Another Thermodynamic Proof to AM-GM

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Abstract: The AM-GM inequality is a classic inequality. Though most proofs utilize Cauchy induction or connections to more advanced inequalities, we show that the AM-GM can also be proven with the laws of thermodynamics. More specifically, we show that a specific arrangement of Carnot engines implies that the AM-GM inequality must hold true. An extension to the weighted AM-GM inequality is also given.

Let a_i be nonnegative real numbers. AM-GM states that

$$\frac{\sum_{k=0}^{n} a_i}{n} \geq \sqrt[n]{\prod_{1}^{n} a_i}$$

There are a couple AM-GM proofs in literature, most of which use some form of induction , such as Cauchy Induction [1] or other more advanced inequalities. A different thermodynamic proof which uses the monotonic nature of entropy also exists [2]. Here I present a proof that utilizes Carnot engines.

Consider *N* heat reservoirs with heat capacities C, each with a starting temperature T_i . The initial energy in the system is $C\sum_1^N T_i$. Consider connecting the heat reservoirs with Carnot engines such that the ith reservoir is connected to the i+1th reservoir, with the Nth one connected to the 1st one. Note that since the net change in entropy is zero and $\delta Q = CdT$,

$$\sum_{1}^{N} \frac{CdT_i}{T_i} = 0$$

Integrating the left hand side, and setting the bounds to be from T_i to T_0 (the final equilibrium temperature), we find that $T_0 = \sqrt[N]{\prod_1^N T_i}$. Thus, at the end of the process, the total energy within the system is $NC\sqrt[N]{\prod_1^N T_i}$

Since no energy is inputted into the system and can only be outputted in the form of work, the final energy must be less than or equal to the initial energy. Simplifying, we arrive at our desired inequality:

$$\sqrt[N]{\prod_1^N T_i} \leq \frac{\sum_1^N T_i}{N}$$

Note that temperature, as used in the Carnot cycle, is defined on the Kelvin scale, which is always nonnegative. As we can see, equality occurs when all temperatures are equal, where no more work can be harvested from the system. The weighted AM-GM inequality could also be proven, but will be done with each reservoir having a heat capacity C_i.

References

[1] Cauchy, A.-L. (1821). *Cours d'analyse de l'cole Royale Polytechnique, premire.*

[2] Unknown. (n.d.). *A Thermodynamic Proof of AM-GM.* Retrieved from Massachussets Institute of Technology: https://www.mit.edu/~ehjin/files/AMGM.pdf