Quantum Algorithms
Lecture 20
Physically realizable transformations of density matrices II

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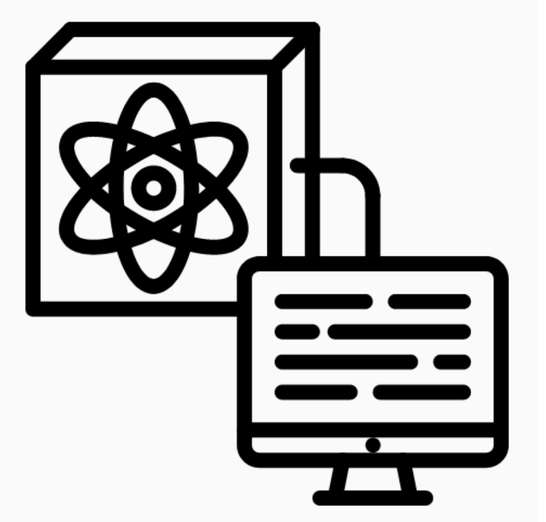
Measurements

Quantum + classical

An important type of interaction between quantum and classical parts is measurement of a quantum register. It yields a classical “record” (outcome), while the quantum register may remain in a modified state, or may be destroyed.

Remark: after the measurement, Quantum system gets into basis state, so the superposition is lost.



Two-part system

Quantum part + classical part

Density matrix that represents both parts.

$$\rho = \sum_{j,k,l} \rho_{jkll} (|j\rangle\langle k|) \otimes (|l\rangle\langle l|) = \sum_{l} w_l \gamma^{(l)} \otimes |l\rangle\langle l|$$

$$w_l = \sum_j \rho_{jjll} \qquad \gamma^{(l)} = w_l^{-1} \sum_{j,k} \rho_{jkll}$$

w_l - prob. of classical state l. $\gamma^{(l)}$ - density matrix in case of state l.

Two-part system

This is what we get – if classical state is l, then the quantum system is in state $\gamma^{(l)}$.

$$\rho = \sum_l w_l \cdot (\gamma^{(l)}, l) = \sum_l (w_l \gamma^{(l)}, l)$$

$$w_l = \sum_j \rho_{jjll} \qquad \gamma^{(l)} = w_l^{-1} \sum_{j,k} \rho_{jkll}$$

Mutually exclusive possibilities

We decompose space into pairwise orthogonal subspaces. Each corresponds to specific classical outcome. If our system is in one subspace, it is in another subspace with 0 probability.

Projective measurement

Number of subspace L_j is put into classical register as j.

$$\text{if } |\xi\rangle \in L_j, \text{ then } |\xi\rangle\langle\xi| \to (|\xi\rangle\langle\xi|, j)$$

The measurement is a linear map

$$R: L(N) \to L(N) \times \{1, \dots, r\} = \bigotimes_{j=1}^{r} L(L_j)$$

Such linear maps will also be called superoperators.

$R\rho = (\rho, j)$ for any $\rho \in L(L_j)$

Physically realizable superoperator

Superoperator of the type

$$L(N) \to L(N) \times \{1, \ldots, r\}$$

$$R: \ \rho \mapsto \sum_j \left(\Pi_{\mathcal{L}_j} \rho \Pi_{\mathcal{L}_j}, j\right)$$

Projective measurement

A superoperator: $$\rho \mapsto \sum_j \mathbf{P}(\rho, \mathcal{L}_j) \cdot \left(\gamma^{(j)}, j\right)$$

$$\gamma^{(j)} = \frac{\Pi_{\mathcal{L}_j} \rho \Pi_{\mathcal{L}_j}}{\mathbf{P}(\rho, \mathcal{L}_j)}$$

$P(\rho, L_j)$ is the probability of getting a specified outcome j. Then our quantum system will be in according state $\gamma^{(j)}$.

Projective measurement

If we measure the pure state $\rho = |\xi\rangle\langle\xi|$, then $\gamma^{(j)} = \left|\eta_j\right\rangle\left\langle\eta_j\right|$, where

$$|\eta_j\rangle = \frac{\Pi_{\mathcal{L}_j}|\xi\rangle}{\sqrt{\mathbf{P}(|\xi\rangle, \mathcal{L}_j)}}$$

In last expression division by $\sqrt{P(|\xi\rangle, L_j)}$ is done as a normalization, so we get mathematically a valid quantum system.

Example of a measurement

Copy a qubit (relative to the classical basis) and apply the decoherence superoperator to the copy.
$\Pi_{L_0} = |0\rangle\langle 0|$, $\Pi_{L_1} = |1\rangle\langle 1|$

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \mapsto \rho_{00} \cdot \Big(|0\rangle\langle 0|,\, 0 \Big) + \rho_{11} \cdot \Big(|1\rangle\langle 1|,\, 1 \Big)$$

The measurement superoperator in such case looks like this.

Destructive measurement

Measured system is discarded. Quantum state ρ becomes classical state: $\sum_j P(\rho, L_j) \cdot |j\rangle\langle j|$.

In general:

$$\rho \to \sum_k Tr(\rho X_k) \cdot |k\rangle\langle k|$$

X_k are projections (matrices with zeros and some elements on diagonal being 1), such that $\sum_k X_k = I$

POVM

Positive operator-valued measure.

POVM = set of operators $\{X_k\}$, for which $\sum_k X_k = I$.

POVM measurement:

$$\rho \to \sum_k Tr(\rho X_k) \cdot |k\rangle\langle k|$$

Nondestructive POVM m.

The operators X_k commute with each other.

Linear combination of projections Π_j with nonnegative coefficients = “conditional probabilities”.

POVM measurement

Any POVM measurement can be represented as an isometric embedding into a larger space, followed by a projective measurement.

We add additional qubits to our system, and then we do a projective measurement, so we store the classical outcome in new added qubits.

Quantum teleportation

We want to teleport a qubit that has a state $|\psi\rangle = \binom{a}{b}$.

We prepare a pair of entangled qubits

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

This pair is also known as ERP pair.

Quantum teleportation

The protocol uses 3 qubits in total. Here is the state of the system:

$$(a|0\rangle + b|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{1}{\sqrt{2}}\big(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\big)$$

First two qubits are kept by Alice, and last qubit – by Bob. We want to teleport the state of first qubit to the third qubit.

Quantum teleportation

Alice applies CNOT to her two qubits, the resulting state becomes:

$$\frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle\right)$$

Quantum teleportation

Then Alice applies Hadamard operator to her first qubit, the resulting state becomes (all three lines are equal, amplitudes have been rearranged):

$$\tfrac{1}{2}\big(a|000\rangle + a|100\rangle\big) + \tfrac{1}{2}\big(a|011\rangle + a|111\rangle\big) + \tfrac{1}{2}\big(b|010\rangle - b|110\rangle\big) + \tfrac{1}{2}\big(b|001\rangle - b|101\rangle\big)$$

$$\tfrac{1}{2}\big(a|000\rangle + b|001\rangle\big) + \tfrac{1}{2}\big(a|011\rangle + b|010\rangle\big) + \tfrac{1}{2}\big(a|100\rangle - b|101\rangle\big) + \tfrac{1}{2}\big(a|111\rangle - b|110\rangle\big)$$

$$\frac{1}{2}|00\rangle\big(a|0\rangle + b|1\rangle\big) + \frac{1}{2}|01\rangle\big(a|1\rangle + b|0\rangle\big) + \frac{1}{2}|10\rangle\big(a|0\rangle - b|1\rangle\big) + \frac{1}{2}|11\rangle\big(a|1\rangle - b|0\rangle\big)$$

Quantum teleportation

$$\frac{1}{2}|00\rangle\big(a|0\rangle + b|1\rangle\big) + \frac{1}{2}|01\rangle\big(a|1\rangle + b|0\rangle\big) + \frac{1}{2}|10\rangle\big(a|0\rangle - b|1\rangle\big) + \frac{1}{2}|11\rangle\big(a|1\rangle - b|0\rangle\big)$$

Alice measures her two qubits, each outcome is with equal probability 1/4. The state of the remaining qubit (hold by Bob) will be in a state, that depends on the measurement outcome:

1. "00": $|v_{00}\rangle = a|0\rangle + b|1\rangle$
2. "01": $|v_{01}\rangle = a|1\rangle + b|0\rangle$
3. "10": $|v_{10}\rangle = a|0\rangle - b|1\rangle$
4. "11": $|v_{11}\rangle = a|1\rangle - b|0\rangle$

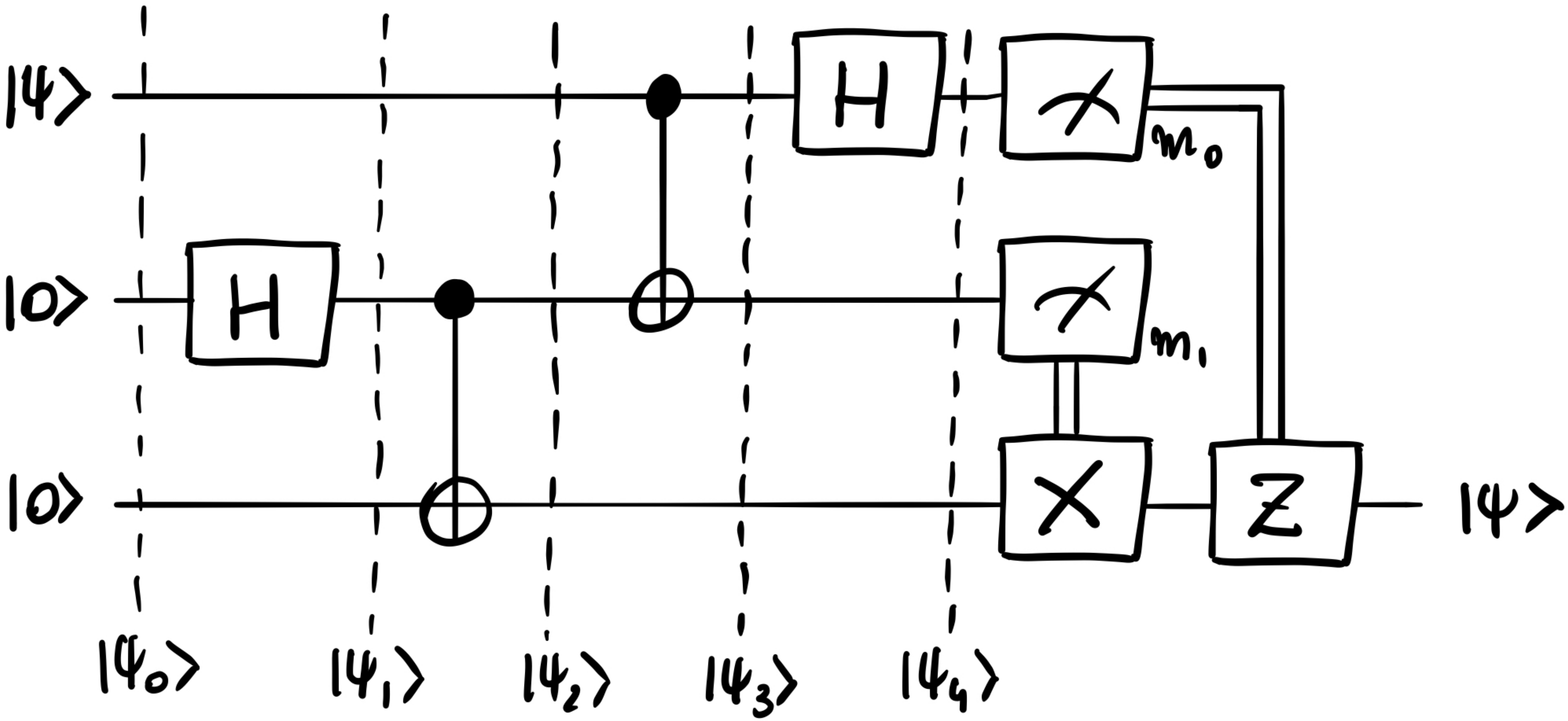
Quantum teleportation

1. "00": $|v_{00}\rangle = a|0\rangle + b|1\rangle$
2. "01": $|v_{01}\rangle = a|1\rangle + b|0\rangle$
3. "10": $|v_{10}\rangle = a|0\rangle - b|1\rangle$
4. "11": $|v_{11}\rangle = a|1\rangle - b|0\rangle$

Alice sends measurement outcome (two classical bits), so Bob can restore teleported state $|\psi\rangle = \binom{a}{b} = a|0\rangle + b|1\rangle$:

1. in case of “00” it is in required state already.
2. in case of “01” apply NOT-gate.
3. in case of “10” apply Z-gate.
4. in case of “11” apply X-gate and then Z-gate.

Quantum teleportation

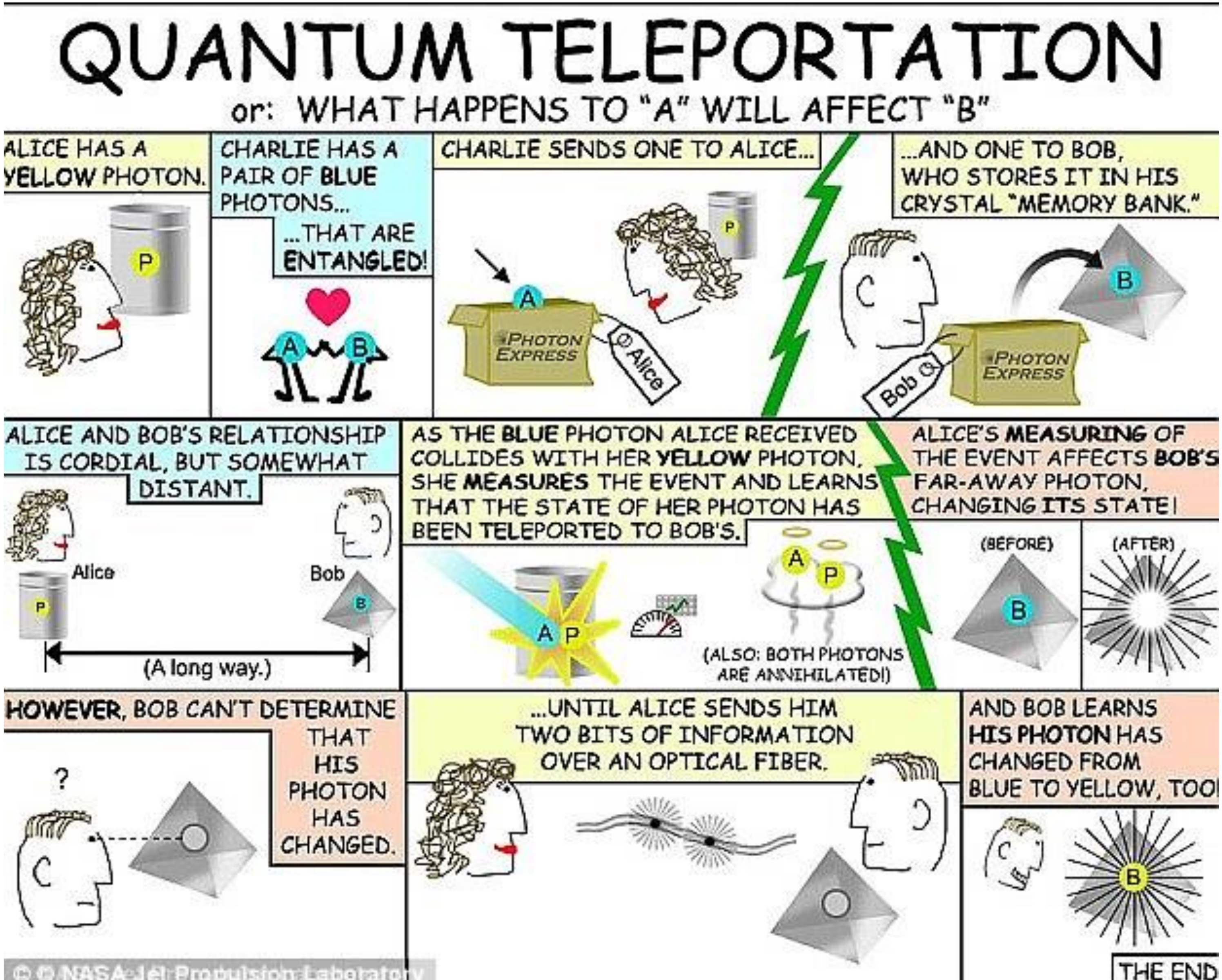


Quantum teleportation

Summary:

- To perform teleportation, two entangled qubits are needed – one held by Alice, one by Bob.
- It is very secure, since we just send two random classical bits of information that are not related to the state of teleported qubit.
- Qubit is not copied, it is teleported (because the state of initial qubit is “destroyed”.
- Quantum teleportation is practically implemented, e.g., in China researchers managed to teleport qubit to satellite (1000 km)

Quantum teleportation - fun



The superoperator norm

Norm for superoperators

Norm for superoperators of type $L(N, M)$ could be defined as (analogy to operator norm):

$$\|T\|_1 = \sup_{X \neq 0} \frac{\|TX\|_{\mathrm{tr}}}{\|X\|_{\mathrm{tr}}}$$

We can use this norm to measure distance between physically realizable transformations of density matrices.

The norm is applied to the difference between two physically realizable superoperators, which is not physically realizable.

Distance between superoperators

Physically realizable superoperators $P, R \in L(N, M)$.

We expect that the distance between P and R is the same as the distance between $P \otimes I_{L(G)}$ and $R \otimes I_{L(G)}$, whatever additional space G we choose. But this is not always the case if we use L1 norm of difference as the distance function.

In these paragraphs, authors try to tell why another measure is necessary.

Example 11.1

L1 matrix norm of a matrix is equal to the maximum of L1 norm of a column of the matrix.

In the example authors observe the properties on Trace norm and L1 norm to obtain the case when L1 norm depends on additional subspace.

Another investigation

The following operator was observed:

$$Q\colon\ \rho \to (T\rho) \otimes \sigma^z$$

Conditions, that Q satisfies:

- $Tr(QX) = 0$
- $(QX)^{\dagger} = QX^{\dagger}$

Any superoperator with these properties can be represented as $c(P - R)$, where P and R are physically realizable, and c is a positive real number.

Diamond norm

When $dimG \geq dimN$, then:

$$\|T \otimes I_G\|_1 = \|T\|_\Diamond$$

So the quantity $\|T\|_\Diamond$ is a norm and does not depend on G.

Two more properties:

- $\|T\|_\Diamond \geq \|T\|_1$
- $\|T \otimes R\|_\Diamond = \|T\|_\Diamond \|R\|_\Diamond$

Diamond norm is also called stable superoperator norm.

Arbitrary superoperator

$T: L(N) \to L(M)$ can be represented in the form $T = Tr_F\,(A \cdot B^\dagger)$, where $A, B \in L(N, M \otimes F)$.

$A \cdot B^\dagger$ denotes the superoperator $X \to AXB^\dagger$.

Without loss of generality we may assume that $dimF = (dimN)(dimM)$.

Diamond norm

$$\|T\|_\diamond = \inf\Big\{\|A\|\,\|B\| \;:\; \mathrm{Tr}_{\mathcal{F}}(A\cdot B^\dagger) = T\Big\}$$

This quantity does not depend on the choice of the auxiliary space F, provided at least one representation $T = Tr_F\,(A_0 \cdot B_0^\dagger)$ exists.

For the minimization of $\|A\|\|B\|$, it suffices to consider operators with norms $\|A\| \leq \|A_0\|$ and $\|B\| \leq \|B_0\|$. The set of such pairs (A, B) is compact, hence the infimum is achieved.

Theorem

If $dimG \geq dimN$, then $\|T \otimes I_G\|_1 = \|T\|_\lozenge$.

To prove equality, authors prove inequality on both sides ($\leq$ and $\geq$).

Operator T is considered as $T = Tr_F\,(A \cdot B^\dagger)$.

By checking carefully, $\|T \otimes I_G\|_1 \leq \|T\|_\lozenge$ can be seen as obtained from the properties of Trace norm.

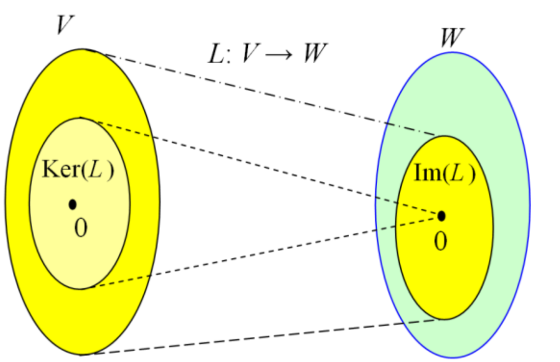
The proof for $\|T \otimes I_G\|_1 \geq \|T\|_\lozenge$ is more complicated.

Kernel

In mathematics, the kernel of a linear map, also known as the null space or nullspace, is the linear subspace of the domain of the map which is mapped to the zero vector.

A vector v is in the kernel of a matrix A if and only if $Av = 0$. Thus, the kernel is the span of all these vectors.

Similarly, a vector v is in the kernel of a linear transformation T if and only if $T(v) = 0$.



Proof for $\|T \otimes I_G\|_1 \geq \|T\|_\Diamond$

Assumptions: $\|T\|_\Diamond = 1$ achieved by $\|A\| = \|B\| = 1$.

There exist three density matrices $\rho, \gamma \in L(N)$ and $\tau \in L(F)$ such that $Tr_M(A\rho A^\dagger) = Tr_M(B\gamma B^\dagger) = \tau$.

$$\mathcal{K} = \mathrm{Ker}(A^\dagger A - I_{\mathcal{N}}), \qquad \mathcal{L} = \mathrm{Ker}(B^\dagger B - I_{\mathcal{N}}),$$

$$E = \left\{ \mathrm{Tr}_{\mathcal{M}}(A\rho A^\dagger) : \rho \in \mathbf{D}(\mathcal{K}) \right\}, \qquad F = \left\{ \mathrm{Tr}_{\mathcal{M}}(B\gamma B^\dagger) : \gamma \in \mathbf{D}(\mathcal{L}) \right\}$$

$D(L)$ – density matrices on subspace L.

$$\tau \in E \cap F$$

Next step in the proof is to prove that $E \cap F \neq \emptyset$.

Proof for $\|T \otimes I_G\|_1 \geq \|T\|_\diamond$

To prove that $E \cap F \neq \emptyset$.

There is no Hermitian operator $Z \in L(F)$ such that $Tr(XZ) > Tr(YZ)$ for all pairs of $X \in E$ and $Y \in F$.

After proving this we can assume:

$$Tr_M(A\rho A^\dagger) = Tr_M(B\gamma B^\dagger) = \tau \in D(F)$$

Because $\dim G \geq \dim N$, we can get purifications of ρ and γ as $|\xi\rangle, |\eta\rangle \in N \otimes G$.

We set $X = |\xi\rangle\langle\eta|$, so $\|X\|_{tr} = 1$.

Proof for $\|T \otimes I_G\|_1 \geq \|T\|_\lozenge$

We set $X = |\xi\rangle\langle\eta|$, so $\|X\|_{tr} = 1$.

The last part of the proof concludes that $\left\|\left(T \otimes I_{L(G)}\right)X\right\|_{tr} \geq 1$.

Overall, to follow the proof, careful understanding of norms is important.

Connection to fidelity

Let $T = Tr_F\,(A \cdot B^{\dagger})$, where $A, B\colon N \to F \otimes M$. Then

$$\|T\|_{\diamondsuit}^{2} = \max\Big\{F\Big(\mathrm{Tr}_{\mathcal{M}}(A\rho A^{\dagger}),\, \mathrm{Tr}_{\mathcal{M}}(B\gamma B^{\dagger})\Big) \;:\; \rho, \gamma \in \mathbf{D}(\mathcal{N})\Big\}$$

where $D(N)$ denotes the set of density matrices on N.

Fidelity and QIP

The result of connection between superoperator norm and fidelity has been used in the study of the complexity class QIP.

Quantum Interactive Polynomial time:
The class of decision problems such that a "yes" answer can be verified by a quantum interactive proof. Here the verifier is a BQP algorithm, while the prover has unbounded computational resources. The prover and verifier exchange a polynomial number of messages, which can be quantum states.

Thank you for your attention!