partial_W(e)$ if $M^*_w + e \in \mathcal{I}^*_w$ or there exists $e' \in M^*_w$ such that $e \succ^*_w e'$ and $M^*_w + e - e' \in \mathcal{I}^*_w$. By the definition of $\mathcal{I}^*_W$, we have $M^*_w + e \in \mathcal{I}^*_w$ if and only if $M^* + e \in \mathcal{I}^*_W$. In addition, if $M^*_w + e \notin \mathcal{I}^*_w$, then $M^*_w + e - e' \in \mathcal{I}^*_w$ holds for $e' \in M_w$ if and only if $M^* + e - e' \in \mathcal{I}^*_W$. Because $\succ_W$ is consistent with $\succ^*_w$, these imply that e is free for $w = \partial_W(e)$ in M^* if and only if e is not dominated by M^* in $\mathcal{M}_W$. Similarly, we can show that e is free for the associated firm $f := \partial_F(e)$ in M^* if and only if e is not dominated by M^* in $\mathcal{M}_F$. Thus, the equivalence holds. ☐

Lemma 12. *For any subset $S^* \subseteq E^*$, we can compute $C_{\mathcal{M}_W}(S^*)$ and $C_{\mathcal{M}_F}(S^*)$ in $O(k^* \cdot |E^*|)$ time, where k^* is the maximum level of nesting of laminar families $\mathcal{L}^*_a$ $(a \in W \cup F)$.*

Proof. We only explain the computation of $C_{\mathcal{M}_W}(S^*)$ because that of $C_{\mathcal{M}_F}(S^*)$ is similar.

Let $\mathcal{L}$ be the union of $\{\mathcal{L}^*_w\}_{w \in W}$ and define $q : \mathcal{L} \to \mathbf{Z}_+$ by setting $q(L) = q^*_w(L)$ for each $w \in W$ and $L \in \mathcal{L}^*_w$. Then, $\mathcal{L}$ is a laminar family on E^* and the matroid $(E^*, \mathcal{I}^*_W)$ is defined by $\mathcal{L}$ and q. The maximum level of nesting of $\mathcal{L}$ is again k^*.

Referring to [2], we represent $\mathcal{L}$ by a forest G whose node set is $\{ v_L \mid L \in \mathcal{L} \}$. Node v_L is the parent of $v_{L'}$ in G if $L \subseteq L'$ and there is no $L'' \in \mathcal{L}$ such that $L \subsetneq L'' \subsetneq L'$. Note that $\mathcal{L}$ contains the set $\{x_i, y_i, z_i\}$ for every $e_i \in E$, which is inclusion-wise minimal in $\mathcal{L}$. Therefore, the node $v_i \coloneqq v_{\{x_i,y_i,z_i\}}$ is a leaf for any $e_i \in E$, and any leaf has this form.

We compute the sequence $T^0, T^1, \ldots, T^{|E^*|}$ of sets in the definition of $C_{\mathcal{M}_W}(S^*)$ as follows. For each v_L, we store a pointer to its parent, the value of $q(L)$, and the value of $|T^{\ell-1} \cap L|$. For each $e^\ell \in E^*$, we have $T^{\ell-1} + e^\ell \in \mathcal{I}^*_W$ if and only if there is no ancestor node v_L of v_i with $q(L) = |T^{\ell-1} \cap L|$, where v_i is the leaf with $e^\ell \in \{x_i, y_i, z_i\}$. Then, we can check whether $T^{\ell-1} + e^\ell \in \mathcal{I}^*_W$ in $O(k^*)$ time by following the path of the parent pointers from v_i. When $T^\ell = T^{\ell-1} + e^\ell$, we update the stored values $|T^{\ell-1} \cap L|$ to $|T^\ell \cap L|$ for each $L \in \mathcal{L}$ with $e^\ell \in L$. This is also performed in $O(k^*)$ time by following the path of the parent pointers. $\square$

Proof of Theorem 2. As we have Theorem 1, what is left is to show the time complexity. The set E^* of contracts in I^* satisfies $|E^*| = 3|E|$. The maximum level of nesting of laminar families $\mathcal{L}^*_a$ in I^* is $k + 1$. By Theorem 10 and Lemmas 11 and 12, then the second step of Algorithm 1 is computed in $O((k + 1) \cdot |E^*|^2) = O(k \cdot |E|^2)$ time. Since the first and third steps can be performed in $O(k \cdot |E|^2)$ time, clearly, Algorithm 1 runs in $O(k \cdot |E|^2)$ time. $\square$

Remark 2. Our analysis depends on the fact that the feasible set family defined by laminar constraints forms the independent set family of a base-orderable matroid. Actually, we can extend Theorem 1 to a setting where the family of feasible sets of each agent $a \in W \cup F$ is represented by the independent set family $\mathcal{I}_a$ of an arbitrary base-orderable matroid. To construct I^* in this case, we define E^* and $\{P^*_a\}_{a \in W \cup F}$ as in Section 3 and define the feasible set family $\mathcal{I}^*_a$ by $\mathcal{I}^*_a = \{ S^* \subseteq E^*_a \mid |\{x_i, y_i, z_i\} \cap S^*| \leq 1 \text{ for any } e_i \in E_a \text{ and } \pi(S^*) \in \mathcal{I}_a \}$. We can easily show that $(E^*_a, \mathcal{I}^*_a)$ is also a base-orderable matroid and apply the arguments in Sections 5 and 6, except Lemma 12. Given a membership oracle for each $\mathcal{I}_a$ available, Algorithm 1 runs in $O(\tau \cdot |E|^2)$ time in this case, where τ is the time for an oracle call.

7 Strategy-proofness for a Special Case

Using Algorithm 1, we present a mechanism that is strategy-proof for workers and attains the approximation ratio $\frac{3}{2}$ in a special case of SMTI-LC, where

- each worker is assigned at most one contract and
- each firm has a strict preference.

We now define a setting SMTI-OLC, which is a many-to-one variant of SMTI-LC. (Here, OLC stands for one-sided laminar constraints). An instance of SMTI-OLC is described as $I = (W, F, E, \{P_w\}_{w \in W}, \{\mathcal{L}_f, q_f, P_f\}_{f \in F})$. To consider strategies of workers, we slightly change the assumption on each P_w. In Section 2, it is assumed that P_w contains all contracts in E_w. Here, we allow each worker to submit a preference list P_w that is defined on any subset of E_w and regard contracts not appearing in P_w as unacceptable for w. Let E° be the set of contracts appearing in $\{P_w\}_{w \in W}$. For an SMTI-OLC instance I, a stable matching is defined similarly as in Section 2, where a matching M should satisfy $M \subseteq E^\circ$ as well as the feasibilities of all firms, and a contract $e \in E \setminus M$ blocks a matching M only if $e \in E^\circ$. The auxiliary instance $I^* = (W, F, E^*, \{P^*_w\}_{w \in W}, \{\mathcal{L}^*_f, q^*_f, P^*_f\}_{f \in F})$ is defined similarly as in Section 3. By restricting I on E°, we can transform I into an equivalent instance in the form in Section 2; hence, all the results in Sections 5 and 6 extend to this setting. See Observation 17 in Appendix B for the details.

A *mechanism* is a mapping from SMTI-OLC instances to matchings. We define the *worker-strategy-proofness* of a mechanism. Let A be a mechanism. For any instance I and any worker w, let I' be an instance obtained from I by replacing w's list P_w with some other list P'_w. Let M and M' be the outputs of A for instances I and I', respectively. Then, A is *worker-strategy-proof* if w weakly prefers M to M' with respect to P_w regardless of the choices of I, w, and P'_w, where we say that w *weakly prefers* M to M' w.r.t. P_w if either (i) w is not assigned a contract on P_w in M' or (ii) w is assigned a contract on P_w in both M and M' and does not strictly prefer the one assigned in M' w.r.t. P_w.

If ties appear in firms' preference lists, no worker-strategy-proof mechanism can perform better than a simple tie-breaking method, which attains an approximation ratio of 2 (see Appendix B). However, if ties appear only in workers' preference lists, we can present a worker-strategy-proof mechanism whose approximation ratio is $\frac{3}{2}$.

Our mechanism is regarded as a possible realization of Algorithm 1. We should choose the worker-optimal stable matching of the auxiliary instance I^* in the second step of Algorithm 1. A stable matching M is called *worker-optimal* if, for any other stable matching N, every worker weakly prefers M to N. Our mechanism is described as follows.

1. Given an instance I (in which ties appear in only workers' lists), construct I^*.

2. Find the worker-optimal stable matching M^* of I^*.

3. Let $M = \pi(M^*)$ and return M.

In the proof of Theorem 10 (Fleiner [8, p.113]), it is shown that one can find the $\mathcal{M}_1$-optimal $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|E| \cdot \mathrm{EO})$ time. The arguments in Section 6 then imply that one can find the worker-optimal stable matching of I^* in $O(k \cdot |E|^2)$ time. As we have Theorem 2, showing the strategy-proofness of the above-mentioned mechanism establishes the following theorem.

Theorem 13. *For a restriction of* SMTI-OLC *in which ties appear in only workers' lists, there is a worker-strategy-proof mechanism that returns a stable matching M with $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$ in $O(k \cdot |E|^2)$ time, where M_{OPT} is a maximum-cardinality stable matching and k is the maximum level of nesting of laminar families $\mathcal{L}_f$ $(f \in F)$.*

To prove the strategy-proofness, we use the following lemmas.

Lemma 14. *Let I be an* SMTI-OLC *instance with $E = \{\, e_i \mid i = 1, 2, \ldots, n \,\}$ and let I^* be the auxiliary instance. If ties appear in only workers' lists in I, then the worker-optimal stable matching M^* of I^* satisfies $M^* \cap \{\, z_i \mid i = 1, 2, \ldots, n \,\} = \emptyset$.*

Proof. Suppose, to the contrary, that $z_i \in M^*$ for some index i. Then $N := M^* - z_i + y_i$ is a matching of I^* and $w := \partial_W(z_i) = \partial_W(y_i)$ prefers N to M. We intend to show that N is stable in I^*. Take any $e \in E^* \setminus N = (E^* \setminus M^*) + z_i - y_i$. If $e = z_i$, then it does not block N because $y_i \succ^*_w z_i$. If $e \neq z_i$, then the assignment of $\partial_W(e)$ does not change in M^* and N, and hence e can block N only if $f := \partial_F(e) = \partial_F(z_i)$ and $z_i \succ^*_f e \succ^*_f y_i$. This is impossible because no contract lies between z_i and y_i in P^*_f as the list P_f of the firm f is strict. Thus, N is a stable matching of I^*, which contradicts the worker-optimality of M^*. $\square$

Lemma 15. *In a restriction of* SMTI-OLC *in which all agents have strict preferences, a mechanism that returns the worker-optimal stable matching is worker-strategy-proof.*

We present the proof of Lemma 15 in Appendix B, where we show that the setting in the statement can be reduced to the model of Hatfield and Milgrom [16].

Proof of Theorem 13. As we have Theorem 2, what is left is to show that our mechanism is worker-strategy-proof. Let $I = (W, F, E, \{P_w\}_{w \in W}, \{\mathcal{L}_f, q_f, P_f\}_{f \in F})$ be an instance of the setting in the statement and let $E = \{\, e_i \mid i = 1, 2, \ldots, n \,\}$. Furthermore, let I' be obtained from I by replacing P_w with some other list P'_w. Let M^* and N^* be the worker-optimal stable matchings of the auxiliary instances defined from I and I', respectively. Note that the two auxiliary instances have no ties and they differ only in the preference list of w. Then, Lemma 15 implies that w weakly prefers M^* to N^* with respect to P^*_w. In other words, either (i) w is not assigned a contract on P^*_w in N^*, or (ii) w is assigned a contract on P^*_w in both M^* and N^* and does not strictly prefer the one assigned in N^* w.r.t. P^*_w. By Lemma 14, w is not assigned a contract of type z_i in M^* or N^*. Then, the definition of P^*_w implies that w weakly prefers $\pi(M^*)$ to $\pi(N^*)$ w.r.t. P_w. Thus the mechanism is worker-strategy-proof. $\square$

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A 2-approximability by Arbitrary Tie-breaking

For MAX-SMTI (i.e., the problem of finding a maximum-cardinality stable matching in the stable marriage model with ties and incomplete lists), which is a special case of MAX-SMTI-LC, it is known that we can obtain a 2-approximate solution by breaking ties arbitrarily and computing a stable matching of the resultant instance [30]. We show that this fact extends to the setting of MAX-SMTI-LC.

Proposition 16. *For an* SMTI-LC *instance* $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$*, define* I' *by replacing each* P_a *with any strict preference* P'_a *that is consistent with* P_a *(i.e., obtain* I' *from* I *by tie-breaking). Then, any stable matching* M *of* I' *is a stable matching of* I *and satisfies* $|M| \geq \frac{1}{2}|M_{\mathrm{OPT}}|$*, where* M_{OPT} *is a maximum-cardinality stable matching of* I.

Proof. Let E be the set of contracts in I. First, we show that M is a stable matching of I. By definition, M is clearly a matching in I. Suppose, to the contrary, that some contract $e \in E \setminus M$ blocks M in I. Then, e is free for $w \coloneqq \partial_W(e)$, which implies that $M_w + e$ is feasible for w or there exists $e' \in M_w$ such that $e \succ_w e'$ and $M_w + e - e'$ is feasible for w, where $\succ_w$ is defined by P_w. As P'_w is consistent with P_w, it implies that e is free for w also in I'. We can similarly show that e is free for $f \coloneqq \partial_F(e)$ in I'. Then, e blocks M in I', which contradicts M being a stable matching of I'.

Next, we show that $|M| \geq \frac{1}{2}|N|$, where $N \coloneqq M_{\mathrm{OPT}}$. Let $(E, \mathcal{I}_W)$ and $(E, \mathcal{I}_F)$ be defined as in Section 5 (after Lemma 8). Then, we have $M, N \in \mathcal{I}_W \cap \mathcal{I}_F$. Suppose, to the contrary, that $|M| < \frac{1}{2}|N|$. As $(E, \mathcal{I}_W)$ is a matroid, the matroid axiom (I3) implies that there exists a subset $A_1 \subseteq N \setminus M$ such that $|A_1| = |N| - |M|$ and $M \cup A_1 \in \mathcal{I}_W$. Similarly, as $(E, \mathcal{I}_F)$ is a matroid, there exists a subset $A_2 \subseteq N \setminus M$ such that $|A_2| = |N| - |M|$ and $M \cup A_2 \in \mathcal{I}_F$. By $|M| < \frac{1}{2}|N|$, we have $|A_1| + |A_2| = 2(|N| - |M|) > |N| \geq |N \setminus M|$. Since $A_1, A_2 \subseteq N \setminus M$, this implies $A_1 \cap A_2 \neq \emptyset$. Then, there exists $e \in A_1 \cap A_2$, which satisfies $M + e \in \mathcal{I}_W$ and $M + e \in \mathcal{I}_F$. Then, e is free for both the worker $\partial_W(e)$ and the firm $\partial_F(e)$ in I'. This contradicts the stability of M in I'. □

B Complements for Section 7 (Strategy-proofness)

This section presents omitted details and additional results related to Section 7.

We investigate approximation ratios for the problem MAX-SMTI-LC that can be attained by strategy-proof mechanisms. First, note that our setting SMTI-LC is a generalization of the stable marriage model of Gale and Shapley [12]; hence, Roth's impossibility theorem [34] implies that there is no mechanism that returns a stable matching and is strategy-proof for agents on both sides. As with many existing works on strategy-proofness in two-sided matching models, we consider one-sided strategy-proofness in the setting of many-to-one matching.

We review the setting of SMTI-OLC defined in Section 7. In SMTI-OLC, each worker is assigned at most one contract and hence has no laminar constraints. Only firms have laminar constraints. Let $I = (W, F, E, \{P_w\}_{w \in W}, \{\mathcal{L}_f, q_f, P_f\}_{f \in F})$ be an SMTI-OLC instance. For each worker w, we allow P_w to be defined on any subset of E_w and regard contracts not appearing in P_w as contracts unacceptable for w. Let E° be the set of acceptable contracts, i.e., $E^\circ = \{\, e \in E \mid e \text{ appears in } P_w, \text{ where } w = \partial_W(e) \,\}$.

A set $M \subseteq E$ is called a *matching* if $M \subseteq E^\circ$, $|M_w| \leq 1$ for every worker $w \in W$, and M is feasible for every firm $f \in F$. For a matching M, a contract $e \in E \setminus M$ *blocks* M if it is free for both $\partial_W(e)$ and $\partial_F(e)$, where we say that e is *free* for the associated worker $w \coloneqq \partial_W(e)$ if e appears in P_w (i.e., $e \in E^\circ$) and w is assigned no contract in M or prefers e to the contract assigned in M.

We remark that SMTI-OLC can be seen as a special case of SMTI-LC, though the assumption on workers' preference lists is a bit different from that of SMTI-LC. From an SMTI-OLC instance I, define $I^\circ = (W, F, E^\circ, \{\mathcal{L}^\circ_a, q^\circ_a, P^\circ_a\}_{a \in W \cup F})$ as follows. For each worker $w \in W$, set $\mathcal{L}^\circ_w = \{E^\circ_w\}$, $q^\circ_w(E^\circ_w) = 1$, and $P^\circ_w = P_w$. For each firm $f \in F$, set $\mathcal{L}^\circ_f = \{\, L \cap E^\circ \mid L \in \mathcal{L}_f \,\}$, $q^\circ_f(L \cap E^\circ) = q_f(L)$ for each $L \in \mathcal{L}_f$, and let P°_f be the restriction of P_f on E°_f (i.e., delete elements in $E_f \setminus E^\circ_f$ from P_f). Then, I° is an instance of SMTI-LC in Section 2. Moreover, we can observe the following fact by definition.

Observation 17. *For an* SMTI-OLC *instance I, a subset $M \subseteq E$ is a stable matching of I if and only if it is a stable matching of the corresponding* SMTI-LC *instance I°.*

Because the auxiliary instance I^* of I is again an SMTI-OLC instance, I^* also can be transformed into an equivalent SMTI-LC instance; hence, we can compute the worker-optimal stable matching of I^* in the manner described in Section 6. Thus, the mechanism in Section 7 runs with the time complexity given in Theorem 13.

B.1 Without Ties

We show Lemma 15 in Section 7, which states that, if the preference lists of all agents are strict in SMTI-OLC, then a mechanism that always returns the worker-optimal stable matching is strategy-proof for workers. This fact and similar ones have been shown in several previous works [13, 28]. We provide a proof for the completeness.

For this purpose, we introduce the model of Hatfield and Milgrom [16], which we call *the HM model*, using our notations and terminologies. An instance of the HM model is given by $(W, F, E, \{P_w\}_{w \in W}, \{C_f\}_{f \in F})$. The difference from SMTI-OLC is that P_w should be strict and each firm has a choice function $C_f : 2^{E_f} \to 2^{E_f}$ instead of the triple $\{\mathcal{L}_f, q_f, P_f\}$. A function $C_f : 2^{E_f} \to 2^{E_f}$ is called a *choice function* if $C_f(S) \subseteq S$ for any $S \subseteq E_f$.

A stable matching in the HM model is defined similarly to that in SMTI-OLC, where the definitions of feasible sets and free contracts for firms are modified as follows. We say that $M \subseteq E$ is *feasible* for $f \in F$ if $C_f(M_f) = M_f$, and we say that $e \in M \setminus E$ is *free* for $f := \partial_F(e)$ if $e \in C_f(M_f + e)$. Let us call this stability *HM-stability* to distinguish it from the stability in SMTI-OLC.[3]

Hatfield and Milgrom [16] showed that the following two conditions of each choice function $C_f : 2^{E_f} \to 2^{E_f}$ are essential for strategy-proofness.[4] [5]

Substitutability: $S \subseteq T \subseteq E_f$ implies $S \setminus C_f(S) \subseteq T \setminus C_f(T)$.

Law of aggregate demand: $S \subseteq T \subseteq E_f$ implies $|C_f(S)| \leq |C_f(T)|$.

Theorem 18 (Hatfield and Milgrom [16])**.** *In the HM model, if each C_f satisfies substitutability and the law of aggregate demand, then the mechanism that always returns the worker-optimal HM-stable matching is worker-strategy-proof.*

We can reduce SMTI-OLC to the HM model if the preference lists of all agents are strict. Let $I = (W, F, E, \{P_w\}_{w \in W}, \{\mathcal{L}_f, q_f, P_f\}_{f \in F})$ be an SMTI-OLC instance without ties. For each firm $f \in F$, let $(E_f, \mathcal{I}_f)$ be a laminar matroid defined by $\mathcal{L}_f$ and q_f and let $\succ_f$ be a strict linear order on E_f representing P_f. From an ordered matroid $(E_f, \mathcal{I}_f, \succ_f)$, define $C_f : 2^{E_f} \to 2^{E_f}$ as in Section 6.1. Then, we say that an instance $I' = (W, F, E, \{P_w\}_{w \in W}, \{C_f\}_{f \in F})$ of the HM model is *induced from* I. The following facts are known in previous works.

Proposition 19. *For an* SMTI-OLC *instance I without ties, the choice functions in the induced instance I' satisfy substitutability and the law of aggregate demand.*

Proof. Fleiner [8, 9] showed that choice functions defined from ordered matroids as in Section 6.1 satisfy substitutability (called *comonotonicity* in [9]). The law of aggregate demand easily follows from the monotonicity of rank functions of matroids (see, e.g., [32, 37]). $\square$

Proposition 20. *For an* SMTI-OLC *instance I without ties, a set $M \subseteq E$ is an HM-stable matching of the induced instance I' if and only if M is a stable matching of I.*

Proof. By the definition of C_f, we have $C_f(M_f) = M_f$ if and only if $M_f \in \mathcal{I}_f$. Note that the definition of C_f is identical to the matroid greedy algorithm (see, e.g., Oxley [32]). Then, if there is a weight function $w : E_f \to \mathbf{R}_+$ such that $w(e) > w(e') \Leftrightarrow e \succ_f e'$, the set $C_f(M_f + e)$ is the maximum weight independent subset of $M_f + e$ (see also [36, Proposition 1]). This fact implies that, when $M_f \in \mathcal{I}_f$, we have $e \in C_f(M_f + e)$ if and only if $M_f + e \in \mathcal{I}_f$ or there exists $e' \in M_f$ such that $e \succ_f e'$ and $M_f + e - e' \in \mathcal{I}_f$. Then, the statement follows. $\square$

Proof of Lemma 15. By combining Theorem 18 and Propositions 19 and 20, we can immediately obtain Lemma 15. $\square$

[3] Hatfield and Milgrom [16] defined stability by the nonexistence of blocking coalitions rather than blocking pairs. Such a definition is identical to ours if the choice functions of firms satisfy substitutability [13, 28].

[4] To be more precise, Hatfield and Milgrom [16] implicitly assumed a condition of choice functions called *the irrelevance of rejected contracts*. Aygün and Sönmez [1] pointed out that this condition is important for the results of [16], and they also showed that substitutability and the law of aggregate demand together imply this condition.

[5] In the original model of Hatfield and Milgrom [16], it is assumed that a choice function of a firm always returns a set that does not contain multiple contracts associated with the same worker. However, this assumption is not necessary to obtain their results.

It is known [9, 16] that, if choice functions of firms satisfy substitutability and the law of aggregate demand in the HM model, then all stable matchings have the same cardinality. Therefore, for a restriction of SMTI-OLC in which all agents have strict preferences, the mechanism returning the worker-optimal stable matching is worker-strategy-proof and solves MAX-SMTI-OLC exactly.

B.2 With Ties

Here, we consider approximation ratios for MAX-SMTI-OLC attained by worker-strategy-proof mechanisms with the presence of ties.

First, we introduce the results of Hamada et al. [15] on MAX-SMTI, which is a special case of SMTI-OLC in which every agent is assigned at most one contract and there is no constraint. Workers and firms here correspond to men and women in their terminologies [15].

Theorem 21 (Hamada et al. [15, Theorem 2])**.** *For* MAX-SMTI*, there is a worker-strategy-proof mechanism that returns a* 2*-approximate solution. On the other hand, for any* $\epsilon > 0$*, there is no worker-strategy-proof mechanism that returns a* $(2 - \epsilon)$*-approximate solution.*

It has also been shown in [15, Corollary 3] that the second statement of Theorem 21 holds even if ties appear in only firms' preference lists.

Theorem 22 (Hamada et al. [15, Theorem 4])**.** *For a restriction of* MAX-SMTI *in which ties appear in only workers' preference lists, there is a worker-strategy-proof mechanism that returns a* $\frac{3}{2}$*-approximate solution. On the other hand, for any* $\epsilon > 0$*, there is no worker-strategy-proof mechanism that returns a* $(\frac{3}{2} - \epsilon)$*-approximate solution.*

Since SMTI-OLC is a generalization of SMTI, the second statements (i.e., the hardness parts) of Theorems 21 and 22 extend to the setting of SMTI-OLC. Furthermore, our result Theorem 13 in Section 7 is a generalization of the first statement of Theorem 22 to SMTI-OLC.

Now, we show that the first statement of Theorem 21 is also generalized to SMTI-OLC.

Theorem 23. *For* SMTI-OLC*, there is a worker-strategy-proof mechanism that returns a stable matching* M *with* $|M| \geq \frac{1}{2}|M_{\mathrm{OPT}}|$ *in* $O(k \cdot |E|^2)$ *time, where* M_{OPT} *is a maximum-cardinality stable matching and* k *is the maximum level of nesting of* $\mathcal{L}_f$ $(f \in F)$*.*

Proof. Define a mechanism A as follows. Given an instance I of SMTI-OLC, break ties such that, among indifferent contracts, contracts with smaller indices have higher priorities. Let I' be the resultant instance, and let $A(I)$ be the worker-optimal stable matching of I'.

By Proposition 16, the matching $M \coloneqq A(I)$ satisfies $|M| \geq \frac{1}{2}|M_{\mathrm{OPT}}|$. Furthermore, the time complexity follows from Theorem 10 and Lemma 12. We complete the proof by showing the worker-strategy-proofness of A.

Let J be an instance obtained from I by replacing the preference list of some worker w with some other list. Then, $N \coloneqq A(J)$ is the worker-optimal stable matching of J', where J' is obtained from J by breaking ties according to the above-mentioned tie-breaking rule. Then, I' and J' differ only in the preference lists of w. Let P_w and P'_w be the preference lists of w in I and I', respectively. By Lemma 15, w weakly prefers M to N with respect to P'_w. As P'_w is consistent with P_w, we can see that w weakly prefers M to N also with respect to P_w. Thus, the mechanism A is worker-strategy-proof. □

By Theorems 13 and 23, we can conclude that both Theorems 21 and 22 extend to SMTI-OLC, i.e., the setting of many-to-one matching with one-sided laminar constraints.