Signals and systems

Signals, systems and tools
Signals, systems and telecommunications

Exercises 4: Discrete-time systems and z-transform [*]

Franky De Bruyne

13 septembre 2024

1. The triangular pulse $x[n]$ is defined by

$$x[n] = \begin{cases} n & 0 \leq n \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Sketch and find an expression for the functions

(a) $y[n] = x[n+3] + x[n-3]$

(b) $y[n] = x[n] + x[-n]$

2. Find the even and odd components of the discrete-time signal

$$x[n] = \begin{cases} 4-n & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

[*] Exercises **marked with an** * **are there FYI only** as they concern **non causal signals**.

3. Compute the response of a system with impulse response $h[n]$ to the signal $x[n]$.

$$x[n] = u[n], \quad h[n] = a^n \, u[n], \quad 0 < a < 1$$

Use the convolution formula. What is the steady-state response, i.e. when $t \to \infty$.

4. Compute the response of the FIR filter

$$y[n] = \frac{1}{2}(x[n] + x[n-1] + x[n-2])$$

to the input

$$x[n] = u[n] - u[n-4].$$

(a) Use convolution : analytical and graphical method

(b) Use the z-transform method

5. Compute the z-transforms of the following sequences and determine the region of convergence.

(a) $x[n] = \delta[n - n_0]$

(b) $x[n] = u[n - n_0]$

(c) $x[n] = a^{n+1} \, u[n+1]$

(d) $x[n] = a^{n+1} \, u[n]$

(e) $x[n] = na^n \, u[n]$

(f) $x[n] = na^{n-1} \, u[n]$

(g*) $x[n] = a^{-n} \, u[-n]$

6. Compute the z-transforms of the following sequences. Compute the poles and zeros and determine the region of convergence.

(a) $x[n] = \left(\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) u[n]$

(b) $x[n] = \begin{cases} a^n & 0 \leq n \leq N-1, \quad a > 0 \\ 0 & \text{otherwise} \end{cases}$

(c*) $x[n] = a^{|n|}$

7. Compute the inverse z-transforms of the following transfer functions. Use the partial fraction method by considering the transfer function as function of z and z^{-1}. Compute the first terms of the impulse response using long division and compare.

(a) $X(z) = \dfrac{8z - 19}{(z-2)(z-3)}, \quad |z| > 3$

(b) $X(z) = \dfrac{z^{-1}}{2 - 3z^{-1} + z^{-2}}, \quad |z| > 1$

8. The response of an LTI system to the step input $x[n] = u[n]$ is

$$y[n] = 2\left(\frac{1}{3}\right)^n u[n].$$

Compute

(a) the impulse response of the system

(b) the response to the input

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

9. Compute the solution to the following difference equations :

(a) $y[n] - \frac{1}{2}y[n-1] = x[n], \quad x[n] = \left(\frac{1}{3}\right)^n u[n],\ y[-1] = 1$

(b) $3\,y[n] - 4\,y[n-1] + y[n-2] = x[n],$

$x[n] = \left(\frac{1}{2}\right)^n u[n],\ y[-1] = 1, y[-2] = 2$

(c) $y[n] + y[n-1] - 4\,y[n-2] - 4\,y[n-3] = 3\,x[n],$

$x[n] = u[n],\ y[-1] = 1, y[-2] = y[-3] = 0$

(d) $y[n+1] - 2\,y[n] = 3^{-n}\,u[n], \quad y[0] = \dfrac{2}{5}$

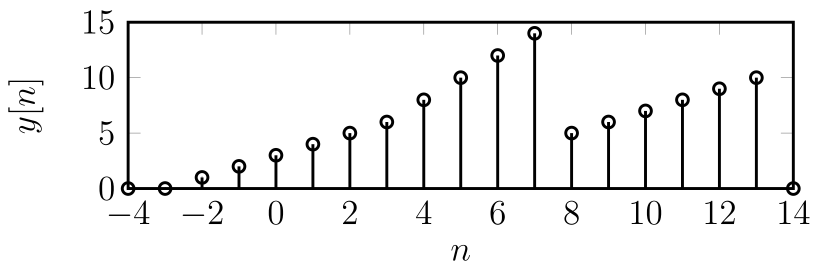
(e) $y[n+2] - 7\,y[n+1] + 10\,y[n] = 16n\,u[n], \quad y[0] = 6, y[1] = 2$

Solutions :

1. Expressions

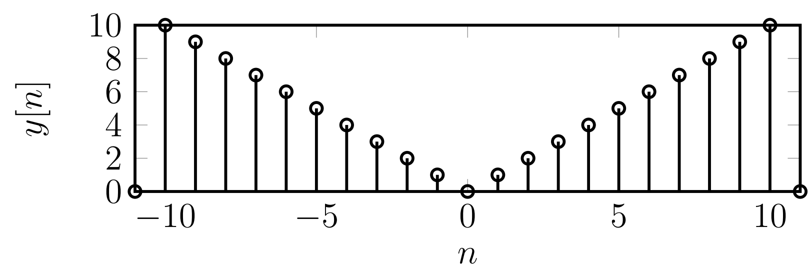
   (a)

$$y[n] = \begin{cases} n+3 & -3 \leq n \leq 2 \\ 2n & 3 \leq n \leq 7 \\ n-3 & 8 \leq n \leq 13 \\ 0 & \text{otherwise} \end{cases}$$

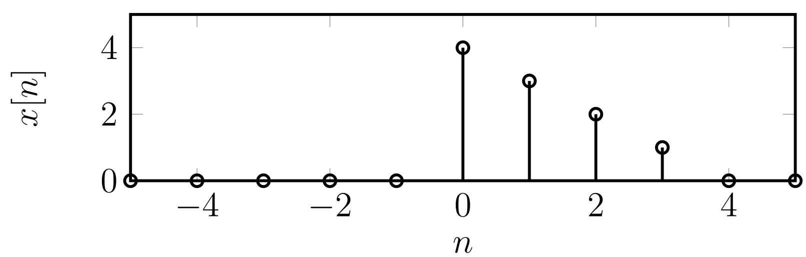


   (b)

$$y[n] = \begin{cases} n & 1 \leq n \leq 10 \\ 0 & n = 0 \\ -n & -10 \leq n \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

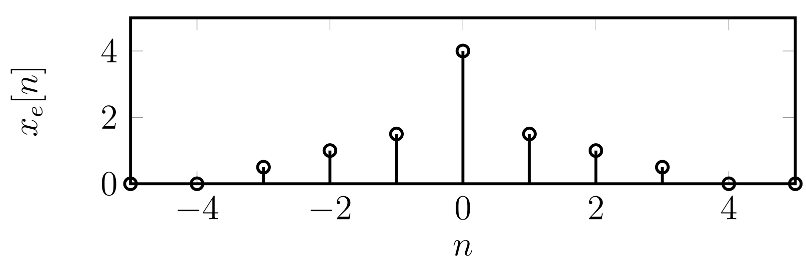


2. Decompose in the even and odd components



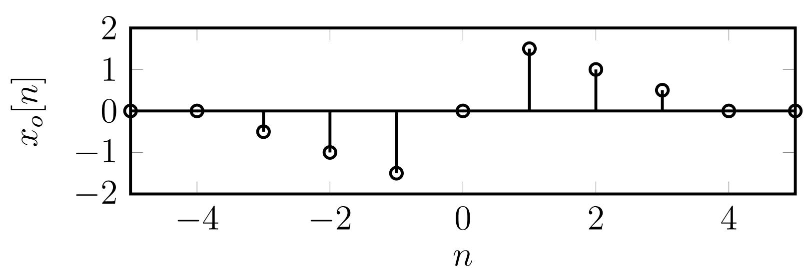
Even component

$$x_e[n] = \begin{cases} 2 + 0.5n & -4 \leq n \leq -1 \\ 4 & n = 0 \\ 2 - 0.5n & 1 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



Odd component

$$x_o[n] = \begin{cases} -2 - 0.5n & -4 \leq n \leq -1 \\ 0 & n = 0 \\ 2 - 0.5n & 1 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



3. Response

$$y[n] = \sum_{k=0}^{n} a^{n-k} \underset{n\to\infty}{\to} \frac{1}{1-a}$$

4. Response

$y[0] = 0.5$
$y[1] = 1,$
$y[2] = 1.5$
$y[3] = 1.5$
$y[4] = 1$
$y[5] = 0.5$
$y[k] = 0, k > 5$

5. Z-transforms

(a) $X(z) = z^{-n_0}, |z| > 0$

(b) $X(z) = \dfrac{z^{-n_0}}{1 - z^{-1}}, |z| > 1$

(c) $X(z) = \dfrac{z^2}{z-a} - z = \dfrac{a\,z}{z-a}, |z| > |a|$

(d) $X(z) = \dfrac{a\,z}{z-a}, |z| > |a|$

(e) $X(z) = \dfrac{a\,z}{(z-a)^2}, |z| > |a|$

(f) $X(z) = \dfrac{z}{(z-a)^2}, |z| > |a|$

(g) $X(z) = \dfrac{1}{(1-az)}, |z| < \dfrac{1}{|a|}$

6. Z-transforms

(a) $X(z) = \dfrac{2\,z(z - \frac{5}{12})}{(z-\frac{1}{2})(z-\frac{1}{3})}, \quad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1}{z^{N-1}} \dfrac{z^N - a^N}{z - a} = \dfrac{1}{z^{N-1}} \prod_{k=1}^{N-1} (z - a\, e^{j\frac{2k\pi}{N}}), \quad |z| > 0$

(c) $X(z) = \dfrac{z(a - \frac{1}{a})}{(z-a)(z-\frac{1}{a})}, \quad |a| < |z| < \dfrac{1}{|a|}$

7. Inverse z-transforms

(a) $x[n] = -\dfrac{19}{6}\,\delta[n] + (\dfrac{3}{2}\,2^n + \dfrac{5}{3}\,3^n)\,u[n]$

(b) $x[n] = \left(1 - \left(\dfrac{1}{2}\right)^n\right)\, u[n]$

8. Responses

(a) $h[n] = 6\,\delta[n] - 4\left(\dfrac{1}{3}\right)^n u[n]$

(b) $y[n] = \left(8\left(\dfrac{1}{3}\right)^n - 6\left(\dfrac{1}{2}\right)^n\right) u[n]$

9. Solutions to the difference equations

(a) $y[n] = \left(\dfrac{7}{2}\left(\dfrac{1}{2}\right)^n - 2\left(\dfrac{1}{3}\right)^n\right) u[n]$

(b) $y[n] = \left(\dfrac{3}{2} + \dfrac{1}{2}\left(\dfrac{1}{3}\right)^n - \left(\dfrac{1}{2}\right)^n\right) u[n]$

(c) $y[n] = \left(-\dfrac{1}{2} - \dfrac{1}{6}\,(-1)^n + \dfrac{8}{3}\,2^n\right) u[n]$

(d) $y[n] = \left(2^n - \dfrac{3}{5}\left(\dfrac{1}{3}\right)^n\right) u[n]$

(e) $y[n] = (4\,2^n - 3\,5^n + 4n + 5)\,u[n]$