Signals and systems

Signals, systems and tools
Signals, systems and telecommunications

Exercises 3: Fourier frequency analysis *

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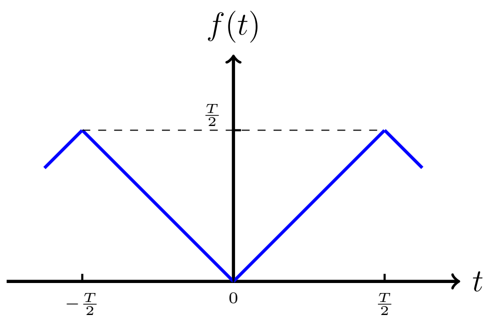
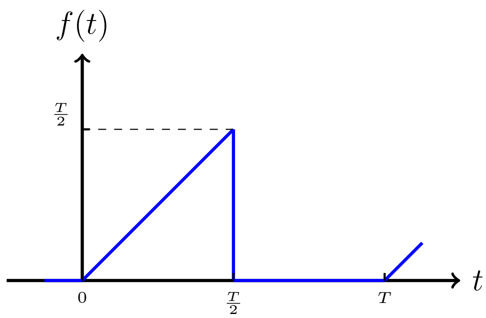
1. Compute the Fourier series of the following functions :

   (a)



*All exercises below are **FYI only** and should be considered as an illustration of the course.

(b)



2. Discuss whether it is possible to obtain the Fourier transform of the following signals from the Laplace transform :
   (a) $x_1(t) = u(t)$
   (b) $x_2(t) = e^{-2t}\,u(t)$
   (c) $x_3(t) = e^{-|t|}$

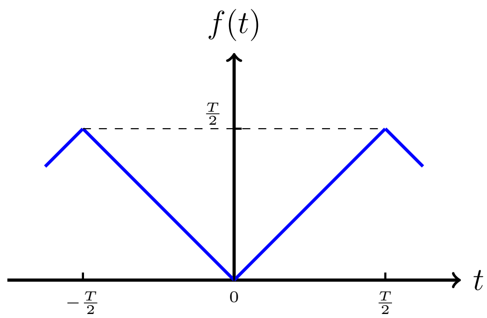
Solutions :

1. Fourier series :
   (a) $f(t) = \dfrac{T}{4} - \displaystyle\sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2T}{k^2\pi^2} \cos\left(\frac{2k\pi t}{T}\right)$
   (b) $f(t) = \dfrac{T}{8} + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,T}{2k\pi} \sin\left(\frac{2k\pi t}{T}\right) - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{T}{k^2\pi^2} \cos\left(\frac{2k\pi t}{T}\right)$
2. Fourier transforms :
   (a) no
   (b) yes, $F(\omega) = \frac{1}{2+j\omega}$
   (c) yes, $F(\omega) = \frac{2}{1+\omega^2}$

Detailed solutions for the Fourier series :

1 Function 1a

Compute the Fourier series of the function $f(t)$ periodic of period T



1.1 Computation of the c_k coefficients

The coefficient c_0 is computed separately; it corresponds to the average of the signal $f(t)$ over a period T

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |t|\,dt = \frac{2}{T}\int_0^{\frac{T}{2}} t\,dt = \frac{T}{4}$$

1.1.1 Computation of the c_k coefficients from the definition

The c_k coefficients are obtained from

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2k\pi t}{T}}\,dt.$$

For the function under consideration, one has

$$\begin{aligned}
c_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |t| e^{-j\frac{2k\pi t}{T}}\,dt \\
&= \frac{1}{T}\left(\int_{-\frac{T}{2}}^{0} -te^{-j\frac{2k\pi t}{T}}\,dt + \int_{0}^{\frac{T}{2}} te^{-j\frac{2k\pi t}{T}}\,dt\right) \\
&= \frac{1}{T}\int_{0}^{\frac{T}{2}} t\left(e^{j\frac{2k\pi t}{T}} + e^{-j\frac{2k\pi t}{T}}\right)dt \\
&= \frac{2}{T}\int_{0}^{\frac{T}{2}} t\cos(\frac{2k\pi t}{T})dt
\end{aligned}$$

Here we have used

$$\int_{-\frac{T}{2}}^{0} -te^{-j\frac{2k\pi t}{T}}\,dt = \int_{-\frac{T}{2}}^{0} \bar{t}e^{j\frac{2k\pi \bar{t}}{T}} - d\bar{t} = \int_{0}^{\frac{T}{2}} \bar{t}e^{j\frac{2k\pi \bar{t}}{T}}\,d\bar{t}$$

with a change of variables $\bar{t} = -t$.

The integral is computed by parts :

$$\int uv' = uv - \int vu'$$

with $u = t$ and $v' = \cos(\frac{2k\pi t}{T})$. One has $u' = 1$ and

$$v = \frac{T}{2k\pi}\sin(\frac{2k\pi t}{T}).$$

Thus

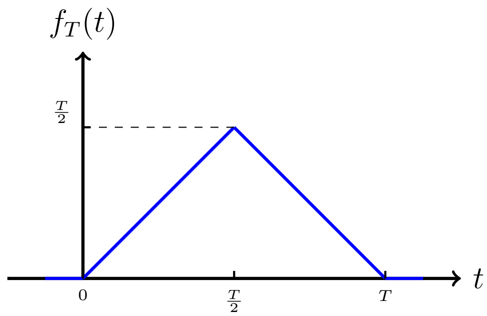
$$\begin{aligned}
c_k &= \frac{2}{T}\int_{0}^{\frac{T}{2}} t\cos(\frac{2k\pi t}{T})dt \\
&= \frac{2}{T}\frac{T}{2k\pi}\left[t\sin(\frac{2k\pi t}{T})\right]_0^{\frac{T}{2}} - \frac{2}{T}\frac{T}{2k\pi}\int_{0}^{\frac{T}{2}}\sin(\frac{2k\pi t}{T})dt \\
&= -\frac{2}{T}\left(\frac{T}{2k\pi}\right)^2\left[-\cos(\frac{2k\pi t}{T})\right]_0^{\frac{T}{2}} \\
&= \frac{T}{2k^2\pi^2}\left[\cos(\frac{2k\pi t}{T})\right]_0^{\frac{T}{2}} \\
&= \frac{T}{2k^2\pi^2}(\cos(k\pi) - 1)
\end{aligned}$$

One has

$$c_k = \begin{cases} 0 & k \text{ even} \\ -\frac{T}{k^2\pi^2} & k \text{ odd} \end{cases}$$

1.1.2 Computation of the c_k coefficients from the Laplace transform

Consider the function $f(t)$ is restricted to the interval $[0, T]$, i.e.



$$f_T(t) = t\,u(t) - 2(t - \frac{T}{2})\,u(t - \frac{T}{2}) + (t - T)\,u(t - T)$$

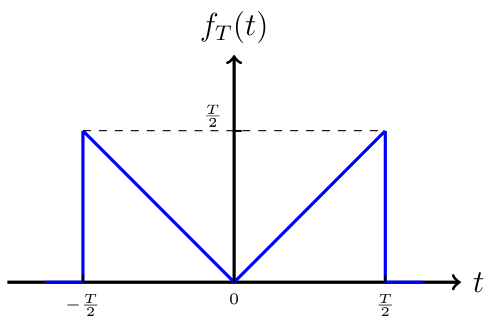
The Laplace transform of the function $f_T(t)$, the restriction of $f(t)$ on one period is :

$$\begin{aligned} F_T(s) &= \mathcal{L}(f_T(t)) \\ &= \frac{1}{s^2}\left(1 - 2e^{-\frac{T}{2}s} + e^{-Ts}\right) \\ &= \frac{1}{s^2}\left(1 - e^{-\frac{T}{2}s}\right)^2 \end{aligned}$$

The Fourier coefficients are obtained as

$$\begin{aligned} c_k &= \frac{1}{T}\left(F_T(s)\big|_{s=j\frac{2k\pi}{T}}\right) \\ &= \frac{1}{T}\frac{1}{-\frac{4\pi^2k^2}{T^2}}\left(1 - e^{-jk\pi}\right)^2 \\ &= \begin{cases} 0 & k \text{ even} \\ -\frac{T}{k^2\pi^2} & k \text{ odd} \end{cases} \end{aligned}$$

One could also consider $f(t)$ restricted on the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$, i.e.



the function $f_T(t)$ is then non-causal ; it has a causal part and an anti-causal part. The computation of c_k is therefore more complicated.

1.2 Fourier computation

By definition, one has

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j\,\frac{2k\pi t}{T}}$$

Consider the harmonics k of $f(t)$, i.e. the contributions of k and $-k$ ($|k| \geq 1$ and k odd) in the sum. One has

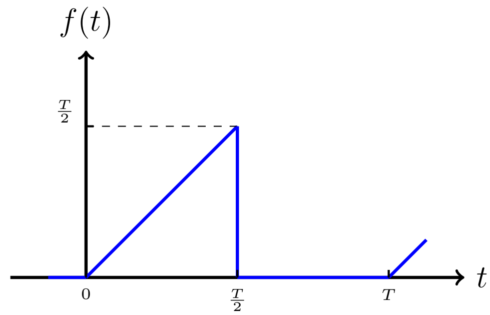
$$\begin{aligned} c_k \, e^{j\,\frac{2k\pi t}{T}} + c_{-k} \, e^{-j\,\frac{2k\pi t}{T}} &= -\frac{T}{k^2\pi^2} \left(e^{j\,\frac{2k\pi t}{T}} + e^{-j\,\frac{2k\pi t}{T}} \right) \\ &= -\frac{T}{k^2\pi^2} \left(2\cos\left(\frac{2k\pi t}{T}\right) \right) \end{aligned}$$

The Fourier series of the function under consideration is therefore

$$f(t) = \frac{T}{4} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2T}{k^2\pi^2} \cos\left(\frac{2k\pi t}{T}\right)$$

2 Function 1b

Compute the Fourier series of the function $f(t)$ periodic of period T



2.1 Computation of the c_k coefficients

The coefficient c_0 is computed separately ; it corresponds to the average of the signal $f(t)$ over a period T

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^{\frac{T}{2}} t\,dt = \frac{T}{8}$$

2.1.1 Computation of the c_k coefficients from the definition

The c_k coefficients are obtained from

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2k\pi t}{T}}\,dt$$

For the function under consideration, one has

$$c_k = \frac{1}{T}\int_0^{\frac{T}{2}} te^{-j\frac{2k\pi t}{T}}\,dt$$

The integral is computed by parts :

$$\int uv' = uv - \int vu'$$

with $u = t$ and $v' = e^{-j\frac{2k\pi t}{T}}$. One has $u' = 1$ and

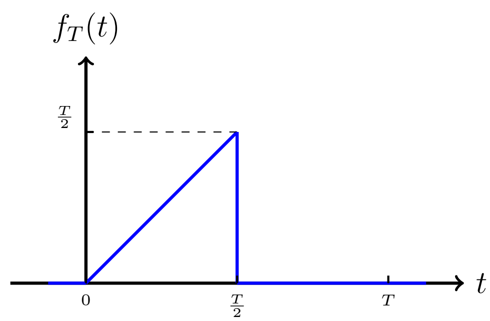
$$v = -\frac{1}{j\frac{2k\pi}{T}}e^{-j\frac{2k\pi t}{T}}.$$

Thus

$$
\begin{aligned}
c_k &= -\frac{1}{T}\frac{T}{j2k\pi}\left[te^{-j\frac{2k\pi t}{T}}\right]_0^{\frac{T}{2}} - \frac{1}{T}\int_0^{\frac{T}{2}} \frac{1}{-j\frac{2k\pi}{T}} e^{-j\frac{2k\pi t}{T}} dt \\
&= -\frac{T}{j4k\pi}e^{-jk\pi} - \frac{1}{T}\left[\left(\frac{1}{-j\frac{2k\pi}{T}}\right)^2 e^{-j\frac{2k\pi t}{T}}\right]_0^{\frac{T}{2}} \\
&= -\frac{T}{j4k\pi}e^{-jk\pi} + \frac{T}{4k^2\pi^2}(e^{-jk\pi} - 1) \\
&= \begin{cases} -\frac{T}{j4k\pi} & \text{k even} \\ \frac{T}{j4k\pi} - \frac{T}{2k^2\pi^2} & \text{k odd} \end{cases}
\end{aligned}
$$

2.1.2 Computation of the c_k coefficients from the Laplace transform

Consider the function $f(t)$ is restricted to the interval $[0, T]$, i.e.



$$
f_T(t) = t\,u(t) - (t - \frac{T}{2})\,u(t - \frac{T}{2}) - \frac{T}{2}\,u(t - \frac{T}{2})
$$

The Laplace transform of the function $f_T(t)$, the restriction of $f(t)$ on one period is :

$$
\begin{aligned}
F_T(s) &= \mathcal{L}(f_T(t)) \\
&= \frac{1}{s^2}\left(1 - e^{-\frac{T}{2}s}\right) - \frac{T}{2}\frac{1}{s}e^{-\frac{T}{2}s}
\end{aligned}
$$

The Fourier coefficients are obtained as

$$
\begin{aligned}
c_k &= \frac{1}{T}\left(F_T(s)|_{s=j\frac{2k\pi}{T}}\right) \\
&= \frac{1}{T}\frac{1}{-\frac{4\pi^2k^2}{T^2}}\left(1-e^{-jk\pi}\right)-\frac{1}{2}\frac{1}{j\frac{2k\pi}{T}}\,e^{-jk\pi} \\
&= \frac{T}{4k^2\pi^2}(e^{-jk\pi}-1)-\frac{T}{j4k\pi}e^{-jk\pi} \\
&= \begin{cases} -\frac{T}{j4k\pi} & \text{k even} \\ \frac{T}{j4k\pi}-\frac{T}{2k^2\pi^2} & \text{k odd} \end{cases}
\end{aligned}
$$

2.2 Fourier computation

By definition, one has

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2k\pi t}{T}}$$

The c_k coefficients can be separated as

$$c_k = c_k^s + c_k^c$$

with

$$c_k^s = (-1)^{(k+1)}\frac{T}{j4k\pi}$$

and

$$c_k^c = \begin{cases} 0 & \text{k even} \\ -\frac{T}{2k^2\pi^2} & \text{k odd} \end{cases}$$

Consider the harmonics k of $f(t)$, i.e. the contributions of k and $-k$ ($|k| \geq 1$ and k odd) in the sum. One has

$$
\begin{aligned}
c_k^s\, e^{j\frac{2k\pi t}{T}} + c_{-k}^s\, e^{-j\frac{2k\pi t}{T}} &= (-1)^{(k+1)}\frac{T}{j4k\pi}\left(e^{j\frac{2k\pi t}{T}}-e^{-j\frac{2k\pi t}{T}}\right) \\
&= (-1)^{(k+1)}\frac{T}{j4k\pi}\left(2j\sin\left(\frac{2k\pi t}{T}\right)\right) \\
&= (-1)^{(k+1)}\frac{T}{2k\pi}\left(\sin\left(\frac{2k\pi t}{T}\right)\right)
\end{aligned}
$$

Similarly

$$\begin{aligned}
c_k^c\, e^{j\frac{2k\pi t}{T}} + c_{-k}^c\, e^{-j\frac{2k\pi t}{T}} &= -\frac{T}{2k^2\pi^2}\left(e^{j\frac{2k\pi t}{T}} + e^{-j\frac{2k\pi t}{T}}\right) \\
&= -\frac{T}{2k^2\pi^2}\left(2\cos\left(\frac{2k\pi t}{T}\right)\right) \\
&= -\frac{T}{k^2\pi^2}\left(\cos\left(\frac{2k\pi t}{T}\right)\right)
\end{aligned}$$

The Fourier series of the function under consideration is therefore

$$f(t) = \frac{T}{8} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, T}{2k\pi} \sin\left(\frac{2k\pi t}{T}\right) - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{T}{k^2\pi^2} \cos\left(\frac{2k\pi t}{T}\right)$$