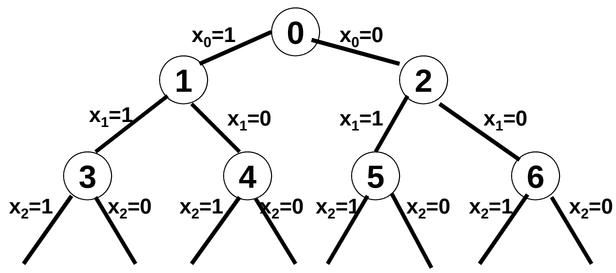
1.204 Lecture 16

Branch and bound:
Method, knapsack problem

Branch and bound

- **Technique for solving mixed (or pure) integer programming problems, based on tree search**
  - Yes/no or 0/1 decision variables, designated x_i
  - Problem may have continuous, usually linear, variables
  - $O(2^n)$ complexity
    - Relies on upper and lower bounds to limit the number of combinations examined while looking for a solution
    - Dominance at a distance
      - Solutions in one part of tree can dominate other parts of tree
      - DP only has local dominance: states in same stage dominate
    - Handles master/subproblem framework better than DP
    - Same problem size as dynamic programming, perhaps a little larger: data specific, a few hundred 0/1 variables
  - Branch-and-cut is a more sophisticated, related method
    - May solve problems with a few thousand 0/1 variables
    - Its code and math are complex
    - If you need branch-and-cut, use a commercial solver

Branch and bound tree



- Every tree node is a problem state
  - It is generally associated with one 0-1 variable, sometimes a group
  - Other 0-1 variables are implicitly defined by the path from the root to this node
    - We sometimes store all {x} at each node rather than tracing back
  - Still other 0-1 variables associated with nodes below the current node in the tree have unknown values, since the path to those nodes has not been built yet

Generating tree nodes

- Tree nodes are generated dynamically as the program progresses
  - Live node is node that has been generated but not all of its children have been generated yet
  - E-node is a live node currently being explored. Its children are being generated
  - Dead node is a node either:
    - Not to be explored further or
    - All of whose children have already been explored

Managing live tree nodes

- Branch and bound keeps a list of live nodes. Four strategies are used to manage the list:
  - Depth first search: As soon as child of current E-node is generated, the child becomes the new E-node
    - Parent becomes E-node only after child's subtree is explored
    - Horowitz and Sahni call this 'backtracking'
  - In the other 3 strategies, the E-node remains the E-node until it is dead. Its children are managed by:
    - Breadth first search: Children are put in queue
    - D-search: Children are put on stack
    - Least cost search: Children are put on heap
  - We use bounding functions (upper and lower bounds) to kill live nodes without generating all their children
    - Somewhat analogous to pruning in dynamic programming

Knapsack problem (for the last time)

$$\max \sum_{0 \le i < n} p_i x_i$$

$$s.t.$$

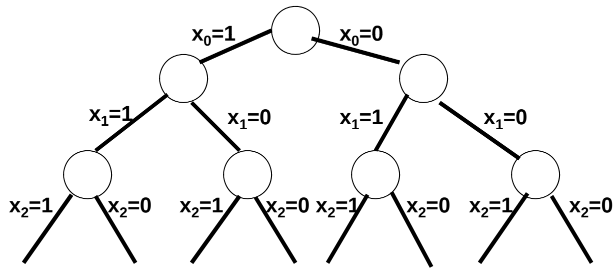
$$\sum_{0 \le i < n} w_i x_i \le M$$

$$x_i = 0,1$$

$$p_i \ge 0, w_i \ge 0,\ 0 \le i < n$$

The x_i are 0-1 variables, like the DP and unlike the greedy version

Tree for knapsack problem



Node numbers are generated but have no problem-specific meaning.
We will use depth first search.

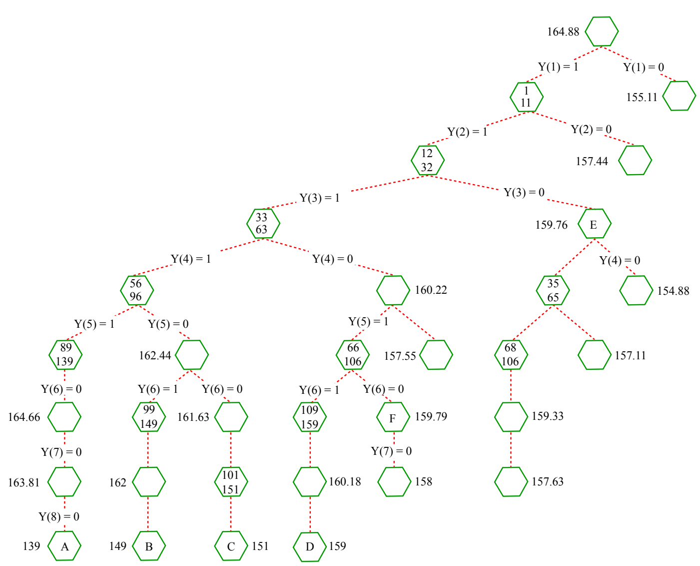
Knapsack problem tree

- **Left child is always x_i= 1 in our formulation**
  - **Right child is always x_i= 0**
- **Bounding function to prune tree**
  - **At a live node in the tree**
    - **If we can estimate the upper bound (best case) profit at that node, and**
    - **If that upper bound is less than the profit of an actual solution found already**
    - **Then we don't need to explore that node**
  - **We can use the greedy knapsack as our bound function:**
    - **It gives an upper bound, since the last item in the knapsack is usually fractional**
  - **Greedy algorithms are often good ways to compute upper (optimistic) bounds on problems**
    - **E.g., For job scheduling with varying job times, we can cut each job into equal length parts and use the greedy job scheduler to get an upper bound**
  - **Linear programs that treat the 0-1 variables as continuous between 0 and 1 are often another good choice**

Knapsack example (same as DP)

Item	Profit	Weight
0	0	0
1	11	1
2	21	11
3	31	21
4	33	23
5	43	33
6	53	43
7	55	45
8	65	55

- **Maximum weight 110**
- **Item 0 is sentinel, needed in branch-and-bound too**



Knapsack Solution Tree

Figure by MIT OpenCourseWare.

Source: Horowitz/Sahni previous edition

Knapsack solution tree

- Numbers inside a node are profit and weight at that node, based on decisions from root to that node
- Nodes without numbers inside have same values as their parent
- Numbers outside the node are upper bound calculated by greedy algorithm
  - Upper bound for every feasible left child (x_i=1) is same as its parent's bound
  - Chain of left children in tree is same as greedy solution at that point in the tree
  - We only recompute the upper bound when we can't move to a feasible left child
- Final profit and final weight (lower bound) are updated at each leaf node reached by algorithm
  - Nodes A, B, C and D in previous slide
  - Solution improves at each leaf node reached
  - No further leaf nodes reached after D because lower bound (optimal value) is sufficient to prune all other tree branches before leaf is reached
- By using floor of upper bound at nodes E and F, we avoid generating the tree below either node
  - Since optimal solution must be integer, we can truncate upper bounds
  - By truncating bounds at E and F to 159, we avoid exploring E and F

KnapsackBB constructor

```
public class KnapsackBB {
   private DPItem[] items;     // Input list of items
   private int capacity;       // Max weight allowed in knapsack
   private int[] x;            // Best solution array: item i in if xi=1
   private int[] y;       // Working solution array at current tree node
   private double solutionProfit= -1;  // Profit of best solution so far
   private double currWgt;     // Weight of solution at this tree node
   private double currProfit;  // Profit of solution at this tree node
   private double newWgt;      // Weight of solution from bound() method
   private double newProfit;   // Profit of solution from bound() method
   private int k;              // Level of tree in knapsack() method
   private int partItem;       // Level of tree in bound() method

   public KnapsackBB(DPItem[] i, int c) {
        items= i;
        capacity= c;
        x= new int[items.length];
        y= new int[items.length];
   }
```

KnapsackBB knapsack()

```
public void knapsack() {
  int n= items.length;        // Number of items in problem
  do {                        // While upper bound < known soln,backtrack
    while (bound() <= solutionProfit) {
      while (k != 0 && y[k] != 1)  // Back up while item k not in sack
        k--;                       // to find last object in knapsack
      if (k == 0)                  // If at root, we're done. Return.
        return;
      y[k]= 0;                     // Else take k out of soln (R branch)
      currWgt -= items[k].weight;  // Reduce soln wgt by k's wgt
      currProfit -= items[k].profit;  // Reduce soln profit by k's prof
    }                              // Back to while(), recompute bound
    currWgt= newWgt;               // Reach here if bound> soln profit
    currProfit= newProfit;         // and we may have new soln.
    k= partItem;                   // Set tree level k to last, possibly
                                   // partial item in greedy solution
    if (k == n) {                  // If we've reached leaf node, have
      solutionProfit= currProfit;  // actual soln, not just bound
      System.arraycopy(y, 0, x, 0, y.length);  // Copy soln into array x
      k= n-1;  // Back up to prev tree level, which may leave solution
    } else                         // Else not at leaf, just have bound
      y[k]= 0;                     // Take last item k out of soln
  } while (true);                  // Infinite loop til backtrack to k=0
}
```

KnapsackBB bound()

```
 private double bound() {
   boolean found= false;      // Was bound found?I.e.,is last item partial
   double boundVal= -1;       // Value of upper bound
   int n= items.length;       // Number of items in problem
   newProfit= currProfit;     // Set new prof as current prof at this node
   newWgt= currWgt;
   partItem= k+1;            // Go to next lower level, try to put in soln
   while (partItem < n && !found) { // More items & haven't found partial
     if (newWgt + items[partItem].weight <= capacity) {  // If fits
       newWgt += items[partItem].weight;      // Update new wgt, prof
       newProfit += items[partItem].profit;   // by adding item wgt,prof
       y[partItem]= 1;        // Update curr soln to show item k is in it
     } else {                 // Current item only fits partially
       boundVal= newProfit + (capacity –
           newWgt)*items[partItem].profit/items[partItem].weight;
       found= true; }         // Compute upper bound based on partial fit
     partItem++;              // Go to next item and try to put in sack
   }
   if (found) {       // If we have fractional soln for last item in sack
     partItem--;      // Back up to prev item, which is fully in sack
     return boundVal; // Return the upper bound
   } else {
     return newProfit;// Return profit including last item
 } }
```

KnapsackBB main()

```
   public static void main(String[] args) {
        // Sentinel- must be in 0 position even after sort
        DPItem[] list= {new DPItem(0, 0),
                       new DPItem(11, 1),
                       new DPItem(21, 11),
                       new DPItem(31, 21),
                       new DPItem(33, 23),
                       new DPItem(43, 33),
                       new DPItem(53, 43),
                       new DPItem(55, 45),
                       new DPItem(65, 55),
        };
        Arrays.sort(list, 1, list.length);    // Leave sentinel in 0
        int capacity= 110;
        // Assume all item weights <= capacity. Not checked. Discard
        // Assume all item profits > 0. Not checked.  Discard.
        KnapsackBB knap= new KnapsackBB(list, capacity);
        knap.knapsack();
        knap.outputSolution();
   }
}
// main() almost identical to DPKnap.
// DPItem identical, outputSolution() almost identical to DP code
```

Depth first search in branch and bound

- Depth first search used in combination with breadth first search in many problems
  - Common strategy is to use depth first search on nodes that have not been pruned
    - This gets to a leaf node, and a feasible solution, which is a lower bound that can be used to prune the tree in conjunction with the greedy upper bounds
      - If greedy upper bound < lower bound, prune the tree!
    - Once a node has been pruned, breadth first search is used to move to a different part of the tree
  - Depth first search bounds tend to be very quick to compute if you move down the tree sequentially
    - E.g. our greedy bound doesn't need to be recomputed
    - Linear program as bounds are often quick too: few simplex pivots

Next time

- Breadth first search in branch and bound trees
- Fixed facility location problem
  - Mixed integer problem
  - Uses linear program (LP) as subproblem
  - We solve the LP with a shortest path algorithm!
- The depth first search for the knapsack problem is mostly pedagogical
  - Sometimes depth first search works well enough for your particular problem and data
  - Usually you need to be a bit more sophisticated

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