EM Notes

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Contents

1 Manifesto

The bulk of the notes are in the section - memorable problems. These are meant to motivate the reader and me to read forward in the notes. It's kind of like a "Oh cool! I wonder how you would do that?" I don't give you the answers, I give you the tools. Of course, you could also read the solution of the memorable problems but it wouldn't do you much good!

These notes assume you understand basic calculus, and that you have a good grasp of mechanics (just the basic stuff).

2 Novelties

"I am not saying that it's right. I'm just saying that it seems cool" - Me.

2.1 Newton's 3rd law proof

Quite a heuristic, hand-wavy proof.

Let us first consider a single mass m without any external forces. Clearly, the net acceleration must be zero. Now consider splitting the mass into two masses, m_a and m_b.

Now consider the system of two particles, a and b, with masses m_a and m_b. Let particle a exert a force F_1 on b and F_2 from b on a. By Newton's second law, $F_{net} = ma_{CM}$, so at an arbitrary time Δt later, the center of mass of the system could have a very large velocity. However, since the external force on the system is zero, the acceleration of the center of mass of the system is also zero. Thus, $F_1 = F_2$.

2.2 Kepler's 3rd Law Proof

We have that from Kepler's second law: $\frac{dA}{dt} = \frac{L}{2m}$.

Thus, the total period is $T = \frac{A}{L/2m}$. Since the area of an ellipse is πab, we have:

$$T = \frac{\pi ab}{L/2m}$$

The total angular momentum of the system can be determined as follows: Consider the object when it is at a distance b from the center of the ellipse. Then, the total angular momentum is:

$$L = mbv_{\perp}$$

The velocity can be found from the energy equation (at this point, the focus is a distance a from the object):

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = \frac{GMm}{2a} \implies v = \sqrt{\frac{GM}{a}}$$

Note that this velocity is perpendicular to b.

Plugging this in:

$$T = \frac{\pi ab}{(mb\sqrt{\frac{GM}{a}})/2m} = 2\pi\sqrt{\frac{a^3}{GM}}$$

3 Electric Fields

3.1 Memorable Problems

1. A charge q is placed at the corner of a cube. What is the flux through the cube?

2. A dipole has its center at the origin, with the xy plane cutting through it. What is the net flux through the xy plane?

3. An uniform electric field $\vec{E}$ passes through a hemisphere with radius R and its bottom is cut off. What is the net flux through the hemisphere?

4*. Two solid spheres, each of radius R and carrying uniform volume charge densities ρ and $-\rho$, respectively, are placed so that they partially overlap. Their centers are separated by a distance d. Find the electric field within the overlap and show that it is constant.

Solution 4: Don't look at it if you haven't tried it yet, it's a good problem. The diagram (credit to Griffiths) is here: You may think that we would have to find the

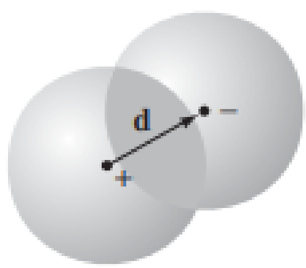


Figure 1: An Amazing diagram

electric field from the oddly shaped partial sphere (with a small section cut out), but realize that we really don't have to! Instead, we can imagine the intersection portion to be filled with both positive and negative charges, then the electric field within the intersection part is simply the superposition of the $\vec{E_+}$ and $\vec{E_-}$ from the left and right spheres.

We're 80% of the way done now. Let us analyze what occurs when we take an arbitrary point r_1 away from the left sphere's center and r_2 away from the negative sphere's center. Note that the vector addition of the two forces its $\vec{E}_{net}$ to lie on the line connecting the two sphere's center. Thus,

$$\vec{E}_{net} = \frac{\rho r}{3\epsilon_0} + \frac{\rho(d-r)}{3\epsilon_0} = \frac{\rho d}{3\epsilon_0}$$

where r is the distance from the positive sphere. Thus, the answer is

$$E = \frac{\rho d}{3\epsilon_0}$$

3.2 Gauss's Law

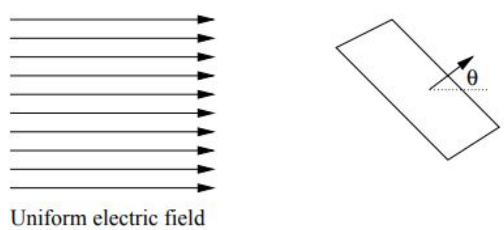
Be sure to exploit symmetry and use the following facts:

3.2.1 Flux

Flux is defined as $\Phi = \vec{v} \cdot dA$ for fluids. In electric fields, it is defined as

$$\Phi = \vec{E} \cdot dA$$

In integral form, it is $\int = \vec{E} \cdot dA$. A proof of this is by imagining any random closed surface and equating the flux through it by adding the flux from an imaginary sphere within it and the flux through it, more of it is explained on page 8 here: MIT Notes



Qualitatively, the flux is proportional to the number of field lines that pass through a given surface.

Another interesting note to make is that

$$\int \Phi \vec{E} \cdot dA = q_{enclosed}/\epsilon_0$$

Thus, you can choose any arbitrary surface to enclose a charge q and it would still have the same flux through it.

1. Electric flux through any closed object is zero

2. A charge $+q$ will induce a charge $-q$ on a conductor (not sure about this one)

3. A spherical conductor has uniform charge on its outside, and no charge within it

Though for (2), we know that since charges affect charge distributions, it is important for test charges to be small enough when finding electric field.

Corona Discharge For more information, see this. The proof in most textbooks, as noted in the above article is (usually) wrong for generalization, but correct for small cases. Here is the proof from HRK Vol.2:

Proof: Let there be two spherical conductors connected by a fine, long wire. Let them be raised to some potential V. Since we are assuming it is static, we have $V = \frac{q_1}{R_1} = \frac{q_2}{r_2}$. Note we divided coulombs constant on both sides. For charge density we have $\frac{\sigma_1}{\sigma_2} = \frac{q_1/4\pi R_1^2}{q_2/r\pi R_2^2} = \frac{R_2}{R_1}$. We now claim that if we let $R_1 \ll R_2$, then $\sigma_1 \to \infty$. Since $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ (gauss's law). Letting $q = 4\pi r^2\sigma$, we have $E = \frac{\sigma}{\epsilon}$, so $E \to \infty$ as well

Remark: For lightning rods, the electric field is greatest at the tip. This means that electrons (from lightning) will experience a very large force from the tip and thus, are more likely to strike the tip than other areas (the tip of the rod is charged by induction of the lighting).

We are commonly told that electric field lines *must* begin on positive charges and end on negative charges. However, note that since $E \propto q$, then a charge of $+2q$ having $2k$ field lines means that a charge of $-q$ will have k field lines going into it. Thus, k field lines will be extending to infinity!

3.2.2 Proof of Gauss's Law

Consider a small cube of length a and a electric field line (which is linear and has a constant magnitude for this small dimension) that goes through it. The flux through the cube is clearly zero, since the two faces the field line goes through both have the same angle $\vec{A}$ with $\vec{E}$ and opposite signs. This is true for any two faces of the cube. Furthermore, this also means that $q = 0$ within the cube, since it would clearly distrot the electric field to make it nonuniform, even in the small dimension of the cube.

Now, extending it to any object, we simply cut the object into many different 'cubes', and thus,

$$\sum \Phi = \Phi_1 + \Phi_2 + \dots \Phi_n = 0$$

Here begins the interesting part (credit to [2]). If we have any arbitrary closed solid, holding a net charge q within it, then what is the flux? We can simply carve out several sphere out of the object, such that the charge in all the spheres combined is equal to the net charge q. Thus:

$$\Phi_{solid} = \Phi_{sphere} + \Phi_{solidwithcavity}$$

For the sphere, $\sum \Phi = A \sum E = \sum \frac{q_i}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{\sum q_i}{\epsilon_0} = q/\epsilon_0$. The solid with spheres cut out, on the other hand, has no net charge within it and by our first proof, consequently has $\Phi = 0$. Therefore,

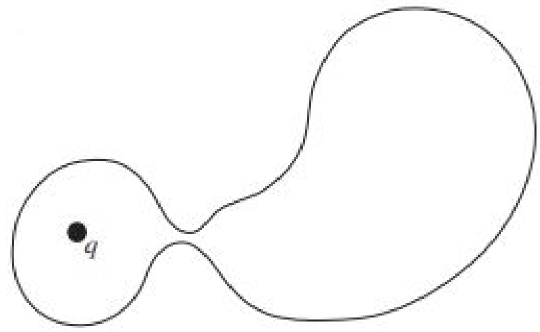
$$\Phi_{solid} = \frac{q}{\epsilon_0}$$

Note that this proof assumed that superposition holds for the flux of an object - that it is the independent sums of the fluxes through several pieces of an object. This is due to the property of dot products, that:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

Where since both terms at the end are scalar quantities, we do scalar addition.

Another way to prove that surfaces with no charge have no net flux is by considering the following (credit to Purcell and Morin):



Clearly the net flux through the entire figure is q/ϵ_0. We have the equation, with Φ_L being the flux through the left lobe and Φ_R being the flux through the right lobe.

$$\Phi_{net} = \Phi_L + \Phi_R$$

In the limit of the gap being infinitesimally small, we find that:

$$\Phi_L = \Phi_{net}$$

This is because $\int E \cdot dA$ for the left lobe becomes nearly equal to q/ϵ_0 when the small opening to one side becomes closed.

3.3 Potentials

3.3.1 Memorable Problems

1. A small conducting sphere is lowered into a cylindrical conductor without touching it. Which of the following is conserved? (A) Potential of cylinder (B) Potential of sphere (C) Net charge of cylinder (D) Net charge of Sphere

2. What if it did touch the cylinder?

3.

3.3.2 Potential due to Conductor

Theorem 1: A conductor will quickly arrange its charge to lie on the outside of its surface to be static

This is done when the charge is all on the outside of the conductor. We prove this by noting that the electric field **must** be zero within the shell. If it is nonzero within the shell and nonzero at outside the shell, there will be a net force on the particles. On the other hand, if it is zero on the outside of the shell, then using a gaussian surface, we find

$$\int E \cdot dA = Q/\epsilon_0 = 0$$

Since $dA \neq 0$, $E = 0$, and then the RHS $= 0$, too. Thus, the charge must all distribute to the edge of the conductor (only for spheres is the charge uniformly distributed)

3.3.3 MC Answers (could be wrong)

1. B
2. A
3. D
4. A
5. Q, [R,P]
6. B, E
7. B, C
8. B
9. A, B
10. A, C
11. B
12. B
13. D
14. D (?) - I feel like this is the electrostatic uniqueness theorem, which would imply C?
15. C, C (?)

3.4 Scaling Arguments

Problem (Morin 5.12): A cube has uniform mass density. What is the ratio of gravitational potential energy of a mass at the corner to the center of the cube?

Hint: Split the cube in such a way that it is symmetric. Also look at the section title.

Solution: We split the cube into eight congruent cubes, each with half the sidelength of the original. Let V be the potential energy at the corner of the smaller cubes, so the center has $PE = 8V$.

Claim: For any arbitrary shape, with a total mass M, there is a unique mass also with mass M that can be placed in the space such that the potential is the same.

The proof should be somewhat trivial: when the mass M is very close to the potential of interest, $U \to \infty$, but when M is very far away, $U = 0$.

Claim: The potential energy scales with k^2, where k is the factor it is scaled by.

The potential energy from a small piece of mass dm is:

$$U = -\int_0^M \frac{Gdmm'}{d} = U(M)$$

If the dimensions are scaled up by a factor of k, then:

- The mass is $\times k^3$
- The distance from each small mass dm to m is $\times k$

Note that scaling means the superposition of these two. If we first scale up the mass, then $U' = Uk^3$. Next, scaling the distance: $U'' = Uk^2$, since $dU \propto \frac{1}{d}$.

This means that if the potential energy is V for a small cube, then it is $2^2V = 4V$ for the corner of a large cube.

Problem (200 Estonia Physics problems): A homogeneously charged cone with height h has a potential of ψ_0 at its tip. If the cone's tip of height h is cut off, and displaced to ∞, what is the new potential at the tip of the cone?

Challenge: What if the cone was not homogeneously charged (e.g. $\rho \propto 1/r$)?

3.5 Some Shortcuts

Two charges, q_1 and q_2 are separated by a distance d apart, where $d \neq 0$. Where in space is the electric force zero? The potential zero?

There are two cases: The point is outside the two charges or that it is between the two charges.

Case1: Between the two charges

Potentials:

$$V_1 + V_2 = 0 \implies \frac{q_1}{r} + \frac{q_2}{d-r} = 0 \implies r = \frac{q_1 d}{q_1 - q_2}$$

Clearly, since we assumed that the potential is zero in between the charges, we have the inequality: $0 < r < d$. Thus,

$$0 < \frac{q_1}{q_1 - q_2} < 1 \implies q_1 < 0 q_2 > 0 or q_1 > 0 q_2 < 0$$

In other words, q_1 and q_2 must have opposite signs.

For the other scenario where $V = 0$ outside of the two charges, we have the same situation:

$$0 < \frac{q_2}{q_1 + q_2} < d$$

Solving them, we find that q_1 and q_2 must have opposite signs.

Note that while we derived all of these from scratch (or at least most of them), it should be pretty intuitive. If they have the same sign, then we are pretty must saying

$$k_1 q_1 + k_2 q_2 = 0$$

Which clearly can't be satisfied (A linear combination of two positive numbers can't be zero, if the constants are positive, too).

E-fields: Instead of deriving them, we will consider intuitively, whether there exists a point in space where the electric field is zero:

Same Signs If two charges have the same sign, then clearly at regions outside of the two charges, $E \neq 0$. Thus, $E = 0$ only along the line that connects the two charges together.

We give a continuity argument here: let the charge on the left be q_1 and the charge on the right be q_2. WLOG, let $q_1, q_2 > 0$. Then at regions close to q_1, but between the two charges E_{net} points to the right. But at regions close to q_2, but between the charges, E_{net} points to the left. Thus, since $E(r)$ is continuous ($E = \frac{q_1}{r^2} + \frac{q_2}{(d-r)^2}$, then there must exist a point in space between the two charges where $E_{net} = 0$.

Different Signs Two charges, q and $-kq$ ($k > 1$ lie on the x-axis, with q at $x = 0$ and $-kq$ at $x = d$. Clearly, $E \neq 0$ in the region between the charges, since it will always have a vector component along the x-axis pointing towards the positive charge. We again give a continuity argument: near the positive charge q at $x < 0$, the net E points to the left. But at distances $r \gg d$, it appears as though the two charges are a single charge of $(1-k)q < 0$. Thus, the electric field would point towards the right. There must be a point in between these two extremes where $E = 0$, then.

Note: If $k = 1$, then $E \neq 0$ anywhere in space. At a horizontal distance x_0 and vertical distance y_0 from the $+q$ charge (at $x_0 < 0$), we have that the electric field is:

$$E = \frac{q}{x_0^2 + y_0^2} - \frac{q}{(x_0 + d)^2 + y_0^2} = 0 \implies (x+d)^2 = x^2 \implies d = 0$$

Not that another solution was $x = -d/2$, but as noted above $E \neq 0$ in $0 < x < d$.

3.6 Force Due to Charges

Problem: Consider a spherical conductor with a charge density of ρ. What is the pressure on the spherical shell due to the charges on the conductor? **Hint**: Consider a small area element dA.

Hand-wavy Solution: Consider a small area element dA.

To the left, $E = 0$, whereas to the right, $E = \rho/\epsilon_0$. So which electric field do we even use? To figure this out, we let the electric field at dx be E_1. Note that $0 < E_1 < \frac{\rho}{\epsilon_0}$ due to Gauss's law (draw a spherical Gaussian surface that goes through this small area element - you will find that the net charge within it is less than the total charge of the sphere, but not zero).

Now we claim that if we draw the Gaussian surface such that it divides the element to $dx/2$ on both sides, the electric field through that surface is congruent to the electric field felt by the small area element. This is because, if we assume the charge is uniformly distributed through the sphere, then $\rho A dx \cdot E_1$ is the force on the center of mass. By

Gauss's law,

$$E_1 4\pi(r - dx/2)^2 = \rho(4\pi r^2)/2 \implies E_1 \approx \frac{\rho}{2\epsilon_0}$$

Note that we neglected the dx term, since $r \gg dx$.

The rest is easy now:

$$\frac{dF}{dA} = \frac{E_1(\rho dx dA)}{dA} = E_1 \rho dx$$

4 Properties of Materials

At very low temperatures, materials can exhibit *superconductivity* where $R = 0$. **Conductors:** Materials that tend to conduct electricity and nearly every atom contributes. Net charge is always at the outside, as any nonzero $\vec{E}$ within it will cause free electrons to move around. **Semiconductors:** Have some (1 in $10^{1}2$) atoms that are conductive. **Insulators:** These are more interesting. They typically do not conduct electricity, but when a sufficiently strong electric field acts on them, they can *breakdown* (acts as a conductor) or have dipoles within them, resulting in an electric field.

4.1 Circuits

4.1.1 Memorable Problems

Problem 1

14. A resistor is in the shape of a truncated right circular cone (Fig. 29-22). The end radii are a and b, and the length is L. If the taper is small, we may assume that the current density is uniform across any cross section. (a) Calculate the resistance of this object. (b) Show that your answer reduces to $\rho L/A$ for the special case of zero taper ($a = b$).

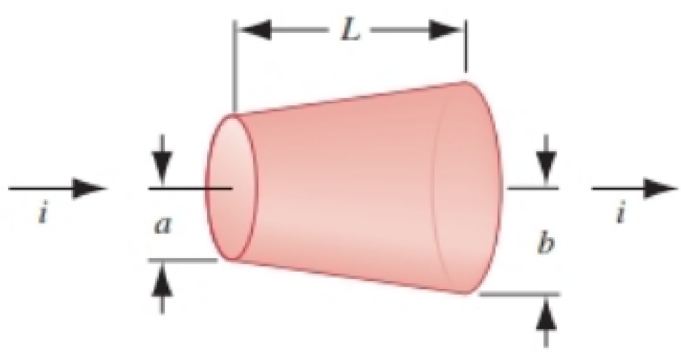


FIGURE 29-22. Problem 14.

Solution: This involves a good amount of calculus. We begin by noting that

$$j = \frac{i}{(\pi(a + (b-a)x/L)^2}$$

We get this from noting that the slanted part of the frustum can be modelled as a line with slope $\frac{b-a}{L}$, so at a distance x from the left side, parallel to the slanted side, we have $r(x) = a + (b-a)x/L$.

Now note that $E = jp$ and $\Delta V = \int E \cdot dx$. Thus,

$$\frac{ip}{\pi}\int_0^L \frac{1}{((a + (b-a)x/L)^2}dx$$

Now we do a u subsitution, letting $u = a + (b-a)x/L$, so $du = (b-a)dx/L$. Note that we have also changed the bounds, since when $x = L, 0$ respectively, $u = b, a$. Thus, we have:

$$\Delta V = \frac{ip}{\pi}\int_a^b \frac{1}{u^2}du$$

$$\Delta V = \frac{ip}{\pi}[-\frac{1}{u}]_a^b$$

$$\Delta V = \frac{ip}{\pi}[\frac{1}{a} - \frac{1}{b}]$$

If we simplified this by having a common denominator we would get our final answer of

$$\frac{Lip}{\pi ab}$$

Problem 2 A resistor is in the shape of a spherical shell, with an inside surface of radius of a covered with a conducting material and an outside surface of radius b covered with a conducting material. Assuming a uniform resistivity p, calculate the resistance between the conducting surfaces.

$$j(r) = \frac{i}{4\pi r^2} \to E(r) = \frac{ip}{4\pi r^2} \to \Delta V = \int_a^b$$

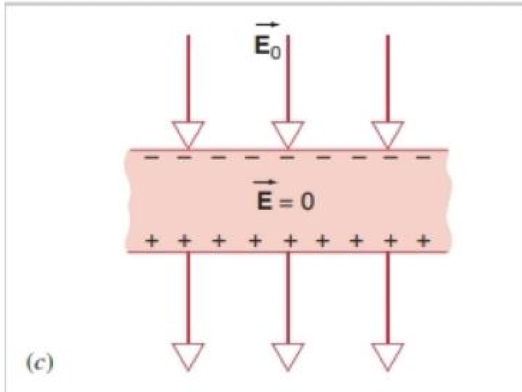
4.1.2 Currents

An electric field E_0 on a conductor will result in an electric field E' from the conductor that points in the opposite direction. Since the inside of the conductor has $E = 0$, then $E' = E_0$.

If we connected a wire from the top of the conductor to the bottom, then we would effectively have a circuit. We define a few quantities:

$$I = \frac{dq}{dt} \qquad \vec{J} = I/\vec{A} \qquad I = \int \vec{J} \cdot d\vec{A}$$

Note that J is known as the *current density* and is defined as if the current was the movement of positive charges (vector direction is opposite to the movement of electrons)



Electrons have irregular motion and are pushed by $\vec{E}$ while also colliding with the lattice (conductor). However, there is no net acceleration (I'll have to see a proof of this later) and the net effect is that the energy goes to the lattice (conductor). Thus, the conductor has an increase in temperature. Overall, we say that electrons have a drift velocity of v_d

Drift Speed and current density By definition, $J = \frac{I}{A} = \frac{q}{At}$. Since $q = \rho v_d At$, where ρ is the charge density (assumed uniform), we have

$$J = \frac{q}{At} = \frac{\rho v_d At}{At} = \rho v_d$$

The drift velocity for electrons is extremely slow ($\approx 14cm/h$), so how does electricity go so fast? See this video

Note: The current is constant throughout the circuit (which we used in our proof that $I = q/t$). Why? If you model the flow of electrons as fluid flow, then you will notice that it is continuous. Thus, an equal volume of electrons pass through a certain area in any second. Of course, this will only hold true for simple circuits.

4.1.3 Resistance

$$J = \sigma \vec{E}$$

σ is the conductivity, which is measured in Siemens per unit (S/m)

This makes sense, since in between collisions w/conductors, electrons are being accelerated by the electric field, so $v_d \propto \vec{E}$ seems legit.

$p = 1/\sigma$, where p is the resistivity and has units of ohms $\cdot$ meters. Thus, $\vec{E} = p\vec{J}$. For some materials, p is a constant, and thus $p \propto E^0$. These materials are known as **ohmic** materials.

Ohm's law: The resistivity of a material is independent of the direction and magnitude of the electric field

Note: For a very large E, all materials will diolate Ohm's law, so it is valid only for a certain range

Now we continue with our derivation for resistance. We have:

$$E = p\vec{J}$$

$$p = E/\vec{J} = \frac{\Delta V/L}{I/A} = R\frac{A}{L}$$

Rewriting, we have

$$R = p\frac{L}{A}$$

Thus, the resistance for *ohmic* materials is a constant and only depends on the resistivity, not the current or the voltage. You may have wondered if this would be a violation of $V = IR$, since shouldn't the resistance increase with an increase of voltage? Note that, this law is only true if:

- The resistor is a steady state (same constant current throughout), which occurs in only very short time intervals

Also note that semiconductor devices are typically non-ohmic and thus have an increase or change in resistance, thus the V vs I plot is not linear with slope of $p\frac{L}{A}$ (which ohmic devices would).

A major misconception is that ohm's law is $V = IR$. Google will also tell you this, but it's is **not** true. Ohm's law tells us that $V \propto I$, or that resistance is a constant. However, as illustrated in this article, it is not the case.

Also note that for ohmic devices, we have

$$I = \frac{\Delta V}{R} = \frac{\Delta V}{\rho x/A} = \sigma A\frac{\Delta V}{\Delta x} = -\sigma A\frac{dV}{dx}$$

The negative sign indicates that positive charge flow from high to low potentials (because electric field lines point from high to low)

This is highly analogous to the equation for heat transfer, given by

$$\frac{dQ}{dt} = -kA\frac{dT}{dt}$$

The constant k is thermal conductivity. As noted in HRK, in pure metals, heat transfer is carried out by electrons within the material. Thus, it makes intuitive sense why these formulas are so similar.

Resistivity grows approximately linear with temperature for temperature ranges from -200 to 600 degrees Celsius. The exact formula is $p - p_0 = p_0\alpha_{avg}(T - T_0)$. For larger temperatures, more terms of $(T - T_0)^2, \quad (T - T_0)^3$ will need to be added.

Ohm's law states that resistivity or consequently, resistance is constant for Ohmic materials. However, this doesn't mean that non-Ohmic materials can't have their resistivity changed under certain conditions. Resistivity comes from collisions with the ionic cores, that are

- Increasing the temperature, thus increasing the vibration of the cores and the probability that the electrons will hit the cores
- Adding amounts of impurities (this is also known as doping for semiconductors, which usually increases conductivity)
- straining it severely and thus increasing lattice imperfections

To analyze the action of electrons further using kinetic gas theory, we call electrons a 'gas'. For this gas, we have the free mean path λ and the average time between collisions τ. The free mean path is the average path distance an electron travels before it hits a cation core and the time τ is the average time it takes an electron to hit one. Note that τ can only be regarded as a constant if $\vec{E}$ is small enough.

Let the velocity gained after a collision with an ionic core be v_r, with r to emphasize that the velocity's direction is random. Now assume that there is a nonzero electric field $\vec{E}$. So t seconds after a collision, $Mv = \frac{1}{N}\sum(Kv_r + e\vec{E}t)$. Note that we are adding these like vector quantities. Since v_r is random, as noted before, its vectoral sum will be 0. Thus, we have $v_{avg} = \frac{eEt_{avg}}{M}$. This v_{avg} is essentially the drift velocity.

Note: You may have asked a very important question as we were going through this: if acceleration due to the $\vec{E}$ is constant, then wouldn't the average velocity of a single electron over a time period of t_{avg} be $\frac{eE}{2t_{avg}}$, since acceleration is uniform?

Now we continue by attempting to solve for p, the resistivity by relating it to τ or t_{avg}. We have $p = \frac{E}{j}$, so forcing p into our equation, we have that $p = \frac{m}{\tau n e^2}$, where n is the numerical density. We can also substitute $ne = \rho$ for charge density.

Notice that the equation has a dependence on τ, so you may think that this is a contradiction to our assumption of ohmic materials (we did our nearly all of our derivations with this assumption). However, notice that $\tau = \frac{\lambda}{v_{avg}}$. λ is independent of the electric field. The average *speed* of an electron is very high, so assuming a sufficiently small enough electric field, v_{avg} is independent of $\vec{E}$. Note that we were careful to specify that Ohm's law doesn't hold for a very large $\vec{E}$.

4.1.4 Insulators in Electric fields

When $\vec{E}$ is applied, the dipoles align themselves to the electric field in such a way that it weakens the electric field within the insulator.

The greater the electric field, the more aligned the dipoles are within an insulator and thus the greater the electric field from the insulators. For some insulators, called linear materials, their polarization electric field is proportional to the external one. Thus, they satisfy

$$\vec{E}_{ind} = \frac{1}{k_e}\vec{E}_{ext}$$

where k_e is the dielectric constant, being $k > 1$ for insulators. For some reason, $k_e = \infty$ for conductors. But how does that make any sense? If $k_e = \infty$ for a conductor then wouldn't that mean the polarization field would be $E_{conductor} \approx 0$? But at the same time we know that $E_{conductor} = E_{external}$ for within a conductor...

The below explanation is likely(?) wrong: The answer to this is simple, the E actually stands for the net electric field within the insulator. SO the larger k_e, the smaller the E field is. So it does make sense that $k_e = \infty$ for conductors. And for vacuums, there is no resistance so it does make sense that $E = E_0$.

Induced Dipole Moments

We begin by modelling atoms as a cloud of negative charge along with a positively charged nucleus. In the absence of $\vec{E}$, the centers of the positive charge and the negative charges coincide. However, When a strong electric field is applied to the atom, it pushes the positive nucleus in the direction of $\vec{E}$, and pulls the electron cloud towards the source of $\vec{E}$.

This is called an induced dipole, since one end is partially positive and the other is partially negative.

4.1.5 Snell's Law proof through Fermat's least time principle

Fermat's principle of least time is frequently used in optics problems. For a proof of it, see this Snell's law

4.1.6 Induced Charge

I took these notes from Griffith's section 2.5.2. Let there be some conductor in any arbitrary shape. Now we place a charge of $+q$ within a cavity of the conductor, then let's take a look at the electric field at:

- Within the cavity
- Inside the conductor but outside the cavity
- Outside the conductor

To begin, let's consider how the presence of the charge $+q$ affects the charge distribution of the spherical conductor

If we drew a Gaussian surface just within the cavity, then we would have

$$\int \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Thus, $\vec{E}$ is not zero and seems to behave as though the conductor does not exist at all!

4.1.7 Constant current

The current before entering a resistor is I_i, and when it exits it is I_o. So how do I_i and I_o compare? Well, there are a few cases, $I_i > I_0$, $I_i < I_0$, and $I_0 = I_i$. Let us consider the first two. In this case, there would either be a net increase in charge or net decrease in charge if we took a section of circuit including the resistor's two ends. This violates conservation charge, since it is being spontaneously created or destroyed within the resistor.

Thus, only the third case follows the law of conservation of charge.

4.1.8 MC Answers

1. B,D
2. C
3. C Within the cavity, the electric field due to the conductor is 0, since a shell has $E = 0$ within it. Similarly, the $+q$ charge's E field outside of the conductor is offset by the induced $-q$ charge along the edge of the conductor.
4. B
5. E,E,C ($E\sigma = j = \frac{I}{A}$, so as A increases, E decreases)
6. B (current is direction of positive charge, and positive charges go from high to low potential)
7. C,B,C
8. B (If it is ohmic, then V vs I is a straight line, with a slope of R)
9. D (ohmic means p is independent of $\vec{E}$, so $R = p\frac{L}{A}$ means R is also independent of $\vec{E}$
10. C,B
11. C (continuity equation - I is constant, so $I/A = v_d ne$ means v_d is too)
12. C (more vibrations, so cation cores will vibrate at larger amplitude, increasing likelihood electrons will bump into them, so it will take less time and thus more resistivity)
13. A,C

4.1.9 Summary

A summary not listed in any particular order.

- $j = \frac{dI}{dA}$ and current is constant throughout resistor due to continuity
- Current is typically denoted as the direction of positive charge and is negative when regarding negative charges (electrons), but electrons are the ones doing the real moving.
- Conductors will always arrange charges in such a way that $E = 0$ within it. The leftover charge is on the surface, and will be distributed symmetrically if the conductor is a sphere
- $V = IR$ is commonly referred as Ohm's law, but it really states that p is independent of E. Materials that follow this are ohmic, though a large enough E field will violate the law.
- The resistivity grows linearly as temperature increases. Though this only holds true for a certain range of temperatures.
- For electrons to flow there must be an electric field to induce a voltage. This commonly comes from a battery.
- $V = \int \vec{E} \cdot d\vec{dA}$, $I = \frac{dq}{dt}$
- When temperature increases, the mean collision time, τ decreases.
- Electrons can be modelled as a gas with a constant drift velocity v_d, which is much lower than their average *speed*.

5 Capacitance

Capacitors consist of two 'plates' that are of equal and opposite charge. These plates need not be rectangular and can be of any geometry.

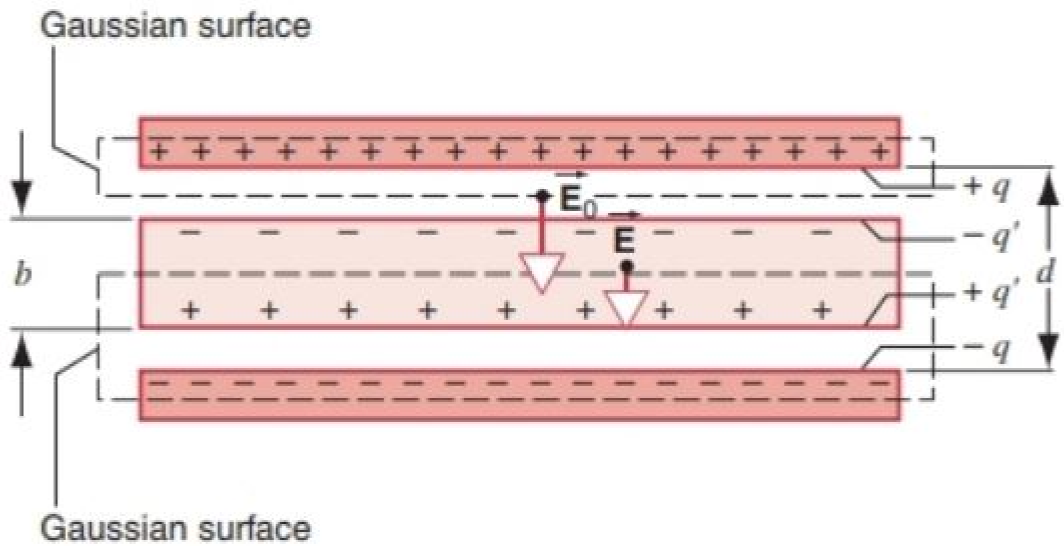
When we connect a battery to two metal plates, the positive terminal of the battery makes that plate positive and vice versa. In electrostatic situations, since both the wire and the plates are conductors, V_+ is the same for both the wire and the capacitor. The same goes for V_-. $\Delta V = V_+ - V_-$. Through experiments, we find:

$$q = C\Delta V$$

C is the capacitance and is measured in farads. Later in my notes I will (hopefully) draw a sound analogy between fluid flow and electric current in circuits. I fear that it will only go so far to only some circuit parts, though.

In the fluid flow analogy, C is essentially the volume and ΔV is something like the pressure, and q is the amount of gas. Thus, the more pressure and volume something has, the greater amount of 'gas' we can fill it with. Not bad, eh?

5.1 Memorable Problems



1. A)*Find the electric field between the plate and the dielectric [1]. B) Find the net $\vec{E}$ within the dielectric. C) What is the potential ΔV of the capacitor?

2. By neglecting the fringe effects of a parallel plate capacitor, are we over- or under-estimating the capacitance?

3. Why does a neutral capacitor, now connected to a battery, must have opposite plated charges of $+q$ and $-q$? Do plate sizes matter?

4. ** A battery far away from A and B maintains a constant voltage ΔV across the infinite grid. Find the net charge on any junction. Show that the potential at any junction is the average of the potential at the four nearest junctions[2].

---

[1] Isn't the electric field uniform? So then we can use this to deduce: $\int E \cdot dA = EA = q/\epsilon = 0 \rightarrow E = 0$ for a closed surface enclosing no charge? Note that this is actually incorrect, so don't agree with this logic!

[2] By junction, we mean by the intersection points, such as junction A and junction B

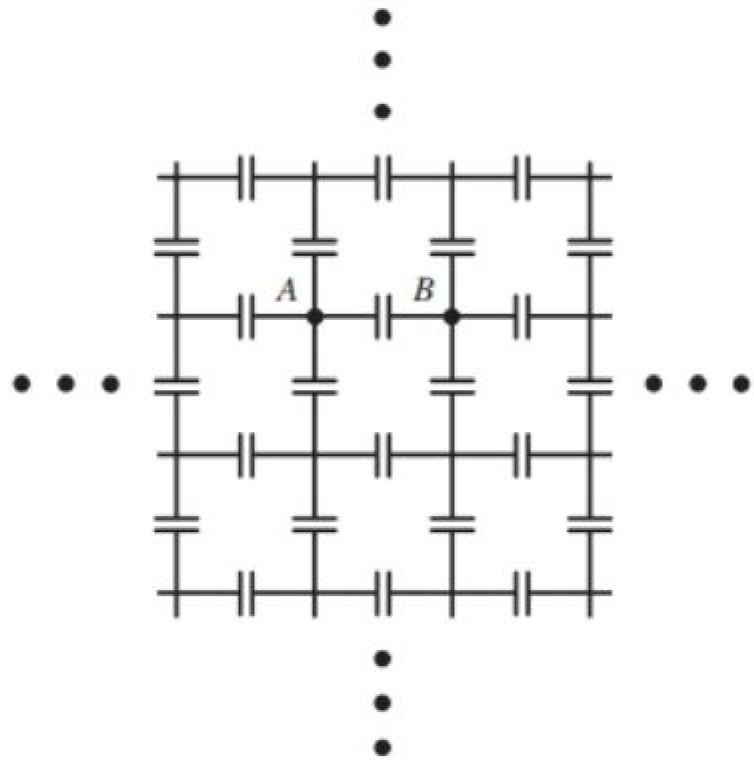


Figure 2: Diagram for problem 4

5.2 Capacitance for some objects

In this part, we will find the capacitance for some common objects. To do so, we will first find a way to determine ΔV. Observe:

$$\Delta V = V_+ - V_- = \int_\infty^+ \vec{E} \cdot d\vec{s} - \int_\infty^+ \vec{E} \cdot d\vec{s}$$

Letting $\int Eds = f(x)$, we have

$$f(+) - f(\infty) - (f(-) - f(\infty))$$

$$= f(+) - f(-) = \int_-^+ \vec{E} \cdot d\vec{s}$$

Note that since our potential was the potential between the two plates, we could have just used our definition for potential instead, but I just wanted to make it a bit more clearer.

Now we will use $\Delta V = \int_-^+ \vec{E} \cdot d\vec{s}$ along with $q = C\Delta V$ to find the capacitance for each of the geometries below. Note that for each, the capacitance is of the form $\epsilon_0 \cdot [L]$, where $[L]$ is some quantity of length.

Infinite plates

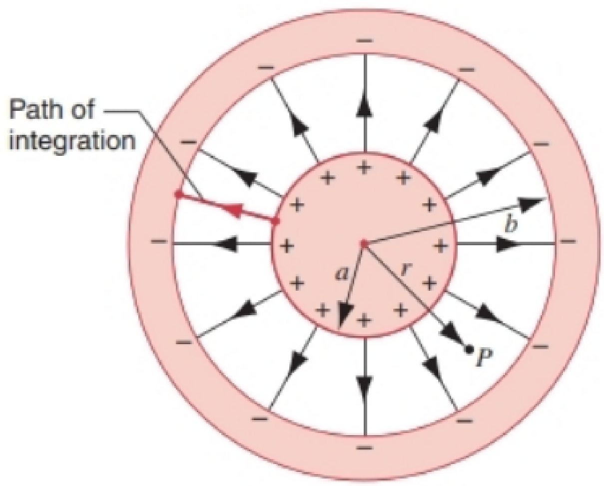
An infinite sheet has a uniform electric field $\vec{E} = \frac{\sigma}{2\epsilon_0}$. Since we have two of them, we then have:

$$\Delta V = \frac{q}{A\epsilon_0}\int_{-}^{+} d\vec{s} = \frac{qd}{A\epsilon_0}$$

Plug and Chug:

$$C = \frac{q}{\Delta V} = \frac{\epsilon_0 A}{d}$$

Spherical Shell (?)



This is simple.

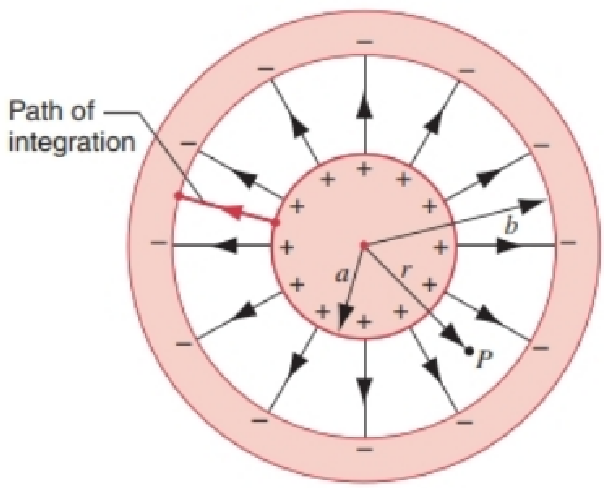
$$\Delta V = \int_a^b \frac{q}{4\pi\epsilon_0 r^2}dr = \frac{q}{4\pi\epsilon_0}[\frac{1}{a} - \frac{1}{b}]$$

Plugging it in, we have

$$C = \frac{4\pi\epsilon_0 ab}{b-a}$$

Note that it wouldn't matter what path we really take (in terms of validity of calculations), since the two faces (conductors) are equipotentials. Though this one is easiest in terms of calculation.

Coaxial Cylinders



Using a Gaussian cylinder, with radius $a < r < b$, we find

$$E2\pi rl = q/\epsilon_0 \rightarrow E = \frac{q}{2\pi\epsilon_0 rl}$$

$$\Delta V = \int_a^b Edr = \frac{q}{2\pi\epsilon_0 l} ln(b/a)$$

Plug and chug, we now have $C = 2\pi\epsilon_0 l/ln(a/b)$

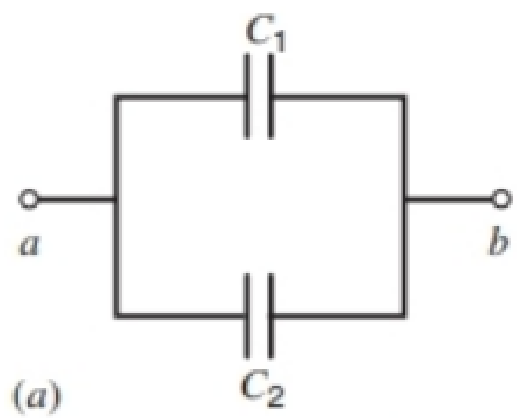
Problem What is the capacitance for a spherical conductor with radius R?

Solution:

5.3 Capacitance in circuits

In parallel

A circuit consists of a battery which imposes a net voltage ΔV_0 across it. Under electrostatic conditions, both capacitors would be "fully charged" and would have a voltage of ΔV_1, where ΔV_1 is the voltage between the end nodes a and b of the capacitors shown below. $\Delta V_1 \leq \Delta V_0$ due to the resistors between the battery and the capacitors.



For these two as an example, we have $q_1 = C_1\Delta V_1$ and $q_2 = C_2\Delta V_1$ The net capacitance is $C_{net}\Delta V_1 = (C_1 + C_2)\Delta V_1 \rightarrow C_{net} = C_1 + C_2$. Thus, $C_{net} = \sum C_n$ for those in parallel.

Another way is to use energy, which is introduced later in here. For the $C_{net} \equiv$ this situation, we would require for both to have the same energy:

$$\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2} = \frac{Q_3^2}{C_3} \rightarrow \Delta V_1(Q_1 + Q_2) = \Delta V_3(Q_3)$$

Since $\Delta V_1 = \Delta V_3$ due to them being the potential difference of points a and b, we then have $Q_1 + Q_2 = Q_3$. The rest of the proof is congruent to the one above.

In Series Two important properties:

- $\Delta V_0 = \Delta V_1 + \Delta V_2 + \ldots \Delta V_n$. This should make sense, as we can see each capacitor's voltage being added one by one to get the final voltage at the other end of the battery

- Each capacitor has a charge q. This is due to induced charges. Try it out!

Now we see:

$$\Delta V_n = q/C_n \to \sum \Delta V_n = V_0 = q/C_{net} \to \sum q/C_n = q/C_{net}$$

$$\sum \frac{1}{C_n} = \frac{1}{C_{net}}$$

Note: You may have wondered why it was q/C_{net} instead of $2q/C_{net}$. After all, the net positive charge is $+2q$, right? Well, consider this from the battery's point of view: It can only 'see' a $+q$ plate and a $-q$ plate, and the part in between is isolated from the circuit. In addition, any charge transfer by the second property listed above, will result in an electrostatic condition where all capacitors have the same charge q', and the battery would not be able to differentiate this from two plates of a single capacitor.

Now we've only been giving an *intuitive* explanation instead of one through facts. Let's verify that an equivalent capacitor conserves energy:

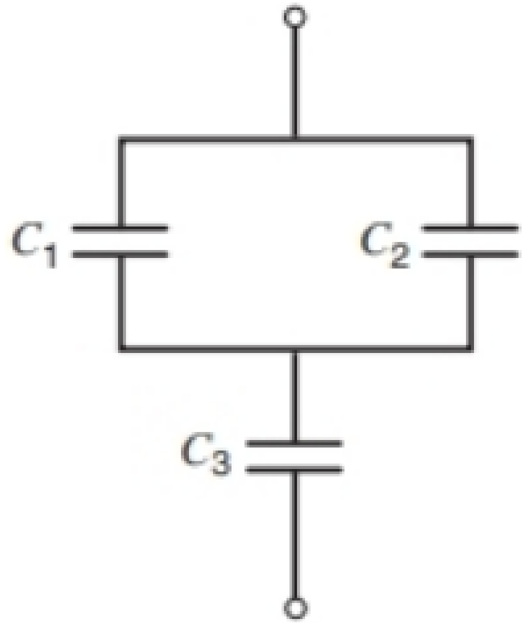
$$U_i = U_f \to \frac{q^2}{C_1} + \frac{q^2}{C_2} + \ldots \frac{q^2}{C_n} = \frac{q'^2}{C_n et} = q^2 (\sum \frac{1}{C_n}$$

Proof: We prove this in the general case by utilizing two equations, energy and voltage which must be the same for both cases:

$$\begin{cases} \frac{Q^2}{C_{net}} = q^2(\sum \frac{1}{C_n} \\ \\ \frac{Q}{C} = q \sum \frac{1}{C_n} \end{cases} \tag{1}$$

Solving, we find that $Q = q$ and $\frac{1}{C_{net}} = \sum \frac{1}{C_n}$ which are the only *unique* solutions to these two.

Exercise: Find the charge on each of the capacitors.



5.4 RC Circuits

Stolen from University Physics, Openstax

For a simple RC circuit consisting of a resistor and a capacitor (we can generalize this to any RC circuit, since it's always possible to combine all resistors to a single one, and the same for capacitors). Since I feel lazy, you should just go to the link above to see the full derivation. Anyway, the main takeaway is:

$$I = I_0 e^{-t/RC}$$

5.5 Stored Energy

Suppose two plates have $+q$ and $-q$ respectively. $\Delta V = q/C$. Suppose we transfer a small charge dq from one plate to another, then $dU = dq\Delta V = dq\frac{q-dq}{C} = dq\frac{q}{C}$, since $(dq)^2 \ll dq$ so it is negligible. Integrating,

$$U = \frac{q^2}{2C} = \frac{1}{2}C(\Delta V)^2$$

For large capacitance parallel plates, when we double the distance, the energy also doubles, so this leads us to conclude that the energy is really stored in the electric field. In the special case of parallel plates, we have that the energy density u is

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}C(\Delta V)^2}{Ad} = \frac{\epsilon_0}{2}E^2$$

In the last step we used our previous result that $C = \frac{\epsilon_0 A}{d}$ and $\frac{\Delta V}{d} = E$ For some reason, this also works for other capacitors in general, too.

Problem: A capacitor on the left begins with a charge of q_0. The circuit is connected and the two capacitors end with electrostatic equilibrium having the same potential. What is the final voltage and what is the final and initial energy? Why are they different?

We will begin working in the case where the capacitors settle down to an electrostatic state. Thus, note by conservation of charge,

$$q_0 = q_1 + q_2 \to C_1 \Delta V_0 = C_1 \Delta V_f + C_2 \Delta V_f \to \Delta V_f = \frac{C_1}{C_1 + C_2} \Delta V_0$$

Now multiplying both sides by $q_1 + q_2 = q_0$, we obtain from the definition $q\Delta V = \Delta U$:

$$U = \frac{C_1}{C_1 + C_2} U_0$$

Note that the confusing part may be C_1, the net capacitance. After all, our derivations of the net capacitance began with the assumption of a battery. But here, it is a capacitor acting as a battery. So are our derivations still correct?

Well note that since the final state *must* end with a static equilibrium, then it will be essentially congruent to the situation we assumed in our derivations.

You may have been confused that we used both $qV = U$ and $q\Delta V = U$ in this. An analogy to this is the change in PE when an object falls from a height h_1 to h_2. In that case we would only care about $h_1 - h_2$. It's all relative! (It's quite similar to the proof that a hanging mass on a spring exhibits SHM - you can simply change the value of x where $PE_{gravity} = 0$).

Seems legit? See this solution now: Note that the potential energy for a conductor is $Q^2/2C$. So for the initial state, the total charge is Q_0 and capacitance is C_1. After electrostatic equilibrium, the total charge is still Q_0 and the net capacitance is $\frac{C_1 C_2}{C_1 + C_2}$. Thus, we have

$$\Delta V_f C_n = \Delta V_0 C_1 \to \Delta V_f = \frac{C_1}{C_n} \Delta V_0$$

Note that this is actually different!

The reason why the second solution is wrong, as you may have guessed, is

$$C_n = \frac{C_1 C_2}{C_1 + C_2}$$

But why? Think about it like this: if we treated the capacitors like they were in series, then $\Delta V_1 + \Delta V_2 = \Delta V_0 \to \Delta V_1 \neq \Delta V_2$, and then $E \neq 0$. For parallel ones, $\Delta V_1 = \Delta V_2 = \Delta V_0 \to E = 0$. Thus, the parallel capacitor model is the correct one.

Problem (HRK) A solid conducting sphere has a radius R and carries a charge q. When you gently pull it to a radius of R', then what is the change in potential energy? Another conducting sphere has a radius of $r < R$ and is placed within it. What should r be for it to hold half the total potential energy?

Solution:

$$U = \frac{q^2}{C} = \frac{q^2}{2\pi\epsilon_0 r}$$

The derivation of C is:

$$\Delta V = -\int E \cdot ds = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} dr = \frac{q}{4\pi\epsilon_0 r} \rightarrow C\Delta V = q \rightarrow C = 4\pi\epsilon_0 r$$

Thus, we only need to plug it in our first formula.

For the second part, there is no electric field within the smaller sphere, so the potential energy inside is equal to the potential energy right outside it.

Question: Let there be a conducting spherical shell with radius of r. Since $E = 0$ within it, then $\Delta V = -\int_a^b E \cdot dr = 0$, where $a, b < r$. But then $q\Delta V = U \rightarrow U = 0$, which we know can't be true! What did we do wrong?

5.6 Dielectrics

Faraday discovered that adding a dielectric in between two capacitor plates would increase the capacitance - if a constant voltage ΔV was maintained:

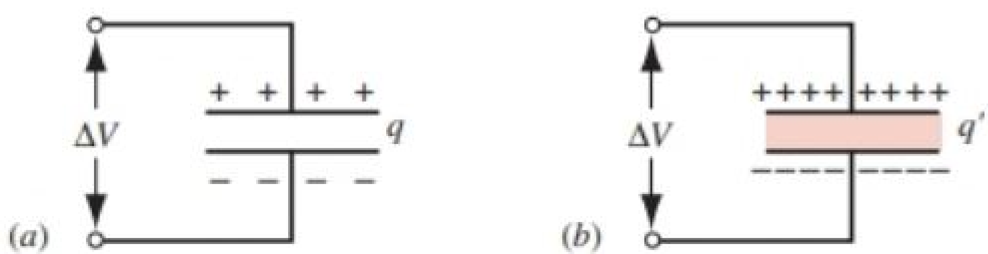


Figure 3: Left: without dielectric, Right: with dielectric

Since $\Delta V = \int_+^- E \cdot d\vec{s}$ and $\vec{E}_{net} = \frac{E_0}{k_c}$, we have that the two electric fields must be equal in both cases. Thus,

$$E = E' \rightarrow ql = q'l/k_c \rightarrow q' = k_c q$$

where l is some constant. And We also have $q = C\Delta V \rightarrow C' = k_c C_0$

If, on the other hand, we went through these steps below then it would be different:

1. Charge both capacitors (two different circuits) to a charge q
2. Remove batteries and insert dielectric into one of them

You may feel that it just became very complicated, but using $C' = k_c C_0$, and $q = C\Delta V$, we find that $\Delta V' = \Delta V/k_c$

It may be tempting to point out that energy conservation is violated since $\Delta V' < \Delta V$ (since $k_c > 1$), but note that adding the dielectric requires doing work against the electric field!

Remark: We defined $\Delta V = \int_+^- E \cdot d\vec{s}$ so the path is going from positive charge side to negative charge side of the capacitor which is in parallel with $\vec{E}$ to make $\Delta V \geq 0$ in any configuration

5.6.1 Using Gauss's Law

For two large plates forming a capacitor, $E = q/A\epsilon_0$, since $E = 0$ at points not between the plates. When we add in a dielectric, we know that $E' = E/k_c$, where E' is the electric field within the capacitor. Furthermore, we have

$$\epsilon_0 \int \vec{E}' \cdot d\vec{A} = q - q' \rightarrow E' = \frac{q - q'}{A\epsilon_0}$$

And now using the relation $E' = E/k_c$, we have

$$E' = \frac{q - q'}{A\epsilon_0} = \frac{q}{A\epsilon_0 k_c} \rightarrow q' = q(1 - \frac{1}{k_c})$$

5.7 MC Answers

1. B
2. C, B
3. A, B
4. C
5. B (fringe E fields are lesser in magnitude, and aren't directed perpendicular to the surface, so a straight line path of $\Delta V = \int E \cdot ds$ means ΔV is overestimated and thus, C is underestimated)
6. B The shell has E fields terminating at a point of ∞, which we can imagine to be another shell with charge $-q$. Exact same problem as sample 30-4
7. B (In the limit where $C_1 = C_2$ for lower bound, and $C_2 \gg C_1$
8. D
9. A,A,D
10. D ($C = \frac{A\epsilon_0}{d} \quad U = \frac{1}{2}C(\Delta V)^2$
11. B $U = \frac{q^2}{2C}$
12. (A,C) (A would require changing the geometry, C is simply altering voltage without any change in geometry)
13. B,A
14. C, A, A

    Going through the choices for part a:

    - A - No change (dielectric cannot transfer charge)
    - B - increase (ΔV decreases)
    - C - Decrease ($\Delta V = E'd = \frac{E_0 d}{k_c} < E_0 d$
    - D - Decrease ($F \propto E = \frac{q'}{A\epsilon_0} \quad q' = q\left(1 - \frac{1}{k_c}\right)$
    - E - Decrease $U = \frac{1}{2}C(\Delta V)^2$

    Since potential energy is decreased when the slab is inserted and

    $$F = -\frac{dU}{dx}$$

    then we know that F must be in the same direction of motion (pull). When it is pulled out, $F < 0$, so it's opposing the motion.

15. (A,B,D,E), B, A

5.8 Lingering Doubts

[3] Parallel Capacitors, no?

Problem 1: Two capacitors C_1 and C_2 are initially charged with the same voltage ΔV_0. Switches S_1 and S_2 are then closed. What is the final voltage between the two points now?

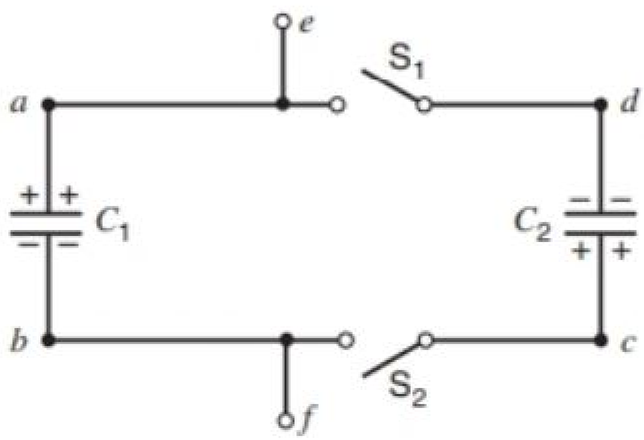


Figure 4: Two capacitors, but with opposite orientations

My Questions: Why should there be an equivalent capacitance in this case? Can you derive it from there?

Find the final charges q_1 and q_2 on C_1 and C_2 respectively. Why are they different? Shouldn't they be the same?

Solution: Why is the equivalent capacitance valid? In this scenario, the equivalent capacitance needs to hold for charge, voltage, and energy (since it's related to voltage). And if we looked through our derivation, all three of these are equivalent in the original and equivalent case. Nothing, even implicitly assumed, hints that the two capacitors must be oriented in the same direction.

While the charge on the capacitors after the switches are closed should vary according to time, we are only interested in the *final*, electrostatic situation[4] where ΔV is a constant. Thus,

$$E = \nabla \cdot V \rightarrow E = 0$$

(Q:both capacitors begin with a potential with ΔV_0, so then shouldn't $E = 0$? Why is there a net movement of charge?) This is the reason why ΔV is the same for both capacitors, not q. If $E \neq 0$, then we would have a net movement of positive charge in

[3]This is a subsection on problems that confused or puzzled me.

[4]This is under the assumption that it is *possible* for it to end up in an electrostatic situation.

the direction of $\vec{E}$. This is similar to the reason why the corona discharge derivation assumed both were at the same *potential*, not charge.

With that being said, we will apply these derivations to help us find ΔV_f. WLOG, let $C_1 > C_2$. By charge conservation:

$$\Delta V_0(C_1 - C_2) = \Delta V_f(C_1 + C_2) \to \Delta V_f = \frac{C_1 - C_2}{C_1 + C_2}\Delta V_0$$

And it then follows that:

$$q_1 = C_1 \Delta V_f \qquad q_2 = C_2 \Delta V_f$$

Remark: Note that voltage is defined as difference between the positive and the negative terminal. So the negative end has a negative potential, and positive end has a positive potential. In other words, the potential is different between the two so there is a net movement of charge and $E \neq 0$. So if both capacitors were oriented in the same direction, then you would be right: $\Delta V_f = \Delta V_0$ and no net charge is transferred from one to another.

Problem 2: An initial voltage of ΔV_i is applied between terminals a and b. Find the ΔV_o between the two right terminals for (a)a being positive (b) b being positive.

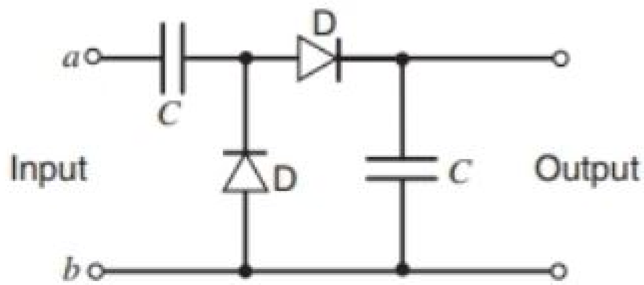


Figure 5: Diagram for problem 2

Questions: For part a, shouldn't the current also flow through the diode to the left, forming a loop? How would we calculate that?

Solution: To answer the above question is simple: after leaving the capacitor on the right, it has two options: two the negative node of the battery or to a a capacitor C. Thus, these two are essentially in parallel and we have:

$$C_{net} = \frac{1}{\frac{1}{0} + \frac{1}{C}} = 0$$

So we can effectively remove the diode on the left[5] since there will be no current through it. Thus, the two capacitors are in series with $C_{net} = C/2 \to \Delta V_i C/2 = q$, so there is a charge of q on each capacitor.

For the output node, the top one should be positive and bottom one should be negative (since we removed the left diode). Thus, it has $\Delta V_o = q/C = \Delta V_i/2$

[5] why is this?

For part b, only the left capacitor is charged (why?). Thus, it doesn't matter if you assume that the top one is positive or the bottom one is negative for the output - $\Delta V_o = 0$.

Problem 3: A battery far away from A and B maintains a constant voltage ΔV across the infinite grid. Find the net charge on any junction. Show that the potential at any junction is the average of the potential at the four nearest junctions [6].

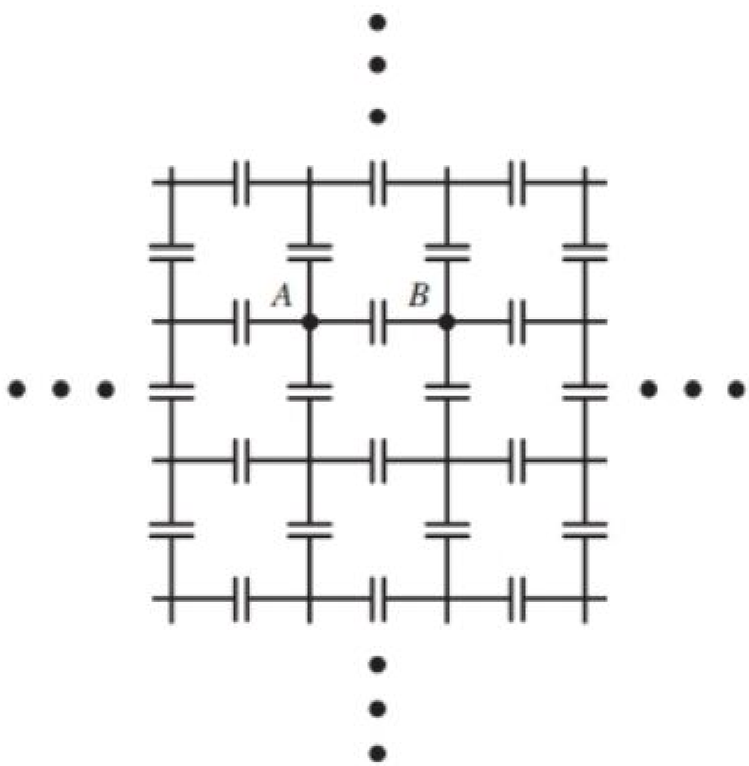


Figure 6: Diagram for problem 3

Questions: What sort of symmetry exists here? How would we find the potential at each node when each is composed of four half capacitors?

Solution/Attempt: Let the center node be called x and the nodes about it be a_1, a_2, a_3 and a_4. The charge on each half capacitor is q_i for the ith one. Since the net charge on each node is zero (it's isolated from the circuit), then we have

$$q_1 + q_2 + q_3 + q_4 = 0 \to \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 = 0$$

Now note that the potential at each node *must* be the same. This is because if it was not uniform throughout, then we would have a current! Thus, the potential difference ΔV between x and a_2, say, is equal to the ΔV_2! Now, letting x have a potential of V_x, we have:

$$V_x = \frac{(\Delta V_1 + V_x) + (\Delta V_2 + V_x) + (\Delta V_3 + V_x) + (\Delta V_4 + V_x)}{4} = \frac{4V_x}{4} = V_x$$

[6]Junction's definition is having three or more circuit paths meet at a single point

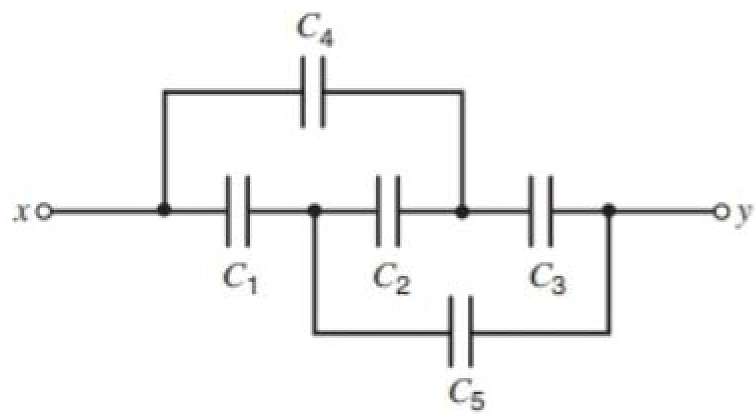


Figure 7: Problem 4

Problem 4: Find the equivalent capacitance between points x and y. Let $C_1 = C_3 = C_4 = C_5 \neq C_2$.

6 DC Circuits

Essentially a battery that supplies a direct current (DC) to each of the circuit elements. The battery will begin at the transient state before entering in a state where current is constant

6.1 Memorable Problems

1. Why does a bird not get shocked when it touches a high-voltage wire?
2. Two resistors can be used in parallel, in series, or separately. Their equivalent resistances are: $\}3R, 4R, 12R, 16R\}$. Find their resistances.
3. ***A battery is connected to two diagonal ends of a very large square grid of resistors R. What is the equivalent resistance of this grid?

Solution 1: Let the original current in the wire be i_3. Now we place a bird on the wire, and is effectively a resistor in parallel of resistance R. The current i_1 through the bird is $\Delta V/R$ and through the wire $i_2 = \Delta V/r$. We also have that $i_3 = i_1 + i_2 = \Delta V\left(\frac{1}{R} + \frac{1}{r}\right)$ Now if $r \ll R$, then we have that $i_3 = i_2$ and $i_1 = 0$.

Remark: Our equation for $\Delta V/R$ holds true only if the bird has both feet on the wire. If it was also touching a wire of lower potential, then ΔV would be large and $i_1 \neq 0$. In other words, we would have a dead bird on our hands.

6.2 Electromotive force

E is defined as $\frac{dW}{dq}$ and has units of volts. In the steady current state, a charge of dq passes through all the circuit elements and is effectively congruent to taking a charge dq from a positive terminal of the battery to the negative terminal. Note that E is in the direction of the current, too.

Be sure to check out HRK pages 705-706 to see a proof that you can analyze a circuit by going either clockwise or counterclockwise - it doesn't matter!

Relationship between emf and ΔV

Let the potential at b be V_b. Going ccw about the circuit:

$$V_b - \mathcal{E} + iR + ir = V_b$$

This is equal to V_b, since we can imagine the circuit as a bunch of hills - if we start and end at the same spot then our potential should still be the same! Solving, we obtain:

$$i = \frac{\mathcal{E}}{r + R}$$

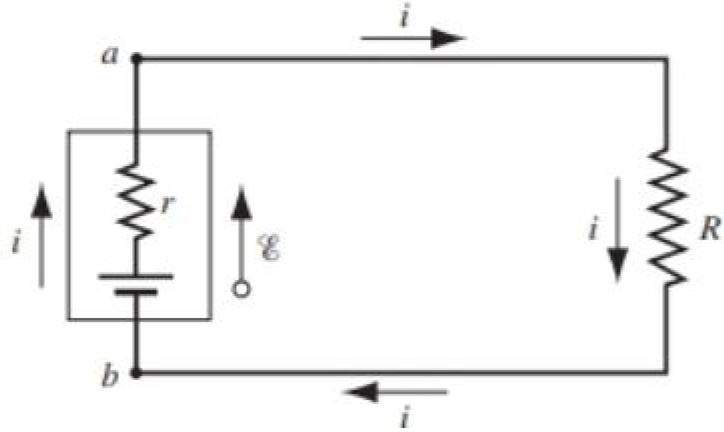


Figure 8: A battery with some internal resistance

Now we solve for V_{ab} or the voltage between a and b Again going counterclockwise:

$$V_b + iR = V_a \to V_a - V_b = V_{ab} = iR = \left(\frac{R}{r+R}\right)\mathcal{E}$$

Notice that V_{ab} is essentially the voltage across the battery. Thus, for

$$V_{ab} = \mathcal{E}$$

we would need to either have $R = \infty$ or $r = 0$. The first case would require for the circuit to be open - current is zero.

Problem: Find V_{ab} in the circuit and find the currents i_1, i_2, i_3.

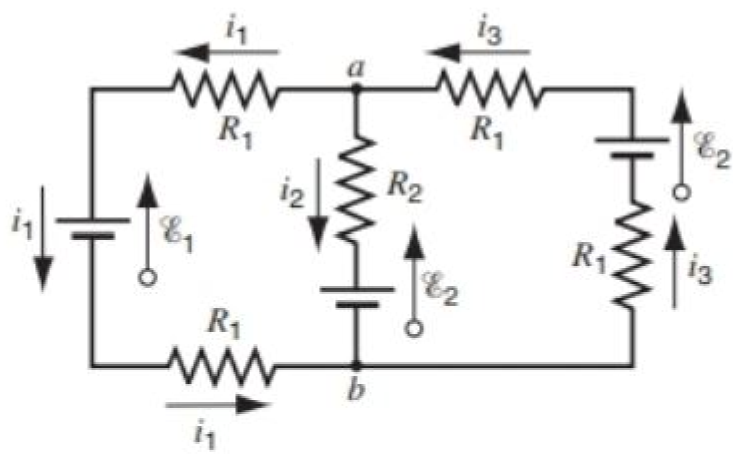


Figure 9: A complicated circuit

Solution: Note that V_{ab} is independent of the path taken. An analogy is a mountain - there are multiple paths that go from the bottom of the bottom to the top, but at the end, the net gravitational PE change is the same (assuming equal elevation at the bottom).

Thus,

$$V_{ab} = \mathcal{E}_1 + 2i_1R_1 = \mathcal{E}_2 - i_2R_2 = \mathcal{E}_2 - 2R_1i_3$$

Note that we can use this information to find the currents, since at junction a, we have

$$i_3 = i_1 + i_2$$

6.3 Electric fields in circuits

Another way to write $\mathcal{E}$ is

$$\mathcal{E} = \frac{dW}{dq} \to \int \mathcal{E} dq = \int dW \to \mathcal{E} = \oint \frac{F}{q} \cdot ds$$

It may be tempting to claim that $F/q = \vec{E}$, but realize that F is not necessarily the force caused by the electric field - it could be mechanical, thermodynamic, or magnetic, too (I feel unsatisfied by this answer).

Apparently, in the initial transient state, the battery imposes a surface charge across the wires. (I hope I can see a proof of this later) That way, a steady current can follow afterwards. We have:

$$E = \rho j \to j = \frac{i}{A} \to E = \frac{\rho i}{A} \approx \frac{q}{4\pi\epsilon_0 r^2}$$

If you assumed the wires were cylindrical and plugged in some values, you would find that q required to set up the electric field is relatively small.

In essence, the battery begins with an explosion of electrons to establish the $\vec{E}$ that determine the flow of the electrons. From $E = \rho\frac{i}{A}$, you could tell that a lot of charges would have to accumulate on the resistors in order to set up E, since ρ is large.

6.4 Resistors in circuits

Parallel: We have a few equations:

$$i_3 = i_1 + i_2 \quad \Delta V/R_1 = i_1 \quad \Delta V/R_2 = i_2 \quad \Delta V/R_{net} = i_3$$

Substituting the last three equations into the first one, we obtain:

$$\frac{1}{R_{net}} = \sum \frac{1}{R_i}$$

This is the general case, since we could have extended it to any number of resistors in parallel.

Series: This one is simple:

$$V_a - i(\sum R_i) = V_b \to \Delta V = i(\sum R_i) = iR_{net} \to R_{net} = \sum R_i$$

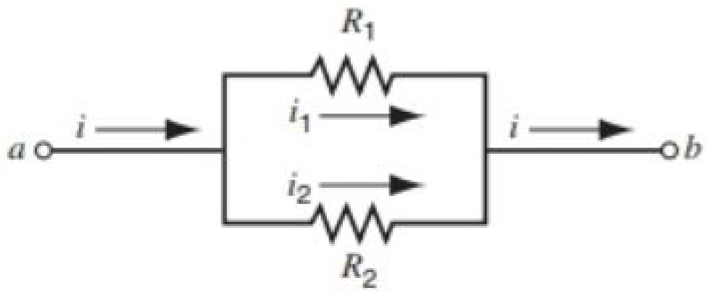


Figure 10: Resistors in parallel

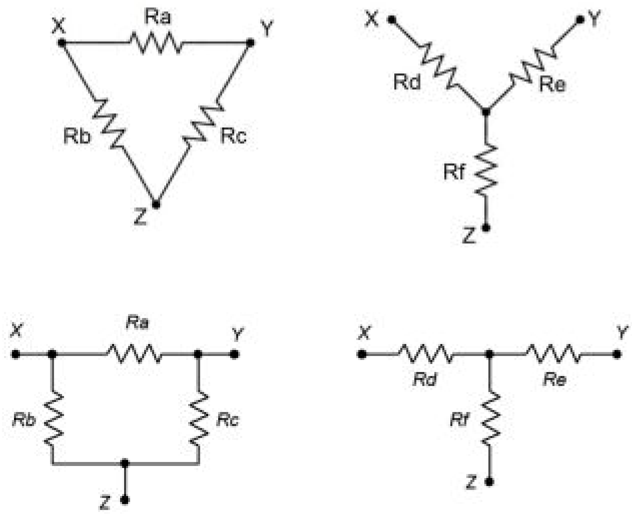
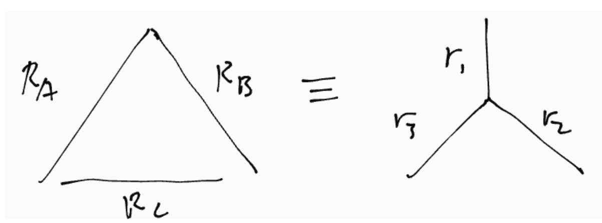


Figure 11: Taken from here

6.5 Y-Delta Transformation

Delta to Y



We have a system of equations, with $\|$ meaning that they are parallel. Note that for each of them, our nodes of the battery are connected between different points.

$$\begin{cases} r_1 + r_3 = R_A \parallel R_B + R_C \\ r_1 + r_2 = R_B \parallel R_A + R_C \\ r_2 + r_3 = R_C \parallel R_A + R_B \end{cases} \tag{2}$$

Adding these, we find

$$r_1 + r_2 + r_3 = \frac{R_A R_B + R_A R_C + R_B R_C}{R_A + R_B + R_C}$$

Using these, we find that

$$r_1 = \frac{R_A R_B}{R_A + R_B + RC} \quad r_2 = \frac{R_B R_C}{R_A + R_B + RC} \quad r_3 = \frac{R_A R_C}{R_A + R_B + RC}$$

Y to Delta

Algebra (bleh)

We first divide

$$r_1/r_3 = R_B/R_C \implies R_B = \frac{r_1 R_C}{r_3}$$

Doing the same to solve for R_A:

$$r_1/r_2 = R_A/R_C \implies R_A = \frac{R_C r_1}{r_2}$$

Plugging R_A and R_B into the equation for r_1, we have:

$$r_1 = \frac{(r_1 R_C)^2}{(r_2 r_3)(\frac{r_1 R_C}{r_3} + \frac{r_1 R_C}{r_2} + R_C)}$$

Simplifying, we find that

$$R_C = \frac{r_1 r_2 + r_1 r_2 + r_2 r_3}{r_1}$$

By symmetry (the denominator is going to be the resistor opposite to it, with $R_C \to r_1, R_A \to r_2, R_B \to r_3$), we have:

$$R_A = \frac{r_1 r_2 + r_1 r_2 + r_2 r_3}{r_2} \quad R_B = \frac{r_1 r_2 + r_1 r_2 + r_2 r_3}{r_3} \quad R_C = \frac{r_1 r_2 + r_1 r_2 + r_2 r_3}{r_1}$$

Genius

Consider converting both circuits into ones with conductances instead of resistors. Then, $G_1 = 1/R_1$, $G_2 = 1/R_2$, etc. Note that this implies that conductors in series have:

$$G_{net} = \frac{1}{\sum \frac{1}{G_i}}$$

and conductors in parallel have:

$$G_{net} = \sum G_i$$

Similarly, note that $G\Delta V = I$, or that current and voltage seem to be switched. Thus, we would replace voltage sources with current sources.

Question: would this mean that $\sum \Delta V = 0$ at each node?

Attempt:** Based on our definitions for I and ΔV for the conductor circuit, if I is conserved in the original circuit (which it must be due to Kirchhoff's laws), then ΔV is conserved in this one, too. But then would this imply that ΔV is also the same at all nodes in the original circuit? After all, ΔV across a conductor is the same ΔV across the resistor!

Now note that since the equivalent conductor does not depend on the input resistance, then we can suppress two of the currents by connecting them to the ground, since without any current, the current sources act as perfect conductors.

6.6 Proof of Thevenin Equivalence

One voltage Source We begin with a single voltage source in series with a resistor. To prove that the net current, and hence the effective resistance is deterministic, the number of paths from one end of the voltage source to the other must equal the number of unique currents.

We can either branch off a line or connected it to another forming a node in either case. Branching it off increases the number of unique currents and paths by 1. While conjoining ≥ 2 wires together does not increase the number of unknown currents or the number of paths. This is because the net current going out of the node is $\sum i_k$.

Let there be k voltage sources, n current sources, and l resistors. By the superposition principle, we can suppress all other voltage and current sources, leaving only one voltage or current source we need to deal with.

6.7 Current in RC

A circuit has a battery with an emf of $\mathcal{E}$, connected to a resistor R and a capacitor C in series. We will find the current now:

$$\mathcal{E} - iR - \frac{q}{C} = 0 \quad \text{(Kirchoff's voltage law)}$$

Now we rearrange to get

$$\frac{dq}{dt} = \frac{\mathcal{E}C - q}{RC} \rightarrow \int_0^q \frac{dq}{\mathcal{E}C - q} = \frac{1}{RC}\int_0^t dt = t/RC$$

$$\rightarrow \ln\left(\frac{\mathcal{E}C - q}{\mathcal{E}C}\right) = -\frac{t}{RC} \rightarrow q(t0 = \mathcal{E}C(q - e^{-t/RC})$$

Taking the derivative with respect to dt on both sides, we get:

$$\frac{dq}{dt} = i = \frac{\mathcal{E}}{R}e^{-t/RC}$$

Discharging the capacitor

From the above figure, we can see that:

$$\frac{q}{C} + iR = 0 \rightarrow q = q_0 e^{-t/RC}$$

We integrated from q_0 to some charge $q < q_0$ for the charge side. Essentially, we applied the same process as the previous example.

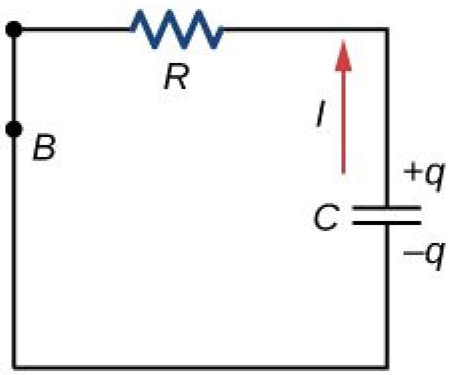


Figure 12: Stolen from Open Stax

6.8 MC Answers

1. C
2. C
3. B
4. C
5. B
6. C
7. A ($i = \frac{\mathcal{E}}{r+R}$, A ($\Delta V(r) = \frac{Er}{r+R}$
8. D
9. A (?)
10. C**
11. B ($P = IV = \frac{V^2}{R} = I^2R$. Here, R is a constant, so we use the second one to get $V \propto \sqrt{P}$
12. A ($I = \frac{\Delta V}{r+R} \to P = \frac{R\Delta V}{r+R}$, B $P = \frac{\Delta V}{1+R/r}$
13. C, C, A (current through resistor is $\Delta V/R$, dependent only on the resistor itself)
14. A ($\Delta V = iR$), A
15. E

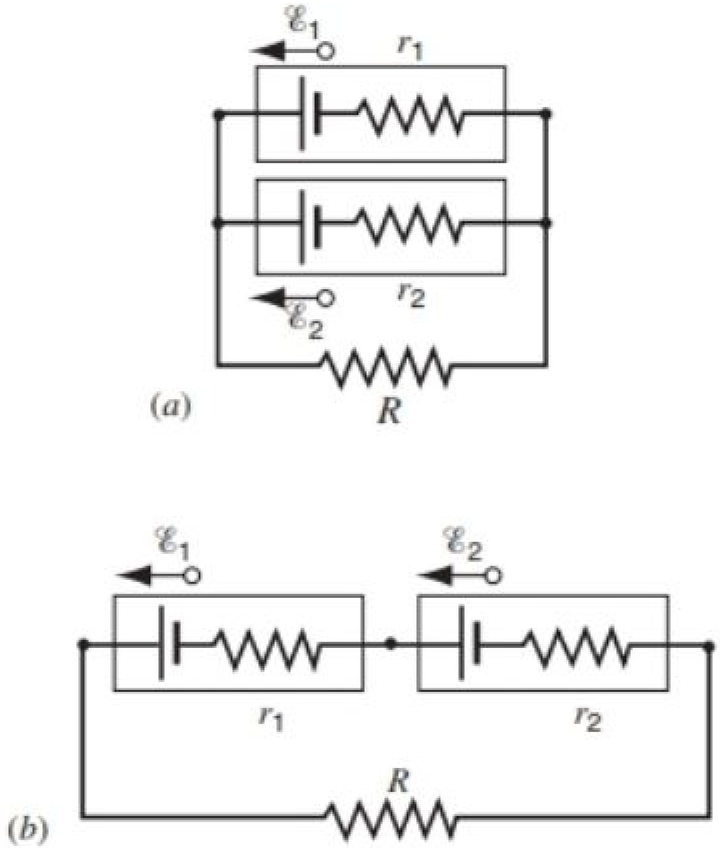


Figure 13: Batteries in parallel and in series

6.9 Lingering Doubts

Two batteries, $\mathcal{E}_1$ and $\mathcal{E}_2$ have internal resistances of r_1 and r_2 respectively. Find the current through R in each of the configurations shown:

Questions: How would you do the case where the batteries are in parallel? Would we do mesh analysis?

Possible attempt: We do mesh analysis and draw the loops I_1 and I_2 for the top and bottom loops respectively. Now we choose our two closed loops to use kirchoff's law:

$$\begin{cases} \mathcal{E}_1 - I_1 r_1 - \mathcal{E}_2 - (I_1 - I_2) r_2 = 0 \\ \mathcal{E}_2 + (I_1 - I_2) r_2 - I_2 R = 0 \end{cases} \tag{3}$$

Solving this system of equations through hairy algebra bash, we finally get our answer of:

$$I_2 = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 R + r_2 R + r_1 r_2}$$

Easier System of Equations

$$\begin{cases} \mathcal{E}_1 - I_1 r_1 - R I_3 = 0 \\ \mathcal{E}_2 - (I_3 - I_1) r_2 - R I_3 = 0 \end{cases} \tag{4}$$

Note that I_3 is the current flowing through R, I_1 is the current flowing through r_1 and $I_3 - I_1$ is the current flowing through r_2.

If we solved this system of equations, then we would get the same answer, only that it is simpler to solve.

Reflection Question: Why is it that when $r_1 = r_2 = 0$, we have that $I_3 = 0$?

Problem 2: In the figure below, find the equivalent resistance between F and G.

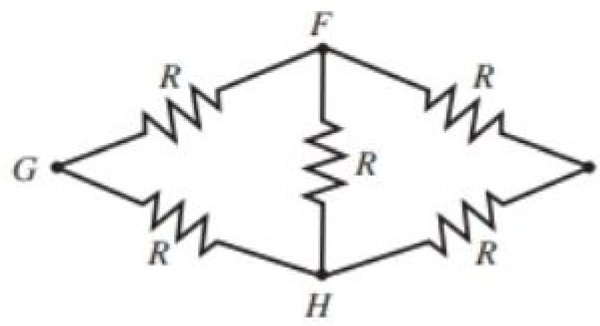


Figure 14: A somewhat complicated circuit

Questions:

1. There are three possible paths to take in this setup, so is it valid to treat these three paths to be parallel to one another?

2. Another way is to solve using Kirchhoff's second law - three equations and three variables. For more variables, is there a quick way to solve it using matrices?

Solution: The reason why we cannot treat the three paths as being *independent*, parallel ones, is that the current must be the same for resistors in series. However, note that since there are intersections of the paths at points not at the battery node, then the current through the bottom right resistor does not equal the current of any other resistor!

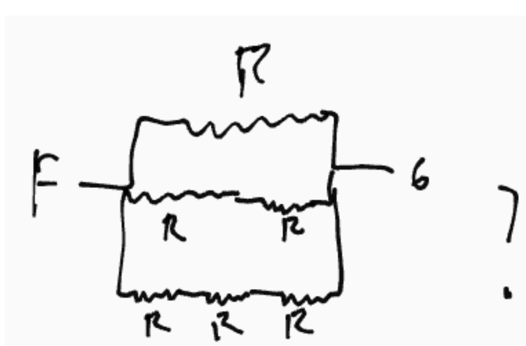


Figure 15: A bad circuit

Instead, we would need to replace our circuit with an equivalent circuit, where the current through each resistor is identical, and thus by $\Delta V = iR$, the resistance must also be:

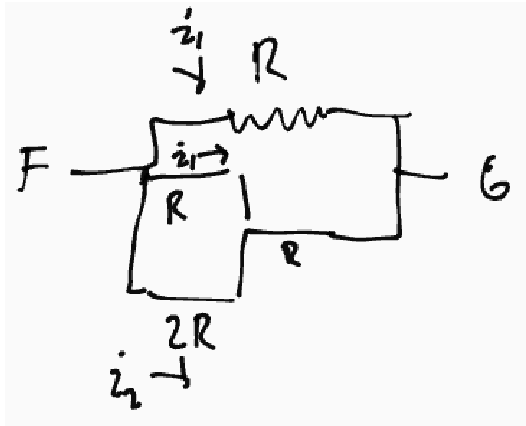


Figure 16: A good circuit

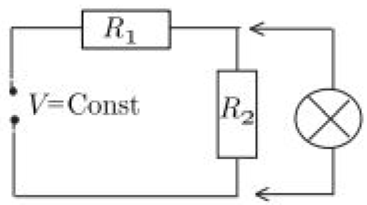


Figure 17: Can get hard if you make it so!

Problem: (Kalda) $R_1/R_2 = 4$ and after the lamp is connected, the current through R_1 increases by 0.1 A. Find the current through the lamp.

Solution (?) Note that the voltage is the same for both. Substituting $R_1 = 4R_2$, and letting i_3 be the final current through R_1 and i_2, i_3 be the current through R_2 and R_3 respectively after the lamp is connected. Thus:

$$\begin{cases} 5R_2(i_3 - 0.1) = 4R_2 i_3 + R_2 i_3 \\ R_2 i_3 - 0.5R_2 = R_2 i_1 \end{cases} \tag{5}$$

Since $i_2 = i_3 - i_1$, we find that $i_2 = i_3 - (i_3 - 0.5) = 0.5$

Problem: (Kalda) A uniform wire of cross-sectional area $A_0 = 1mm^2$ had a millimetre scale marked on it: an array of streaks with interstreak distance $a_0 = 1mm$ covered the entire length of the wire. The wire was stretched in a non-uniform way, so that the interstreak distance a is now a function of the distance l from one end ofthe wire (as measured after the stretching), see figure. The new length of the wire is $L = 4$ m. Using the graph, determine the electrical resistance R of the stretched wire assuming that the resistivity of the wire material is $1 \cdot 10^-6$. Assume density is constant for the stretching

Questions: We know that $R = \rho\frac{L}{A}$, so then does that imply that since the only changed quantity is L, we can simply plug and chug? Looking at our original derivation, it assumed that the electric field within the wire was uniform and the resistivity was a constant. These assumptions seem to be valid for this situation, though?

7 Fun Problems

A section for fun and challenging problems that I **already** solved.

Problem 1: A cylindrical, open-faced, capacitor has its two cylindrical plates connected to a battery that maintains a voltage ΔV. The bottom end is slightly submerged in a dielectric fluid. What height does the fluid rise to?

Problem 2: A particle has an initial velocity v_0 at an angle θ. Find the area under the trajectory curve and the angle θ which maximizes this.

Problem 3: **(Kalda Projectiles)** A projectile needs to overcome a trapezoidal roof with one height a, another height c and the distance between the two peeks is b.

Problem 4: A particle has a force exerted on it given by $F = -kx$. Find $x(t)$ given that it begins with $v_0 = 0$ and x_0.

Problem 5: Two masses, m and M are separated by a distance d and attracted by their mutual graviational force. How long does it take for them to collide? And where do they collide at?

8 Magnetic Fields

While I can't prove it right now, the magnetic field is a direct consequence of relativity[7] - electrons moving relative to one another will not experience a magnetic force. While both the magnetic and electric field are from the electronmagnetic field, the magnetic one is not conservative, since it depends on the velocity of the particle.

From experimental observations alone, one can conjecture:

$$F_b = q\vec{v} \times \vec{B}$$

where $\vec{B}$ is the magnetic field. Note that this implies that $F_b \perp v$, so F_B cannot do any work of the particle.

8.1 Memorable Problems

Problem 1: A particle begins with an initial velocity v_0 at an angle θ with $\vec{B}$. Find the pitch, radius, and period of its path.

Solution: We split $\vec{v}$ into two components, v_x and v_y, where $v_x = v \sin\theta$ and $v_y = v\cos\theta$. Thus,

$$F_B = q(v_x + v_y) \times \vec{B} = qv_x\vec{B}$$

This produces circular motion, since $F_B \perp v_x$ at all times. Thus,

$$F_B = qv_x\vec{B} = \frac{mv_x^2}{R} \to \omega = v_x/R = \frac{q\vec{B}}{m} \to T = 2\pi/\omega = \frac{2\pi m}{qB}$$

[7] Aargh! The dissatisfaction of just being *told* and not understanding is extremely frustrating! I will now mark on my calendar a deadline to learn relativity to help aid me in creating a new section in my notes

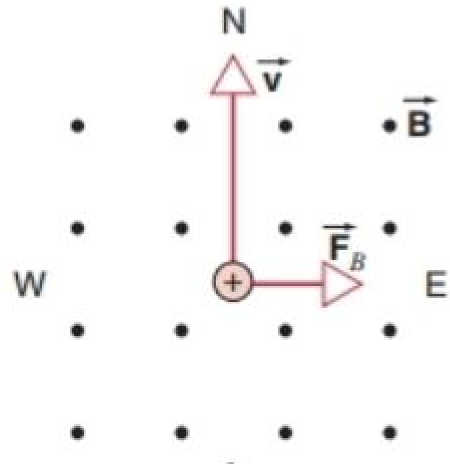


Figure 18: The dots represent the magnetic field pointing out of the page - if they were out then they would be crosses.

Note that $v_y \parallel \vec{B}$, so $F_By = 0$. This means that in the direction parallel to $\vec{B}$, the particle undergoes constant velocity. Since the pitch is the distance between subsequent circles, then

$$P = v_y T = \frac{2\pi m v_y}{qB}$$

Then we just substitute v_x and v_y in polar form.

8.2 Applications

Consider a charge q given an initial velocity boost v in the x-y plane. The magnetic field is parallel to the z-axis. If we assume that the particle undergoes circular motion then we have that:

$$m\frac{v^2}{r} = |q|vB \to r = \frac{mv}{|q|B} \to \omega = v/r = \frac{|q|B}{m} \to f = 2\pi\frac{|q|B}{m}$$

8.3 Uniform Fields In Depth

For nonuniform magnetic fields, the object's motion is not a circle and can be quite complicated depending on how the magnetic field varies. For uniform ones, the object's motion is always a circle, since there will always be a solution to r of:

$$\frac{mv^2}{r} = |q|vB \to r = \frac{mv}{|q|vB\sin\theta}$$

However, this is in fact, incorrect. Let the velocity vector have components of $v\sin\theta$ and $v\cos\theta$. Then,

$$F_B = |q|\vec{v} \times \vec{B} \to |q|(\vec{v}\sin\theta + \vec{v}\cos\theta) \times \vec{B}$$

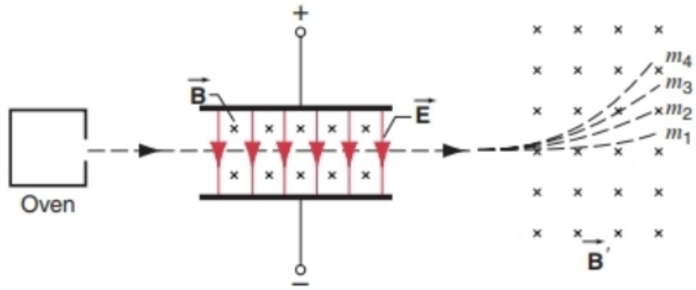


Figure 19: A technique to measure the mass to charge ratio

Since $\vec{v}\cos\theta \parallel \vec{B}$, then

$$\vec{v}\cos\theta \times \vec{B} = \vec{v}\cos\theta \vec{B}\sin 0 = 0$$

Thus,

$$F_B = |q|\vec{v}\sin\theta \times B = |q|\vec{v}\sin\theta B\sin(\pi/2) = |q|vB$$

So the v_x parallel to $\vec{B}$ will be unaffected, whereas v_y will undergo uniform circular motion. The superposition of these two types creates a helix shape.

In the region where both $\vec{E}$ and $\vec{B}$ exist, we define the *Lorentz Force* to be

$$F_L = q\vec{E} + q\vec{v} \times \vec{B}$$

Note in Figure 19, the magnetic field opposes the electric field. Thus, electrons that leave undeflected satisfy:

$$qvB = qE \to v = E/B$$

This guarantees that the electrons that enter the magnetic field all have the same velocity. So they must satisfy:

$$qvB = m\frac{v^2}{r} \to r = \frac{mv}{qB}$$

Thus, the greater the mass, the larger the radius. For mass spectrometry, a sensor is placed at the end of the magnetic field region (all the way to the right) and detects the distribution of mass through the deflection of the masses in the final magnetic field. The smaller the mass, the greater the deflection.

8.4 Hall Effect

The hall effect is where a magnetic field deflects a current in a wire, resulting in the *charge carriers* to move to the left or right of the wire according to the direction of $\vec{F}_B$.

In the final equilibrium state, part b of figure 20 shows that the electric field force exactly cancels the magnetic field force. Note that the width $w \ll l$, where w and l are the width and length of the wire, respectively. This is because we are assuming that $\vec{E}$ is uniform throughout the wire, so equilibrium is possible. This means

$$qE_H = qv_dB \to E_H = Bv_d$$

Note that since $j = -env_d \rightarrow v_d = \frac{\vec{j}}{en}$ Thus, letting t be the thickness of the plate and taking advantage that $E_H w = \Delta V$ we have

$$E_H = \frac{\Delta V_H}{w} = \frac{Bi}{wtne} \rightarrow n = \frac{Bi}{et\Delta V_H}$$

Quantum Hall Effect

Note that this implies that $\frac{\Delta V_H}{i} = \frac{1}{nte}B$, or that the graph of *Hall Resistance* against the magnetic field B is linear. However, under experimental observations at high magnetic fields and low temperatures, the plot resembles a series of steps. Thus, this is known as the *quantized hall effect*, since electrons can only be at a finite number of orbitals. The hall resistance exists at values of $\{\frac{h}{e^2}, \frac{h}{2e^2}, \frac{h}{3e^2} \dots\}$, where h is Planck's constant.

8.5 F_B on current-carrying wires

Consider a wire of length l with a current pointing in the up direction. A magnetic field points out the page, then what is the direction of the magnetic field?

This is simple to solve, as $F_B = q\vec{v}_d \times \vec{B}$, so using the right hand rule, we find that the magnetic force is to the right of the wire. To find the vector $\vec{F}_B$, we do:

$$\vec{d}F_B = dq\frac{d\vec{L}}{dt} \times \vec{B} = \frac{dq}{dt}d\vec{L} \times \vec{B} = id\vec{L} \times \vec{B}$$

where $\vec{L}$ is the length of the wire and points in the same direction of v_d, the velocity of positive charge or in the same direction as current (more rigorously, $\vec{j}$).

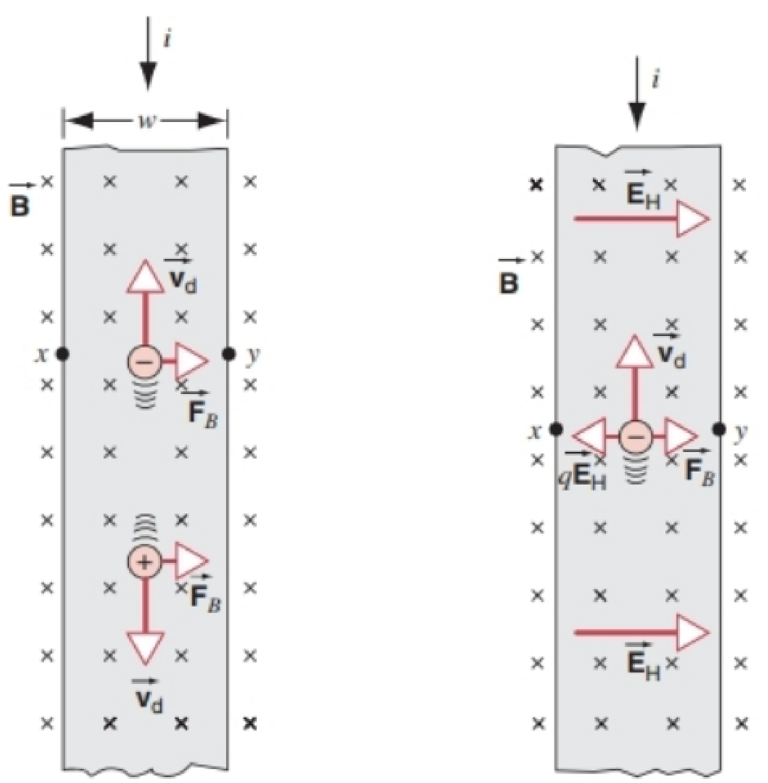


Figure 20: An electron being deflected by the magnetic field

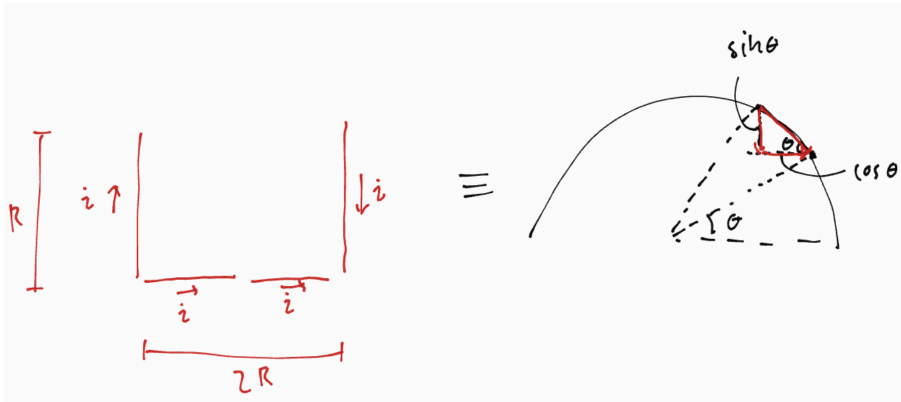


Figure 21: A congruent case

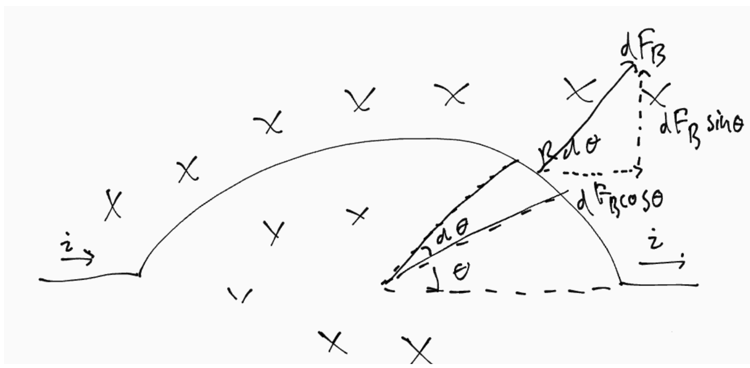


Figure 22: A FBD of a small segment

Problem: A current i flows through a semicircle-shaped wire of radius R. If a magnetic field $\vec{B}$ is applied pointing in the page, then what is the net force from the magnetic field on the wire?

Solution: Consider a small arc that subtends an angle $d\theta$ and makes an angle θ with the horizontal. From the diagram, we can split $d\vec{F}_B = d\vec{F}_B \sin\theta + d\vec{F}_B \cos\theta$. Note that due to symmetry, $d\vec{F}_B \cos\theta$ component will cancel. But for completeness, we will still calculate this term:

$$\int dF_B = \int i\vec{L}B = \int (i\vec{L}B\sin\theta + i\vec{L}B\cos\theta)d\theta = i\vec{L}B\Big[-\cos\theta + \sin\theta\Big]_0^\pi = 2i\vec{L}B$$

Remark: You can reduce the semicircle into four wires, two from each side - the horizontal and vertical components respectively. The $\sum F_w \equiv \sum F_{\text{semi}}$ on the wires, as shown in figure 21. Note that this implies that any shape that spans a width of $2R$ can replace the semicircle, since it will always be able to be projected onto the straight line. Proving it uses the same technique that we did.

8.6 Torque on a wire loop

A motor consists of a rectangular-framed loop of wire placed in a strong electric field. Note that while the magntic force, $\vec{F}_B$ is a constant, the torque caused by it is not.

Letting the center of the rectangular frame be the pivot, then we have (with θ being the angle that the rectangle makes with the horizontal if we were looking down at it):

$$\tau = 2F_B \cos\theta(b/2) = (iLB)\cos\theta b = AiB\cos\theta$$

Note that A is the area of the closed loop.

Challenge: Find the net torque on a circular loop of wire in a uniform magnetic field parallel to the loop (perpendicular to $\vec{A}$).

Solution: Proof without words:

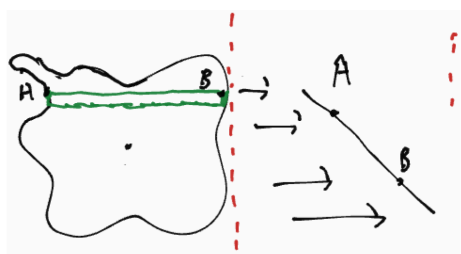
$$\vec{\tau} = 2\int_0^{\pi} F_B \times r = 2\int_0^{\pi} (i\vec{L} \times \vec{B} = \int_0^{\pi} 2iR\sin\theta d\theta B) \times r = 2iRd\theta B(r\sin^2\theta) = \pi R^2 iB$$

Note that this agrees with the result that we just found.

Challenge:** Is this true for any generally-shaped loop? What if the loop isn't 2-dimensional? Prove it. How would you determine what axis the loop rotates about?

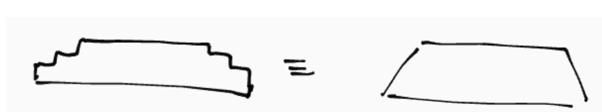
Solution:

Basis Consider an arbitrary 2-dimensional loop of wire. We will slice this wire into infinitely rectangles each with a width of dx and sufficient length to approximate the length of the loop of wire at that point. These rectangles are oriented such that their long side is oriented as shown in the diagram below.



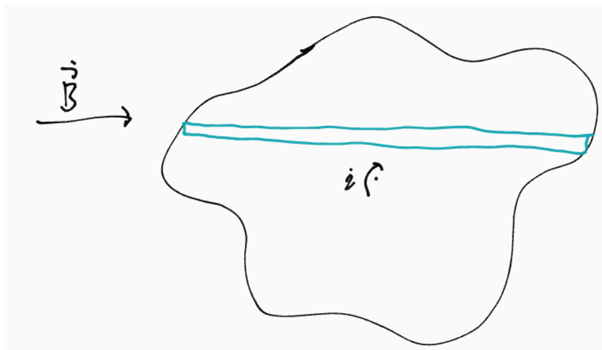
Note that for the drawing on the right, we depicted the loop of wire to be a 1-D line when viewed from above, with the magnetic field pointing in a single direction. You may argue that the magnetic field does not have to point in the same plane as this 1-D line and can have a component that is at a nonzero angle with respect to this plane that the line resides in. You are correct, but note that this extra component does not matter. This is because this component of the magnetic field is parallel to the wire loop, so the net force is zero and no torque is generated. I believe that this was proven somewhere in this section.

When we derived the net torque on a rectangular loop of wire, notice that the two sides which in the diagram is perpendicular to the plane of the loop contribute no torque. Furthermore, letting the torque contribution from each small segment be a long rectangle will be valid, since the long side of the rectangle provides no torque anyway.



Final Step

Now we claim that adding the torque contributions from each of the long rectangles sums up to be the total torque exerted on the loop of wire. But is this legitimate? This is true because the force on each side of the rectangle is equal.

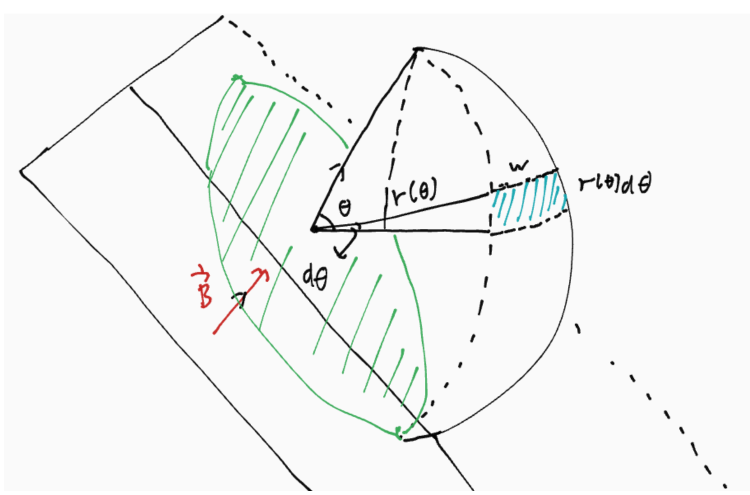


For example, in the diagram above, if we place the pivot anywhere in space, the torque will still be the same, $\tau = Fl$ (assuming the axis is perpendicular to the page). Letting the left end of the stick (where F is) be $x = 0$. Then if the pivot is at $x = b$, $\tau = Fb + F(l - b) = Fl$

Thus, since the force on the rectangle's two sides is the same, then we can place an axis at any point and sum up the torques from each of the rectangles.

$$d\tau = idA \times B \implies \tau = iA \times B$$

3D loops Consider an arbitrary loop of wire as shown below. We can treat this by summing the torque contributions from each small area dA. Since each of these areas are small, we can approximate these to be 2-dimensional rectangles. The net torque from each of these is $idA \times B$. This implies that if we project the area onto the plane perpendicular to the magnetic field, we can find the net torque! [8]



[8] Given a curve $f(x, y, z)$, along with boundary conditions, what is the area of the projected curve? Might need more math background to do this

Axis of Rotation

This is more of a mechanics question. For 2D loops, we know that the axis of rotation must be perpendicular to the magnetic field lines. Since the COM must have $a = 0$ due to $F_{net} = 0$, then the axis must go through the center of mass. So it is indeed uniquely determined by this procedure. For 3D loops, it is simply a 2D problem as we can project the area onto the magnetic field (see previous part).

8.7 MC Answers

1. A,C
2. (A,C), C
3. B? (Initially, $F_B = 0$, but $F_E \neq 0$, so it begins by going down. Since $q < 0$, then F_B points to the right).
4. B,C,D,E
5. C
6. B,C,D,E
7. D
8. B, C
9. B, C
10. A (since the center consists of sharp points, signs of a high acceleration. If it was low, then it would be a more of a rounder, longer shape)
11. ** B,B Note that the charge carriers don't matter, since either way, the negative charges move to the right side, or the positive charges move to the left side. Either way, there is a net negative charge on the right wing.
12. D This uses the exact same idea we derived earlier.
13. D This would only add an extra constant factor $f(\theta)$ to the net force. So there is no change.
14. ???
15. E, D
16. (A,B,E), (D,E,C) If by stronger nonuniform, they mean that it is still parallel to one direction, then indeed the answer would be E,D. But if it isn't then (A,B) and C are possible, too.

9 F_B from moving charges

The magnetic field from a single moving charge is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{r^3} \tag{6}$$

where r is the scalar distance from the charge and the point we are measuring the magnetic field $\vec{B}$. r is the vector distance, which points from the charge to our point. Note that since $\vec{r} = \hat{r}r$, $B \propto \frac{1}{r^2}$

However, note that a problem arises when we consider the charge interacting with its own magnetic field! The field is clearly not uniform, so its motion cannot be a straight line - which is not helpful for magnetostatics t where $\vec{B}$ does not vary with time.

9.1 Memorable Probs

1. A messy coil of wire is has a counterclockwise current applied to it. Will the wire attempt to form a circle or bunch up further?

2. An infinitely long wire is bent such that it forms a semicircle on one end (like a very long U). What is the magnetic field at the center of the semicircle? And what is the magnetic field at a point very far away from the semicircle?[9]

3. A long cylindrical wire has a hole in it. The rest of the wire has uniform current density, and the shaded part has a total current i. Find the magnetic field at the center of the hole and prove that it is uniform in the hole.[10]

4. ** Prove that the fringe effect is necessary for a uniform magnetic field.

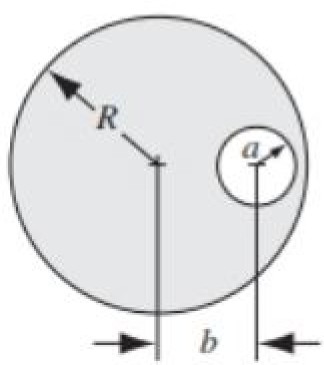


Figure 23: A wire with a hole in it

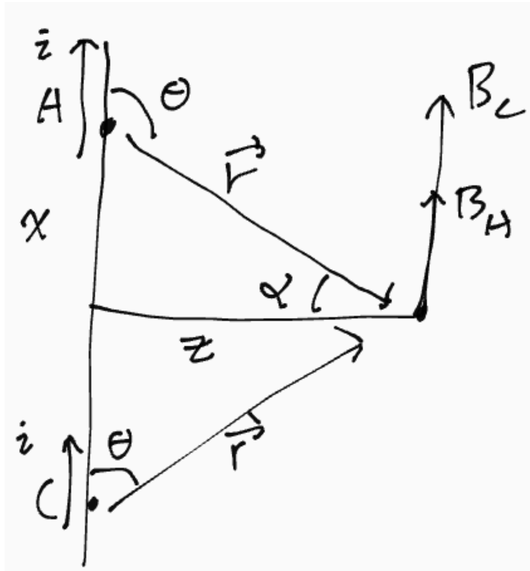


Figure 24: The magnetic field contribution from each point on the wire adds up in the same direction

9.2 Force from a current-carrying wire

Let there be a wire of length l. At a perpendicular distance z from $l/2$, we want to find $\vec{B}$.

Attempt:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}(dq\frac{\mathrm{d}l}{\mathrm{d}t} = id\vec{L} \times \vec{r})$$

An intuitive explanation for $dq\frac{\mathrm{d}l}{\mathrm{d}t} = id\vec{L} \times \vec{r}$ is by considering a mechanical analogy: fluid flow. In fluid flow, we have $dm \equiv dq$. Thus, we must prove that:

$$dmv = \frac{\mathrm{d}m}{\mathrm{d}t}dl$$

Note that $dm = \rho l A = \rho(vdt)A$ and so then:

$$\frac{\mathrm{d}m}{\mathrm{d}t}dl = \frac{\rho v dt A}{dt} = \rho dl A v = dmv$$

Now continuing with our derivation, the scalar form of B is

$$B = \frac{i\mu_0}{4\pi}\int \frac{\sin\theta}{r^2}dx = \frac{i\mu_0}{4\pi}\int \frac{z}{(x^2+z^2)^{3/2}}dx$$

Now, neglecting the constant and only evaluating the integral, we have ($x = z\tan\alpha \implies dx = z\sec^2(\alpha)d\alpha$):

$$\frac{z}{z^3}\int \frac{1}{(\tan^2\alpha + 1)^{3/2}} z\sec^2\alpha d\alpha = \frac{1}{z}\int \cos\alpha d\alpha$$

[9]This is actually easy, but I was stumped on it in the beginning

[10]hint: superposition

Putting the constant back in, we have:

$$B = \frac{i\mu_0}{4\pi z}\int \cos\alpha d\alpha = \frac{i\mu_0}{4\pi}\Big[\sin\alpha\Big]_{\sin^{-1}(-\frac{L}{2\sqrt{L^2/4+z^2}})}^{\sin^{-1}(\frac{L}{2\sqrt{L^2/4+z^2}})}$$

Finally, using the second fundamental theorem of calculus:

$$B = \frac{i\mu_0}{4\pi z}\frac{L}{\sqrt{\frac{L^2}{4}+z^2}} \tag{7}$$

For $L \to \infty$, $B = \frac{i\mu_0}{2\pi z}$. This seems to imply that there exists a gauss's law version for magnetic field - and that is Ampere's Law! (which we will learn soon).

The method we used:

$$dB = \frac{idL\sin\phi}{r^2}$$

is known as the Biot-Savart law. Fancy name, but it's really just analyzing small pieces then integrating to solve for the whole.

For a straight wire, the magnetic field is in the shape of concentric circles. The direction can be determined with the right hand rule: curl your hand around the wire, and point you thumb in the direction of the current (defined positive). The curl of your hand is the direction of $\vec{B}$.

Circular loop In the diagram below, we can see that $\phi = \pi/2$, so $\sin\phi = 1$. Also, the horizontal component of $dB_x = 0$ due to symmetry. Thus, we have:

$$\int d\vec{B} = \int \frac{i\mu_0}{2\pi z}\frac{ds}{r^2}\cos\alpha = \frac{\mu_0 i}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

The final step was done by noting that all terms are constants except for ds, which $\int ds = 2\pi R$.

To find the magnetic field B at the center of the arc is simple:

$$d\vec{B} = \frac{i\mu_0}{4\pi}\frac{ds}{r^2} \implies B = \frac{\mu_0 i}{2R}$$

Another way is to take the general B equation and take $\lim_{R\to\infty}$ or $R \gg z$. This is because this makes the point seem to be very close to the center.

If instead, it subtended an angle θ ,then we would have

$$B = \frac{i\mu_0}{4\pi}\int_0^\theta \frac{ds}{r^2} = \frac{i\mu_0}{4\pi R}\theta$$

Problem: A sheet of metal has a width w and a very long length. What is the magnetic field at a distance z from the center of the sheet? The current in the sheet is i. What would the field be due to an infinite sheet? (solution on page 757)

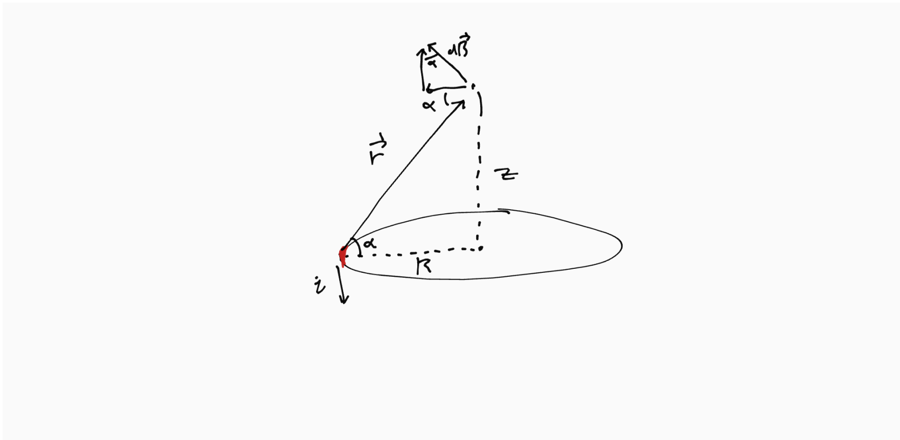


Figure 25: A quick diagram

9.3 Two current carrying wires

Consider two current-carrying wires of current i_1 and i_2. Let their lengths $l \gg 1$. From the right hand rule, we can easily find:

- Parallel wires attract
- Anti parallel wires repel

The force on each wire is:

$$F_B = il \times B = i_1 l \frac{\mu_0 i_2}{2\pi r} = \frac{\mu_0 i_1 i_2}{2\pi r} l$$

While we did calculate the force F_{12} (the force on 1 from 2), by Newton's second law $F_{12} = F_{21}$.

9.4 Solenoids

A solenoid is made of tightly wound circular coils, each of a radius R. As shown in the diagram above, there is a current i. What is the magnetic field at a distance d from the center of the solenoid?

We begin by treating the solenoid as made of N turns of wire, each with a current i (they are concentric loops not connected to one another). We define $n = N/L$, where L is the length of the solenoid. n is the turn density. Since each turn has a current i, then $i_{net} = nidx$ Now, we analyze a small loop of width dx and a distance x from the center. Then we have:

$$dB = \frac{\mu_0}{2} \frac{(nidx)R^2}{(R^2 + (x-d)^2)^{3/2}}$$

To integrate, we first realize that

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}}$$

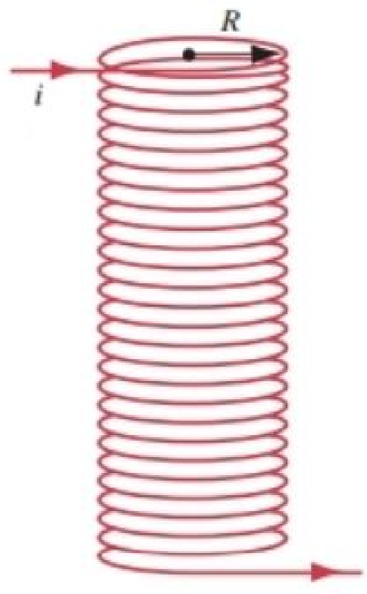


Figure 26: A solenoid

We did this by using a trig substitution of $x = a\tan\theta$. Check out Khan Academy for a video on this.

Now going back to our previous problem, we can make a u subsitution of $u = (x - d) \implies du = dx$. This allows us to apply the identity:

$$B(d) = \int dB = \frac{\mu_0 ni R^2}{2}\int_{-L/2}^{L/2} \frac{du}{(R^2+u^2)^{3/2}} = \frac{\mu_0 ni}{2}\Big[\frac{x-d}{\sqrt{R^2+(x-d)^2}}\Big]_{-L/2}^{L/2}$$

$$\frac{\mu_0 ni}{2}\Big[\frac{L/2-d}{\sqrt{R^2+(L/2-d)^2}} + \frac{L/2+d}{\sqrt{R^2+(L/2+d)^2}}\Big] \tag{8}$$

In the limit of a very long solenoid, $L \gg R$, we have that

$$B(d) = \mu_0 ni$$

In other words, the magnetic field is completely uniform within the solenoid. This also means that $B = 0$ outside an infinitely long solenoid. You can play around with a solenoid graph here: Desmos

9.5 Ampere's Law

Now comes the big daddy. Using the Biot-Savart law was cumbersome, and overly mathy. Now we have Ampere's law, which can help simplify situations with high symmetry, just like Gauss's law.

Ampere's law tell us:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i \tag{9}$$

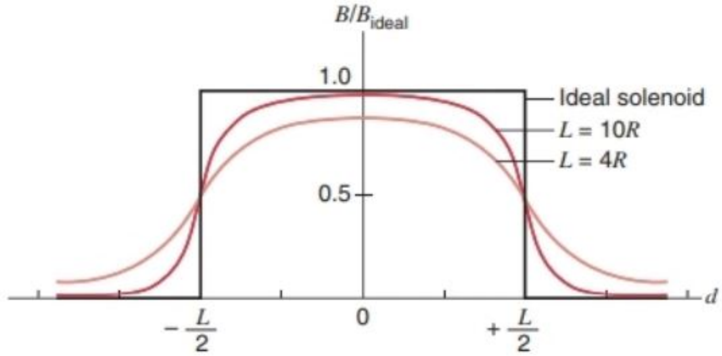


Figure 27: nonideal vs ideal solenoid

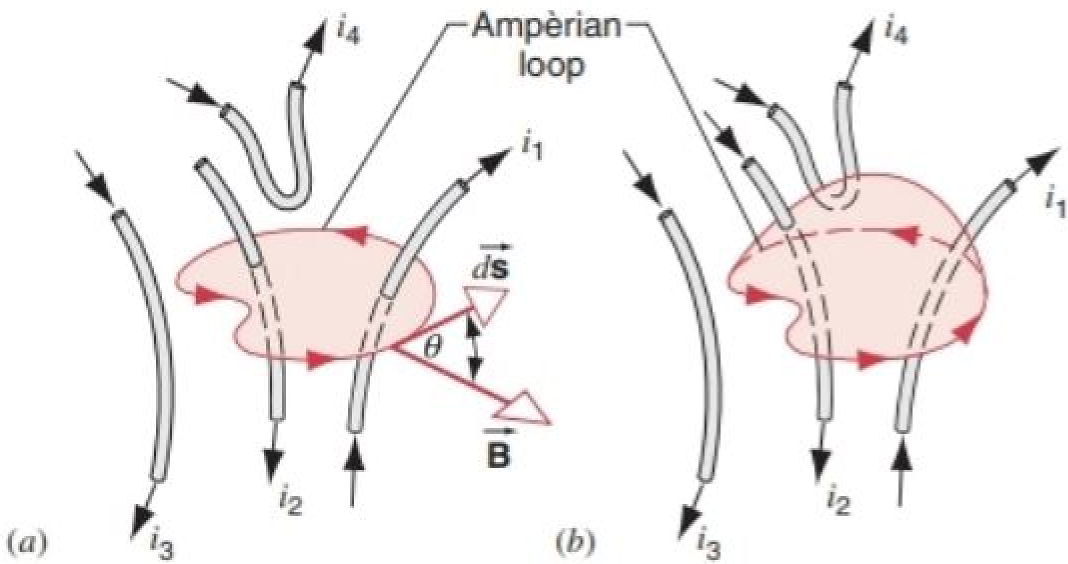


Figure 28: a) Amperian loop drawn, $i = i_1 - i_2$ b) Stretching the Amperian loop does not alter the value of the current

Similar to Gauss's law which uses a closed surface, Ampere's law uses a closed loop. Note that for Ampere's law to be useful in solving for B, B must be able to be factored out of the integral. Note in 28 b., stretching the loop to include i_4 adds both an $+i_4$ and $-i_4$ component.

Note that directions are taken with the right hand rule - since the closed loop has a direction associated with it. Since for this one in 28, it is ccw, currents pointing up are positive.

9.5.1 Applications

In all these, you can assume that we will be using 9. You are recommended to try these problems first, before looking that the solutions. I kept the solutions brief for this purpose.

Infinitely long wire We have:

$$B(2\pi r) = i\mu_0 \implies B = \frac{i\mu_0}{2\pi r}$$

We draw a circular loop around the wire, whose direction is in the same as $\vec{B}$. Note that we were able to factor B out of the $\oint$ due to symmetry. **Within a wire** Let the wire have radius R and the part within the wire has radius $r < R$. Thus, $i_r = i_R\frac{r^2}{R^2}$. Then

$$B(2\pi r) = \mu_0 i_R \frac{r^2}{R^2} \implies B = \frac{\mu_0 i r}{2\pi R^2} \propto r$$

Within a solenoid We construct a rectangle, with a length h. Let $n = N/L$, or the loop density. Then the number of loops is hn. The net current, since each loop has a current i is then hni. Then we have:

$$\oint \vec{B} \cdot d\vec{s} = Bh = \mu_0(hni) \implies B = \mu_0 ni$$

Within a toroid A toroid is simply a solenoid but bent into a circle. We can take an arc of radius $r_a < R < r_b$, such that $B \neq 0$ that is in the same direction as B. Thus, we have:

$$BR\theta = \mu_0(i\frac{N}{2\pi R}R\theta) \implies B = \mu_0 i\frac{N}{2\pi R}$$

Note that the field within the toroid is not uniform.

Outside a solenoid* While we often state, as I did earlier, that the magnetic field outside of a solenoid is zero, let's actually calculate it: We draw a circle tangential to the solenoid. Thus,

$$B2\pi R = \mu_0 i \rightarrow B = \frac{\mu_0 i}{2\pi r}$$

The B within the solenoid was $B = \mu_0 i\frac{N}{L}$. Taking the ratio:

$$\frac{B_o}{B_i} = \frac{1}{2\pi rn}$$

Note that n is the number of turns per unit length. But $1/n$ is the length per number of turns, or assuming that the distance between each coil is constant, is the distance between each turn, D. Thus,

$$\frac{B_o}{B_i} = \frac{D}{2\pi r} \ll 1$$

Outside an infinite plane

9.6 Different ref. frames

Consider a charge q at rest, with a wire of current i a distance r from q inducing a magnetic field B. We call this reference frame S_1.

Note that in S_1, only B exists, and $F_E = 0$, since $\lambda_+ + \lambda_- = 0$ as the wire is neutral.

Now let our reference frame be at a speed v_d such that the wire appears to have no current. Then, the electrons appear to be at rest (random thermal motion will cancel out any net force), while the positive ionic cores are moving at a speed v_d in the other direction.

In this second scenario, S_2, the charge still must be in equilibrium, so $F_{net} = 0$. This means that $F_B = F_E$. But we just said the rod was neutral, how could F_E exist? The answer is length contraction.

In situation S_1, $\lambda_- = \lambda_+$, while in S_2, $\lambda_+ > \lambda_-$.

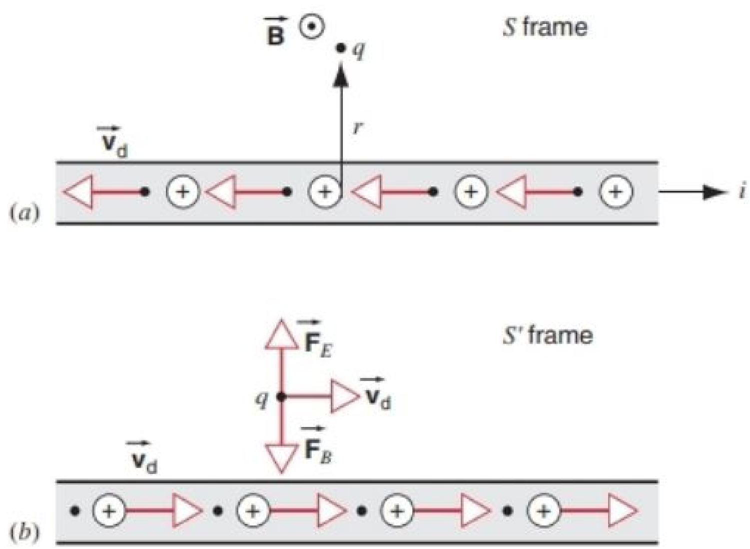


Figure 29: Different reference frames have powerful implications

9.7 MC Answers

1. D,C
2. A,C
3. A (rotating is effectively having a cw current)
4. D
5. E,A
6. C
7. A
8. C - consider a circular loop. Let the uniform magnetic field is pointing into the page, then from symmetry, $F_B = 0$. Since the force always points from the center of the circle, $\tau = 0$.
9. C
10. D From equation 8, we would find $B \propto z^0$ as $d \to \infty$, it seems to be equal to zero!? But if we take the special case of a circular loop, we would find $B \propto z^{-3}$. Since a solenoid is made up of multiple circular loops whose magnetic fields can be added together since they're in the same direction, then would they all be $B \propto z^{-3}$? This would make the net $B \propto z^{-3}$.

    Two contradictory yet seemingly correct approaches: So which is wrong here?

    The answer is that the solenoid formula is only valid for points
11. C, the Amperian surface is an infinitely large plane, with one side on the axis, and the others infinitely far away.
12. C
13. D
14. C, since without sufficient symmetry, then B would also exist outside the solenoid.

10 Faraday's Law of Induction

A changing magnetic field produces a current is essentially what the law states. Faraday's law:

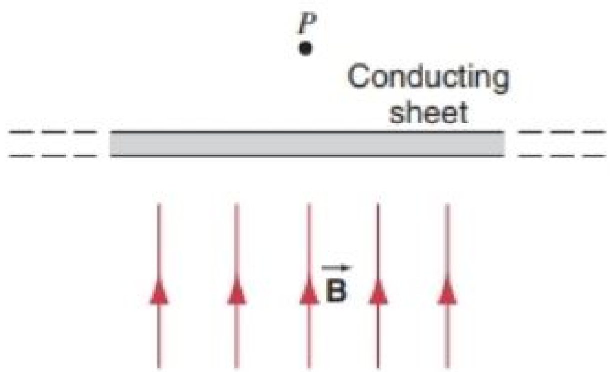
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where

$$\Phi = \int \vec{B} \cdot d\vec{A}$$

10.1 Memorable Problems

1. A copper disk is rolling down a ramp. In the middle of the ramp is a uniform magnetic field. Describe the balls' motion as it rolls down. E1) What if the magnetic field was angled with respect to the plane of the disk? E) Would anything change if the disk slid down instead of rolling (assume everything else is constant, such as area)? 2. In the diagram below, investigate what occurs when the magnetic field changes.



3. An semi-infinite long rectangle has its short side cut off, with its long side stretched to infinity. A rod is placed parallel to the short side. The short side has a battery that supplies a constant emf, then find the velocity of the rod as a function of time. What is its terminal velocity? Assume the rectangle is immersed in a uniform magnetic field B.

4. Consider a small area element on a rotating disk. If only that small area element is in a uniform magnetic field, what is the torque exerted by the magnetic field? Assume a conductivity σ.

Solution 1: The balls' motion can be split among four parts:

1. Outside of magnetic field
2. Entering magnetic field
3. Immersed in magnetic field
4. Exiting magnetic field

Outside of the field, its acceleration is $\frac{g\sin\theta}{1+\beta}$.

As it is entering, magnetic flux is increase, so the current must point counterclockwise, meaning that the magnetic field generates a force to the left, parallel to the plane (by symmetry).

As it is exiting, magnetic flux is decreasing, so the current must point clockwise, and the magnetic field still results in a force that points to the left parallel to the plane.

Completely within the magnetic field, there is no net motion and no emf generated.

E1) If the magnetic field were at an angle , then it would cause the disk to spin, resulting in a net torque but no net force as it is exiting and entering.

E2) The rolling makes no difference to the portions where the disk is interacting with the magnetic field.

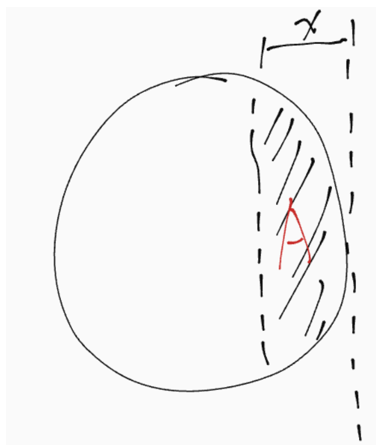
Now let's work things out symbolically. Let the magnetic field be B perpendicular to the disk's plane. Right before it enters the magnetic field, let the disk have a transnational velocity of v_0. The disk has a radius of R.

We have by Faraday's law:

$$i = \mathcal{E}/R = \frac{d\Phi}{dt}\frac{1}{R}$$

Since (letting x be the horizontal distance that the disk is immersed into the magnetic field)

$$A(x) = R^2 \sec\left(\frac{R-x}{R}\right)$$



Noting that $a = \frac{g\sin\theta}{1+\beta}$, with $I = \beta m R^2$, then we would have:

$$x(t) = v_0 t + \frac{1}{2}at^2$$

Thus,

$$\mathcal{E} = \frac{d}{dt}(BA(x)) = B\frac{d}{dt}(\sec(1 - x(t)/R)) = B\tan(1 - x(t)/R)\sec(1 - x(t)/R)(\frac{v_0 + at}{R})$$

Using this, we can find the current $i = \mathcal{E}/R$, which is too long to write out for now. But now, let's find the force on the loop of wire!

10.2 Questions about Lenz's law proof

So a moving magnetic field will produce a current in a loop equal to $\frac{d\Phi}{dt}$. But then how would we find the direction of that current? Let's guess that the current's direction is such that the magnetic field it produces aligns with the direction of increasing magnetic field (e.g. if it was a bar magnet with its north face heading heading from the right towards a loop of wire then the current would point clockwise (when seen from the right).

Question: Why are we ignoring the south pole? Wouldn't it affect the magnet, too?

10.3 Faraday's Law in-depth

Note that in Gauss's law, the dA is always in the direction of the outward normal, whereas in Faraday's law, the dA direction can have two possible directions.

Note that reversing the dA sign will reverse the sign of $\frac{d\Phi}{dt}$, so then wouldn't that change the direction of $\mathcal{E}$?. The answer to this is using a right-hand rule. Using your right hand, point your thumb in the direction of $d\vec{A}$, then curl your fingers around it. This is the direction of the current. Thus, the direction of current is independent of where you define the direction of $d\vec{A}$.

To summarize, if the $\mathcal{E}$ is positive, then its projection onto $d\vec{A}$ will be parallel. If negative, then it will be antiparallel. So the choice of $d\vec{A}$ is yours!

10.4 Motional EMF

Let there be a rectangular loop of wire with a width of D and a length of x. When we move the wire with a constant velocity, we find that

$$\Phi = BxD \implies \frac{d\Phi}{dt} = BvD$$

Now we will derive the power loss for such a loop of wire:

$$\mathcal{E} = BvD \implies P = \frac{\mathcal{E}^2}{R} = \frac{(BvD)^2}{R}$$

The other way is

$$P = F_1 v = (iDB)vc = BDv\frac{\mathcal{E}}{R} = \frac{(BvD)^2}{R}$$

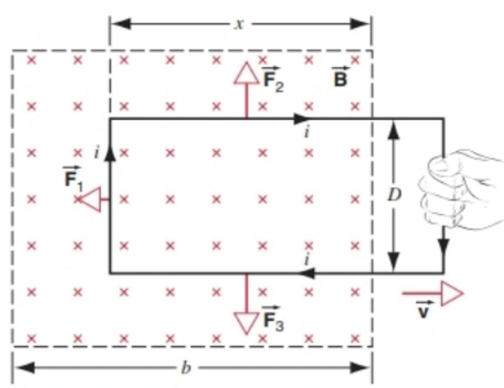


Figure 30: Moving loop of wire sets up a current in the loop

Problem: How should power dissipation be minimized in a changing magnetic field similar to the one shown above?

Solution: Let there be a uniform square metal sheet with constant resistivity ρ. For now, let us assume that the eddie currents (this is what they are called when formed on

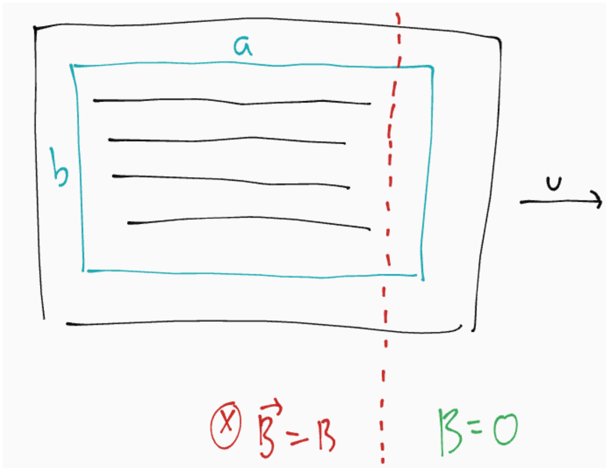


Figure 31: A labelled diagram for the problem

a flat sheet), are in the shape of squares. At this instant, let the metal sheet be moving out at a velocity of v. Then, we have that:

$$\Phi = xbB \implies \frac{d\Phi}{dt} = Bvb$$

Since $R = \rho L/A$, then assuming that the metal sheet has a small thickness t and that the current travels in way similar to a wire with a radius of t (a very bad worse than an order-of-magnitude approximation), we have:

$$R = \rho(2a + 2b)/(\pi t^2)$$

Thus,

$$P = \frac{\mathcal{E}^2}{R} = \frac{\pi(Bv)^2 t^2}{2}\frac{b^2}{a+b}$$

Now, assuming that we cut the metal sheet into tiny strips and separate each strip with an ideal insulator, we have that the above equation is the power for each strip. Since we have w/b strips (w is the width of the sheet, and b is the width of each strip), then

$$P_{net} = \frac{\pi(Bv)^2 t^2}{2}\frac{b^2}{a+b}\frac{w}{b} \approx \frac{\pi(Bv)^2 t^2}{2}\frac{b}{a} \ll 1$$

So if we wanted to reduce eddy current losses, we would insert insulators parallel to the direction of the velocity.

Remarks: While we began our analysis with assuming that the currents would flow in rectangular loops, when we compress it down into nearly 1-D lines, the circular loops would become rectangular. I also don't know a better approximation than πt^2, but perhaps you can imagine it to be a bunch of rectangular loops of wires.

In addition, eddy currents have many applications, including *magnetic breaking*.

10.4.1 Open Circuits

Starter: Consider a rod of width w and length l moving perpendicularly through a uniform magnetic field B, with a velocity v. What is the value of $\frac{d\Phi}{dt}$?

The answer is not zero! You may initially think that since $\Phi = wlB = \text{constant}$, then $\frac{d\Phi}{dt} = 0$. But there is in fact a way to overcome this hurdle.

Let us first imagine that we close the circuit - by adding a loop of wire that connects the top and bottom of the rod. But then if we consider an isosceles triangle, then we have:

$$\Phi = Bx^2 \tan(\theta/2)$$

Where $dx/dt = v$. Clearly, taking the derivative on both sides yields an expression that depends on x. So then how would we resolve this paradox? A conjecture:

Conjecture 1: When an unclosed loop of any shape is instantaneously moving at some velocity v, it can be thought of adding a loop of wire in such a way that if it were to continue with that velocity v, the loop would continue to be closed.

Questions: If this works, then why does it work? How would we know that adding the loop around it to complete the circuit doesn't affect the value of $\frac{d\Phi}{dt}$?

Answer: The "emf" is the net work done per charge when it makes a full revolution. Note that since the magnetic field is static, it cannot do any work on any part of the wire than the stick.

Example 1: A metal rod of length l is moving perpendicularly to a magnetic field with $v(t) = 5 + 3t^2$. Assuming a nonzero resistance R, find the induced current in the rod.

Solution:

$$\Phi = Blx \implies \frac{d\Phi}{dt} = Blv = Bl(5 + 3t^2) = \mathcal{E}$$

$$\implies i = \frac{Bl(5 + 3t^2)}{R}$$

Example 2: The same metal rod in E1 now has $l(t) = 8t^2$. What is the induced current as a function of time? Assume the same uniform magnetic field as the previous example. a) Constant velocity

Solution: We have two approaches, one using a rectangular loop temporarily at some time t, and another using a loop that conforms to the motion of the rod at all times.

METHOD 1: We want to draw a wire in such a way that it conforms to the length of the rod at a particular position at every time t. Since $l = 8t^2$ and $t = x/v$, we have:

$$l(x) = \frac{8}{v^2}x^2 \implies y(x) = \frac{4}{v^2}x^2$$

where y is the function of position for the top half of the loop. Then integrating,

$$A = \int y(x)dx = \frac{4}{v^2}\frac{x^3}{3}$$

Then,

$$\frac{d\Phi}{dt} = \frac{d}{dt}(\frac{8Bv}{3}t^3) = 8Bvt^2$$

METHOD 2: We introduce two methods in one, a wrong one and a right one. First try to see if you can spot what's wrong in the incorrect one.

At this instant, let the rod be moving at a velocity v and have a length $l(t)$ at a particular time t. Since $A = lx$ and $x = vt$, we have that

$$\frac{d\Phi}{dt} = B\frac{dA}{dt} \to \frac{dA}{dt} = \frac{(8t^2)(vt)}{dt} = 24vt^2$$

So what's wrong here? This is a very important question to ask - it's the reason why we introduced this conjecture in the first place.

The reason why this is wrong is that our formula for area $A = l(t)(vt)$ is an approximation (graph it out! - you'll find that it creates a rectangle). The correct formula is $A = l(t)vdt$, which aligns with our assumption that this is true at this *instant*. The rest of the derivation follows from above.

Example 3: A rod of length R and constant angular velocity ω is rotating in a uniform magnetic field pointing in the page. Find the $\mathcal{E}$ between the two ends of the rod.

Solution: We create a loop of wire that conforms to the rod's motion (so basically a circle). Then we connect a wire from the center to an edge of the circle. Then we can easily calculate the value of $\frac{d\Phi}{dt}$:

$$\frac{d\Phi}{dt} = B\frac{dA}{dt} = B\frac{d(\pi(\frac{1}{2}r^2\theta)}{dt} = \frac{1}{2}B\omega R^2$$

–

10.4.2 Fluid Analogy

Let us begin by placing ourselves in the rod's reference frame - the magnetic field has a velocity v. Assume that the direction of the magnetic field is the direction of fluid flow.

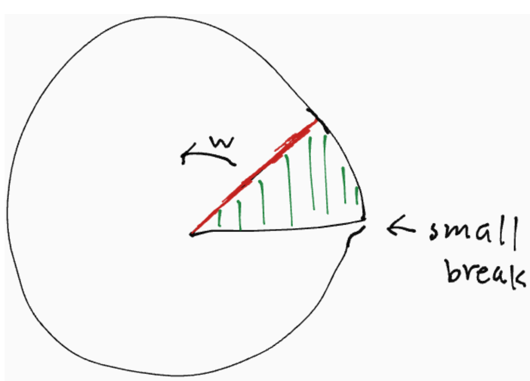


Figure 32: Wire traversing a circular loop

10.4.3 Motors

Example 1: A rectangular loop of wire with width a and length b is rotating with an angle as a function of theta $\theta(t) = \pi/2 + 14t$ with respect to the magnetic field. Find the force $F(t)$ from the rising steam on the loop of wire that causes this rotation. The loop is in a magnetic field which varies from $0 < l < b$ as Bl^3 into the page.

Solution: For such complicated problems, it is best to drop some conditions which make it difficult, for example assuming a uniform magnetic field instead of one that varies on length. Or a constant angular velocity. Then from there, we tackle that one individual component of the problem, which is much easier than tackling multiple at once.

Assume a uniform magnetic field, then

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Bab\frac{d(\cos\theta)}{dt} = Bab(\sin\theta)\frac{d\theta}{dt} = Bab\sin(14t)(14)$$

Note that this implies that the current generated in the loop is an AC current or that it is sinusoidal. This is because as its area projection on the magnetic field's perpendicular increases, flux increases, so current opposes the flux. But when the area projection decreases, flux decreases, so current is in the same direction as the magnetic field.

By conservation of energy, the power by the force $F(t)$ must be equal to the power dissipated by the current. Note that if there was an angular acceleration, then the two would not be equal. Then, $P - \mathcal{E}^2/R = \frac{d}{dt}(\frac{1}{2}I\omega^2)$. Using this,

$$\frac{d}{dt}\int(\tau d\theta = F\frac{b}{2}d(\pi/2 + 14t)) = \frac{d}{dt}(7Fbdt) = 7Fb$$

From now on, it's pure math:

$$P = \frac{\mathcal{E}^2}{R} \implies 7Fb = \frac{(14Bab\sin(14t))^2}{R} \implies F = \frac{(14Ba\sin(14t))^2 b}{7R}$$

Adding the unnecessarily complicated magnetic field seems boring now, so I'll skip it. But you should try it out for a fun challenge ;).

Exercise: Show that the torque on a spinning rectangular loop of wire with constant angular speed ω from a magnetic field always opposes its motion (e.g. infinite energy cannot be generated from this generator)

An electric works like a motor, but in reverse. An electric current is supplied through a (for simplicity) rectangular loop of wire such that the torque provided by the magnetic field is always in the same direction (how is this possible?).

But then would this imply infinite energy? The answer is that the magnetic field induces a current, known as the back current in the loop of wire, given by the same equation we derived earlier:

$$\mathcal{E} = Bab\sin(\omega t)\omega$$

So the faster the motor is, the greater the back current, until $\mathcal{E}_{ind} = \mathcal{E}_{gen}$

10.5 Connection To Electric Fields

We will consider a simple case to give you an intuition that $\mathcal{E} = \oint E \cdot ds$. Consider a circle that is filled with a constantly increasing magnetic field pointing into the page. Then, if we place a loop of wire within the uniform magnetic field with radius r (centered at the circle of the magnetic field), it would have an emf of

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

. This implies that electrons accelerate as they move around the loop of wire - which is only possible if there is an Electric field.

By symmetry, $|(|\vec{E})$ must be the same at all points along the circle and there must not be a radial component, since if we draw a Gaussian surface (like a cylinder), then

$$\int E \cdot dA = q/\epsilon_0 = 0$$

At all points along the cylinders' surface, the electric field is the same (by symmetry), so we can factor out an E from the integral to find $E = 0$.

When the electrons travel once around the wire, the work done by the emf must be equal to the work done by the electric field (since they are basically the same thing - emf's are caused by an electric field). Thus,

$$\mathcal{E}q = E(2\pi r)q \implies \mathcal{E} = E(2\pi r)$$

Now we do something illegal and equate $E(2\pi r) = \oint E \cdot ds$

But it should make sense, since $\mathcal{E} \equiv V = \int E \cdot ds$ and in order for the wire to form a current, it must form a loop. Thus, in this case $\int$ will now be $\oint$ so it must form a *closed* loop.

10.6 Mutual Inductance

Problem: Say that you have a closed loop C_1 with a current I_1. And another loop C_2 and a current I_2. They are placed near each other, what is the inductance of C_1?

You may have noticed that the inductance comes in two parts: the induction due to itself (self-inductance), and also the inductance from the second loop (mutual-inductance).

Recall that inductance was defined as:

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \implies Li = \Phi$$

Thus, the second loop causes an extra flux through the first one, resulting in the 'mutual inductance'.

To be rigorous with what we mean by mutual inductance, we must first show that the mutual inductance M_{12}, the mutual inductance on 1 by 2, to be a constant. You should try this yourself, and we'll do it below if you want to check:

Magnetic flux definition:

$$\Phi = \int B \cdot dA$$

Magnetic field by the loop 2:

$$B = \int \frac{\mu_0 i}{4\pi} \frac{dl}{r^2} \propto I_2$$

Thus, $M = (\int B \cdot dA)/I_2$, which in the equations above, depends only on the geometry of the two loops.

10.7 Different Reference Frames

Consider from a previous example, a rectangular loop of wire moving at a constant velocity v out of a magnetic field. Try two reference frames, and look at the hints if you feel stuck or that the situation is very ordinary. 1) The loop is moving at a velocity v 2) The loop is stationary 3) Neither is stationary

10.8 BONUS: Observations

Consider the situation similar to a previous example: a rod moving at a constant angular velocity within a magnetic field. Let this be S1 (situation 1). Now consider if we replaced the rod with a uniform magnetic field that fills up the space of the rod and the uniform magnetic field with a flat metal sheet. Let this be S2 (situation 2). Would the $\mathcal{E}$ be different now?

The answer is no. This is because after some time Δt, in both S1 and S2, the moving object sweeps out the same area. Since,

$$\frac{d\Phi}{dt} = \frac{dA}{dt} B$$

then we clearly must have that the resulting emf is the same for both. But wait! What do we mean by resulting emf? The net voltage between different points is different! This is a simple problem but illustrates an important point - always think about what you are *actually* solving for and what the quantities *truly mean*.

Question: We claim that adding a wire to a rod rotating at a constant angular velocity in a uniform magnetic field has no effect on the net $\mathcal{E}$. But why is this?

First, let's consider a rod moving at a constant veloicty v in a magnetic field pointing into the page. Since both positive and negative charges have $v_{net} = v$ within the rod, the positive charges move up the rod and the negative charges move down (in reality it would just be electrons moving down, but my point still stands). Thus, we would have a net positive charge at the top and a net negative at the bottom. This continues until F_E balances with F_B. When it's connected to a rod, the same thing occurs, only that there is no longer a E.

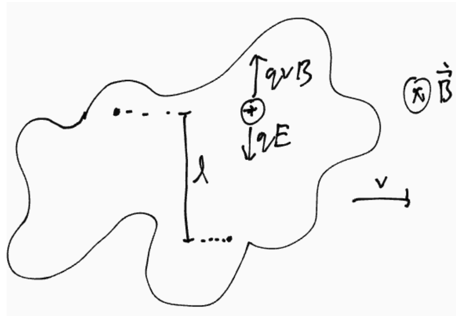


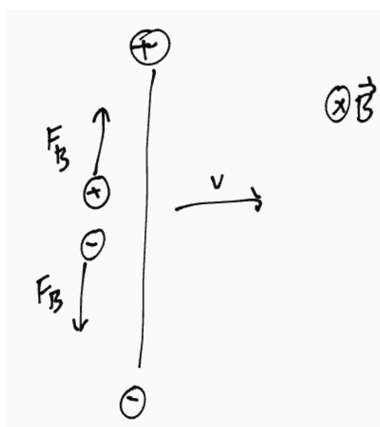
Figure 33: For this case, the emf between the two points is $\mathcal{E} = lBv$

So if we find that there is no difference in $\mathcal{E}$ for the two situations between the ends of the rod, then we are done - at least for this example.

First we tackle the isolated rod. Note that $F_B = qBv$ for every positive charge along the rod (first order approximation). When $\mathcal{E}$ has reached its final state,

$$F_E = F_B \implies qE = qBv \implies E = Bv$$

Since $V = \mathcal{E} = \int E \cdot dl = E \int dl = El = Blv$, the emf for both cases is indeed the same!



Note that this method extends to any arbitrary object. Let the conductor be instantaneously moving at a velocity v. Let's assume that within the arbitrary conductor, there is a charge q, which is moving at a velocity v in the direction of the conductor's motion. Following the same procedure, $E = Bv$, and the $\mathcal{E} = \int E \cdot dl = EL$ where L is the displacement (perpendicular to $\vec{B}$) of two points on the conductor.

And for the two points of interest, we could simply construct a rectangular loop of wire connected to these two points and find $\frac{d\Phi}{dt} = Bvl$

Question: But what if the object is changing dimensions? In that case, the imaginary rectangular loop of wire would no longer be valid?

10.9 MC Answers

1. A (must be continuous)
2. A
3. D At a certain time t and radius r, $\frac{d\Phi}{dt} = \frac{(k/r)(\pi r^2)}{dt} = \frac{d}{dt}(k\pi(ct)) \propto t^0$
4. C $\frac{d\Phi}{dt} = \mathcal{E} = i(t)R \implies \Phi = R\int i(t)dt = Rq \implies \Delta\Phi = \Phi_f - \Phi_i = R(q_f - q_i)$ So is the change in flux really proportional to R, the resistance? Would that make any sense? Not really, because if the resistance increased, the flux would not increase, but rather the net charge going through a cross section will decrease. But if the net charge through a cross section increases, resistance is constant, so the flux should increase (it actually happens the other way around - increased flux = increased charge through a cross section). Thus, the answer is **C** .
5. A, note that in the
6. A, since assuming that the coils are placed close to one another, $B = \frac{\mu_0 i}{2r}$, so $\frac{d\Phi}{dt} \propto \frac{di}{dt}$
7. C
8. A
9. B
10. C
11. A, A (magnetic field is stronger at the lower part), C
12. E, ($\mathcal{E} = iR = BA\omega\sin(\omega t)$, where ω is the angular frequency of the person's cranking)
13. C, $\mathcal{E} \propto \omega \implies P \propto \omega^2$
14. C, this is congruent to asking if the angular velocity has any effect on the net torque on a loop of wire. But note that torque is caused by the magnetic force, not the changing magnetic flux, since $F_B = iL \times B$ on each side of a rectangular loop.
15. A, For the torque to always point in the direction of rotation, the current must flip after each time when the loop of wire is parallel to the $\vec{B}$. So we find

$$P \propto \frac{\mathcal{E}^2}{R} = \frac{(\mathcal{E} - \mathcal{E}_{ind})^2}{R}$$

Note that

$$\frac{d\omega}{dt} \propto \tau = iAB\sin\theta$$

And

$$\mathcal{E}_{ind} = \frac{d\Phi}{dt} = A\frac{d}{dt}(B\cos(\omega t)) = A(B\sin(\omega t))(iAB\sin(\omega t)$$

$$\propto i = \frac{\mathcal{E} - \mathcal{E}_{ind}}{R} \implies \mathcal{E}_{ind} \propto \mathcal{E}$$

So assuming that overcharged battery means a higher emf, then $P \propto \mathcal{E}^2$ so it must be A.

16. C

17. A (process of elimination, clearly if we place a circular loop of wire in the solenoid, facing towards the axis, then $\mathcal{E} \neq 0$, so $E \neq 0$. B,C,D, and E would all mean that there is no current in such a loop of wire,

18. Clearly, E is incorrect since if we placed a loop of wire in the changing magnetic field, then it would have an emf - indicating $E \neq 0$. C and D cannot be correct, since by symmetry, the electric field is symmetric about the axis of the wire, and if we had a rectangular loop of wire, with one of its sides parallel to the wire, would mean that $E_{left} = E_{right}$, so $\mathcal{E}_{net} = 0$. Thus, the only possible answers are A and B. Intuitively, the answer is A, since B implies a stronger electric field at further distances away. But at the same time (for a rectangular loop of wire with the left side a distance r from the current and a length of l and width D):

$$\oint E \cdot ds = \frac{\mu_0 k D}{2\pi}\ln\left(\frac{r+l}{r}\right)$$

So the greater l is, the greater E is, since

$$\oint E = D(E(r) - E(r+l))$$

Note that if we streched the loop of wire to infinity, keeping the left end a distance r away, then $\oint E \cdot ds \to \infty$, so then

$$D(E(r) - E(\infty)) = \frac{\mu_0 k D}{2\pi}\ln(\infty) = \infty \implies |E(\infty)| = \infty$$

The last step was done by assuming that $E(r)$ is continuous.

Claim: $E(r)$ is continuous and increases with distance.

Proof: Since we have

$$|D(E(r+l) - E(r))| = \frac{\mu_0 k D}{2\pi}\ln\left(\frac{r+l}{r}\right)$$

Let $l = dr$. Then,

$$|D(E(r+dr) - E(r))| = \frac{\mu_0 k D}{2\pi} \ln(1 + dr/r) \approx \frac{\mu_0 k D}{2\pi}(dr/r)$$

Simplifying and dividing by dr on both sides,

$$|\frac{dE(r)}{dr}| = \frac{\mu_0 k}{2\pi r} \implies |E| = \frac{\mu_0 k}{2\pi} \ln(r)$$

Thus, E increases with distance and the answer is B.

Remark: Would this really make sense? If the electric field increases with distance, then would that not imply a very large electric field very far away? Let's revisit our argument and attempt to make sense of it.

11 Magnetic Properties of Materials

We define the magnetic dipole moment to be $\mu = Ai$. For a circular loop of wire, the magnetic field can be written as (at a distance far away):

$$\frac{\mu_0 \vec{\mu}}{2\pi z^3}$$

This is also the magnetic field produced by any magnetic dipole. Since the torque on a loop of wire is given by

$$iA \times B = \vec{\mu} \times B$$

Thus, we can find the potential energy:

$$U = -\int \mu B \sin\theta(-d\theta) = -\mu \cdot B$$

The $-d\theta$ term comes from the fact that we define $U = 0$ when the magnetic moment is completely aligned with the magnetic field. So the angle θ is defined to be the angle to the left or right

This makes sense, since the potential energy should be the lowest when $\theta = 0$, since there is no torque and any perturbation will only bring the loop back. On the other hand, $\theta = \pi$ would mean that any small perturbation causes the loop to gain a large amount of kinetic energy from the torque. However, the potential energy is not conservative in general, so most of the time, U is irrelevant.

11.1 Memorable Problems

1 (warm up). A circular loop of wire carries a current i, with a uniform magnetic field pointing into the page. a) What should i point in (ccw or cw) for the loop of wire to be stable? b) Assuming it is stable, if we tilt the loop of wire such that one end dips down and one end levels up, what is the angular frequency of the resulting oscillations (assume only a small tilt)? c) Do oscillations imply stability? Does it work the other way around?

11.2 Forces on Dipoles from Nonuniform Magnetic Fields

Within a uniform magnetic field, dipoles cannot have a net force exerted on them - a great analogy are electric field dipoles. For many scenarios, finding analogies between the two is very useful for intuition.

One way to 'prove' this is by considering a circular loop of wire with a constant current i immersed in a uniform magnetic field. Note that the net magnetic force from the magnetic field on the wire loop must be zero (see F_B from moving charges section).

Exercise: Two current carrying circular loops of wire have currents in the same direction. a) Are they attracting or repelling? b) Calculate the force on each loop of wire

11.2.1 Induced Dipoles

Consider two circular loops of wire placed very close to one another. They have currents of the same magnitude but of opposite direction. What occurs when a North pole moves towards it? Try to answer this yourself.

The answer is that they get repelled. This is because the induced current aligns with the current of the loop whose B repels the magnet but opposes the other one. Thus, the B is strengthened in the opposition to the magnet!

Note that one way to verify this is

$$-\frac{\mathrm{d}}{\mathrm{d}z}U = -\frac{\mathrm{d}}{\mathrm{d}z}(-\mu B) = \mu\frac{\mathrm{d}z}{\mathrm{d}B}$$

Since (in the two loops reference frame) $\mu < 0$ and $\frac{\mathrm{d}}{\mathrm{d}z}B < 0$ (the magnitude of B decreases with distance from the bar magnet). then $F > 0$ mean

11.3 Within Atoms

Atoms have an intrinsic magnetic moment, which we can (classically) argue to be caused by the current loops created by the electrons. Consider an electron that spins about a radius of r, velocity v and charge e. Thus, the current is: $i = \frac{e}{2\pi r/v}$ And magnetic moment: $\frac{e}{2\pi r/v}\pi r^2 = \frac{erv}{2} = \frac{e}{2m}mvr = \frac{e}{2m}l$

The fundamental unit for the magnetic dipole moment is the Bohr magneton. This is because angular momentum is measured in $h/2\pi$. The Borh magneton is:

$$\mu_B = \frac{e}{2m}\frac{h}{2m} = \frac{eh}{4\pi m}$$

Electrons have intrinsic spin equal to μ_B. Thus, the total spin of an atom is the vector sum of the orbital spin and the electrons' intrinsic spins. Note that in the equation above, to calculate the intrinsic spin, we use $m = m_p$ or the proton/neutron mass. *Nuclear magnetic resonance* is when a nucleus is exposed to a specific EM wave frequency that causes it to flip its magnetic dipole.

11.4 Magnetization

Similar to electric fields, materials exposed to a magnetic field generate an induced magnetic field. Recall that for electric fields, the polarization field is:

$$E = E_0/k_e$$

where k_e is a number greater than or equal to 1.

We define a quantity, the magnetization, $\vec{M} = \frac{\vec{\mu}}{V}$

The net magnetic field is:

$$\vec{B} = B_0 + B_M$$

This, in general is difficult to calculate, but for a long solenoid, we have that $B_M = \vec{\mu}_0 \vec{M}$. In weak fields, M increases linearly with B_0, so $B = k_m B_0$ Thus, $B_m = \mu_0 M = (k_m - 1)B_0$ For non ferromagnetic materials, $k_m - 1$ is very small, so the magnetic contribution is very small.

11.4.1 Types of magnetism

Paramagnetism

This occurs when each atom has a permanent magnetic dipole moment that is nonzero. From our first section, we discovered that such dipole moments would align with the magnetic field, since the U vs x graph is concave down with a local minimum when their directions align. Since in the beginning, such materials are in random thermal motion, the net dipole moment is zero.

So the dipoles align with the magnetic field when the material is exposed to one. This induced magnetic field, $B_M = \mu_0 \vec{M}$, is inversely proportional to the temperature. This is known as Curie's law:

$$M = C\frac{B_0}{T}$$

C is known as Curie's constant. Note that this formula only works if B_0/T is small.

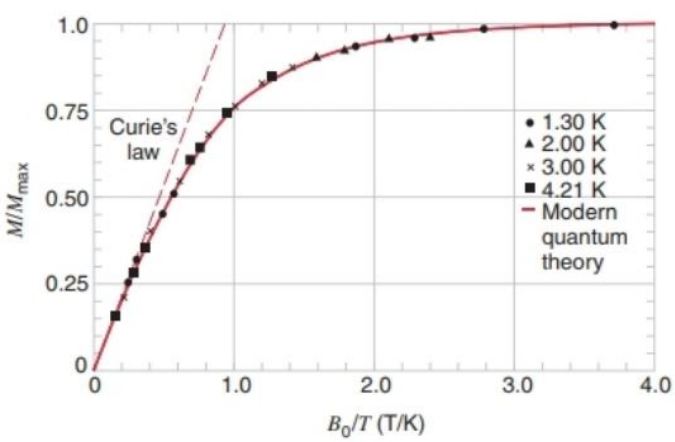
Problem: Determine the maximum value of M. Consider its definition: $M = \frac{\vec{\mu}}{V}$.

Solution: Its maximum occurs when all the dipoles are perfectly aligned with the magnetic field. If there are N dipoles of the same material, then

$$M_{max} = \frac{\sum \vec{\mu}}{V} = \frac{N\mu}{V}$$

This is because each of the $\vec{\mu}$ are parallel, so vector addition results in adding their scalar quantities.

This point is known as 'saturation', as increases in the magnetic field don't increase the magnetization.



This can result in (I seriously need to learn some thermodynamics) *adiabatic demagnetization*, where magnetizing the dipoles results in a release of heat to the environment. If the sample is then isolated, and demagnetized, its temperature will decrease (assuming adiabatically) since its magnetic energy increases (recall $U = -\mu \times B$).

Diamagnetism

Definition: a material with no magnetic dipole has an induced one by a magnetic field that causes it to repel. **Ferromagnetism**

Similar to paramagnetism, atoms have permanent magnetic moments, but after the application of the magnetic field, the dipoles have a strong enough interaction between them that they do not randomize again.

At a certain temperature, however, known as the Curie temperature, the ferromagnetic material becomes paramagnetic. It does not increase linearly with B_0 or $\vec{M}$ even at small values and its relative permeability, k_m, generally depends on the applied field.

Unlike paramagnetism, saturation occurs not when all the dipoles are aligned but rather when the *magnetic domains* are aligned. Each iron atom consists of multiple magnetic domains where coupling of atomic magnetic dipoles produces near perfect alignment - with or without a magnetic field. When exposed to a magnetic field, two things can occur:

1. Dipoles of neigboring domains swing into alignment of a particular domain
2. dipoles of nonaligned domains swing in alignment with the magnetic field.

When the domain is removed, the domain walls do not move completely back to their original positions.

Solution 1: a) In general, the current density should align with the magnetic field, so it must be clockwise.

b) We will assume[11] that $\tau = iA \times B$. Thus,

$$|\tau| = iAB\sin\theta \approx iA\theta = I(\omega^2\theta) \implies \omega^2 = \frac{\pi r^2 iB}{I}$$

Since $I = \frac{1}{2}mR^2$ (by the perpendicular axis law), we find: $\omega = \sqrt{\frac{2\pi iB}{m}}$

c) Oscillations imply that there is always a force pointing towards the equilibrium - where the force is zero ($\theta = 0$ for this case). So if $x < x_{eq}$, then the slope of the U vs x graph is negative to have a positive force. Similarly, if $x > x_{eq}$, then $\frac{dU}{dx} > 0$ for a negative force. Stability and oscillations are two sides of the same coin - one will always imply the other. But note that objects tend to be stable only for a certain range.

12 Inductors

Problem: Consider a solenoid with n turns. You are given that the emf can be written in this form:

$$\mathcal{E} = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

[11] I hope that someday I will find/discover a convincing proof of this

Find the value of L in terms of some constants, n and the cross-sectional area. Assume that the equation deals only with the magnitudes, and not the signs. **Hint:** Faraday's Law of Induction

Note that the value of L that you obtained is a negative number! What does this negative sign signify? To gain some insight, let us consider inductors as a circuit element. Let the current be moving from its left to the right. If $\frac{\mathrm{d}i}{\mathrm{d}t} > 0$, then the electric field will point clockwise when viewed from the right. The induced current must point to the left, but then how does this involve the induced electric field?

If we change the coils from ccw to cw or back would that change the direction of the induced current?

The answer is no. If we swapped from ccw to cw, then the direction of the magnetic field is in the opposite direction, thus the value $\frac{\mathrm{d}\Phi}{\mathrm{d}t}$ is the negative of its original value. While this does mean that the induced current will flow from ccw to cw, the cw direction is still against the original current. Thus, the induced current will still be to the left.

12.1 Memorable Problems

Problem 1: A circuit with a very large inductance value and low resistance is turned on. As the current slowly reaches its transient state, a switch is flipped and an arc of electricity flies from the switch to the circuit as it is being flipped. Why? (Hint: time constant)

13 AC Currents

Consider a driven circuit consisting of inductors, resistors, and capacitors in series. How should we analyze such a circuit?

Perhaps one of the best ways is through superposition. Note that the general equation for such a circuit (by Kirchoff's laws):

$$\mathcal{E} = L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C} + iR$$

13.1 MC Answers

1. D

2.

14 Math

14.1 ODE

"To the ode we pray, for the ode we will pay"

Say we want to find the solution to a differential equation of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t)$$

14.2 Hyperbolic Trig Functions

We define hyperbolic trig functions to be such that $x^2 - y^2 = 1$. While there are many functions that can substitute x and y, the hyperbolic ones are:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

These have some interesting properties, such as:

$$\frac{\mathrm{d}\sinh(x)}{\mathrm{d}x} = \cosh(x) \quad \frac{\mathrm{d}\cosh(x)}{\mathrm{d}x} = \sinh(x)$$

In addition, the arc length over any finite integral for the $\cosh x$ function is always equal to the area under the curve:

$$\int_a^b \cosh(x) = \int_a^b \sqrt{1 + (\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x))^2 dx}$$

14.3 Linear Algebra

14.3.1 Vectors

An example of a vector $\vec{v}$ in $\mathbb{R}^2$

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

A linear combination of a set of vectors $\{v_1, v_2, \ldots v_n\}$ is defined to be

$$c_1v_1 + c_2v_2 + \ldots c_nv_n$$

The span is defined as the set of vectors that can be created through a linear combination of some number of vectors.

Claim: The span of n unique, independent vectors is $\mathbb{R}^n$. **Proof:** This is trivial to see - say that we want to create a vector $V \in \mathbb{R}^n$ with dimensions of $a_1, a_2, \ldots a_n$. Each one of these dimensions has a linear equation to it - so we have n linear equations and n unknowns (being $c_1, c_2, \ldots c_n$). Thus, it is possible to generate a single or set of solutions to these equations.

Linearly dependent/independent

We will be using the following vectors as examples:

$$\vec{v_1} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} c \\ d \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} e \\ f \end{bmatrix}$$

If $\vec{v}_1 = k\vec{v}_2$, then span$(v_1, v_2) = kv_1$, so adding $v_2 = kv_1$ adds nothing to the span - the span is still just the scaled up version of v_1. On the other hand, if $\vec{v}_3 \in$span(v_1, v_2), then the same thing happens! This is not hard to see or prove - if $c_1v_1 + c_2v_2 = v_3 \rightarrow (c_a + kc_1)v_1 + (c_b + kc_2)v_2 = c_ac_1 + c_bc_2 + kc_3$. This is $\equiv$span(v_1, v_2, v_3) $\equiv$span(v_1, v_2)

Another way to phrase this is: $0 \in$span$(v_1, v_2, \ldots v_n)$ if some constant $c_i \neq 0$. I will leave the proof as an exercise (translation: prove it yourself! Right now!)

Example: If we have n vectors each of dimension $\mathbb{R}^m$, where $m < n$, then prove that at least one pair must be dependent

Proof: FSC, assume that all the vectors are independent, then m of them must also be independent. Clearly, span$(v_1, v_2, \ldots v_m) = \mathbb{R}^m$. This is because for a linear combination to give a vector V of m dimensions, we must have at least m equations. For the other vectors, however, are $\in \mathbb{R}^m$, which implies that $v_{m+1}, v_{m+2}, \ldots v_n \in \mathbb{R}^m$ =span$(v_1, v_2, \ldots v_m)$ or that it is possible to express these $n - m - 1$ vectors in terms of the other m ones.

Subspaces A *subspace* of $\mathbb{R}^n$ is a subset that has both *closure under scaling* and *closure under addition*. For example a subspace V would have respectively:

- $v_1 \in V \rightarrow cv_1 \in V$ (closure under scaling)
- $v_1, v_2 \in V \rightarrow v_1 + v_2 \in V$ (closure under addition)

Note that both rules imply that the zero vector *must* $\in V$.

An example of a subspace is the zero vector. Another is $\mathbb{R}^a, a < n$ for $\mathbb{R}^n$. **Example(s)** Determine if the below subsets are subspaces in $\mathbb{R}^3$

1. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x, y \equiv 0 \mod 2 \right\}$
2. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x = 0 \right\}$
3. S =span(v_1, v_2, v_3), where $v_i \in \mathbb{R}^3$ for any v_1, v_2, v_3

Problem: Prove or disprove that span$(v_1, v_2, \ldots v_n)$ is always a subspace of span$(v_1, v_2, \ldots v_m)$ where $m > n$. If it is incorrect, then how would you change it to make it right?

Basis of subspaces

Let V be a subspace and define it as V =span$(v_1, v_2, \ldots v_n)$ where the only solution to $c_1v_1 + c_2v_2 + \ldots c_nv_n = 0$ is $c_1 = c_2 = c_3 = \ldots c_n = 0$.

Then we define the basis of V, S, to be a set of linearly dependent vectors such that adding any other vector v_{n+1} to S will make it dependent. So in other words, S is a span of vectors that spans V which has the properties stated in the sentence above.

Problem: Find the basis of $\mathbb{R}^2$.

Solution: We claim that the basis of $\mathbb{R}^2$ is

$$\text{span}\left(\left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\}\right) \quad \text{(The two vectors are independent)}$$

Note that $\text{span}(v_1, v_2) = \mathbb{R}^2$. Note that taking away any vector won't satisfy this condition since the span of any single vector is simply $\mathbb{R}^1$. Thus, we are done.

Dot Product The dot product of two numbers is defined as:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 \dots a_n b_n$$

Note that $a \cdot b = ab\cos\theta \to \cos\theta = \frac{a \cdot b}{ab}$ **Problem:** Prove that the dot product is associative and commutative.

Problem: Prove Cauchy Schwartz's identity:

$$|v_1||v_2| \geq v_1 \cdot v_2$$

Solution: We have:

$$0 \leq (|v_1|v_2 - |v_2|v_1)^2 = 2(v_1 v_2)^2 - 2|v_1||v_2|v_1 v_2 \to |v_1 v_2| \leq |v_1||v_2|$$

Problem: Prove the Triangle inequality:

$$|\vec{x} + \vec{y}| \leq \vec{x} + \vec{y}$$

Solution:

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + 2\vec{x}\vec{y} + |\vec{y}|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

$$\implies |x + y| \leq |x| + |y|$$

Note that the second step was done in combination of the Cauchy Schwartz inequality:

$$\vec{x} \cdot \vec{y} \leq |xy| \leq |x||y|$$

14.4 Divergence and Curl

Divergence: The divergence of a function is defined as:

$$\vec{\nabla} \cdot \vec{F}$$

It essentially assigns each point with a scalar quantity. And we have

I highly recommend checking out this article for some awesome intuition!

The most intuitive analogy is in fluid mechanics. When there is a 'source', $\vec{\nabla} \cdot \vec{F} > 0$ in the region/point we are analyzing (density decreases). When there is a 'sink', $\vec{\nabla} \cdot \vec{F} < 0$ (density increases since it build up in the sink) and when there are neither sources nor sinks then the fluid is *continuous*. This is because sources have vectors directed away

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \left(\frac{\partial f}{\partial \mathbf{x}} \right)^{\mathsf{T}}$$

Figure 34: Stolen from Wikipedia

from them and sinks have vectors directed towards them. For $\vec{\nabla} \cdot F = 0$, then we have that the density is constant.

Curl: Qualitatively, curl is the tendency of a group of particles to rotate about a region in a vector (velocity) field. For counterclockwise rotation, we would want $\frac{\partial v}{\partial y} < 0$ so for $y > 0$, the velocity

Problem: Find the divergence of the function:

$$\vec{v} = \frac{\hat{r}}{r^2}$$

Solution: We write

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{\vec{x} + \vec{y} + \vec{z}}{r^3}$$

$$\to \nabla \cdot \vec{v} = \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial z}\frac{z}{r^3}$$

Note that

$$\frac{\partial}{\partial x} x(x^2 + y^2 + z^2) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-3x^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

By symmetry, the other equations with $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ will simply have y or z replace x. Thus, by symmetry we find that

$$\nabla \cdot \vec{v} = 0$$

However, you may have noticed, when sketching out the graph, that the divergence is certainly not zero at the origin! Instead, it's infinite, which we will see why soon[12].

14.4.1 Stoke's Theorem

The proof of it is long and hideous, and utilizes math out of the scope of these notes (and my knowledge admittedly). So here it is:

$$\int\int (\nabla \times F) \cdot d\vec{A} = \oint F \cdot dl$$

[12] hopefully

This is true for any continuous vector field in $\mathbb{R}^3$ Note that $\int\int$ stands for the surface integral (of some vector field) and $\oint$ stands for the closed loop. Thus, for an electric field we have:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \rightarrow \int_a^b E \cdot dl = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \rightarrow \oint E \cdot dl = 0$$

The last step is done by definition since $a = b$ in a closed loop (a is the starting position, b is the final position in our line integral. In a closed loop, the particle begins and ends at the same position). Thus, applying Stoke's theorem, we have

$$\int\int (\nabla \times \vec{E}) \cdot d\vec{A} = \oint E \cdot dl = 0$$

$$\rightarrow \nabla \times E = 0$$

While we proved this only for a point charge, we can break up any charged solid into a bunch of point charges and sum:

$$\nabla \times E = \sum \nabla \times E_i = \sum 0 = 0$$

So $\nabla \times E$ is zero for any orientation of charges, regardless.

A nice video on the intuition on Stokes's law (not the electric field derivation) is introduced in this video. Another way to see this is by drawing the field lines from a charge q and noting that there is no rotation about any point if a group of particles was spread throughout the space.

14.4.2 Linear Systems

Matrices are simply linear sysems of equations but on steroids. They are not necessarily the best method, but are useful for computers to compute solutions to large complex linear systems.

For example,

14.5 Dirac Delta Function

The dirac-delta function is defined as

$$\begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{10}$$

Furthermore,

$$\int_{-\infty}^{\infty} D(x)dx = 1$$

You may wonder why define such an odd function, but its applications are:

- Instantaneous impulses

15 Temperature

This will be a short section. If we want to be rigorous with our definition of 'temperature', then we define it to be a property such that if two objects have the same temperature and are placed next to one another, no heat flows between them. This is effectively the 0th law of thermodynamics.

The standard of temperature is set at the triple point of water, where thermometers will agree.

15.1 Thermal Expansion

When subject to an increase in temperature, it makes sense that a substance increases in dimensions. But why? Imagine the solid to be a set of springs. When you increase the temperature, the thermal vibrations also increase, which stretches the bonds further out.

But is this argument enough/convincing enough? No, imagine if such a temperature change resulted in just a bigger amplitude of oscillation. Then would the object increase in size? Try to figure this out yourself. [13]

16 Ideal Gasses

We study gasses as being particles, ignoring quantum or relativistic effects.

Properties of ideal gasses:

- Gasses are in random motion and obey Newton's Laws
- The number of gasses is 'large' - well governed by statistical behavior, for example averages work quite well (e.g. pressure on a wall)
- The gasses have negligible volume compared to their environment
- No forces act on the molecule other than when it collides
- All collisions are elastic and instaneous

Question: A wall has an area A. Gasses around the wall have an average speed v_0 and have a density ρ. What is the force on the wall? (or pressure) The gas is enclosed in a cubical box with side length L.

The force on the wall is:

$$\frac{\Delta p}{\Delta t} = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}$$

[13]**Hint**: Average change?

How would we calculate the average value for v_x^2? Note that this is not the same as the square of the average value of v_x (test a few numbers e.g. 1,2,3,4,5). We can calculate this quite easily with a clever observation:

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

Note that since the motion is entirely random, on average

$$v_x^2 = v_y^2 = v_z^2 \implies v_x^2 = \frac{1}{3}v^2$$

.

Extension: Find the average value of v_x.
Bogus solution:

$$v^2 = v_x^2 + v_y^2 + v_z^2 \implies v_x^2 = \frac{1}{3}v^2 \implies v_x = \frac{1}{\sqrt{3}}v^2$$

Why: Why is the above solution incorrect? The answer has to do with averages. Note when we say v_x^2, we mean $(v_x^2)_{av}$. So clearly, we cannot take the square root of the entire quantity! The two operations $_av$ and ^{2} are not associative (order matters)!

So then how would we find $(v_x)_{av}$? Note that the only time a gas will hit a wall is when its v_x is directed towards it. Furthermore, the set of all possible velocities the gas can take on takes on a sphere (same velocity magnitude vector can point in any direction). We chop off half (by the previous reasoning), and now the problem is to simply find the weighted average of the hemisphere.

To do understand what we are doing, imagine the hemispherical shell to have a uniform mass density. The mass of each small part is effectively the 'velocity' of the particle. We now try to find the vertical distance of the center of mass from the bottom of the hemisphere. This will give us the average velocity, since we are averaging the masses which are effectively the velocities. It is well known that this is $R/2 \equiv v/2$.

16.1 Memorable Problems

1. Why don't all gasses in the air have the same velocities? What observation supports this?

2.

16.2 Mean Free Path

For this analysis we will assume that the velocity of the molecule is much larger than Let the effective radius of each molecule be d. Then if the centers of two molecules come within $2d$ of each other, then there will be a collision.

If we want to find the average path the molecule of gas travels before it hits another molecule, then we realize that:

$$\lambda = \frac{\text{Length of a path in a time t}}{\text{Total number of collisions in that path}}$$

Thus, in a time t, the molecule travels a total length of vt, where v is the average velocity of the molecule. The total number of molecules it encounters is the number of gas molecules in a cylinder of radius d and length vt:

$$(\frac{N}{V})(\pi d^2)(vt)$$

Note, however, in this calculation we assumed that the other gasses are on average, stationary, This would only be valid if we were in the reference frame of the average relative velocity between two gass molecules (the one that sweeps out the cylinder and the next one it is just about to hit). First we plug in the symbols into the equation, obtaining:

$$\frac{vtV}{N(\pi d^2)(v_{rel}t)}$$

To calculate the average relative velocity:

Consider two gasses, of v_1 and v_2 respectively. The relative velocity between them is the vector sum $v_1 - v_2$. Note that we can write:

$$v_{rel} = \sqrt{v_{rel} \cdot v_{rel}} = \sqrt{(v_1 - v_2) \cdot (v_1 - v_2)}$$

$$= \sqrt{v_1^2 + v_2^2 - 2v_1 \cdot v_2}$$

Note that since the angle θ between v_1 and v_2 is equally likely to be in $[0, 2\pi]$, on average, $< 2v_1 \cdot v_2 >= 0$. Thus, because on average $v_1 = v_2$,

$$v_{rel} = \sqrt{v_1^2 + v_2^2} = \sqrt{2}v$$

The maxwell-boltzmann distribution governs the distribution of molecule speeds. It relates $N(v)$ to v. $N(v)$ is not the number of molecules but is given such that $N(v)dv$ is the number of molecules that have a velocity v in the range of v and $v + dv$.

The equation is very hairy and not fun to write.

We can also derive the maxwell-boltzmann distribution for energies. Note that the number of molecules between E and $E + dE$ is the same as the number of molecules between v and $v + dv$. Thus, $N(E)dE = N(v)dv$. Then we would use $v = \sqrt{2Em}$ to solve for dv/dE

16.3 Volume Correction

For true gasses, they take up a nonzero amount of volume, which reduces the amount of free space in the container. Consider a single gas molecule B that is instantaneously moving at a velocity v. In front of it let there be a molecule. Assume all molecules of the gas are hard spheres of radius r. Thus, B cannot move within a distance of $2r$ within the center of molecule A. But why is it a hemisphere? Not a sphere? This question still bugs me, and I have a feeling it has something to do with the way we are analyzing things (translation: statistics).

However, it should be clear that each molecule takes up the same amount of space, V_0, so the volume taken up by gasses should be $NV_0 \propto n$.

16.4 Pressure Correction

Molecules exert an attractive force on all nearby molecules, which we can approximate to be only within a distance R away. A molecule in the middle of a container would have no net force, since by randomness, there are just as many molecules on one side of it as the other (uniformly distributed).

However, molecules within a distance R away from a wall have a net attractive force away from the wall. This is proportional to the number of molecules within a hemisphere of radius R, or n/V. The net reduction in pressure for the *entire* wall is proportional to all the number of molecules along the *entire* wall, which is also proportional to n/V. Multiplying the two, we have the net reduction in pressure is $\propto (\frac{n}{V})^2$.

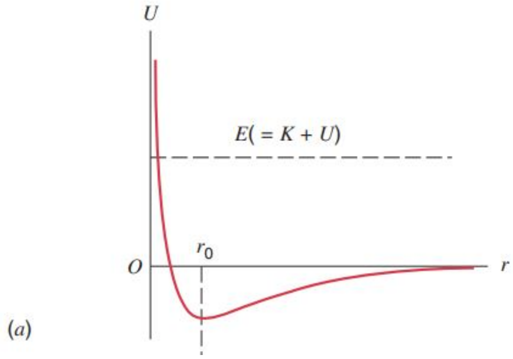
Combining these two, we have:

$$(P - a(\frac{n}{V})^2)(V - bn) = nRT$$

This is known as the Van-der-Waals equation for real gasses.

16.5 Intermolecular Forces

Approximating gassses as hard spheres certainly is a bold assumption! What is the justification for this? The answer lies in the interactions between gas molecules. An atom/molecule can be approximated as having a positive center and negative shell of electrons. When far apart, they have an attractive force, which becomes stronger until a certain distance. Closer than that distance, the force becomes more and more repulsive until it is so strong, that the molecules cannot move closer anymore and they "bounce" away.



As shown in the picture above, the intersection of E and the $U(r)$ curves is the effective diameter of two identical gas molecules. Let this distance be r_1 This is because if $U(r) > E$, then $KE < 0$, which is not possible since $KE = \frac{1}{2}mv^2$.

Since energy is conserved, and the 'balls' cannot move closer than their diameter, a hard sphere approximation is pretty good. Though it neglects the fact that the spheres do not need to touch to interact (there is a nonzero force after r_1, even though the molecules are a distance greater than d between them).

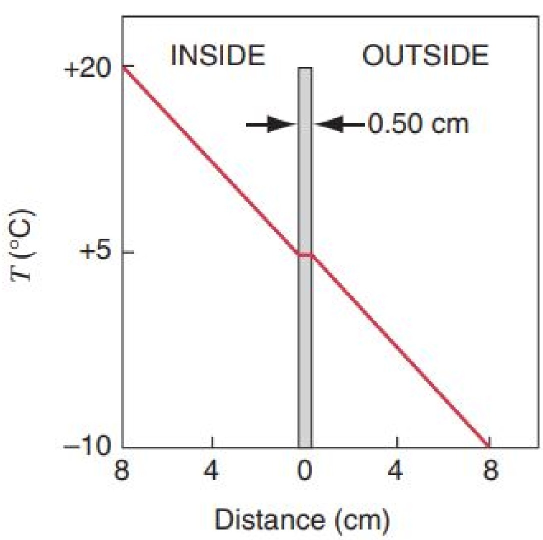
17 Thermodynamics

"There's no way he would do that to EM"

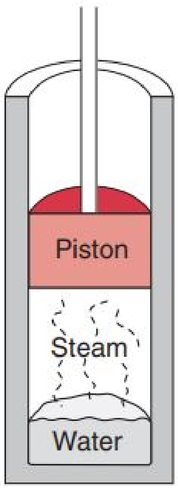
Heat is the change in internal energy of a system, which is microscopically observed as a change in temperature and microscopically observed to be more energetic vibrations of an electron.

17.1 Memorable Problems

1. Given the window has a thermal conductivity k_w, the air k_a, find the rate of heat transfer through the window. Assume a steady state.



2. A piston is fitted on top of a water-steam mixture, which does not leak. Given that the piston has an area A, is falling at a constant speed v (decreasing the volume of the water and steam), the P_{atm}, the molar mass of water, m_w, and the density of water ρ, find a) The rate gas is condensing into water b) The rate of heat loss c) The rate of change of the total internal energy.



17.2 Convection

Convection between two substances obeys the following formula:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -hA(T_1 - T_2)$$

Note that h is not a unitless constant and this does not depend on the distance between the two objects, which I think are probably assumed to be closed together.

Problem (Wang): A coffee cup has a heat capacity C. It begins with an initial temperature T_0 and the air has a constant temperature T_a. Find the $T(t)$.

17.3 Radiation

A black body is a theoretical object that absorbs all incoming EM waves and emits the largest possible amount of heat. The total heat flux density is given by:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \sigma T^4$$

Where $\sigma = 5.67 \cdot 10^{-8}$ is the Stefan-Boltzmann constant.

17.4 Conduction

If two sides of a thin slap of thickness dx have a temperature difference dT, then the heat transfer between the two is:

$$H = -kA\frac{\mathrm{d}T}{\mathrm{d}x}$$

where k is the conductance of the material. Note that this is very similar to the other equation. The negative sign comes from the fact that heat flows in the direction of decreasing temperature.

In a *steady state*, the heat transfer is constant, and the temperature on the two ends are also a constant.

Problem: Two conductors each of thickness L_1 on the left and L_2 on the right are placed together. The conductor of L_1 has a temperature T_1 on its left end and the conductor L_2 has a temperature T_2 on its right end. The materials have a conductivity of k_1 and k_2 respectively. Find the heat flow between them.

Hint 1: What is the temperature between the plates?

Hint 2: Let $i \equiv H$, $V \equiv \Delta T$, and $\rho = \sigma \equiv 1/k$

Note that heat is very analogous to currents - which should not be a surprise given that both are due to exchanges of electrons. Try to solve the following problem with only an EM analogy:

Problem: A solid pipe of inner radius r_1 and outer radius r_2 has a temperature difference of ΔT between the inside and outside. If the material has a conductivity of k, find the heat flow through the pipe.

17.5 First Law Of Thermo

Remember back in mechanics that one pesky ΔE_{int} term? When a person pushes on a wall and skates away or something? Here, we find that the net internal energy of a system is:

$$\Delta E_{int} = W + Q$$

W is the work done on the system. And Q is the heat added to the system. Work is negative if the system does work and Q is negative if the system transfers heat out. Note that just like in mechanics, we need to define a clear system and stick to that system!

17.6 Heat Capacity

The heat capacity is defined as:

$$C = \frac{Q}{\Delta T}$$

Note that while we don't have a Δ before the Q, it is because Q is a change - a change in energy. The specific heat capacity is more microscopic and offers a better understanding:

$$c = \frac{Q}{m\Delta T}$$

Note, however, that the constant c is not a constant and instead depends on the temperature, pressure, and other conditions. [14] If we write the last equation as a

[14] I swear that I will derive these by providing a good model of gasses or solids

differential:

$$Q = m\int_{T_1}^{T_2} cdT$$

For water, c is approximately constant over the temperature range 0 to 100 degrees Celsius.

For gasses, we can also define the molar heat capacity: $Q = nC\Delta T$, where n is the number of moles.

17.6.1 Heat of Formation

Heat does not always result in an increase in temperature. For example, water can freeze into a solid and release heat while doing so. This heat does not change the temperature of the water, however.

The heat is:

$$Q = Lm$$

L is the heat of formation or the heat of vaporization, L_v.

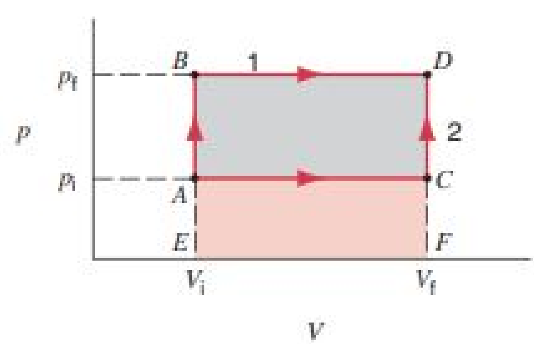
17.7 Ideal Gasses

Consider a gas within a piston. If you push the piston down by dx, then the gas becomes compressed and positive work is done on the gas. If the piston is then let go, the gas expands and negative work is done on the gas, as it loses energy. Defining 'away' as positive, then:

$$W = \int F_{ext}dx = \int(-pA)dx = -\int pdV$$

If we took a look at a PV diagram, then the work would be the area under the curve. Note that the pressure is nonconservative, since to get from one point to another on the PV diagram, there could be multiple different paths, each with a different area.

Note that to perform zero work, we would first let the volume stay constant as we decrease the pressure to zero, then at that pressure, increase the volume to some value then increase the pressure back up.



For the above figure, it would be the path AEFD

For the pressure to be zero at a constant volume, we would need to decrease the temperature to absolute zero!

Challenge: No heat is added to a system. Find the work required to move from (P_1, V_1) to (P_2, V_2) on the PV diagram. **Hint:** Consider $PV = nRT$

Challenge2: Is there any difference if instead we add no heat? Note: I am currently on 23-6. Will read up on gasses now due to its hyperfocus on it.

Note2: I have read up on gasses. I think that HRK is generally just bad at being rigorous on gasses. Darn.

According to the equipartition theorem, energy is equally split across the degrees of freedom for a gas molecule. For example, a monoatomic ideal gas has 3 degrees of translational freedom.

The energy for such one atom of such a gas is

$$\frac{3}{2}nRT$$

This is because $v_{rms} = \sqrt{\frac{3p}{\rho}}$. Plugging it into

$$\frac{1}{2}mv^2 = \frac{1}{2}m(\frac{3p}{Nm/V}) = \frac{3}{2}\frac{pV}{N} = \frac{3}{2}\frac{nRT}{N}$$

Since $R = N_A k$, where N_A is avagadro's constant, this translates to:

$$\frac{3}{2}\frac{nRT}{N} = \frac{3}{2}kT$$

Alternatively, the total energy for such a gas is:

$$\frac{3}{2}nRT$$

Since each degree of freedom has the same amount of energy, in general a gas/molecule with a degrees of freedom has a total energy of

$$\frac{a}{2}nRT$$

Vibrations can also occur, though are more noticeable at high temperatures where violent collisions leave molecules shaking in fear afterwards. A typical diatomic molecule has **two** vibrational modes of freedom.

Note that this $\frac{a}{2}nRT$ is the total internal energy for a gas! Thus, note this connection to $\Delta E_{int} = Q + W$

Heat Capacities for Solids

A solid has 6 degrees of freedom, 3 translational, and 3 others from the forces and potential energies with neighboring molecules. This means that its total energy is $3nRT$. The molar heat capacity is:

$$c = \frac{3nR\Delta T}{n\Delta T} = 3R \approx 25\text{J/(mol K)}$$

This agrees with the Dulong-Petit value.

17.7.1 Heat Capacities for Gasses

The heat capacity for a gas varies depending on what conditions the gas is changing temperature.

We will deal with several cases below:

Constant Volume: At constant volume, zero work is done, but this does not mean zero change in internal energy! This is because $E_{int} = Q + W$. Again, we use a PV diagram. Such a graph is a vertical line, since the volume remains constant.

Let the heat that is added to the system be Q. Let C_V be the molar heat capacity at constant volume, which is the change in temperature per unit heat added in. Thus,

$$C_V = \frac{Q}{n\Delta T} = \frac{\Delta E_{int}}{n\Delta T} \implies \Delta E_{int} = C_V$$

The change in temperature is $\Delta T = C_V n \Delta T$.

Since in general, if there are a degrees of freedom, then the total internal energy is:

$$\Delta E_{int} = \frac{a}{2} nRT$$

This would mean that $C_V = \frac{aR}{2}$.

Constant Pressure: At constant pressure, work is being done. The total work done is $W = -p(V_f - V_i) = -nR\Delta T$. Letting $Q = nC_p\Delta T$, and from the constant volume case, $E = nC_V\Delta T$. Thus, this means that by $\Delta E_{int} = W + Q$:

$$C_V = C_P - nR \implies C_p = nR + C_V$$

Problem: An adiabatic process brings a gas from V_i, P_i to V_f. What is the total change in internal energy? Assume that you are given γ.

Note that there are two ways to solve this.

17.8 Applications of the First Law

17.8.1 Adiabatic

No heat is added into the syste, so $\Delta E_{int} = W$. Thus,

$$\Delta E_{int} = C_V ndT = -pdV$$

Taking the differential on both sides of the ideal gas law:

$$d(PV) = d(nRT) \implies pdV + Vdp = nRdT \implies Vdp = C_V ndT + nRdT = C_p ndT$$

Dividing the first and second equations and solving through integration, we find: $P_i V_i^\gamma = P_f V_f^\gamma$

17.8.2 Cyclical

This is a process that starts and ends at the same state. Assuming that no gas leaks, by the first law of thermodynamics, since the $P_i = P_f$ and $V_i = V_f$, we have that $\Delta E_{int} = 0$

Problem: A cycle is done in the ccw direction, what are the signs of Q and W? The cw direction?

17.8.3 Free Expansion

A gas is in container 1 and diffuses itno container 2. There is no work or heat being added to the gas, other than the gas simply increasing in volume. What is the total change in internal energy? 0. No heat is added, no work is done, so since $\Delta E_{int} = Q + W = 0 + 0 = 0$.

However, note that during the expansion, volume and pressure are not well-defined.

17.9 MC Answers

1. B
2. E ($T = T_C + iR_2$
3. E
4. E
5. B
6. A
7. C?
8. C (apparently there's some second law stuff going on, but the two systems are in thermal equilibrium)
9. D
10. A,B,D
11. A,D,C,A,D (You may wonder why there can be nonzero heat added in when the temperature is constant. This is because if you plot the $PV = nRT$ curve, you will find that there is a net nonzero work being done. Since the net change in internal energy is zero, $Q \neq 0$, too.)
12. D
13.
14.

18 Second Law of Thermo

Consider a piston on a cylinder that is filled with gas on top of a thermal resevoire. Imagine the piston to have a pile of sand on the top of it. Imagine removing a bit of sand from the piston, then the piston shifts up by a bit, and a small amount of heat dQ flows into the cylinder. If we add that bit of sand back in, the piston shifts down by the same amount and the same dQ flows out of the cylinder. Such a process is reversible. This is a quasistatic process, since we are doing it slowly enough to cause the system to remain in equilibrium.

However, consider an irreversible process: A hot metal touching a cold metal and the two coming to thermal equilibrium. In this case, it is irreversible, since heat cannot flow from one to another.

Entropy for a reversible process is defined as:

$$\Delta S = \int_i^f \frac{dQ}{T}$$

Note that if T is approximately a constant, then it reduces to $\Delta S = Q/T$

We can prove that entropy is a state property and derive an equation for gasses (and other substances) as follows:

$$dW + dQ = dE_{int} \implies dQ = pdV + nC_V dT \implies \frac{dQ}{T} = nR\frac{dV}{V} + nC_V\frac{dT}{T}$$

Integrate both sides and you get:

$$\Delta S = nR\ln\left(\frac{v_f}{V_i}\right) + nC_V\ln\left(\frac{T_f}{T_i}\right)$$

Note that this definition means that entropy depends purely on the initial and final states.

18.1 Memorable Problems

1. A gas expands freely to fill a vacuum within an insulated cylinder from a volume V_i to V_f. What is the change in entropy, given the initial temperature is T?

2. Solve problem 1 first. Explain what is wrong with this "solution": The heat $Q = pdV$. Since $PV = nRT \implies PdV = nrdT$. Plugging it into $\int \frac{dQ}{T}$, we obtain $\Delta S = 0$.

18.2 Irreversible Processes

Since Entropy is a state funciton, and depends on initial and final states, the entropy for an irreversible process is the same for a reversible one that takes it from the same initial to final state.

Now let it start off at state 1 (V_i, T_i) and end at state 2 (V_f, T_f).

Problem: Water is on a hot plate initially in equilibrium at 20 degrees Celsius. To make the process reversible, should the hot plate increase a) quickly? b) slowly? c) Somewhere in between?

Solution: Imagine if it was increasing very quickly, such that at some time t, its temperature is $T_1 > T_2$ of the water. Such a process is not quasi-static, so that when the two finally come in equilibrium, it is impossible to bring it back to the state when the hot plate was much hotter than the water (the temperature of the two will fall at the same rate).

Thus, we should do it very slowly.

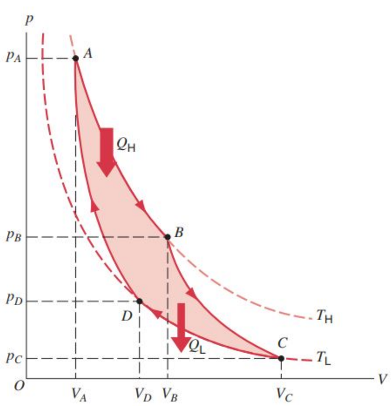
18.3 Carnot Engine

The second law of thermodynamics states that for a irreversible process in a closed system, the net change in entropy is positive.

A Carnot engine does work by moving back and forth between two thermal reservoirs, T_L and T_H, where $T_H > T_L$.

It's easier to conceptualize this by imagining the gas to be in a cylinder with a piston that is free to move without friction. When we place the piston on the hotter reservoir, it gains heat while maintaining a constant temperature. In transitioning to the colder reservoir, assume that it does it adiabatically. Thus, we allow the piston to rise slowly by taking lead shots (or weight) off the piston. As this occurs, work is being done by the gas. Since this is an adiabatic process, $PV^\gamma =$ constant. $\gamma > 1$, so if V increases by a factor f, P decreases by a factor f^γ. This decreases the total product $PV = nRT$, so the temperature drops.

When the temperature $T = T_L$, we move it to a reservoir at T_L. Note by the zeroeth law, since the cylinder is in thermal equilibrium with the reservoir, no heat is exchanged. However, if we add on more lead shots/weights, it gradually moves up the $PV = nRT_L$ curve. Then again, at a certain point, we move it to an adiabatic process where we can make it shrink in volume in order to cycle through the process.



Question: What if the cycle happened ccw? Would you use this ccw engine to do work? Why? **Problem:** Plot a T vs S graph for the Carnot cycle. What do isotherms look like? Adiabatic parts?

18.3.1 Efficiency

The efficiency can be defined as: useful energy out/energy in. The useful energy output is the work done by the engine. The energy put into the engine is simply

$$\frac{W}{|Q_H|} = \frac{|Q_H| - |Q_L|}{|Q_H|}$$

We made the evaluation by noting that for the full cycle, $Q + W = 0$ And the work done by the gas is $W = \int pdV = Q_{in}$

Thus, the efficiency is $1 - \frac{Q_L}{Q_H}$. Since entropy is a state variable, and by the previous exercise, the only change in entropy occurs during the isothermal processes. Thus,

$$\Delta S = 0 \implies \frac{Q_H}{T_H} - \frac{Q_L}{T_L} = 0$$

This then implies that $\frac{Q_L}{Q_H} = \frac{T_L}{T_H}$

Problem: What is the most efficient refrigerator between two reservoirs? What is its efficiency?

Question: When a working substance moves from reservoir T_1 to T_2, why is it that the moment it hits the other reservoir, it has that reservoir's temperature?

18.4 Perfect Engines

Prove that having an engine that transfers heat from a lower temperature to a higher temperature without any work cannot exist. (Hint: Second law of thermo).

18.5 Statistical view

Consider a box with two equal halves. Assume that all molecules are distinct, and the two halves of the boxes are distinct. Then if we have N molecules, there are a total of 2^N ways to arrange them into the two boxes. If we want k molecules on one side and $N - k$ molecules on the other, there are a total of ${}_NC_k$ ways to do this. Note that we assumed the order the molecules are placed in does not matter.

The number of ways for this one case (k on one side) is known as a configuration that has ${}_NC_k$ microstates within it. Thus, notice that for $N \gg 1$, the system would tend towards having $N/2$ on the left and right sides for this symmetric configuration.

18.5.1 Entropy and probability

We will use two facts to derive a relationship between the two:

1. The probability of P(A and B)= $P(A)P(B)$.

2. The entropy of a system is the sum of entropies of individual components.

Let $f(n)$ be the entropy given that the multiplicity of a certain case is n. Then, from property 2, $f(n_1) + f(n_2) = f(n_1 n_2)$

19 Gratings and Spectra

19.1 Initial Questions

1. Say that I have N slits on a wall. What is the intensity as a function of y on a screen far, far away?

2. Consider $N \gg 1$ slits lined uniformly on a wall. If I take αN of the slits and bend the wall by an angle θ, where $0 < \alpha < 1$, what is the intensity as a function of y now?

3. Say that I have three slits separated by distances of a_1 and a_2. Find the maxima and minima of their interference pattern on the wall. Generalize to N slits of distances $a_1, a_2, \ldots a_{N-1}$

4. What should the shape of a 2D surface be such that if the surface has coherent slits that are separated by a constant distance d, it produces a maxima at the origin?

5. A stick has a height H, and the mirror has a height $H/2$. If Bob cuts N thin slits in the mirror that are equally spaced from each other and the ends of the mirror, what is the resulting image of the stick in the mirror?

19.2 Elementary Derivations

1. Consider N slits. What condition must be satisfied for there to be a maxima at an angle θ on the screen (assume parallel rays)?

Hint: This occurs when each of the slits are in phase with each other.

done 1.

2. Consider drawing a phasor diagram. Under what condition will there be destructive interference?

done 1.

3. Let the *mth* maxima occur at an angle θ. What is the corresponding mimima to this maxima? Assume that $N \gg 1$ and that N is an even number. Why is this still a valid approximation if N is odd?

Consider two adjacent slits. Their path difference at the maxima case is $m\lambda = d \sin\theta$. When we add these constructive interferences up, they do not result in any net change in phase (phasor diagram). But when we add in a λ/N to the path difference, then it will create a "loop" in the phasor diagram.

4. Are the two adjacent minima symmetrically located around this maxima? Yes, their path differences are off by $\pm\frac{\lambda}{N}$

19.3 Useful Formulas

The angular difference between a maxima and a minima is:

$$\delta\theta = \frac{\lambda}{Nd\cos\theta}$$

19.4 Dispersion and Resolving Power

19.4.1 Dispersion

Consider two wavelengths of λ and $\lambda + \Delta\lambda$, where $\Delta\lambda \ll \lambda$. If the slits are separated by a distance of d, find the angular separation of the first order maximum of these two wavelengths of light.

$d \sin\theta = m\lambda$

Taking a differential of both sides:

$$d \cos\theta d\theta = m d\lambda \implies \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{m}{d\cos\theta}$$

This quantity, known as the dispersion, measures how discernible our spectroscopy is. The greater θ is, and the smaller the λ is, means that for small λ it is able to project them at great values of θ.

19.4.2 Resolving Power

$$R = \frac{\Delta\lambda}{\lambda}$$

The resolving power is defined in a different way. To discern between two wavelengths of light, Rayleigh's criterion states that the minimum of one wavelength must fall on the maximum of the other. In other words,

$$\delta\theta = \Delta\theta$$

Where $\delta\theta$ is the angular difference between the maximum and the minimum and $\Delta\theta$ is the angular difference between the two maxima.

This gives:

$$\frac{\lambda}{Nd\cos\theta} = \frac{m\Delta\lambda}{d\cos\theta}$$

19.5 MC Answers

1. D - Adding addition slits or N means that the width of each maxima is decreased.
2. C - central maximum angular width depends only on the product Nd, which is the same for both.
3. B - Same formula used in 2)
4. A - Just think
5. A - Similar to 1, if the dark bands are thicker, then they're easier to see.
6. C -

7. C, B -
8. E
9. E

Acknowledgements

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- Halliday Resnick and Krane Physics Volume 2
- University Physics, Openstax
- MIT EM Notes 8.022 by Dr.Sahughes
- *Introduction to Electrodynamics 4th Edition* by David J. Griffiths
- Khan Academy for their multi-variable calculus course