MTL-101

Note Title 3/13/2024

non-homogeneous system

$$Ax = b \quad \text{--- (1)}$$

corresponding homogeneous system

$$Ax = 0 \quad \text{--- (2)}$$

If system (1) is consistent then all its solutions are obtained as

$$x = x_0 + x_h$$

where x_0 is any (fixed) sol^n of (1) &

x_h runs throught all the sol^n of homogeneous system (2)

$x \rightarrow$ sol^n of non-homegeneous system

$Ax = b$

$x_0 \rightarrow$ any (fixed) sol^n of non-hom. system $Ax = b$

$Ax_0 = b$

$A(x - x_0) = Ax - Ax_0 = b - b = 0$

$$x_h = x - x_0$$

$$x = x_0 + x_h$$

— x —

Differential Equations

An eq^n having derivatives of one or more dependent variables with respect to one or more independent variables is called a diff. eq^n.

Radio active decay

$y(t) \rightarrow$ amount of the substance present at the time t

Physical information rate of decay of the substance is proportional to the substance present at that time

$$\frac{dy}{dt} \propto y(t)$$

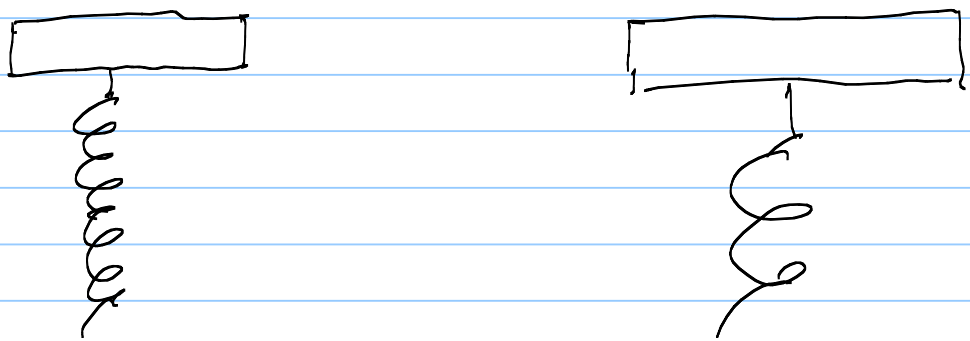
Initial value problem

$$\boxed{\frac{dy}{dt} = -ky} \qquad k > 0 \text{ constant} \quad \text{decay constant}$$

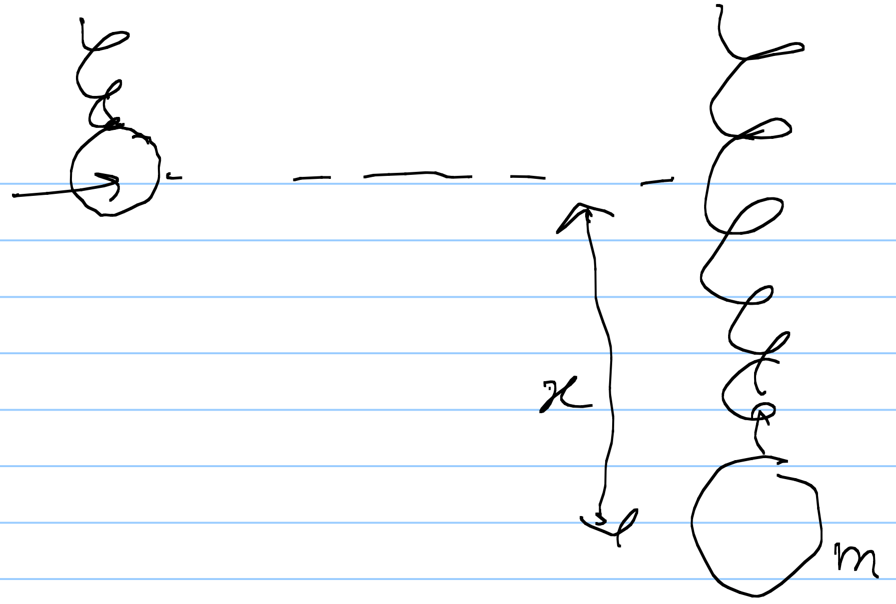
$$\boxed{y(0) = 10}$$

② Vibration of a a spring

fixed support



ball of mass m



$$m \frac{d^2x}{dt^2} = Kx$$

$$\frac{d^2x}{dt^2} \neq \frac{K}{m} x = 0$$

Classification of DE's

DE → differential Equation

ODE A DE having one or more dependent variables and only one independent variable is called ordinary differential eqn (ODE)

e.g.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 3x = 0$$

dep. vari $= x$

indep. variable $= t$

$$\frac{dx}{dt} = 3xy^2$$

$$\frac{dy}{dt} = 2x^2 y$$

$x, y \to$ dep. variables

$t \to$ independent variables

Partial Differential Equations (PDE)

A DE having one or more dependent variables and more than one independent variables are called PDEs.

e.g. ① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u = u(x, y)$

$u \to$ dependent variable

$x, y \to$ independent variables

(2) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$$

$u \rightarrow$ dep. variable

$x, y, t \rightarrow$ independent variables.

Order of a DE

The order of the highest derivative present in the DE is called order of the DE.

e.g. (1) $$\frac{d^2 x}{dt^2} + 2\frac{dx}{dt} + 5x = t^3$$

$\hookrightarrow 2^{nd}$ order ODE

(2) $$\frac{d^4 x}{dt^4} + 4\frac{d^3 x}{dt^3} + 5t^2 = 0$$ $\rightarrow 4^{th}$ order ODE

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow 2^{nd} \text{ order PDE.}$$

— X —

General form of ODE

$y \to$ dependent variable

$x \to$ independent variable

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dx^2}, \ldots \qquad y^{(n)} = \frac{d^n y}{dx^n}$$

The general form of n^{th} order ODE is

$$F(x, y, y', y'', \dots y^{(n)}) = 0 \quad \text{—— (1)}$$

<u>Linear ODE</u> If F is linear in $y, y', \dots y^{(n)}$

then (1) is called linear DE.

The general form of n^{th} order linear DE is

$$a_0(x)\, y^{(n)}(x) + a_1(x)\, y^{(n-1)}(x) + \cdots + a_n(x)\, y(x) = r(x)$$

$a_0 \neq 0$, $a_0, a_1 \dots a_n \rightarrow$ functions of x only

→ A DE which is not linear is called non-linear DE.

Exps ① $$x\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = \underline{\log x}$$

2nd order Linear ODE

$$\underbrace{x\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y}_{= L(y)}$$

$$L \equiv x\frac{d^2}{dx^2} + x^2\frac{d}{dx} + 1$$

$$L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$$

② $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$ → 2nd order non-linear DE

$$F(x, y, y', \dots y^{(n)}) = 0 \quad \text{—} \; (1)$$

<u>Solution of DE</u> : A f^n $\varphi(x)$ is called a solution of DE (1) in (a,b) if $\varphi', \varphi'', \dots \varphi^{(n)}$ exist and they satisfy

$$F(x, \varphi, \varphi', \dots \varphi^{(n)}) = 0 \quad \forall \; x \in (a,b).$$

<u>First Order DE</u>

Implicit form $\quad F(x, y, y') = 0$

Explicit form $y' = f(x,y)$

$f : D \to \mathbb{R}$

$D \subseteq \mathbb{R}^2$

<u>Initial value Problems</u> (IVP)

A DE together with initial conditions is called an initial value problem.

e.g. First order IVP

$$\left.\begin{aligned} &\boxed{\frac{dy}{dx} = f(x,y)} \\ &\boxed{y(x_0) = y_0} \end{aligned}\right\} \quad (2)$$

Sol^{n}s of IVP — A sol^n $y(x)$ of IVP (2) in the interval I is a differentiable f^n satisfying

$$\frac{dy}{dx} = f(x,y) \quad \forall \; x \in I$$

$$y(x_0) = y_0 \quad , \quad x_0 \in I$$

Exp's

(1) No sol^n

$$|y'(x)| + |y(x)| = 0$$

$$y(0) = 1$$

(2) Unique sol^n

$y' = 2x \quad , \quad y(0) = 1$

$y(x) = x^2 + 1$

(3) infinitely many sol^ns

$$\begin{cases} xy' = y - 1 \\ y(0) = 1 \end{cases}$$

$y(x) = 1 + Cx$ is a sol^n of this IVP

$C \to$ arbitrary constant

Solving DEs

(1) Separable DEs

An ODE of the form

$$M(x) + N(y)\, y' = 0 \quad \text{(1)}$$

is called a separable DE.

Let $H_1(x)$ & $H_2(y)$ be the f^ns s.t.

$M(x) = H_1'(x)$, $N(y) = H_2'(y)$

then ① can be written as

$$H_1'(x) + H_2'(y)\, y' = 0 \quad ②$$

By the chain rule $\frac{d}{dx}\left(H_2(y)\right) = H_2'(y)\, y'$

$\therefore$ ② can be written as

$$\frac{d}{dx}\left(H_1(x) + H_2(y)\right) = 0$$

$$\boxed{H_1(x) + H_2(y) = c}$$

$c \to$ constant

Exp. ① $y' = 1+y^2$

$$\frac{dy}{1+y^2} = dx$$

$$y = \tan(x+c)$$

Exps ② $y' = -2xy$

③ $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$, $y(0)=1$

④ $y' = 3y^{2/3}$, $y(0)=0$

$\rightarrow$ $y = 0$ is a sol^n.

Using separation of variables $\longrightarrow$ $y = x^3$ is also a sol^n.

$$\phi_K(x) = \begin{cases} 0 & -\infty < x \leq K \\ (x-K)^3 & K < x < \infty \end{cases}$$

then $\phi_K(x)$ is a sol^n for any K.

<u>RK.</u> Separation of variables method does not yield all the sol^{n}s of the DE.