Linguagens de Programação

Semântica Denotacional

– Soluções –

1. **A.** `y:=2*x+2; z:=x*2; x:=2*x+1`

$$
\begin{aligned}
&I[[\ y := 2*x+2;\ z := x*2;\ x := 2*x+1\ ]](s_0) = \\
&I[[\ z := x*2;\ x := 2*x+1\ ]](I[[\ y := 2*x+2\ ]](s_0)) = \\
&I[[\ z := x*2;\ x := 2*x+1\ ]](s_1) = \\
&I[[x := 2*x+1\ ]](I[[\ z := x*2\ ]](s_1)) = \\
&I[[x := 2*x+1\ ]](s_2) = \\
&s_3
\end{aligned}
$$

onde

$$
\begin{aligned}
s_0 &= \{(x, x_0), (y, y_0), (z, z_0)\} \\
s_1 &= modificar(s_0, y, A[[2*x+2]](s_0)) \\
&= modificar(s_0, y, s_0(2*x+2)) = \\
&= \{(x, x_0), (y, 2*x_0+2), (z, z_0)\} \\
s_2 &= modificar(s_1, z, A[[x*2]](s_1)) \\
&= modificar(s_1, z, s_1(x*2)) = \\
&= \{(x, x_0), (y, 2*x_0+2), (z, x_0*2)\} \\
s_3 &= modificar(s_2, x, A[[2*x+1]](s_2)) \\
&= modificar(s_2, x, s_2(2*x+1)) = \\
&= \{(x, 2*x_0+1), (y, 2*x_0+2), (z, x_0*2)\}
\end{aligned}
$$

B. `x:=2*x+1; y:=x+1; z:=x-1`

$$
\begin{aligned}
&I[[\ x := 2*x+1;\ \ y := x+1;\ \ z := x-1\ ]](s_0) = \\
&I[[\ y := x+1;\ \ z := x-1\ ]](I[[\ x := 2*x+1\ ]](s_0)) = \\
&I[[\ y := x+1;\ \ z := x-1\ ]](s_1) = \\
&I[[\ z := x-1\ ]](I[[\ y := x+1\ ]])(s_1) = \\
&I[[\ z := x-1\ ]](s_2) = \\
&s_3
\end{aligned}
$$

onde

$$
\begin{aligned}
s_0 &= \{(x, x_0), (y, y_0), (z, z_0)\} \\
s_1 &= modificar(s_0, x, A[[2*x+1]](s_0)) \\
&= modificar(s_0, x, s_0(2*x+1)) \\
&= \{(x, 2*x_0+1), (y, y_0), (z, z_0)\} \\
s_2 &= modificar(s_1, y, A[[x+1]](s_1)) \\
&= modificar(s_1, y, s_1(x+1)) \\
&= \{(x, 2*x_0+1), (y, (2*x_0+1)+1), (z, z_0)\} \\
&= \{(x, 2*x_0+1), (y, 2*x_0+2), (z, z_0)\} \\
s_3 &= modificar(s_2, z, A[[x-1]](s_2)) \\
&= modificar(s_2, z, s_2(x-1)) \\
&= \{(x, 2*x_0+1), (y, 2*x_0+2), (z, (2*x_0+1)-1)\} \\
&= \{(x, 2*x_0+1), (y, 2*x_0+2), (z, 2*x_0)\}
\end{aligned}
$$

2. `if x=0 then ( y:=z^2; x:=2*y ) else ( y:=(z+x)^2; z:=z+x; x:=x+2*y )`

   (a) `P = {if x=0 then ( y:=z^2; x:=2*y ) else ( y:=(z+x)^2; z:=z+x; x:=x+2*y ) }`

$$
\begin{aligned}
&I[[P]](s_0) = \\
&\texttt{if } A[[x=0]](s_0) \texttt{ then} \\
&\quad I[[y := z^2; x := 2*y]](s_0) \\
&\texttt{else} \\
&\quad I[[y := (z+x)^2; z := z+x; x := x+2*y]](s_0)
\end{aligned}
$$

Seja $s_0 = \{(x, x_0), (y, y_0), (z, z_0)\}$.
Se $x_0 = 0$ tem-se

$$\begin{aligned}
&I[[y := z^2; x := 2 * y]](s_0) = \\
&I[[x := 2 * y]](I[[y := z^2]](s_0)) = \\
&I[[x := 2 * y]](s_1) = \\
&s_2
\end{aligned}$$

com

$$\begin{aligned}
s_1 &= modificar(s_0, y, A[[z^2]](s_0)) \\
&= modificar(s_0, y, s_0(z^2)) \\
&= \{(x, 0),\ (y, z_0^2),\ (z, z_0)\} \\
s_2 &= modificar(s_1, x, A[[2 * y]](s_1)) \\
&= modificar(s_1, x, s_1(2 * y)) \\
&= \{(x, 2 * (z_0^2)),\ (y, z_0^2),\ (z, z_0)\}
\end{aligned}$$

Se $x_0 \neq 0$ tem-se

$$\begin{aligned}
&I[[y := (z + x)^2; z := z + x; x := x + 2 * y]](s_0) = \\
&I[[z := z + x; x := x + 2 * y]](I[[y := (z + x)^2]](s_0)) = \\
&I[[z := z + x; x := x + 2 * y]](s_1) = \\
&I[[x := x + 2 * y]](I[[z := z + x]](s_1)) = \\
&I[[x := x + 2 * y]](s_2) = \\
&s_3
\end{aligned}$$

com

$$\begin{aligned}
s_1 &= modificar(s_0, y, E[(z + x)^2]](s_0)) \\
&= modificar(s_0, y, s_0((z + x)^2)) \\
&= \{(x, x_0),\ (y, (z_0 + x_0)^2),\ (z, z_0)\} \\
s_2 &= modificar(s_1, z, A[[z + x]](s_1)) \\
&= modificar(s_1, z, s_1(z + x)) \\
&= \{(x, x_0),\ (y, (z_0 + x_0)^2),\ (z, z_0 + x_0)\} \\
s_3 &= modificar(s_2, x, A[[x + 2 * y]](s_2)) \\
&= modificar(s_2, x, s_2(x + 2 * y)) \\
&= \{(x, x_0 + 2 * (z_0 + x_0)^2),\ (y, (z_0 + x_0)^2),\ (z, z_0 + x_0)\}
\end{aligned}$$

(b) `y:=(z+x)^2; z:=z+x; x:=x+2*y`

3. **A.** `t:=x+x; y:=t+x;`

$$
\begin{aligned}
&I[[\ t := x + x;\ \ y := t + x;\ ]](s_0) = \\
&I[[\ y := t + x;\ ]](I[[\ t := x + x;\ ]](s_0)) = \\
&I[[\ y := t + x;\ ]](s_1) = \\
&s_2
\end{aligned}
$$

onde

$$
\begin{aligned}
s_0 &= \{(x, x_0), (y, y_0), (t, t_0)\} \\
s_1 &= modificar(s_0, t, A[[x + x]](s_0)) \\
&= modificar(s_0, t, s_0(x + x)) = \\
&= \{(x, x_0), (y, y_0), (t, x_0 + x_0)\} \\
s_2 &= modificar(s_1, y, A[[t + x]](s_1)) \\
&= modificar(s_1, y, s_1(t + x)) = \\
&= \{(x, x_0), (y, (x_0 + x_0) + x_0), (t, x_0 + x_0)\} \\
&= \{(x, x_0), (y, 3 * x_0), (t, 2 * x_0)\}
\end{aligned}
$$

B. `t:=2*x; t:=2*t; y:=t-x;`

$$
\begin{aligned}
&I[[\ t := 2 * x;\ \ t := 2 * t;\ \ y := t - x;\ ]](s_0) = \\
&I[[\ t := 2 * t;\ \ y := t - x;\ ]](I[[\ t := 2 * x;\ ]](s_0)) = \\
&I[[\ t := 2 * t;\ \ y := t - x;\ ]](s_1) = \\
&I[[y := t - x;\ ]](I[[\ t := 2 * t;\ ]](s_1)) = \\
&I[[y := t - x;\ ]](s_2) = \\
&s_3
\end{aligned}
$$

onde

$$\begin{aligned}
s_0 &= \{(x, x_0), (y, y_0), (t, t_0)\} \\
s_1 &= modificar(s_0, t, A[[2 * x]](s_0)) \\
&= modificar(s_0, t, s_0(2 * x)) = \\
&= \{(x, x_0), (y, y_0), (t, 2 * x_0\} \\
s_2 &= modificar(s_1, t, A[[2 * t]](s_1)) \\
&= modificar(s_1, t, s_1(2 * t)) = \\
&= \{(x, x_0), (y, y_0), (t, 2 * (2 * x_0))\} \\
s_3 &= modificar(s_2, y, A[[t - x]](s_2)) \\
&= modificar(s_2, t, s_2(t - x)) = \\
&= \{(x, x_0), (y, 3 * x_0), (t, 4 * x_0)\}
\end{aligned}$$

Os fragmentos não são equivalentes porque, embora `x` e `y` tenham o mesmo valor no estado final, o mesmo não acontece com a variável `t`.

4. `a:=123; b:=0; while a>0 do ( r:=a*10; b:=b+r; a:=a-100; )`

   Seja

   $B_1 =$ `a:=123;`

   $B_2 =$ `b:=0;`

   $B_3 =$ `while a>0 do ( r:=a*10; b:=b+r; a:=a-100; )`

   $B_4 =$ `r:=a*10;`

   $B_5 =$ `b:=b+r;`

   $B_6 =$ `a:=a-100;`

   Assim,

$$\begin{aligned}
&I[[B_1;\ B_2;\ B_3]](s_0) = \\
&I[[B_2;\ B_3]](I[[a := 123]](s_0)) = \\
&I[[B_2;\ B_3]](s_1) = \\
&I[[B_3]](I[[b := 0]](s_1)) = \\
&I[[B_3]](s_2)
\end{aligned}$$

com

$$
\begin{aligned}
s_0 &= \{(a, a_0), (b, b_0), (r, r_0)\} \\
s_1 &= modificar(s_0, a, A[[123]](s_0)) = modificar(s_0, a, s_0(123)) \\
&= \{(a, 123), (b, b_0), (r, r_0)\} \\
s_2 &= modificar(s_1, b, A[[0]](s_1)) = modificar(s_1, b, s_1(0)) \\
&= \{(a, 123), (b, 0), (r, r_0)\}
\end{aligned}
$$

$$
\begin{aligned}
&I[[B_3]](s_2) = \\
&\texttt{if } \; not\ A[[a > 0]](s_2) \; \texttt{ then } \; s_2 \; \texttt{ else } \; I[[B_3]](\ I[[B_4; B_5; B_6]](s_2)\ )
\end{aligned}
$$

$$
\begin{aligned}
&I[[B_5; B_6]](\ I[[r := a * 10]](s_2)\ ) = \\
&I[[B_5; B_6]](s_3) = \\
&I[[B_6]](\ I[[b := b + r]](s_3)\ ) = \\
&I[[a := a - 100]](s_4) = \\
&s_5
\end{aligned}
$$

com

$$
\begin{aligned}
s_3 &= modificar(s_2, r, A[[a * 10]](s_2)) = modificar(s_2, r, s_2(a * 10)) \\
&= \{(a, 123), (b, 0), (r, 1230)\} \\
s_4 &= modificar(s_3, b, A[[b + r]](s_3)) = modificar(s_3, b, s_3(b + r)) \\
&= \{(a, 123), (b, 1230), (r, 1230)\} \\
s_5 &= modificar(s_4, a, A[[a - 100]](s_4)) = modificar(s_4, a, s_4(a - 100)) \\
&= \{(a, 23), (b, 1230), (r, 1230)\}
\end{aligned}
$$

$I[[B_3]](s_5) =$

$\texttt{if } \mathit{not}\ A[[a > 0]](s_5) \texttt{ then } s_5 \texttt{ else } I[[B_3]](\ I[[B_4; B_5; B_6]](s_5)\ )$

$I[[B_3]](s_5) = I[[B_3]](\ I[[B_4; B_5; B_6]](s_5)\ ) =$

$I[[B_5; B_6]](\ I[[r := a * 10]](s_5)\ ) =$

$I[[B_5; B_6]](s_6) =$

$I[[B_6]](\ I[[b := b + r]](s_6)\ ) =$

$I[[a := a - 100]](s_7) =$

s_8

com

$$s_6 = modificar(s_5, r, A[[a * 10]](s_2)) = modificar(s_5, r, s_2(a * 10))$$
$$= \{(a, 23), (b, 1230), (r, 230)\}$$
$$s_7 = modificar(s_6, b, A[[b + r]](s_3)) = modificar(s_6, b, s_3(b + r))$$
$$= \{(a, 23), (b, 1460), (r, 230)\}$$
$$s_8 = modificar(s_7, a, A[[a - 100]](s_4)) = modificar(s_7, a, s_4(a - 100))$$
$$= \{(a, -77), (b, 1460), (r, 230)\}$$

$I[[B_3]](s_8) =$

$\texttt{if } \mathit{not}\ A[[a > 0]](s_8) \texttt{ then } s_8 \texttt{ else } I[[B_3]](\ I[[B_4; B_5; B_6]](s_8)\ )$

$I[[B_3]](s_8) = s_8$

5. `i:=0; q:=0; while i<n do ( q:=q+2*i+1; i:=i+1; )`

   Seja

   $B_1 =$ `i:=0;`

   $B_2 =$ `q:=0;`

   $B_3 =$ `while i<n do ( q:=q+2*i+1; i:=i+1; )`

   $B_4 =$ `q:=q+2*i+1;`

   $B_5 =$ `i:=i+1;`

Assim,

$$
\begin{aligned}
&I[[B_1;\ B_2;\ B_3]](s_0) = \\
&I[[B_2;\ B_3]](I[[i := 0]](s_0)) = \\
&I[[B_2;\ B_3]](s_1) = \\
&I[[B_3]](I[[q := 0]](s_1)) = \\
&I[[B_3]](s_2)
\end{aligned}
$$

com

$$
\begin{aligned}
&s_0 = \{(i, i_0), (q, q_0), (n, n_0), \ldots\} \\
&s_1 = modificar(s_0, i, A[[0]](s_0)) = modificar(s_0, i, s_0(0)) \\
&\quad = \{(i, 0), (q, q_0), (n, n_0), \ldots\} \\
&s_2 = modificar(s_1, q, A[[0]](s_1)) = modificar(s_1, q, s_1(0)) \\
&\quad = \{(i, 0), (q, 0), (n, n_0), \ldots\}
\end{aligned}
$$

$$
\begin{aligned}
&I[[B_3]](s_2) = \\
&\texttt{if}\ \ not\ A[[i < n]](s_2)\ \ \texttt{then}\ \ s_2\ \ \texttt{else}\ \ I[[B_3]](\ I[[B_4; B_5]](s_2)\ )
\end{aligned}
$$

$$
\begin{aligned}
&I[[B_4; B_5]](s_2) = \\
&I[[B_5]](\ I[[q := q + 2 * i + 1]](s_2)\ ) = \\
&I[[i := i + 1]](s_3) = \\
&s_4
\end{aligned}
$$

com

$$
\begin{aligned}
&s_3 = modificar(s_2, q, A[[q + 2 * i + 1]](s_2)) \\
&\quad = modificar(s_2, q, s_2(q + 2 * i + 1)) \\
&\quad = \{(i, 0), (q, 0 + 1), (n, n_0), \ldots\} \\
&s_4 = modificar(s_3, i, A[[i + 1]](s_3)) = modificar(s_3, i, s_3(i + 1)) \\
&\quad = \{(i, 1), (q, 0 + 1), (n, n_0), \ldots\}
\end{aligned}
$$

$I[[B_3]](s_4) =$

$\texttt{if}\ \ not\ A[[i < n]](s_4)\ \ \texttt{then}\ \ s_2\ \ \texttt{else}\ \ I[[B_3]](\ I[[B_4; B_5]](s_4)\ )$

$I[[B_4; B_5]](s_4) =$

$I[[B_5]](\ I[[q := q + 2 * i + 1]](s_4)\ ) =$

$I[[i := i + 1]](s_5) =$

s_6

com

$$\begin{aligned}
s_5 &= modificar(s_4, q, A[[q + 2 * i + 1]](s_4)) \\
&= modificar(s_4, q, s_4(q + 2 * i + 1)) \\
&= \{(i, 1), (q, 0 + 1 + 3), (n, n_0), \ldots\} \\
s_6 &= modificar(s_5, i, A[[i + 1]](s_5)) = modificar(s_5, i, s_5(i + 1)) \\
&= \{(i, 2), (q, 0 + 1 + 3), (n, n_0), \ldots\}
\end{aligned}$$

$I[[B_3]](s_6) =$

$\texttt{if}\ \ not\ A[[i < n]](s_6)\ \ \texttt{then}\ \ s_6\ \ \texttt{else}\ \ I[[B_3]](\ I[[B_4; B_5]](s_6)\ )$

$I[[B_4; B_5]](s_6) =$

$I[[B_5]](\ I[[q := q + 2 * i + 1]](s_6)\ ) =$

$I[[i := i + 1]](s_7) =$

s_8

com

$$\begin{aligned}
s_7 &= modificar(s_6, q, A[[q + 2 * i + 1]](s_6)) \\
&= modificar(s_6, q, s_6(q + 2 * i + 1)) \\
&= \{(i, 2), (q, 0 + 1 + 3 + 5), (n, n_0), \ldots\} \\
s_8 &= modificar(s_7, i, A[[i + 1]](s_7)) = modificar(s_7, i, s_7(i + 1)) \\
&= \{(i, 3), (q, 0 + 1 + 3 + 5), (n, n_0), \ldots\}
\end{aligned}$$

$\ldots$

$s_{2*(n_0-1)+2} = \{(i, n_0 - 1), (q, 0 + 1 + 3 + \ldots + 2 * (n_0 - 2) + 1), (n, n_0), \ldots\}$

$I[[B_3]](s_{2*(n_0-1)+2}) =$
$\texttt{if}\ \ not\ A[[i < n]](s_{2*(n_0-1)+2})\ \ \texttt{then}\ \ s_{2*(n_0-1)+2}\ \ \texttt{else}\ \ I[[B_3]](\ I[[B_4; B_5]](s_{2*(n_0-1)+2})\ )$

$I[[B_4; B_5]](s_{2*(n_0-1)+2}) =$
$I[[B_5]](\ I[[q := q + 2 * i + 1]](s_{2*(n_0-1)+2})\ ) =$
$I[[i := i + 1]](s_{2*(n_0-1)+2+1}) =$
s_{2*n_0+2}

com

$s_{2*n_0+2} = \{(i, n_0), (q, 0 + 1 + 3 + \ldots + 2 * (n_0 - 1) + 1), (n, n_0), \ldots\}$

$I[[B_3]](s_{2*n_0+2}) =$
$\texttt{if}\ \ not\ A[[i < n]](s_{2*n_0+2})\ \ \texttt{then}\ \ s_{2*n_0+2}\ \ \texttt{else}\ \ I[[B_3]](\ I[[B_4; B_5]](s_{2*n_0+2})\ ) =$
s_{2*n_0+2}

$$\begin{aligned}
0 + 1 + 3 + \ldots + 2 * (n_0 - 1) + 1) &= \sum_{i=0}^{n_0-1} 2 * i + 1 \\
&= 2\left(\sum_{i=0}^{n_0-1} i\right) + n_0 \\
&= 2\frac{(n_0 - 1)n_0}{2} + n_0 \\
&= n_0^2
\end{aligned}$$

Assim, o programa é equivalente a `(i:=n; q:=n`^{2}`)`.

6. $\texttt{x}_1\texttt{,x}_2\ \texttt{:=}\ \texttt{e}_1\texttt{,e}_2$

(a)

$$I[\![\ x := e\ ]\!](s) = modificar(s, x, A[\![e]\!](s))$$

$$I[\![\ x_1, x_2 := e_1, e_2\ ]\!](s) = modificar(modificar(s, x_1, A[\![e_1]\!](s)), x_2, A[\![e_2]\!](s))$$

(b)

$$\begin{aligned} I[\![\ x, y := y, x\ ]\!](s_0) &= modificar(modificar(s_0, x, A[\![y]\!](s_0)), y, A[\![x]\!](s_0)) \\ &= modificar(modificar(s_0, x, s_0(y)), y, s_0(x)) \\ &= s_1 \end{aligned}$$

onde

$$\begin{aligned} s_0 &= \{(x, x_0), (y, y_0)\} \\ s_1 &= modificar(modificar(s_0, x, s_0(y)), y, s_0(x)) \end{aligned}$$

$$\{(x, y_0), (y, x_0)\}$$