$$\mathbb{E}(X_n) = \mathbb{E}\left(M^2_{min(T,T_R,n)} - \sigma^2 min(T,T_R,n)\right) \geq \mathbb{E}(X_0) = \mathbb{E}(1) = 1 \geq A \leq$$

Particle Drift in a Periodic Flow Field

All programs are attached in the end of the report.

Question 1:

Consider the system

$$\frac{dX}{dt} = \alpha \cos k(X(t) - ct) \tag{1}$$

By a change of variable, it suffices to consider

$$\frac{dX}{dt} = a \cos 2\pi(X(t) - t) \tag{2}$$

Question 2:

We use Matlab's ode45 method to numerically solve 2 with a representative set of values of a and $X(0) = 0$, producing the following plots.

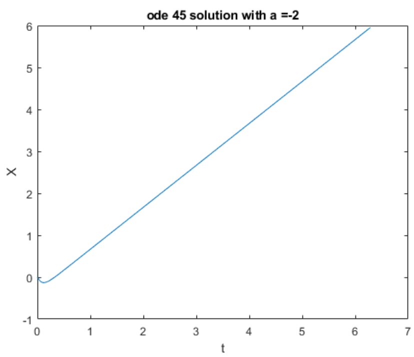

Figure 2.1: ode45 solution for a = -2

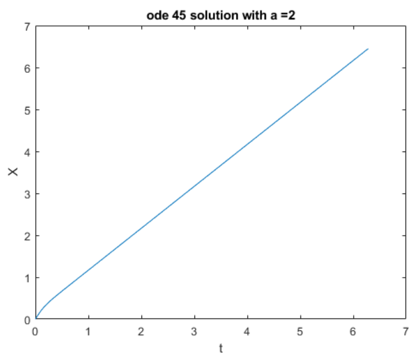

Figure 2.2: ode45 solution for a = 2

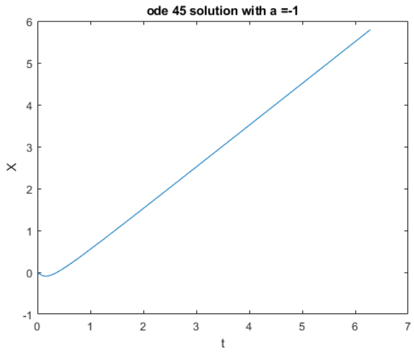

Figure 2.3: ode45 solution for a = -1

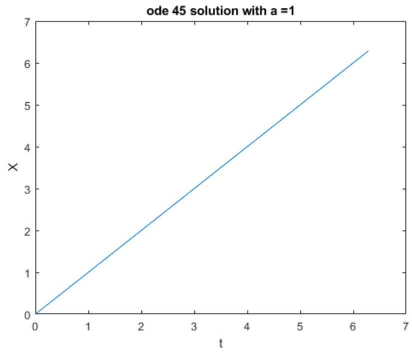

Figure 2.4: ode45 solution for a = 1

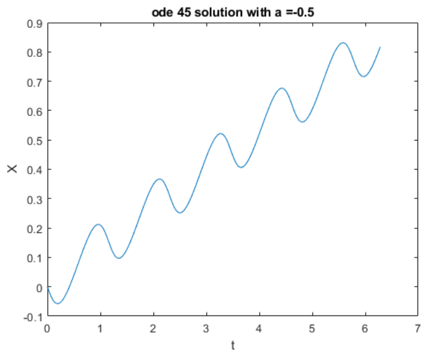

Figure 2.5: ode45 solution for a = -0.5

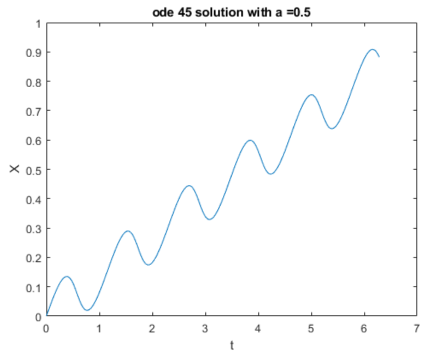

Figure 2.6: ode45 solution for a = 0.5

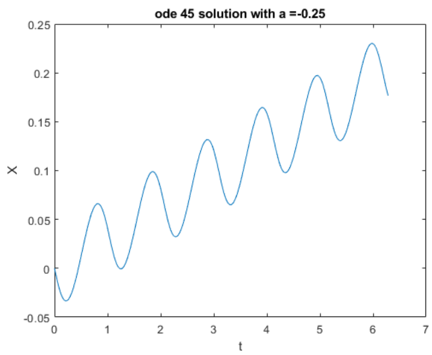

Figure 2.7: ode45 solution for a = -0.25

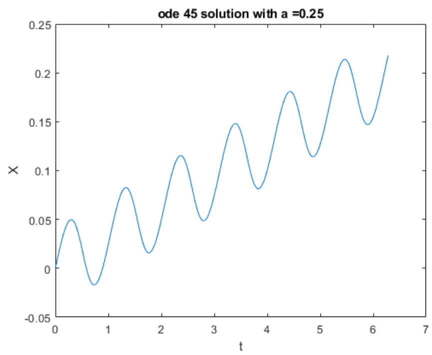

Figure 2.8: ode45 solution for a = 0.25

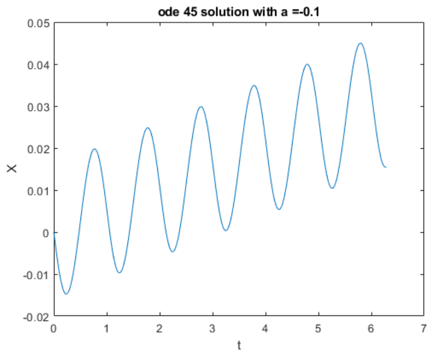

Figure 2.9: ode45 solution for a = -0.1

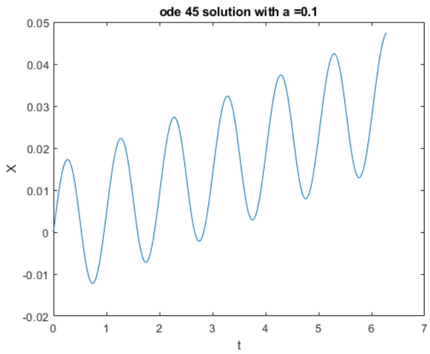

Figure 2.10: ode45 solution for a = 0.1

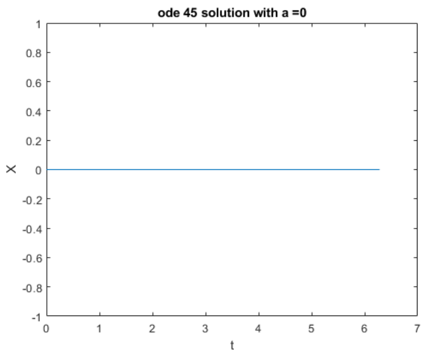

Figure 2.11: ode45 solution for $a = 0$

We justify the accuracy of the graphs by setting different absolute tolerances in the ode45 solver and observing that different absolute tolerances give similar graphs. Figures 2.1 - 2.11 shows that the solution is sinusoidal. The period increases when a increases. Note also that for small values of a, there is a time-averaged mean 'drift' velocity. This fact would be explored in the next section.

If $X(0) \neq 0$, we may substitute $Y(t) = X(t) - X(0)$ to get an equation of the form 2 and similar conclusions hold.

Question 3:

The time-averaged drift velocity, when it exists, is found by fixing two local maxima of the solution and apply the 'difference-quotient' approximation to find the velocity. We tabulate the results of approximations for some values of a below.

Table 1: Time-averaged drift speed v for some values of a

a	-0.5	-0.25	-0.1
v	0.134448	0.03188	0.004997
a	0.1	0.25	0.5
v	0.004954	0.031711	0.133924

The results confirm that, when $|a|$ is small, there is an approximately $\frac{1}{2}a^2$ time-averaged drift velocity.

Question 4:

A material fluid element's motion $X(t)$ in a periodic flow which has velocity $u(x,t)$ satisfies the differential equation

$$\frac{dX}{dt} = u(x,t) \tag{3}$$

Question 5:

For small values of a, we show there is an approximate time-averaged drift velocity $\frac{1}{2}a^2$.

First we substitute $X = t + Y$ in 2 to obtain

$$\frac{dY}{dt} = a\cos 2\pi Y - 1 \tag{4}$$

Rearrange 4 and separate variables to see that

$$-\frac{dY}{1 - a\cos 2\pi Y} = dt \tag{5}$$

When $|a|$ is small, we approximate the RHS of 5 to the second order to obtain

$$-(1 + a\cos 2\pi Y + a^2\cos^2 2\pi Y)dY = dt \tag{6}$$

Integrate both sides to obtain

$$-Y + \frac{a}{2\pi}\sin(2\pi Y) + a^2(\frac{\sin(4\pi Y)}{8\pi} + \frac{Y}{2}) = t \tag{7}$$

The foreseen term $\frac{a^2}{2}$ arises as expected.

Source Code

```
function drift = solPlot(a)
tspan = [0 2*pi];
x0 = 0;
options = odeset('RelTol',1e-8,'AbsTol',1e-10);
derivative = @(t,X)a*cos(2*pi*(X-t));
[t,X] = ode45(derivative, tspan, x0,options);
plot(t,X,'-')
ttl = strcat('ode 45 solution with a = ',num2str(a));
title(ttl);
xlabel('t')
ylabel('X')
s = 0;
X_list = zeros(1,2);
t_list = zeros(1,2);
for i = 2:length(t)-1
if s == 2
break
else
if (X(i)>X(i+1)) && (X(i)>X(i-1))
s = s + 1;
t_list(s) = t(i);
X_list(s) = X(i);
end
end
end
drift = (X_list(2)-X_list(1))/(t_list(2)-t_list(1));


end
```

```
a_list = [-2,-1,-0.5,-0.25,-0.1,0,0.1,0.25,0.5,1,2];
drift_list = zeros(1,11);
for i = 1:11
drift_list(i) = solPlot(a_list(i));
input(' ')
end
```