OPTIMAL VELOCITY IN A RACE

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1. Formulation. We wish to determine how a runner should vary his speed $v(t)$ during a race of distance D in order to run it in the shortest time. Previously we formulated this problem in the following way [**1**]:

The time T to run the race is related to $v(t)$ and D by

$$D = \int_0^T v(t)\, dt. \tag{1.1}$$

The velocity v satisfies the equation of motion

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t). \tag{1.2}$$

Here v/τ is a resistive force per unit mass and τ is a given constant, while $f(t)$ is the propulsive force per unit mass. Initially

$$v(0) = 0. \tag{1.3}$$

The force $f(t)$ is controlled by the runner, but it cannot exceed the maximum value F,

$$f(t) \leqq F. \tag{1.4}$$

The force also affects the quantity $E(t)$ of available oxygen in the muscles per unit mass, since oxygen is consumed in reactions which release the energy used in running. Energy is used at the rate fv, and we assume that oxygen is supplied by breathing and circulation at the rate σ in excess of that supplied in the non-running state. Then the oxygen balance equation is

$$\frac{dE}{dt} = \sigma - fv. \tag{1.5}$$

In (1.5) and elsewhere we measure oxygen in units of the amount of energy it would yield in a reaction. Initially

$$E(0) = E_0. \tag{1.6}$$

Since $E(t)$ can never be negative, we have

$$E(t) \geqq 0. \tag{1.7}$$

The problem now is this:

PROBLEM. *Find $v(t)$, $f(t)$, and $E(t)$ satisfying* (1.2)–(1.7) *so that T, defined by* (1.1) *is minimized. The four physiological constants τ, F, σ, and E_0 and the distance D are given.*

This is a problem in the calculus of variations with differential equations and inequalities as constraints. Since there are two differential equations and three unknown functions, it is called a problem of optimal control theory. The force $f(t)$ may be thought of as the control variable, and it is at the disposal of the runner.

We shall reformulate this problem in Section 2 and solve it in Section 3. Then we shall compare the predictions of the theory with the world records at distances D from 50 yards to 10,000 meters.

2. Reformulation. It is convenient to eliminate both $f(t)$ and $E(t)$ from the problem by expressing them in terms of $v(t)$. To do so we first note that (1.2) gives f directly in terms of v. When we use (1.2) to eliminate f from (1.4), we obtain

$$\frac{dv}{dt} + \frac{v}{\tau} \leqq F. \tag{2.1}$$

Next we use (1.2) to eliminate f from (1.5). Then we can integrate the resulting equation and utilize (1.6) to get

$$E(t) = E_0 + \sigma t - \frac{v^2(t)}{2} - \frac{1}{\tau} \int_0^t v^2(s)\, ds. \tag{2.2}$$

This equation gives E in terms of v, so we can use it to eliminate E from (1.7) with the result

$$E_0 + \sigma t - \frac{v^2(t)}{2} - \frac{1}{\tau} \int_0^t v^2(s)\, ds \geqq 0. \tag{2.3}$$

Now that f and E have been eliminated, the problem is to find $v(t)$ satisfying (1.3), (2.1), and (2.3) so that T, defined by (1.1), is minimized. Instead of minimizing T for given D, we shall consider the equivalent problem of maximizing D with T given. It is easy to see that these two problems yield the same relation between T and D.

3. Solution. Since $v(0) = 0$, the rate of doing work fv is zero initially no matter what value f has. Therefore initially f can be as large as possible without energy consumption, so we may set $f(0) = F$. We assume that $f(t)$ remains equal to F during some initial interval $0 \leqq t \leqq t_1$ where t_1 is to be determined. If it turns out that $t_1 = 0$ then this assumption is vacuous. Upon using this assumption in (2.1) we obtain

$$\frac{dv}{dt} + \frac{v}{\tau} = F, \qquad 0 \leqq t \leqq t_1. \tag{3.1}$$

The solution of (3.1) satisfying (1.3) is

$$v(t) = F\tau(1 - e^{-t/\tau}), \qquad 0 \leqq t \leqq t_1. \tag{3.2}$$

Upon using (3.2) in (2.3) we obtain

$$E_0 + \sigma t - F^2\tau^2 \left(\frac{t}{\tau} + e^{-t/\tau} - 1\right) \geqq 0, \qquad 0 \leqq t \leqq t_1. \tag{3.3}$$

If $\sigma \geqq F^2\tau$, (3.3) is satisfied for all $t \geqq 0$ and then (3.2) is the optimal velocity for all t. However, this is unrealistic, so we shall ignore it and consider only the case $\sigma < F^2\tau$. Then (3.3) holds only for $0 \leqq t \leqq T_c$ where T_c is the unique positive root of (3.3) with the equality holding.

Now if $T \leqq T_c$ we set $t_1 = T$ and (3.2) yields the optimal velocity throughout the race. We shall call such races "short sprints" or "dashes." For them (3.2) and (1.1) yield

$$D = F\tau^2\left(\frac{T}{\tau} + e^{-T/\tau} - 1\right), \qquad 0 \leqq T \leqq T_c. \tag{3.4}$$

The length of the longest dash we shall call D_c, which is given by (3.4) with $T = T_c$.

If $T > T_c$ then we must have $t_1 < T$, and it remains to find t_1 and the function $v(t)$ for $t_1 < t \leqq T$. It is clear that the solution must satisfy $E(T) = 0$, i.e., the oxygen supply must be used up at the end of the race. We shall assume that $E(t) = 0$ throughout an interval $t_2 \leqq t \leqq T$ at the end of the race, the length of which is to be determined. If it turns out that $t_2 = T$, this assumption is also vacuous. Thus we suppose that for some $t_2 \geqq t_1$ we have

$$E(t) = 0, \qquad t_2 \leqq t \leqq T. \tag{3.5}$$

We now use (2.2) for E in (3.5) and differentiate with respect to t to obtain

$$\sigma - \frac{1}{2}\frac{dv^2}{dt} - \frac{v^2}{\tau} = 0, \qquad t_2 \leqq t \leqq T. \tag{3.6}$$

This is an ordinary differential equation for v^2 which has the solution

$$v^2(t) = \sigma\tau + [v^2(t_2) - \sigma\tau]\, e^{2(t_2 - t)/\tau}, \qquad t_2 \leqq t \leqq T. \tag{3.7}$$

The value $v(t_2)$ has not yet been determined, nor has condition (3.5) been satisfied. Instead we have set $dE/dt = 0$, so (3.5) will hold if we make $E(t_2) = 0$.

We have now found $v(t)$ to be given by (3.2) in the interval $0 \leqq t \leqq t_1$ and by (3.7) in the interval $t_2 \leqq t \leqq T$. To find v in the remaining interval $t_1 \leqq t \leqq t_2$ we use these results in (1.1) to write D in the form

$$\begin{aligned} D = &\int_0^{t_1} F\tau(1 - e^{-t/\tau})\, dt + \int_{t_1}^{t_2} v(t)\, dt \\ &+ \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau]\, e^{2(t_2 - t)/\tau}\}^{\frac{1}{2}} dt. \end{aligned} \tag{3.8}$$

We must choose v to maximize D subject to the constraint $E(t_2) = 0$, which was mentioned above. Therefore we consider the functional $D + \lambda E(t_2)/2$ where λ is a Lagrange multiplier, $E(t)$ is given by (2.2) and D is given by (3.8). We equate to zero the first variation of this functional and obtain

$$\int_{t_1}^{t_2} \delta v(t)\,dt + v(t_2)\,\delta v(t_2) \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau] \cdot e^{2(t_2-t)/\tau}\}^{-\frac{1}{2}} e^{2(t_2-t)/\tau} dt$$
$$- \frac{\lambda}{2} v(t_2)\,\delta v(t_2) - \frac{\lambda}{2\tau} \int_{t_1}^{t_2} 2v(t)\,\delta v(t)\,dt = 0. \tag{3.9}$$

In order that (3.9) hold for arbitrary $\delta v(t)$, the coefficient of δv in the integrand must vanish and so must the coefficient of $\delta v(t_2)$. This yields the two equations

$$v(t) = \tau/\lambda, \qquad t_1 \leqq t \leqq t_2, \tag{3.10}$$

$$\lambda = 2 \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau] e^{2(t_2-t)/\tau}\}^{-\frac{1}{2}} e^{2(t_2-t)/\tau} dt. \tag{3.11}$$

From (3.10) we obtain the interesting result that $v(t)$ is constant from t_1 to t_2.

Now by equating $v(t_1)$ given by (3.2) to $v(t_1)$ given by (3.10) we obtain

$$F(1 - e^{-t_1/\tau}) = 1/\lambda. \tag{3.12}$$

Next we rewrite (3.11), using (3.10) for $v(t_2)$ and evaluating the integral. This yields

$$\lambda = 2\left(\sigma - \frac{\tau}{\lambda^2}\right)^{-1} \left[\left\{\sigma\tau + \left(\frac{\tau^2}{\lambda^2} - \sigma\tau\right) e^{2(t_2-T)/\tau}\right\}^{\frac{1}{2}} - \frac{\tau}{\lambda}\right]. \tag{3.13}$$

Then we set $E(t_2) = 0$, using (2.2) for E and in it using (3.2) and (3.10) for $v(t)$. This gives

$$E_0 + \sigma t_2 - \frac{\tau^2}{2\lambda^2} - F^2\tau\left(-\frac{3\tau}{2} + t_1 + 2\tau e^{-t_1/\tau} - \frac{\tau}{2} e^{-2t_1/\tau}\right)$$
$$- \frac{\tau}{\lambda^2}(t_2 - t_1) = 0. \tag{3.14}$$

The three equations (3.12)–(3.14) determine t_1, t_2, and λ, provided that $t_1 \leqq t_2 \leqq T$. If the equations yield $t_1 > t_2$, then the maximum occurs at the endpoint $t_1 = t_2$ and (3.14) shows that $t_1 = t_2 = T_c$.

To find when the latter case occurs, we view (3.13) as a biquadratic for λ and solve it to obtain the four roots $\pm(\tau/\sigma)^{\frac{1}{2}}$ and $\pm(\tau/\sigma)^{\frac{1}{2}}[1 - 4\exp 2(t_2 - T)/\tau]^{\frac{1}{2}}$. The negative roots are not consistent with (3.12), and $(\tau/\sigma)^{\frac{1}{2}}$ violates (3.11), so the only admissible root is

$$\lambda = (\tau/\sigma)^{\frac{1}{2}}[1 - 4e^{-2(T-t_2)/\tau}]^{\frac{1}{2}}. \tag{3.15}$$

We now use (3.15) for λ in (3.12), set $t_1 = t_2 = T_c$, and solve the resulting equation for T. The solution is T^* defined by

$$T^* = T_c + \tau\{\log 2 - \tfrac{1}{2}\log[1 - \sigma F^{-2}\tau^{-1}(1 - e^{-T_c/\tau})^{-2}]\}. \tag{3.16}$$

For $T \geqq T^*$, the equations (3.12), (3.13) or (3.15), and (3.14) yield t_1, t_2 and λ. For $T_c \leqq T \leqq T^*$ we have instead $t_1 = t_2 = T_c$ and (3.12) yields λ.

Once t_1, t_2, and λ have been found, $v(t)$ is given by the three expressions (3.2), (3.7), and (3.10). When these expressions for $v(t)$ are used in (3.8), it yields the following result for D:

$$
\begin{aligned}
D = {} & F\tau^2\left(\frac{t_1}{\tau} + e^{-t_1/\tau} - 1\right) + \frac{\tau(t_2 - t_1)}{\lambda} \\
& + \tau(\sigma\tau)^{\frac{1}{2}}\left[-\frac{1}{\lambda}\left(\frac{\tau}{\sigma}\right)^{\frac{1}{2}} - \tanh^{-1}\frac{1}{\lambda}\left(\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right. \\
& - \left\{1 + \left(\frac{\tau}{\lambda^2\sigma} - 1\right)e^{-(2/\tau)(T-t_2)}\right\}^{\frac{1}{2}} \\
& \left. + \tanh^{-1}\left\{1 + \left(\frac{\tau}{\lambda^2\sigma} - 1\right)e^{-(2/\tau)(T-t_2)}\right\}^{\frac{1}{2}}\right], \qquad T \geqq T_c.
\end{aligned} \tag{3.17}
$$

For $T \leqq T_c$, D is given by (3.4).

In obtaining the optimal solution $v(t)$, we assumed that the inequality (1.4) is an equality in the initial interval $0 \leqq t \leqq t_1$. We also assumed that the inequality (1.7) is an equality in the final interval $t_2 \leqq t \leqq T$. These two assumptions were based on physical reasoning. They could have been avoided by using the following fact from the general theory of optimal control, which is nearly obvious: An optimal solution consists of a finite number of arcs each satisfying one or more of the equalities in the constraints, together with a finite number of arcs satisfying none of the equalities. To use this fact we would have had to consider an arbitrary number of intervals, instead of just three. Then we would have had to show that the optimal solution is the one with just three intervals located as we assumed them to be. Instead of that we could try to show directly that our solution yields the minimum value of T. Since we shall do neither of these two things, we have shown only that our solution yields a stationary value of T rather than a minimum.

4. Determination of the constants from the world records. The four constants τ, F, σ, and E_0 can be found by comparing the predictions of the theory with the world records. To find them we consider first the results for 50 yds., 50 m., 60 yds., 60m., 100 yds., 100m., 200 yds., and 200 m., which we assume to be dashes. Then T is related to D by (3.4), which involves only τ and F. We choose, τ and F to minimize the sum of the squares for these eight dashes of the relative errors,

$$\Sigma(T_{\text{record}} - T_{\text{calculated}})^2 / T^2_{\text{record}}.$$

This minimization is carried out on a computer by varying τ and F and searching for the minimum. Once τ and F are found in this way, we determine σ and E_0 to minimize the sum of the squares of the relative errors for 14 records at distances from 400 m. to 10,000 m. In doing so we calculate T from (3.17) after solving (3.12)–(3.14) for t_1, t_2, and λ. Finally we calculate T_c from (3.3) and use it in (3.4) to get $D_c = 291$ m.

Since D_c is greater than 200 m. and less than 400 m., our assumption that the eight shortest distance races are dashes is confirmed and the calculation of the constants is consistent. The values of the constants and the calculated values of T, t_1, and $T-t_2$ are given in [**1**].

In Figure 1, a curve of the calculated value of the average velocity D/T is shown for $D \leqq 2000$ m., together with points based on the world records.

It is to be noted that the theory yields no final "kick," which runners often use. Instead it yields a slowing down in the last one or two seconds—a negative kick— which can be understood by thinking about driving a car a fixed distance in the shortest time with a limited amount of fuel. Primarily the theory confirms the accepted view that for distances greater than one half mile, a constant speed is best, and it refines that view by specifying the best initial and final velocity variations.

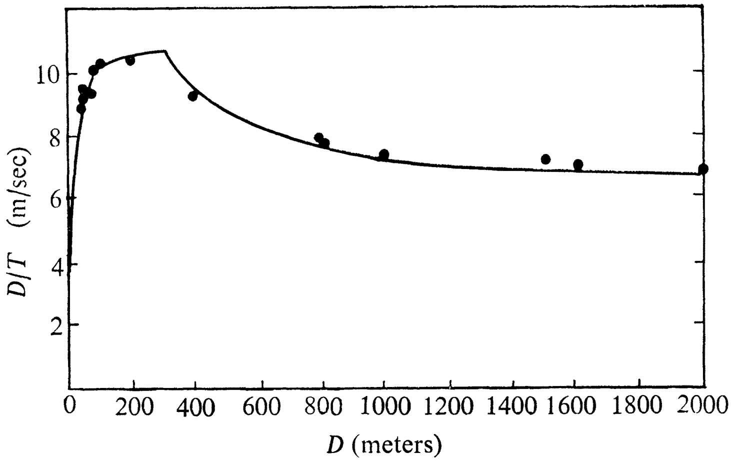

FIG. 1. The average speed D/T for a race of distance D. Points represent the world records and the curve is based upon the theory. For $D <$ 291 meters, D and T are related by (3.4) while for $D >$ 291 meters they are related by (3.17). The constants are those determined by least squares.

A simple energy balance theory to explain the world records was proposed by A. V. Hill in 1928, (see Lloyd [**2**]). His theory is simply that $v(t)$ is constant, and its value is determined by setting $E(T) = 0$. We shall show now that for $D \gg D_c$, our theory yields the same result for D/T as his does. In this case we can replace t_2 by T and t_1 by 0 with an error that tends to zero as D tends to infinity, and we can neglect the kinetic energy term $F^2\tau^2(t_1/\tau + e^{-t_1/\tau} - 1)$ compared to the other terms in (3.14). Then (3.14) becomes $E_0 + \sigma T = v^2(t_1)T/t$, which is just Hill's equation for the case of our resistive force $-v/\tau$. Solving this equation for $v(t_1)$ and noting that $v(t_1) = D/T$ when the end intervals are neglected, we obtain

$$(4.1) \qquad \frac{D}{T} \sim \left(\frac{\tau E_0}{T} + \sigma\tau\right)^{\frac{1}{2}}, \qquad D \gg D_c.$$

This is just Hill's result when the power usage is v^2/τ, which it is in our case, (see Lloyd [2], eq. (5)). From (4.1) we see that the square of the average speed is a linear function of $1/T$. On the other hand, for the short sprints, which are not covered by Hill's theory, T/τ is large so we can omit $e^{-T/\tau}$ from (3.4). Then (3.4) yields D as a linear function of T, which is in fair agreement with the data.

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References

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AN ELEMENTARY PROOF OF KOLMOGOROV'S THEOREM

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Recently in this MONTHLY I. J. Schoenberg [2] has written a detailed and elementary account of the Landau problem concerning sharp inequalities between the supremum norms of derivatives. Professor Schoenberg there suggests a method of proof which offers many new insights; in particular, he presents an elegant discussion of the uniqueness of the extremizing functions and an extension of the problem to complex valued functions. Consistent with his purpose of presenting material accessible to, say, a calculus student, Professor Schoenberg proves only the theorems involving low order derivatives; the method he has developed, though completely general, requires appropriate differentiation formulae and, when high order derivatives are involved, these formulae become quite complicated.

The present paper is written in the same spirit as Professor Schoenberg's and indeed can be considered as an appendix to his paper. Our concern is the Landau problem on $(-\infty, \infty)$, which was first completely solved by Kolmogorov. We prove the theorem for all values of n in an elementary way, using only Rolle's theorem and the Leibniz formula for differentiation of a product. The approach is first to prove the result for functions having an integral period; the general result follows from the special case by an elementary approximation argument.

1. The Euler spline functions and statement of Kolmogorov's theorem. In this section we review briefly the background of our problem as it is developed in Professor Schoenberg's paper. Throughout we use the notations set forth in his paper, and for details concerning this section the reader should consult the earlier paper.