MTL-101

Separable ODEs

$M(x) + N(y)\, y' = 0$

Homogeneous functions

A f^n $f(x_1, x_2, \dots x_n)$ is called homogeneous if $f(tx_1, tx_2, \dots tx_n) = t^d f(x_1, x_2, \dots x_n)$ for some $d \in \mathbb{Z}$ & $t \neq 0$.

f is homogeneous of degree d.

e.g. $f(x,y) = x^2 - 2xy + y^2$ (degree 2)

$f(x,y) = y + x\cos^2(y/x)$ (degree 1)

<u>Homogeneous ODE</u> A first order ODE

$\underline{M(x,y)} + \underline{N(x,y)}\, y' = 0$ is called

homogeneous ODE if the functions $M(x,y)$ & $N(x,y)$ are homogeneous with equal degree.

Homogeneous ODE $\longrightarrow$ reduction to seperable form

$M(x,y) + N(x,y)\, y' = 0$, $M(x,y)$ & $N(x,y)$ are homogenous f^n

of equal degree, say d.

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$M(x, vx) + N(x, vx)\left(v + x\frac{dv}{dx}\right) = 0$$

$$x^d\left[M(1,v) + N(1,v)\left(v + x\frac{dv}{dx}\right)\right] = 0$$

$$M(1,v) + N(1,v)\left(v + x\frac{dv}{dx}\right) = 0$$

$$M(1,v) + N(1,v)\,v + N(1,v)\,x\frac{dv}{dx} = 0$$

separable eqn $\left\{ \dfrac{dx}{x} + \dfrac{N(1,v)}{M(1,v) + vN(1,v)}\, dv = 0 \right.$

Expr Solve $2xy \cdot y' = y^2 - x^2$ (hom. ODE

$$y = vx \quad , \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2} - \frac{1}{2v}$$

$$x\frac{dv}{dx} = -\frac{v}{2} - \frac{1}{2v} = -\frac{(v^2+1)}{2v}$$

$$\frac{2v}{1+v^2}\,dv = -\frac{dx}{x}$$

$$\ln|1+v^2| = -\ln|x| + c^* \qquad \ln c \qquad \ln\frac{c}{|x|}$$

$$1+v^2 = e^{c^* - \ln|x|}$$

$$1+v^2 = \frac{c}{x}$$

$v = y/x$

$$\boxed{x^2+y^2 = cx}$$

$y(1)=1$

$1+1 = c$

$c=2$

$$\boxed{x^2+y^2=2x}$$

<u>Exer 1</u> - Find the curve through the point $(1,1)$ in the xy-plane and having the slope $-y/x$ at

each of its points.

$$\boxed{y' = -y/x} \quad , \quad \boxed{y(1) = 1}$$

Eq^n's reducible to separable form

Exm 2 Solve $(4x + 2y + 5)\, y' + (2x + y - 1) = 0$

Substitute $v = 2x + y$

Exm 3 Solve $y' = \dfrac{x + y - 3}{x - y - 3}$

Substitute $x = x_1 + h$, $y = y_1 + K$ for some h & K

$$\frac{dy_1}{dx_1} = \frac{x_1 + y_1 + h + K - 3}{x_1 - y_1 + h - K - 3}$$

choose h & K s.t. $\left.\begin{array}{l} h + K - 3 = 0 \\ h - K - 3 = 0 \end{array}\right\}$ $h = 3, K = 0$

$\downarrow$

homogeneous ODE.

<u>Exact ODE</u> A first order ODE $M(x,y) + N(x,y)y' = 0$ is called exact if $\exists$ a function $u(x,y)$ s.t.

$$\boxed{\frac{\partial u}{\partial x} = M(x,y)} \qquad \boxed{\frac{\partial u}{\partial y} = N(x,y)}$$

e.g.: Is $(2x+y^2) + 2xy\frac{dy}{dx} = 0$ exact?

$$u(x,y) = x^2 + y^2 x$$

$$\frac{\partial u}{\partial x} = M = 2x+y^2, \quad \frac{\partial u}{\partial y} = N = 2xy$$

Remark. (Recall from calculus) Given a f^n $u(x,y)$ with its partial derivatives, the total differential is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$0 = M(x,y)\,dx + N(x,y)\,dy = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = du$$

$$du = 0$$

$$u(x,y) = \text{constant} = C$$

$$\boxed{u(x,y) = C} \rightarrow \text{(implicit) sol}^{n} \text{ of the ODE.}$$

Working Methodoly of Exact DEs

Given exact ODE $M(x,y) + N(x,y)\, y' = 0$, the f^n $u(x,y)$ can be found either by inspection or by the following method.

① Integrate $\dfrac{\partial u(x,y)}{\partial x} = M(x,y)$ to obtain

$$u(x,y) = \int M(x,y)\,dx + K(y) \qquad \text{— ①}$$

$K(y) \rightarrow$ integration constant (∵ y is constant in integration w.r.t. x)

② To determine $K(y)$; differentiate ① w.r.t. y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x, y)\, dx\right] + K'(y)$$

we have

$$\frac{\partial u}{\partial y} = N(x, y)$$

$$K'(y) = N - \underbrace{\frac{\partial}{\partial y}\int M\, dx}$$

integrate w.r.t. y to get

$$\boxed{K(y) = \int N(x, y)\, dy - \int \underbrace{\frac{\partial}{\partial y}\left(\int M\, dx\right)} dy}$$

$\therefore$ from ①

$$u(x,y) = \int M(x,y)\,dx + \int N(x,y)\,dy$$

$$- \int \left\{ \frac{\partial}{\partial y} \left(\int M\,dx \right) \right\} dy$$

$\boxed{u(x,y) = c}$ → implicit solⁿ of ODE

Exp

Solve

$$(2x + y^2) + 2xy \frac{dy}{dx} = 0$$

$$M = 2x + y^2, \qquad N = 2xy$$

$$\frac{\partial u}{\partial x} = M = 2x + y^2 \quad — (2), \qquad \frac{\partial u}{\partial y} = N = 2xy \quad — (3)$$

Integrate (2) w.r.t x

$$u(x,y) = x^2 + y^2 x + K(y)$$

differentiating w.r.t. y

$$\frac{\partial u}{\partial y} = 2xy + K'(y)$$

using (3) $\quad \frac{\partial u}{\partial y} = 2xy \quad \Rightarrow \quad K'(y) = 0$

$$K(y) = C^*$$

$$u(x,y) = x^2 + y^2 x + c^*$$

sol^n of ODE $\quad u(x,y) = \text{constant} = C_1$

$$x^2 + y^2 x + c^* = c_1$$

$$\boxed{x^2 + y^2 x = c}$$

Exp: Solve $\cos(x+y)\, dx + \left(3y^2 + 2y + \cos(x+y)\right) dy = 0$

Remark- $\frac{\partial u}{\partial x} = M$, $\underbrace{\frac{\partial u}{\partial y} = N}$

integration w.r.t y

$$u(x,y) = \int N\, y + l(x)$$

determine $l(x)$ using $\frac{\partial u}{\partial x} = M$

Test of Exactness

$M(x,y) + N(x,y)\, y' = 0$

Thm Let M, N and their first partial derivative exist and be continuous in a region $D \subseteq \mathbb{R}^2$. Then

① If $M\,dx + N\,dy = 0$ is exact ODE then

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}. \checkmark$$

② If D is $\boxed{\text{convex}}$ and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{ODE is exact}$$

Proof: Let $M\,dx + N\,dy = 0$ is exact.

$\Rightarrow \exists\ u(x,y)$ s.t.

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y\,\partial x}$$

using the continuity of M_y & N_x, we have

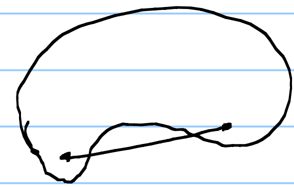
$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Convex set

$z_1, z_2 \in D$

$\theta z_1 + (1-\theta) z_2 \in D$

$\forall\ 0 \le \theta \le 1$



Not convex

Suppose $\boxed{D \to \text{convex}}$ & $\underbrace{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$$H(x,y) = \left(M(x,y),\ N(x,y)\right)$$

$$\text{Curl } H = \left(N_x - M_y\right)\hat{k} \qquad \left(\text{verify}\right)$$

$$\text{Curl } H = 0$$

$$\Rightarrow \quad H = \nabla\phi$$

$$\left(M(x,y),\ N(x,y)\right) = \left(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y}\right)$$

Conservative fields

F is a conservative vector field if

$$F = \nabla f$$

$$M = \frac{\partial \phi}{\partial x}, \qquad N = \frac{\partial \phi}{\partial y}$$

$$\boxed{\phi(x,y) = c}$$

—x—

Integrating Factors (reduction to exact form)

Suppose $M\,dx + N\,dy = 0$ is not exact i.e

$$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right)$$

To find a function $v(x,y)$ s.t.

$$v M\,dx + v N\,dy = 0 \text{ is exact}$$

$$\boxed{\frac{\partial}{\partial y}(vM) = \frac{\partial}{\partial x}(vN)}$$

$$\Leftrightarrow \quad v_y M + v M_y = v_x N + v N_x$$

$$\Leftrightarrow \quad N v_x - M v_y - v(M_y - N_x) = 0 \qquad (4)$$

$v(x,y) \rightarrow$ called integrating factor

<u>Particular Cases</u>

(1) Suppose v is a function of x alone in the DE (4).

Then from (4)
$$N v_x = v(M_y - N_x)$$

or
$$\frac{v_x}{v} = \frac{M_y - N_x}{N} \longrightarrow (5)$$

If $\frac{M_y - N_x}{N}$ is a function of x only then

then (5) is separable ODE
$$\frac{dv}{v} = \left(\frac{M_y - N_x}{N}\right) dx$$

sol^n is $\quad v(x) = e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$

(2) If integrating factor v is a function of y only then using (4)

$$-M v_y - v(M_y - N_x) = 0$$

$$\frac{v_y}{v} = \frac{N_x - M_y}{M} \quad \text{--- (6)}$$

If $\frac{N_x - M_y}{M}$ is a function of y only then (6) is a separable ODE

$$\frac{dv}{v} = \left(\frac{N_x - M_y}{M}\right) dy$$

$$v(y) = e^{\int \left(\frac{N_x - M_y}{M}\right) dy}$$

Exp:

Solve $\underbrace{\left(e^{x+y} + y e^{y}\right)}_{M} dx + \underbrace{\left(x e^{y} - 1\right)}_{N} dy = 0$

$y(0) = -1$

$M = e^{x+y} + y e^{y}$, $N = x e^{y} - 1$

$\frac{\partial M}{\partial y} = e^{x+y} + e^{y} + y e^{y}$, $\frac{\partial N}{\partial x} = e^{y}$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ DE is not exact

$$\frac{M_y - N_x}{M} = \frac{e^{x+y} + ye^y}{M} = 1$$

$\therefore$ integrating factor $v(y) = e^{\int -dy} = e^{-y}$

multiplying DE by integrating factor

exact DE $(e^x + y)\,dx + (x - e^{-y})\,dy = 0$

$\tilde{M}\,dx + \tilde{N}\,dy$

$\dfrac{\partial \tilde{M}}{\partial y} = 1$, $\dfrac{\partial \tilde{N}}{\partial x} = 1$