MTL-101

Note Title 3/19/2024

→ Separable form

→ homogeneous DE

→ Exact DE

→ Integrating factor $Mdx + Ndy = 0$

→ $\frac{M_y - N_x}{N}$ is a f^n of x only

then $I.F. = e^{\int \frac{M_y - N_x}{N}}$

$\rightarrow$ $\dfrac{N_x - M_y}{M}$ is a f^n of y only

then I.F. $= e^{\int \frac{N_x - M_y}{M}}$

— x —

$$\frac{dy}{dt} = ay$$

$$\frac{dy}{y} = a\,dt$$

$$\ln|y| = at + c^*$$

$$\boxed{y = c e^{at}}$$

$$|y| = \underline{\underline{c}}\, e^{at}$$
≥ 0

$$|y e^{-at}| = c \quad \geq 0$$

$$y e^{-at} = c$$

(can even be negative)

$$y = c e^{at}$$

$c \to$ arbitrary constant

Exer If $f : \mathbb{R} \to \mathbb{R}$ is cts; and $|f|$ is contant then f is a constant f^n.

Linear Eq^{n}s A first order ODE is called linear if it can be written in the following form

$$y' + p(x)\,y = Q(x) \qquad (1)$$

and it is called non-linear if it can not be written in the form (1).

Remark. $a_0(x)\,y' + a_1(x)\,y = r(x)$, $\boxed{a_0 \neq 0}$

$$y' + \underbrace{\frac{a_1(x)}{a_0(x)}}_{p(x)} y = \underbrace{\frac{g(x)}{a_0(x)}}_{Q(x)}$$

In (1) if $Q(x) = 0$ then DE is called homogeneous o/w it is called non-homogeneous

DE

$$y' + p(x)y = Q(x) \quad — (1)$$

$$\underset{\uparrow N}{1} \cdot dy + \left(\underset{\uparrow M}{p(x)y} - Q(x)\right) dx = 0$$

$$\frac{\partial M}{\partial y} = M_y = p(x) \quad , \quad \frac{\partial N}{\partial x} = N_x = 0$$

$\rightarrow$ DE (i) is not exact unless $p(x) = 0$

$$\frac{M_y - N_x}{N} = p(x) \rightarrow f^n \text{ of } x \text{ only}$$

$$I.F = e^{\int p(x)\,dx}$$

set $h(x) = \int p(x)\,dx$; $h'(x) = p(x)$

$$I.F. = e^{h(x)}$$

$$\text{I.F.}\left(y' + p(x)\,y\right) = Q(x) \times \text{I.F.}$$

$$e^{h} y' + e^{h} p y - e^{h} Q = 0$$

$$\frac{d}{dx}\left(e^{h} y\right) = e^{h} Q$$

$$e^{h} y = \int e^{h(x)} Q(x)\,dx + C$$

$$y(x) = e^{-h(x)} \int e^{h(x)} Q(x)\,dx + C e^{-h(x)}$$

$$\boxed{\text{I.F. } y(x) = \int \text{I.F. } Q(x)\, dx + C}$$

Exp.

$$y' - y = e^{2x}$$

$$p(x) = -1 \quad , \quad Q(x) = e^{2x}$$

$$\text{I.F.} = e^{\int p(x)\,dx} = e^{-x}$$

$$e^{-x} y(x) = \int e^{-x} e^{2x}\, dx + C$$

$$= e^{x} + C$$

$$\boxed{y(x) = e^{2x} + ce^{x}}$$

__Exm__ Solve $y' + y\tan x = \sin 2x$

__Bernoulli Equations__ (reduction to linear DE)

Consider
$$y' + p(x)\,y = Q(x)\,y^{a} \quad , \quad a \in \mathbb{R} \quad \text{--- } (2)$$

$a = 0, 1 \rightarrow$ DE is linear

$a \neq 0, 1$, $\to$ DE is non-linear

Eq^n (2) can be rewritten as

$$y^{-a} y' + p(x) y^{1-a} = Q(x) \quad \text{———} \quad (*)$$

let

$$v(x) = y^{1-a}$$

$$\frac{dv}{dx} = (1-a) y^{-a} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1-a} y^{a} \frac{dv}{dx}$$

(*) reduces to $\frac{1}{1-a}\frac{dv}{dx} + p(x)\,v(x) = Q(x)$

$$\frac{dv}{dx} + \underbrace{(1-a)\,p(x)}\,v(x) = \underbrace{Q(x)\,(1-a)}$$

$\longrightarrow$ linear in $v(x)$

Exp.

$$\frac{dy}{dx} + y = xy^3$$

$v(x) = y^{-2}$, $\frac{dv}{dx} = -2\,y^{-3}\frac{dy}{dx}$

$$-\frac{1}{2}\frac{dv}{dx} + v(x) = x$$

$$\frac{dv}{dx} - 2 \quad v(x) = -2x$$

$$\text{I.F.} = e^{-2x}$$

$$e^{-2x} \, v(x) = \int \underset{\text{II}}{-2x} \; \underset{\text{I}}{e^{-2x}} \; + C$$

$$= -2\left[\frac{x e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\right] + C$$

$$e^{-2x}\, v(x) = x e^{-2x} - \frac{e^{-2x}}{-2} + C$$

$$e^{-2x}\, v(x) = x e^{-2x} + \frac{e^{-2x}}{2} + C$$

$$v(x) = x + \frac{1}{2} + C e^{2x}$$

$$\boxed{\frac{1}{y^2} = x + \frac{1}{2} + C e^{2x}}$$

Equations reducible to linear DE

$$\frac{d}{dy}(f(y))\frac{dy}{dx} + p(x) f(y) = Q(x)$$

$f(y) \rightarrow$ Knocon function

(In Bernoulli eq^n
$f(y) = y^{1-q}$)

$v(x) = f(y)$

$$\frac{dv}{dx} = \frac{d}{dy}(f(y))\frac{dy}{dx}$$

$$\frac{dv}{dx} + p(x) v(x) = Q(x) \rightarrow \text{linar}$$

Exp. Solve $\cos y \frac{dy}{dx} + \frac{1}{x} \cdot \sin y = 1$

Set $v = \sin y$

—x—

Orthogonal Trajectories

one parameter family of curves

e.g. $x^2 + y^2 = c$, $c \to$ parameter

$m \quad , -1/m$

<u>Orthogonal Trajectories</u> If two family of curves intersect each other at right angles then they are said to be orthogonal trajectories of each other.

<u>Q.</u> How to find OTs of a given family of curves.

Given $F(x, y, c) = 0 \rightarrow$ One parameter family of curves

Step 1 Find the DE s.t $F(x,y,c)=0$ is its solution

$$\frac{dy}{dx} = \phi(x,y)$$

Step 2 ODE of the OTs ~~are~~ given by

$$\frac{dy}{dx} = \frac{-1}{\phi(x,y)} \quad \text{——} \quad (**)$$

Step 3 Solve the DE (**) to find a

one parameter family of curves $G(x,y,c) = 0$.

which is OTs of the family of curves

$$F(x,y,c) = 0 .$$

Exp

$$x^2 + y^2 = c^2$$

$$2x + 2y \frac{dy}{dx} = 0 \quad , \quad \frac{dy}{dx} = \frac{-x}{y}$$

For OTs $\boxed{\frac{dy}{dx} = \frac{y}{x}}$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C^*$$

$$|y| = C|x|$$

$$\boxed{y = cx}$$

Exn Find OT's for $y = cx^2$

— X —

Initial Value Problems (IVPs)

① $|y'| + |y| = 0$, $y(0) = 1$ | No soln

② $y' = 2x$, $y(0) = 1$ | Unique soln

$y = x^2 + 1$

③ $xy' = y - 1$, $y(0) = 1$ | Infinitely many solutions.

$y = 1 + cx$, $c \to$ arbitrary constant

IVP ① $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

<u>Possibilites</u> No sol^n, unique sol^n, infinitely many sol^{ns}

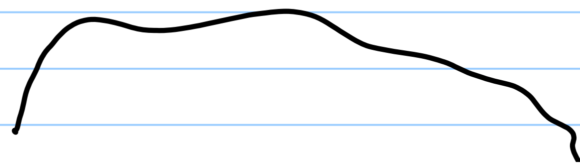
<u>Questions</u>- ① Under what conditions IVP ① has a solution.

② Under what conditions IVP ① has unique sol^n ?

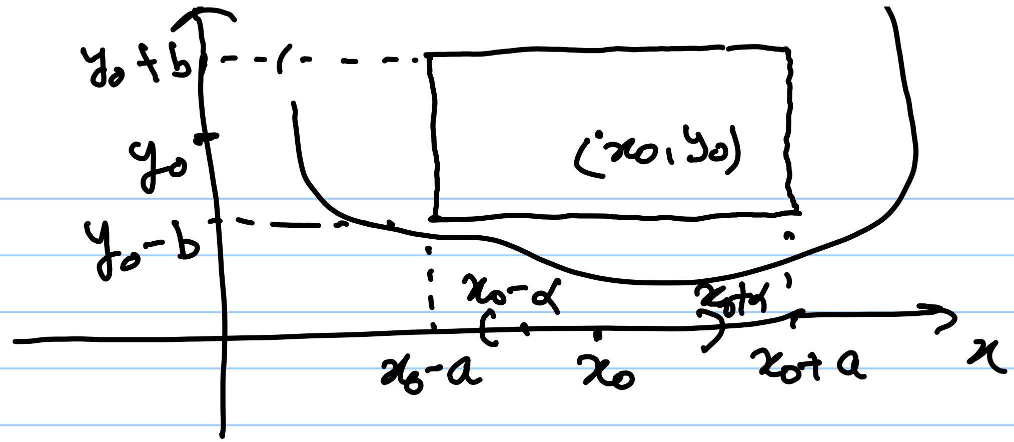
$$f : D \subset \mathbb{R}^2 \to \mathbb{R}$$

$$y' = \underline{\underline{f(x,y)}}$$

$$y(x_0) = y_0$$



Existence Theorem



Consider the IVP

$$y' = f(x,y) \quad , \quad y(x_0) = y_0 \Big\} \longrightarrow \text{IVP(1)}$$

Let $R = \{ (x,y) \mid |x-x_0| < a \ , \ |y-y_0| < b \}$

be a rectangle containing (x_0, y_0) in the domain D. If $f(x,y)$ is continuous in R and $f(x,y)$ is bdd in R s.t.

$$|f(x,y)| \le K \quad \text{for some } K>0$$
$$\forall \ (x,y) \in R$$

Then IVP has atleast one solution in the neighbourhood $|x-x_0| < \alpha$ when

$$\alpha = \min\left\{ a, \frac{b}{K} \right\}.$$

$$a \le b/K \quad , \ \alpha = a$$

Uniqueness Theorem

Consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$

If

① f and its partial derivative w.r.t. y $f_y = \frac{\partial f}{\partial y}$ are continuous for all $(x,y) \in R$

② f & f_y are bounded in R and

$|f(x,y)| \leq K$ for some $K > 0$ $\forall (x,y) \in R$

$|f_y(x,y)| \leq M$ for some $M > 0$, $\forall (x,y) \in R$

Then IVP ① has atmost one solution ;

thus by the existence theorem IVP ① has a unique solution and this sol^n exists for all x s.t. $|x-x_0| < \alpha$, $\alpha = \min\{a, \frac{b}{K}\}$.

Exp- Consider the IVP

$$y' = 1 + y^2 \quad , \quad y(0) = 0$$

and the rectangle $R: |x| < 5$, $|y| < 3$

Find α , which appears in existence & uniqueness thm.

$x_0 = 0 \quad , \quad y_0 = 0$

$f(x,y) = 1 + y^2$

$R: \ |x| < 5 \ , \quad \underline{|y| < 3}$

$a = 5 \quad , \quad b = 3$

$f_y = 2y$

f, f_y are cts in R (verify)

$|f(x,y)| = |1 + y^2| \leq 10 = K$

$$\alpha = \min\left\{ a, \frac{b}{K} \right\} = \left\{ 5, \frac{3}{10} \right\} = 0.3$$

$\rightarrow$ Solⁿ exist & it is unique $\forall x$ s.t. $|x| < 0.3$

<u>Exp:</u> Consider $y' = y^2$, $y(1) = -1$

Find α in the existence & uniqueness thm

$R: |x-1| < a, \quad |y+1| < b$.

Lipschitz Condition

$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

Def^n The function f is said to satisfy Lipschitz condition w.r.t. y in D if $\exists$ a constant $M > 0$ (called the Lipschitz constant) s.t.

$$|f(x, y_1) - f(x, y_2)| \le M |y_1 - y_2| \quad \forall (x, y_1), (x, y_2) \in D$$

Remark $g: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$

g satisfy Lipschitz condition if $\exists$ constant $M \geq 0$

s.t. $\underbrace{|g(x_1) - g(x_2)| \leq M |x_1 - x_2|}$ $\forall\, x_1, x_2 \in I$

$\rightarrow$ g is called Lipschitz continuous if it satisfies Lipschitz condition.