Exer Solve $x^2y'' + xy' - y = 0$

$x = e^s$

$a = 1, \quad b = 1, \quad c = -1$

$$\frac{d^2y}{ds^2} \cancel{+ \frac{dy}{ds}} - y = 0$$

$y = e^{ms}$

auxiliary equation.

$m^2 - 1 = 0 \qquad m = \pm 1$

$y_1(x) = e^s = x \qquad y_2(x) = e^{-s} = \frac{1}{x}$

General $sol^n \rightarrow c_1 x + \frac{c_2}{x}$

→ Non homogenous linear 2^{nd} DE.

$y'' + p(x)\, y' + q(x)\, y = r(x)$

$p(x),\ q(x),\ r(x) \rightarrow$ continuous f^n

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = r(x) \quad -(1)$$

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)I$$

$I \rightarrow$ Identity operator
$Iy = y$

(1) is
$$Ly(x) = r(x)$$

$$\underbrace{L(\alpha y + \beta w) = \alpha L(y) + \beta L(w) \quad \alpha, \beta \in R \quad \text{If } L(y) \text{ \& } L(w) \text{ exist}}_{\text{check for linearity of D.E.}}$$

Associated homogenous DE is

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad -(3)$$
$$Ly(x) = 0 \quad -(4)$$

Th^m- The difference of two sol^{n}s of non homogenous DE (2) is a sol^n of corresponding homogenous DE (3)

Proof- Let y and $\tilde{y}$ be 2 sol^{n}s of non homogenous DE (2) then
$$L(y)(x) = r(x)$$
$$L(\tilde{y})(x) = r(x)$$
$$L(y-\tilde{y})(x) = r(x) - r(x) = 0$$
$y, \tilde{y}$ ~~are~~ is a sol^n of homogenous DE (3) or (4)

Corollary - Any sol^n $y(x)$ of non-homogenous DE ① is given by

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the ~~gre~~ general sol^n of corresponding homogenous DE ③ and $y_p(x)$ is a particular sol^n of the non homogenous DE ①

Proof - $y(x) \rightarrow$ a sol^n of non-hom DE ①

$y_p(x) \rightarrow$ a particular sol^n of non-hom. DE ①

$y(x) - y_p(x) \rightarrow$ sol^n of homogenous DE ③

$y(x) - y_p(x) = y_h(x)$
↳ general sol^n of hom. DE. ③

$$y(x) = y_h(x) + y_p(x)$$

Def^n - General sol^n - A general sol^n of non-homogenous DE

$y'' + p(x)y' + q(x)y = r(x)$ ④ is

a sol^n of the form

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general sol^n of the homogenous DE. $y'' + p(x)y' + q(x)y = 0$.

and $y_p(x)$ is a particular sol^n of non. homogenous DE.

$$y'' + p(x)y' + q(x)y = r(x)$$

↳ A particular solution of non hom. DE (*) is a sol^n obtained from (5) by assigning specific values to the constants occuring in the general sol^n $y_h(x)$ of homogenous DE

General sol^n

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

where $\{y_1, y_2\} \rightarrow$ fundamental set of sol^{n}s of the hom. DE.

→ Methods to find a particular sol^n of Non hom. DE.

(1) Variation of parameters method

↳ Always gives a y_p

$$y'' + p(x)\, y' + q(x)\, y = r(x) \quad - (6)$$

associated with homogenous DE.

$$y'' + p(x) y' + q(x) y = 0 \quad - (7)$$

$\{y_1, y_2\} \rightarrow$ basis of (7)

The general sol^n of (7) is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \quad - (8)$$

The idea of this method is to replace the constants c_1 & c_2 in (8) by the functions say $v_1(x)$ and $v_2(x)$ which are to be determined.

(9) $y_p(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x)$

is a sol^n of non homogenous DE (6)

We have

$$y_p' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'$$

Choose v_1, v_2 s.t. $v_1' y_1 + v_2' y_2 = 0$ (10)

then $y_p' = v_1 y_1' + v_2 y_2'$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Substituting y_p, y_p', y_p'' in non-hom. DE (6)

$$y_p'' + p(x)y_p' + q(x)y_p = r(x)$$

$$v_1 y_1'' + v_1' y_1' + v_2' y_2' + v_2 y_2'' + p(x)(v_1 y_1' + v_2 y_2') + q(x)(v_1 y_1 + v_2 y_2) = r(x)$$

$$v_1(y_1'' + p(x)y_1' + q(x)y_1) + v_2(y_2'' + p(x)y_2' + q(x)y_2) + v_1'y_1' + v_2'y_2' = r(x)$$

$$v_1'y_1' + v_2'y_2' = r(x) \quad \text{— } (11)$$

We solve (10) and (11) for v_1' & v_2'

$$v_1' \underbrace{(y_1 y_2' - y_1' y_2)}_{W(y_1, y_2)(x)} = -y_2 r(x)$$

$W(y_1, y_2)(x) \neq 0$ as y_1, y_2 are L.I. (form basis)

$$v_1' = -\frac{y_2 r(x)}{W}$$

$$v_2' = \frac{y_1 r(x)}{W}$$

$$v_1(x) = -\int \frac{y_2 r(x)}{W} dx \; ; \; v_2(x) = \int \frac{y_1 r(x)}{W} dx$$

$$y_p = -y_1 \int \frac{y_2 r(x)}{W}\,dx + y_2 \int \frac{y_1 r(x)}{W}\,dx$$

↳ a solⁿ of non-hom. DE

The general solⁿ of DE ⑥ is

$$y(x) = y_h(x) + y_p(x)$$
$$= c_1 y_1(x) + c_2 y_2(x) - y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$$

Q. Find the general solⁿ of
$$y'' + y = \operatorname{cosec} x$$

Ans Step 1 – Associated hom. DE is

$$y'' + y = 0$$

$y = e^{mx}$, a.e. $m^2 + 1 = 0$

$m = \pm i$

General solⁿ of $y'' + y = 0$ is

$$y_h = c_1 \cos x + c_2 \sin x$$

$y_1 = \cos x \quad y_2 = \sin x$

A solⁿ of D.E. $y'' + y = \operatorname{cosec} x$

$$y_p(x) = v_1 y_1 + v_2 y_2$$

$v_1 = -\int \frac{y_2 r}{W} dx \qquad v_2 = \int \frac{y_1 r}{W} dx$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 1$

$v_1 = -\int \frac{\sin x \ \text{cosec} x}{1} dx = -x$

$v_2 = \int \frac{\cos x \ \text{cosec} x}{1} dx = \ln|\sin x|$

$y_p(x) = -x\cos x + \ln|\sin x| \cdot \sin x$

General solⁿ of DE. $y'' + y = \text{cosec} x$

$y(x) = c_1 \cos x + c_2 \sin x - x\cos x + \ln|\sin x| \sin x$

Exer Find general solⁿ

$y'' - y' - 2y = e^{-x}$

→ Method of undetermined coefficients

To find the particular solⁿ of

$y'' + ay' + by = r(x) \qquad a, b \in R$

↳ Non homogenous LDE with constant coeff.

$r(x) \rightarrow$ exponential f^n, a power of x, a cosine or sine

OR sum or product of these functions

Note - These functions (exponential, power of x, sine or cosine) have derivatives similar to $r(x)$

Idea - Choose $y_p(x)$ similar to $r(x)$ but with unknown coeff. to be determined so that $y_p(x)$ is a sol^n of DE (11)

(1) Basic rule

Terms in $r(x)$	Choice of $y_p(x)$
$Ke^{\alpha x}$	$Ce^{\alpha x}$
Kx^n	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$K\cos(\omega x+d)$ $\longrightarrow$	$C_1\cos(\omega x+d) + C_2\sin(\omega x+d)$
$K\sin(\omega x+d)$ $\longrightarrow$	$C_1\cos(\omega x+d) + C_2\sin(\omega x+d)$

(2) Multiplication rule

If a term in our choice of $y_p(x)$ happens to be a sol^n of the associated homogenous DE $y''+ay'+by=0$ then multiply this term by x (or by x^2 if the sol^n corresponds to a double root of the auxilliary eq^n of hom. D.E.)

③ Sum/Product rule – If $r(x)$ is a sum/product of functions in the first column of table ①, then choose y_p the sum/product of the functions in the corresp. lines of the second column in table ①

e.g. ① If $r(x) = 2e^x - 10\sin x$, then choice of $y_p(x)$

$$y_p(x) = Ae^x + B\sin x + C\cos x$$

② If $r(x) = x^2e^x + e^{2x}\sin x$

$$x^2 e^x \to (C_1x^2 + C_2x + C_3)\,K_1e^x \;+\; K_2e^{2x}(C_4\sin x + C_5\cos x)$$

$$y_p(x) = Ax^2e^x + Bxe^x + Ce^x + De^{2x}\sin x + Ee^{2x}\cos x$$

③ If $r(x) = x^n e^x \sin(\omega_1 x + \omega_2) + e^{-2x}$

$$x^n \to K_n x^n + K_{n-1}x^{n-1} + \dots K_1 x + K_0$$

$$e^x \to Ke^x$$

$$\sin(\omega_1 x + \omega_2) \to C_1\sin(\omega_1 x + \omega_2) + C_2\cos(\omega_1 x + \omega_2)$$

$$e^{-2x} \to C_3 e^{-2x}$$

$$y_p = A_{11}e^x x^n \sin(\omega_1 x + \omega_2) + A_{12}x^n\cos(\omega_1 x + \omega_2)e^x$$

$+ A_{21} e^{x} x^{n-1} \sin(\omega_1 x+\omega_2) + A_{22} e^{x} x^{n-1} \cos(\omega_1 x+\omega_2)$

$+ \cdots + A_{n+1\,1} \sin(\omega_1 x+\omega_2) e^{x}$

$+ A_{n+1\,2} \cos(\omega_1 x+\omega_2) e^{x}$

eg $y'' - 2y' - 3y = 2e^{x} - 10\sin x$

Corresp. hom. DE is $y'' - 2y' - 3y = 0$

a.e. is $m^2 - 2m - 3 = 0$

$m = -1, 3$

General sol^n of hom. D.E is

$y_h(x) = c_1 e^{-x} + c_2 e^{3x}$

$r(x) = 2e^{x} - 10\sin x$

$y_p(x) = Ae^{x} + B\sin x + C\cos x$

Note - None of e^{x}, $\sin x$, $\cos x$ are the sol^{n}s of homogenous DE.

$y_p'(x) = Ae^{x} + B\cos x - C\sin x$

$y_p''(x) = Ae^{x} - B\sin x - C\cos x$

Substitute y_p, y_p', y_p'' in DE

$y'' - 2y' - 3y = 2e^{x} - 10\sin x$

$(Ae^{x} - B\sin x - C\cos x) - 2(Ae^{x} + B\cos x - C\sin x) - 3(Ae^{x} + B\sin x + C\cos x)$

$= 2e^{x} - 10\sin x$

$$e^x(A-2A-3A)+\sin x(-B+2C-3B)$$
$$+\cos x(-C-2B-3C)$$
$$=2e^x-10\sin x$$

$$-4A=2 \qquad -4B+2C=-10$$
$$-4C-2B=0$$

$$y_p(x)=\frac{-e^x}{2}+2\sin x-\cos x$$

$$y(x)=y_p(x)+y_h(x)$$

<u>Exer</u> Solve (using method of undetermin-ed coeff.)

$$y''-3y'+2y=2x^2+e^x+2xe^x+4e^{3x}$$