Supplement: A General Incentives-Based Framework for Fairness in Multi-agent Resource Allocation

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Here we present the supplementary material complementing the main paper, including full proofs for the theoretical results and extended discussion.

A THEORETICAL RESULTS

In this section, we present the full proofs for the core theoretical properties of GIFF. Throughout this section, let $\mathbf{Z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ denote the current payoff vector, and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_{\geq 0}$ denote the increments from a feasible allocation in the current round. For a fairness function $F : \mathbb{R}^n \to \mathbb{R}$ we define:

$$\Delta_{\text{joint}} := F(\mathbf{Z} + y) - F(\mathbf{Z}), \tag{1}$$

$$\Delta_i^{\text{local}} := F(\mathbf{Z} + y_i e_i) - F(\mathbf{Z}) \tag{2}$$

$$S := \sum_{i=1}^{n} \Delta_i^{\text{local}} \tag{3}$$

where e_i is the i-th unit vector. We call Δ_{joint} the *realized fairness improvement*, Δ_i^{local} the *local fairness gain* for agent i, and S the *sum of local gains*, which we term as GIFF's **surrogate**.

Our proofs rely on the following assumption:

Assumption 1 (Nonnegative increments). *Any coordinate's change in utility is nonnegative for any allocation: $y_i \geq 0$ for all i.*

Assumption 2 (Q-value correctness). *For every agent i and action a, the estimated Q-value equals the true utility increment:*

$$Q(o_i, a) \;=\; \Delta z_i(a),$$

where $\Delta z_i(a)$ denotes the change in payoff z_i that results from assigning action a to agent i. In other words, Q-values are perfectly accurate predictors of payoff increments.

Our results fall into three families:

1. A *Local–Gain Lower Bound*, showing that the surrogate never overstates realized fairness improvements.
2. A *Monotonicity in β*, showing that surrogate fairness is nondecreasing as the fairness weight β increases.
3. *Slack bounds*, quantifying the gap between surrogate and realized fairness to turn surrogate guarantees into realized guarantees.

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A.1 Local–Gain Lower Bound

Our first main result establishes that GIFF's per-agent local gains form a certified lower bound on the realized joint fairness change. The result covers four canonical metrics: α-fairness, negative variance, generalized Gini (GGF), and maximin.

Theorem 1 (Local–Gain Lower Bound). *Let $\mathbf{Z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ be a payoff vector and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_{\geq 0}$ be a nonnegative increment vector. For each fairness function F below, the realized joint gain dominates the sum of local gains:*

$$\Delta_{\text{joint}} := F(\mathbf{Z} + y) - F(\mathbf{Z}) \quad \geq \quad \sum_{i=1}^{n} \Delta_i^{\text{local}} \tag{4}$$

$$\Delta_i^{\text{local}} := F(\mathbf{Z} + y_i e_i) - F(\mathbf{Z}) \tag{5}$$

under the stated assumptions:

1. *$F = F_\alpha$ (α-fairness): $\alpha \geq 0$ and $z_i,\ z_i + y_i$ lie in the domain of U_α (so $z_i > 0$ and $z_i + y_i > 0$ when $\alpha = 1$).*
2. *$F = F_{\text{var}}$ (negative variance): no further assumptions (beyond $y \geq 0$).*
3. *$F = F_{\text{GGF}}$ (generalized Gini) with nonincreasing weights $w_1 \geq \cdots \geq w_n$.*
4. *$F = F_{\min}$ (maximin): no further assumptions (beyond $y \geq 0$).*

Moreover, equality holds in (1); in (2) equality holds iff at most one $y_i > 0$; and in (4) equality conditions are given in Lemma 4.

Lemma 1 (α-fairness: exact additivity). *Let $F_\alpha(\mathbf{Z}) = \sum_{i=1}^{n} U_\alpha(z_i)$ with $U_\alpha(t) = \frac{t^{1-\alpha}}{1-\alpha}$ for $\alpha \neq 1$ (domain $t \geq 0$) and $U_1(t) = \log t$ (domain $t > 0$). If z_i and $z_i + y_i$ are in the domain for all i, then*

$$F_\alpha(\mathbf{Z} + y) - F_\alpha(\mathbf{Z}) = \sum_{i=1}^{n} \big[F_\alpha(\mathbf{Z} + y_i e_i) - F_\alpha(\mathbf{Z})\big].$$

Proof. By separability, $F_\alpha(\mathbf{Z} + y) - F_\alpha(\mathbf{Z}) = \sum_i [U_\alpha(z_i + y_i) - U_\alpha(z_i)]$, and for a single-coordinate update $F_\alpha(\mathbf{Z} + y_i e_i) - F_\alpha(\mathbf{Z}) = U_\alpha(z_i + y_i) - U_\alpha(z_i)$. Summing over i yields the identity. □

Lemma 2 (Negative variance: nonnegative synergy). *Let $F_{\text{var}}(\mathbf{Z}) = -\mathrm{Var}(\mathbf{Z})$ with $\mathrm{Var}(\mathbf{Z}) = \frac{1}{n}\sum_{i=1}^{n}(z_i - \mu)^2$ and $\mu = \frac{1}{n}\sum_i z_i$. If $y \geq 0$, then*

$$F_{\text{var}}(\mathbf{Z} + y) - F_{\text{var}}(\mathbf{Z}) \quad \geq \quad \sum_{i=1}^{n} \big[F_{\text{var}}(\mathbf{Z} + y_i e_i) - F_{\text{var}}(\mathbf{Z})\big],$$

with equality iff at most one $y_i > 0$.

Proof. Use $F_{\mathrm{var}}(\mathbf{Z}) = \mu^2 - \frac{1}{n}\sum_i z_i^2$. Let $S = \sum_i z_i$ and $Y = \sum_i y_i$. Then

$$F_{\mathrm{var}}(\mathbf{Z}+y) - F_{\mathrm{var}}(\mathbf{Z}) = \frac{2SY+Y^2}{n^2} - \frac{2}{n}\sum_i z_i y_i - \frac{1}{n}\sum_i y_i^2.$$

For a single update y_i, $F_{\mathrm{var}}(\mathbf{Z}+y_i e_i) - F_{\mathrm{var}}(\mathbf{Z}) = \frac{2Sy_i+y_i^2}{n^2} - \frac{2}{n} z_i y_i - \frac{1}{n} y_i^2$. Summing over i gives

$$\sum_i \Delta_i^{\mathrm{local}} = \frac{2SY}{n^2} - \frac{2}{n}\sum_i z_i y_i + \left(\frac{1}{n^2} - \frac{1}{n}\right)\sum_i y_i^2.$$

Subtracting yields $\Delta_{\mathrm{joint}} - \sum_i \Delta_i^{\mathrm{local}} = \frac{Y^2 - \sum_i y_i^2}{n^2} = \frac{2}{n^2}\sum_{i<j} y_i y_j \geq 0$, with equality iff at most one $y_i > 0$. □

Lemma 3 (GGF: joint gain dominates locals). *Let $F_{\mathrm{GGF}}(\mathbf{Z}) = \sum_{k=1}^{n} w_k z_{(k)}$ with nonincreasing weights $w_1 \geq \cdots \geq w_n$ (and $w_{n+1} := 0$), where $z_{(1)} \leq \cdots \leq z_{(n)}$ are the sorted entries of $\mathbf{Z}$. If $y \in \mathbb{R}^n_{\geq 0}$, then*

$$F_{\mathrm{GGF}}(\mathbf{Z}+y) - F_{\mathrm{GGF}}(\mathbf{Z}) \quad \geq \quad \sum_{i=1}^{n}\big[F_{\mathrm{GGF}}(\mathbf{Z}+y_i e_i) - F_{\mathrm{GGF}}(\mathbf{Z})\big].$$

Proof. Use the equivalent form $F_{\mathrm{GGF}}(\mathbf{Z}) = \sum_{k=1}^{n} v_k S_k(\mathbf{Z})$ with $v_k := w_k - w_{k+1} \geq 0$ and $S_k(\mathbf{Z}) = \sum_{j=1}^{k} z_{(j)}$. Since $v_k \geq 0$, it is enough to prove the claim for each S_k and sum with weights v_k.

Fix k and let τ be the k-th smallest value in $\mathbf{Z}$. Set $L = \{i : z_i < \tau\}$, $T = \{i : z_i = \tau\}$, $c = k - |L|$. Any k-subset achieving $S_k(\mathbf{Z})$ must include all of L and exactly c indices from T.

Local updates. For a single-coordinate increase $y_i \geq 0$,

$$S_k(\mathbf{Z}+y_i e_i) - S_k(\mathbf{Z}) = \begin{cases} y_i, & i \in L, \\ y_i, & i \in T \text{ and } c = |T|, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\sum_{i=1}^{n}\big[S_k(\mathbf{Z}+y_i e_i) - S_k(\mathbf{Z})\big] = \sum_{i\in L} y_i \; + \; \mathbf{1}_{\{c=|T|\}} \sum_{i \in T} y_i. \qquad (*)$$

Joint update. For the full increment $y \geq 0$,

$$S_k(\mathbf{Z}+y) = \min_{|J|=k} \sum_{j\in J}(z_j + y_j) \;\geq\; \min_{\substack{C\subseteq T\\ |C|=c}} \left[\sum_{j\in L}(z_j+y_j) + \sum_{j\in C}(z_j+y_j)\right]$$
$$= S_k(\mathbf{Z}) + \sum_{j\in L} y_j + \min_{\substack{C\subseteq T\\ |C|=c}} \sum_{j\in C} y_j.$$

If $c = |T|$, the minimum over C equals $\sum_{j\in T} y_j$; otherwise it is ≥ 0. Thus

$$S_k(\mathbf{Z}+y) - S_k(\mathbf{Z}) \;\geq\; \sum_{j\in L} y_j \; + \; \mathbf{1}_{\{c=|T|\}} \sum_{j\in T} y_j, \qquad (**)$$

which matches or exceeds the sum in (*). This proves the claim for S_k. Multiplying by $v_k \geq 0$ and summing over k yields the inequality for F_{GGF}. □

Lemma 4 (Maximin: joint never underestimates locals). *Let $F_{\min}(\mathbf{Z}) = \min_i z_i$, let $m = \min_i z_i$, and let $S = \{i : z_i = m\}$ be the set of minimizers. Let σ be the second-smallest baseline value (or $+\infty$ if none), and let*

$$\sigma' \;:=\; \min_{j\neq i^\star}(z_j + y_j)$$

be the post-update second-smallest value when $|S| = 1$ with $S = \{i^\star\}$. If $y \geq 0$, then

$$F_{\min}(\mathbf{Z}+y) - F_{\min}(\mathbf{Z}) \quad \geq \quad \sum_{i=1}^{n}\big[F_{\min}(\mathbf{Z}+y_i e_i) - F_{\min}(\mathbf{Z})\big].$$

Equality conditions.

- *If $|S| \geq 2$ (multiple minima), equality holds iff $\min_{i\in S} y_i = 0$. Otherwise the inequality is strict.*
- *If $|S| = 1$ with $S = \{i^\star\}$ (unique minimum), write*

$$\textit{local sum} \;=\; \min\{y_{i^\star},\, \sigma - m\}, \qquad \textit{joint} \;=\; \min\{y_{i^\star},\, \sigma' - m\}.$$

Equality holds iff either
(1) $y_{i^\star} \leq \sigma - m$ (then both sides equal $y_{i^\star}$), or
(2) $\sigma' = \sigma$ (then both sides equal $\min\{y_{i^\star},\, \sigma - m\}$).
If $y_{i^\star} > \sigma - m$ and $\sigma' > \sigma$, the inequality is strict.

Proof. *Local updates.* If $i \notin S$, the minimum stays m, so the local change is 0. If $|S| \geq 2$ and $i \in S$, raising a single tied minimum still leaves some index at m, so the local change is 0. If $|S| = 1$ with $S = \{i^\star\}$, then

$$F_{\min}(\mathbf{Z}+y_{i^\star} e_{i^\star}) - F_{\min}(\mathbf{Z}) = \min\{\, m + y_{i^\star},\, \sigma \,\} - m = \min\{\, y_{i^\star},\, \sigma - m \,\}.$$

Summing over i gives the stated "local sum."

Joint update. If $|S| \geq 2$,

$$F_{\min}(\mathbf{Z}+y) - F_{\min}(\mathbf{Z}) = \min\{\, \min_{i\in S} y_i,\, \sigma - m \,\} \;\geq\; 0,$$

so the inequality holds; equality iff $\min_{i\in S} y_i = 0$. If $|S| = 1$ with $S = \{i^\star\}$,

$$F_{\min}(\mathbf{Z}+y) - F_{\min}(\mathbf{Z}) = \min\{\, y_{i^\star},\, \sigma' - m \,\}.$$

Since $\sigma' \geq \sigma$ (because $y \geq 0$ on non-min entries), we have $\min\{y_{i^\star}, \sigma' - m\} \geq \min\{y_{i^\star}, \sigma - m\}$ with equality exactly under (1) or (2) above. □

Now we are ready to prove the main result.

Proof of Theorem 1. Each case follows from the corresponding lemma, which establishes $\Delta_{\mathrm{joint}} \geq \sum_i \Delta_i^{\mathrm{local}}$ under the stated assumptions, and provides the equality conditions. □

This theorem implies that GIFF's surrogate never *over-promises*: the realized fairness gain is guaranteed to be at least as large as the surrogate, and is exact for α-fairness. Equality conditions for variance and maximin are given in Lemmas 2 and 4. See Subsection A.5 for a detailed discussion of implications.

A.2 Monotone Surrogate Fairness under β

Next, we show that as the fairness weight β increases, the allocation's sum of local fairness gains is nondecreasing. This gives a monotonic control knob: tuning β cannot reduce surrogate fairness, and under uniqueness, any allocation switch yields a strict increase.

Theorem 2 (Monotone increase of the sum of local fairness gains). *Fix a decision round with baseline payoff vector $\mathbf{Z} \in \mathbb{R}^n$. Let $\mathcal{A}$ be the (finite) set of feasible joint allocations $A = (a_1, \dots, a_n)$. For each $A \in \mathcal{A}$, define*

$$U(A) \;:=\; \sum_{i=1}^{n} Q(o_i, a_i) \tag{6}$$

$$S(A) \;:=\; \sum_{i=1}^{n} \Delta F_i(a_i) \tag{7}$$

$$\Delta F_i(a_i) \;:=\; F(\mathbf{Z} + y_i(a_i)\, e_i) - F(\mathbf{Z}) \tag{8}$$

where $y_i(a_i) \geq 0$ is the accounted increment added to z_i if agent i takes a_i, and e_i is the i-th unit vector. (All $\Delta F_i(\cdot)$ use the same baseline $\mathbf{Z}$.)

For $\beta \in [0, 1)$, write $\theta = \beta/(1-\beta)$ and consider

$$G_\theta(A) \;=\; U(A) \;+\; \theta\, S(A).$$

Let $A^*(\theta) \in \arg\max_{A \in \mathcal{A}} G_\theta(A)$ be any maximizer. Then for any $0 \leq \theta_1 < \theta_2 < \infty$ and any choices $A_1 \in \arg\max G_{\theta_1}$, $A_2 \in \arg\max G_{\theta_2}$,

$$S(A_2) \;\geq\; S(A_1).$$

Equivalently, as β increases, the chosen allocation's sum of local fairness gains is nondecreasing.

Proof. By optimality of A_1 at θ_1 and A_2 at θ_2,

$$U(A_1) + \theta_1 S(A_1) \;\geq\; U(A_2) + \theta_1 S(A_2), \tag{1}$$

$$U(A_2) + \theta_2 S(A_2) \;\geq\; U(A_1) + \theta_2 S(A_1). \tag{2}$$

Subtract (1) from (2) to cancel the U terms:

$$(\theta_2 - \theta_1)\,\big(S(A_2) - S(A_1)\big) \;\geq\; 0.$$

Since $\theta_2 > \theta_1$, we conclude $S(A_2) \geq S(A_1)$. □

Corollary 1 (Strict increase at a true switch under uniqueness). *If the maximizer is unique at θ_1 and at θ_2, let $A_1 = A^*(\theta_1)$ and $A_2 = A^*(\theta_2)$. If $A_1 \neq A_2$, then*

$$S(A_2) \;>\; S(A_1).$$

Proof. From Theorem 2, $S(A_2) \geq S(A_1)$. If equality held, then (1) and (2) above would force $U(A_1) = U(A_2)$ as well, so both A_1 and A_2 would maximize G_{θ_1} and G_{θ_2}—contradicting uniqueness. □

Scope, tie-breaking, and the $\beta \to 1$ endpoint. Theorem 2 requires only that (i) the feasible set $\mathcal{A}$ for the round is *finite*, and (ii) all $\Delta F_i(\cdot)$ are computed against the *same* baseline $\mathbf{Z}$ for that round. Changing the feasible set (e.g., constraints) or the baseline mid-sweep can break monotonicity; otherwise the result is agnostic to the choice of F and to how Q is obtained, provided $S(A)$ is well defined. Deterministic tie-breaking is *not* required for nondecreasing $S(A^*(\theta))$, but if you want strict improvement at switches without assuming uniqueness, adopt a consistent rule such as: among G_θ-maximizers, first maximize $S(A)$, then $U(A)$. With this rule, any change in the selected allocation across $\theta_1 < \theta_2$ implies $S(A_2) > S(A_1)$. Finally, interpreting $\beta \to 1^-$ as $\theta = \beta/(1-\beta) \to \infty$, the maximizers converge to $\arg\max_{A \in \mathcal{A}} S(A)$, so the monotonicity statement extends continuously to the endpoint $\beta = 1$.

This monotonicity applies to the surrogate $S(A)$, not necessarily the realized fairness $F(\mathbf{Z})$. To connect the two, we require slack bounds.

A.3 Slack Bounds Between Surrogate and Realized Fairness

Define the *slack* of an allocation as

$$\text{slack} \;:=\; \Delta_{\text{joint}} - S \quad \geq 0.$$

By Theorem 1, slack ≥ 0 for the given fairness metrics. Here we provide per-metric exact formulas or bounds, so that we can certify not just lower bounds but also two-sided guarantees.

Lemma 5 (Slack for α-fairness). *Let $F_\alpha(\mathbf{Z}) = \sum_i U_\alpha(z_i)$ with $U_\alpha(t) = \frac{t^{1-\alpha}}{1-\alpha}$ for $\alpha \neq 1$ (domain $t \geq 0$) and $U_1(t) = \log t$ (domain $t > 0$). Assume all arguments lie in the domain. Then*

$$\Delta_{\text{joint}} \;=\; S \quad \textit{and} \quad \text{slack} = 0.$$

Interpretation. *The surrogate is exact for any nonnegative increment profile.*

Proof. By separability, $F_\alpha(\mathbf{Z} + y) - F_\alpha(\mathbf{Z}) = \sum_i \big[U_\alpha(z_i + y_i) - U_\alpha(z_i)\big]$, while for a single-coordinate update $F_\alpha(\mathbf{Z} + y_i e_i) - F_\alpha(\mathbf{Z}) = U_\alpha(z_i + y_i) - U_\alpha(z_i)$. Summing over i gives $\Delta_{\text{joint}} = S$. □

Lemma 6 (Slack for negative variance). *Let $F_{\text{var}}(\mathbf{Z}) = -\text{Var}(\mathbf{Z})$, and set $Y := \sum_i y_i$. Then*

$$\text{slack} \;=\; \frac{Y^2 - \sum_i y_i^2}{n^2} \;=\; \frac{2}{n^2} \sum_{i<j} y_i y_j \;\in\; \left[0,\; \frac{Y^2}{n^2}\Big(1 - \frac{1}{m}\Big)\right],$$

where $m := |\{i : y_i > 0\}|$. Interpretation. *The gap is a pure "synergy" term that grows when gains are spread across more agents; it vanishes when at most one coordinate increases.*

Proof. Using $F_{\text{var}}(\mathbf{Z}) = \mu^2 - \frac{1}{n}\sum_i z_i^2$ with $\mu = \frac{1}{n}\sum_i z_i$, the calculation in Lemma 2 gives

$$\Delta_{\text{joint}} - \sum_i \Delta_i^{\text{local}} = \frac{Y^2 - \sum_i y_i^2}{n^2} = \frac{2}{n^2} \sum_{i<j} y_i y_j \;\geq 0.$$

For the upper bound, by Cauchy–Schwarz, with m positive entries, $\sum_i y_i^2 \geq Y^2/m$, hence $Y^2 - \sum_i y_i^2 \leq Y^2(1 - 1/m)$. □

Lemma 7 (Slack for GGF (nonincreasing weights)). *Let $F_{\text{GGF}}(\mathbf{Z}) = \sum_{k=1}^{n} w_k\, z_{(k)}$ with nonincreasing weights $w_1 \geq \cdots \geq w_n$ and $w_{n+1} := 0$, and let the increments y be nonnegative. Set $Y := \sum_i y_i$, $m := |\{i : y_i > 0\}|$, and $y_{\max} := \max_i y_i$. Define*

$$q := \min\Big(m,\; \big\lfloor Y/y_{\max} \big\rfloor\Big), \qquad r := Y - q\, y_{\max} \in [0, y_{\max}).$$

Then

$$\Delta_{\text{joint}} \;\leq\; y_{\max} \sum_{k=1}^{q} w_k \;+\; r\, w_{q+1}$$

and hence,

$$\text{slack} \;\leq\; \Big(y_{\max} \sum_{k=1}^{q} w_k + r\, w_{q+1}\Big) - S.$$

If the baseline order of $\mathbf{Z}$ *is strict and preserved after the update, writing* $y_{(1)} \le \cdots \le y_{(n)}$ *aligned with that order,*

$$\Delta_{\text{joint}} \;=\; \sum_{k=1}^{n} w_k\, y_{(k)} \;=\; S, \quad \textit{so}\ \text{slack} = 0.$$

Interpretation. *The* $y_{\max}$*-cap is a simple data-dependent envelope (depending only on* m, Y, $y_{\max}$ *and the weights); it is typically tight when mass is spread.*

Proof. Let $y_{[1]} \ge \cdots \ge y_{[n]}$ be the increments sorted descending. By rearrangement,

$$\Delta_{\text{joint}} \;\le\; \sum_{k=1}^{n} w_k\, y_{[k]}.$$

Maximizing the right-hand side under $0 \le y_{[k]} \le y_{\max}$, $\sum_k y_{[k]} = Y$, and at most m positives fills the top q slots with $y_{\max}$ and places the remainder r in slot q+1, yielding $y_{\max} \sum_{k=1}^{q} w_k + r\, w_{q+1}$. Since slack = $\Delta_{\text{joint}} - S$, the stated slack bound follows. In the no-rank-crossing regime, each baseline rank-k index remains at rank k, so the joint change equals $\sum_k w_k y_{(k)} = S$. □

Lemma 8 (Slack for maximin)**.** *Let* $F_{\min}(\mathbf{Z}) = \min_i z_i$*. Denote* $m^\star := \min_i z_i$ *and* $S := \{i : z_i = m^\star\}$*. Let* $\sigma := \min_{j \notin S} z_j$ *(or* $+\infty$ *if* $S = \{1, \dots, n\}$*), and define* $y_{\max} := \max_i y_i$*.*

(1) *If* $|S| \ge 2$*,*

$$\text{slack} \;=\; \min\bigl\{ \min_{i \in S} y_i,\ \sigma - m^\star \bigr\} \;\in\; [0,\ y_{\max}],$$

with slack = 0 *iff* $\min_{i \in S} y_i = 0$.

(2) *If* $|S| = 1$ *with* $S = \{i^\star\}$*, let* $\sigma' := \min_{j \ne i^\star}(z_j + y_j) \ge \sigma$*. Then*

$$\begin{aligned} \text{slack} = {} & \min\{\, y_{i^\star},\ \sigma' - m^\star \,\} - \min\{\, y_{i^\star},\ \sigma - m^\star \,\} \\ & \in \bigl[\, 0,\ \min\{ y_{i^\star},\ \sigma' - \sigma \} \,\bigr]. \end{aligned}$$

In particular, slack = 0 *if either* $y_{i^\star} \le \sigma - m^\star$ *or* $\sigma' = \sigma$*.*

Proof. Let $\Delta_{\text{joint}} = F_{\min}(\mathbf{Z}+y) - F_{\min}(\mathbf{Z})$ and $\Delta_i^{\text{local}} = F_{\min}(\mathbf{Z}+ y_i e_i) - F_{\min}(\mathbf{Z})$.

(1) If $|S| \ge 2$, every single-coordinate update leaves some entry at $m^\star$, so $\Delta_i^{\text{local}} = 0$ for all i and $S = 0$. The new minimum after the joint update is $\min\{\, m^\star + \min_{i \in S} y_i,\ \sigma \,\}$, hence $\Delta_{\text{joint}} = \min\{\min_{i \in S} y_i,\ \sigma - m^\star\}$ and the stated bounds follow.

(2) If $|S| = 1$ with $S = \{i^\star\}$, then $\Delta_{i^\star}^{\text{local}} = \min\{y_{i^\star}, \sigma - m^\star\}$ and $\Delta_i^{\text{local}} = 0$ for $i \ne i^\star$, so $S = \min\{y_{i^\star}, \sigma - m^\star\}$. After the joint update, the second-smallest value becomes $\sigma' = \min_{j \ne i^\star}(z_j + y_j) \ge \sigma$, hence $\Delta_{\text{joint}} = \min\{y_{i^\star}, \sigma' - m^\star\}$. Subtracting gives the claim and bounds. □

These results allow us to strengthen the monotonicity guarantee: if S increases by more than an upper bound on slack, then realized fairness must also strictly increase. For α-fairness, this is immediate because slack = 0.

A.4 Corollary: Strict Realized Fairness at β-Driven Switches

Combining Theorem 2 with the slack bounds above yields a useful corollary: when reallocations occur as β increases, and the increase in surrogate fairness exceeds the slack bound of the previous allocation, realized fairness must strictly improve.

This gives a precise, testable condition:

- For α-fairness, every true switch increases realized fairness.
- For variance, a computable quadratic bound applies (Lemma 6).
- For GGF, a $y_{\max}$-cap yields a data-dependent bound, with slack vanishing when no rank crossings occur (Lemma 7).
- For maximin, the gap depends on whether the minimum is unique (Lemma 8).

In practice, this means β can be tuned with confidence that surrogate monotonicity is preserved, and—in many metrics—realized fairness improves as well.

A.5 Practical Implications of the Theoretical Results

The preceding theorems are not only of theoretical interest but also provide practical tools for deploying GIFF in real systems:

Certified lower bounds. The Local–Gain Lower Bound guarantees that GIFF's surrogate never overstates realized fairness improvement. This turns the surrogate into a safe proxy: if an allocation is predicted to achieve at least ε fairness gain, the realized gain is provably $\ge \varepsilon$. For α-fairness, the surrogate is exact.

Safe tuning of β*.* The monotonicity theorem ensures that increasing the fairness weight β cannot reduce surrogate fairness. Practitioners can therefore adjust β to explore efficiency–fairness trade-offs without fear of hidden regressions.

Two-sided certificates. Slack bounds provide upper bounds on how much realized fairness can exceed the surrogate. Together with the lower bound, this yields a sandwich:

$$S \;\le\; \Delta_{\text{joint}} \;\le\; S + \text{slack}_{\max}.$$

This makes GIFF auditable: both the minimum guaranteed improvement and the potential gap are known at each round.

Fairness floors as constraints. Because S is conservative, one can impose hard constraints of the form $S \ge \varepsilon$ in the allocation ILP, guaranteeing that realized fairness improvement is at least ε each round.

Telescoping guarantees. Summing the Local–Gain Lower Bound over time gives a cumulative guarantee:

$$F(\mathbf{Z}_T) - F(\mathbf{Z}_0) \;\;\ge\;\; \sum_{t=0}^{T-1} S^{(t)}.$$

This yields an auditable trajectory of fairness progress over a deployment horizon.

Monitoring and debugging. The bounds enable runtime checks. If the observed joint change Δ_{joint} ever falls below the computed S, then assumptions (such as nonnegative increments) have been violated, or an implementation error is present.

In short, the theorems make GIFF *deployable*: the system's fairness behavior becomes predictable, auditable, and tunable in ways that are both theoretically grounded and operationally meaningful.

B APPROXIMATION FOR DISTRIBUTED COMPUTATION:

Computing the counterfactual fairness gains requires access to the Q-values of all agents. However, in many practical settings, collecting the true Q-values from every agent at each time-step may be infeasible due to communication constraints or scalability concerns.

To reduce this overhead, we adopt an approximation where each agent assumes that all other agents evaluate actions similarly to themselves. That is, agent i approximates the Q-values of other agents $j \in \alpha_c$ as:

$$Q(o_i, a) \simeq Q(o_j, a) \quad \forall j \in \alpha_c, \tag{9}$$

where $\alpha_c \subseteq \alpha$ is the subset of agents competing for a.

This assumption enables each agent to use their own Q-value estimates as stand-ins for those of others when computing fairness-aware updates. As a result, the modified Q-values can be computed using only the current utility vector $\mathbf{Z}_t$ and the agent's local Q-values, avoiding the need for global Q-value communication.

C SIMPLE INCENTIVES - EXTENDED (SI-X) BASELINE

To provide a strong point of comparison for our homelessness prevention experiments, we developed a competitive baseline by directly adapting the **Simple Incentives (SI)** framework [2]. The original method was designed to mitigate fairness issues in ridesharing by moving allocations closer to statistical parity. We apply its core variance-minimization logic to the homelessness domain, leading to the variant we term SI-X.

The central idea of the original SI framework is to treat the variance of outcomes across all groups, $\text{var}(\mathbf{Z})$, as a proxy for unfairness. At each step, the goal is to make an allocation that takes a gradient step toward minimizing this variance. The original paper derives a modified score function, s', by subtracting the gradient of the variance from the original utility score, $s(i, a)$:

$$s'(i,a) = s(i,a) - \lambda \frac{\partial}{\partial \mathcal{A}} \text{var}(\mathbf{Z}) = s(i,a) + \frac{2}{|\mathbf{Z}|}\lambda \sum_{z_j \in \mathbf{Z}} (\bar{z} - z_j) \frac{\partial z_j}{\partial \mathcal{A}} \tag{10}$$

where λ is a hyperparameter, z_j is the historical average outcome for group j, and $\bar{z}$ is the average across all groups. The second term is the fairness *incentive*: for each group, its disparity from the mean, $(\bar{z} - z_j)$, is weighted by the impact of the allocation on its outcome, $\frac{\partial z_j}{\partial \mathcal{A}}$.

To apply this framework to the homelessness domain, we make two simplifying assumptions in line with the assumptions made by the original work:

(1) **Simplification**: For an action a involving a single household h, the sum over all groups collapses to just the term for that household's group, $G(h)$, since no other group's metric is affected.
(2) **Approximation**: The complex partial derivative, $\frac{\partial z_{G(h)}}{\partial a}$, is approximated with a simple, intuitive heuristic: the difference between the action's re-entry probability and the group's historical average, $\Pr(h, a) - z_{G(h)}$.

These steps yield the final form for the fairness incentive, $F(a)$, used in our experiments. The modified utility for an action, $s'(a)$, becomes:

$$s'(a) = Q(a) + \beta F(a) \tag{11}$$

$$\text{where } F(a) = \underbrace{(\bar{z} - z_{G(h)})}_{\text{group advantage}} \times \underbrace{(\Pr(h,a) - z_{G(h)})}_{\text{action advantage}} \tag{12}$$

and $Q(a) = -\Pr(h, a)$. The two components of this incentive have a clear intuition:

- **Group Advantage**: The term $(\bar{z} - z_{G(h)})$ is positive for better-off groups (where $z_{G(h)} < \bar{z}$) and negative for worse-off groups.
- **Action Advantage**: The term $(\Pr(h, a) - z_{G(h)})$ is negative for "good" actions (lower re-entry risk than the group average) and positive for "bad" actions.

The product of these two terms, summarized in Table 1, creates a targeted incentive structure that directly reflects the original framework's goal of reducing variance by penalizing actions that increase disparity and rewarding those that reduce it.

D GIFF ALGORITHM

Here we provide general algorithms for GIFF. These can be specialized for specific use cases and improved with practical modifications, like vectorization or short-circuiting when β or δ is zero. The main algorithm is provided in Algorithm 1, with the advantage correction process outlined in Algorithm 2.

Table 1: The effect of the SI-X fairness incentive $F(a)$ for $\beta > 0$.

	Better-off group ($\bar{z} - z_{G(h)} > 0$)	**Worse-off group** ($\bar{z} - z_{G(h)} < 0$)
Bad action ($\Pr(h, a) - z_{G(h)} > 0$)	Discourage (Negative incentive)	Heavily discourage (Positive incentive, but bad action)
Good action ($\Pr(h, a) - z_{G(h)} < 0$)	Lightly encourage (Negative incentive)	Heavily encourage (Positive incentive)

Algorithm 1 General Incentives-based Framework for Fairness (GIFF)

Require: Set of agents α, Current payoff vector $\mathbf{Z}$, Q-values $Q(i, a)$, Fairness function F, Hyperparameters β, δ

Initialize modified utility matrix Q_{GIFF}
for each agent $i \in \alpha$ **do**
    for each available action $a \in A_i$ **do**
        // *1. Compute Local Fairness Gain*
        $\Delta F(a) \leftarrow \text{ComputeFairnessGain}(i, a, Q(i, a), \mathbf{Z}, F)$

        // *2. Compute Advantage Correction (see Algorithm 2)*
        $\Delta Q_{\text{adv}}(a) \leftarrow \text{AdvantageCorrection}(i, a, \{Q\}, \mathbf{Z}, F)$

        // *3. Combine into GIFF-modified Utility*
        $Q_f(a) \leftarrow \Delta F(a) + \delta \cdot \Delta Q_{\text{adv}}(a)$
        $Q_{GIFF}(i, a) \leftarrow (1 - \beta) \cdot Q(i, a) + \beta \cdot Q_f(a)$
    end for
end for
// *4. Solve for Final Allocation*
$\mathcal{A}^* \leftarrow \text{SOLVE\_ALLOCATION}(Q_{GIFF})$
return The fair allocation $\mathcal{A}^*$

Algorithm 2 Counterfactual Advantage Correction

Require: Agent i, Action a, Q-values $\{Q(j, a)\}$, Current payoff vector $\mathbf{Z}$, Fairness function F

function ADVANTAGECORRECTION($i, a, \{Q\}, \mathbf{Z}, F$)
    // *1. Compute agent i's local fairness gain*
    $\Delta F(a) \leftarrow \text{ComputeFairnessGain}(i, a, Q(i, a), \mathbf{Z}, F)$

    // *2. Compute average counterfactual fairness gain*
    Initialize list $\mathcal{F}_{\text{cf}}$
    for each agent $j \neq i$ that can take action a **do**
        $\Delta F^{(j)} \leftarrow \text{ComputeFairnessGain}(j, a, Q(j, a), \mathbf{Z}, F)$
        Append $\Delta F^{(j)}$ to $\mathcal{F}_{\text{cf}}$
    end for
    $\Delta F_{\text{avg}}(a) \leftarrow \text{mean}(\mathcal{F}_{\text{cf}})$

    // *3. Compute final correction term*
    $F_{\text{adv}}(a) \leftarrow \Delta F(a) - \Delta F_{\text{avg}}(a)$
    $\Delta Q(a) \leftarrow Q(i, a) - \min_{a' \in A_i} Q(i, a')$
    $\Delta Q_{\text{adv}}(a) \leftarrow F_{\text{adv}}(a) \cdot \Delta Q(a)$
    return $\Delta Q_{\text{adv}}(a)$
end function

E IMPLEMENTATION DETAILS FOR THE RIDESHARING EXPERIMENT

We used the code provided by the authors of SI [2] directly, modifying the parts where the score is modified by SI, instead using GIFF's approach. The ridesharing experiments were conducted in a simulation environment based on prior work [2, 3].

- **Environment**: The simulation models a fleet of 1,000 vehicles in Manhattan using a real-world NYC dataset. The experiments focus on the morning rush hour from 8 am to 12 pm.
- **Q-Values**: We use the same pre-trained Q-value models from the baseline work to ensure a direct and fair comparison. A central allocator uses these Q-values to assign passengers to drivers.
- **Fairness Metrics**: We evaluate fairness for both passengers and drivers, using the same definitions as the baseline method.
  - **Passenger Fairness**: Passengers are grouped by their origin-destination neighborhood pair. The payoff vector, $\mathbf{Z}_p$, consists of the service rate for each group. Fairness is measured as the negative variance, $-var(\mathbf{Z}_p)$.
  - **Driver Fairness**: Each driver is their own group. The payoff vector, $\mathbf{Z}_d$, is the cumulative number of trips assigned to each driver. Fairness is measured as $-var(\mathbf{Z}_d)$.

F IMPLEMENTATION DETAILS FOR THE HOMELESSNESS EXPERIMENT

This section provides additional details on the experimental setup for the homelessness prevention domain, complementing the description in the main text.

F.1 Data and Preprocessing

The experiment utilizes two primary datasets, based on the work of Kube et al. [1]. The first contains counterfactual re-entry probabilities for 13,940 households. These probabilities were generated using Bayesian Additive Regression Trees (BART) and estimate the likelihood of a household re-entering the homelessness system if assigned to one of four interventions: **Emergency Shelter (ES)**, **Transitional Housing (TH)**, **Rapid Re-housing (RRH)**, and **Homelessness Prevention (Prev)**. The second dataset, contains demographic and background features for each household. These two datasets were merged using the household identifier.

To define the fairness groups for our analysis, we automatically selected relevant features from the household data. We filtered for categorical features with at least two and at most twenty unique values. To ensure statistical stability, we further required that each unique value (i.e., each subgroup) within a feature be associated with at least 50 households in the dataset. This process yielded **38 distinct features** (e.g., race, family size, disability status) that were used to define demographic groups in 38 independent experimental runs.

F.2 Temporal Simulation Setup

To simulate a realistic, dynamic allocation process, we structured the experiment into discrete time windows. Using the `EntryDate` for each household, we divided the dataset into sequential, non-overlapping windows of **30 days**. All households entering the system within a given 30-day period were considered for allocation simultaneously at the end of that window.

F.3 Resource Allocation and Constraints

The experiment was conducted in a **constrained allocation setting**, where the total number of available slots for each intervention was fixed over the entire simulation. These totals were derived from the historical assignments in the original dataset:

- **Prevention (Prev)**: 6202 slots
- **Emergency Shelter (ES)**: 4441 slots
- **Transitional Housing (TH)**: 2451 slots
- **Rapid Re-housing (RRH)**: 846 slots

The total available slots were not divided evenly across time. Instead, for each 30-day window, a number of slots for each intervention was made available proportionally. The number of slots in a given window was scaled based on the number of households arriving in that window relative to the total number of households. This ensures that resource availability realistically matches demand over time. Within each time window, an ILP solver was used to find the optimal assignment of households to the available intervention slots, based on the (potentially fairness-modified) utility scores.

F.4 Fairness Metric and Evaluation Protocol

The fairness objective in this experiment was to minimize the disparity in average re-entry probabilities across demographic groups. We used the **Gini index** as our measure of inequality. Since our framework is formulated as a maximization problem, the specific fairness function provided to the methods was the **negative Gini index** (`fairness_function = lambda x: -gini(x)`).

The full experiment consisted of a loop over the 38 selected demographic features. For each feature, we ran the entire temporal simulation for both the GIFF and SI-X methods. For each method, we performed a sweep across a range of the fairness hyperparameter (β) to trace the trade-off between overall efficiency (total re-entry probability) and fairness (Gini index). The results from all 38 runs were then aggregated to produce the final distributions shown in the main paper.

G PRACTICAL NOTES

G.1 A Note on Computational Overhead:

Assuming that evaluating the fairness function on a payoff vector takes constant time, computing the GIFF-modified Q-value for one agent involves two main steps for each available action. First, the fairness gain for an action is computed in constant time. Without advantage correction, this is the same computational complexity as it takes to enumerate all Q-values. Second, the advantage correction term requires evaluating the counterfactual fairness gain for up to n candidate agents, resulting in $O(n)$ time per action. With an average of m actions available per agent, the overall computational overhead for calculating the modified Q-values for one agent is $O(m \cdot n)$. The total time for all agents, then, is $O(m \cdot n^2)$, which is much smaller than evaluating fairness over the total combinatorial search space of $O(m^n)$ joint actions.

G.2 Clarification of the Q-value Assumption in Theoretical Analysis

Our theoretical results in Appendix A, particularly the local-gain lower bound (Theorem 1) and the monotonicity guarantee (Theorem 2), are derived under the idealizing assumption of Q-value correctness (Assumption 2). Specifically, we assume that the Q-value for an agent's action, $Q(o_i, a)$, represents the exact, single-step utility increment, y_i, that the agent will realize from that allocation in the current timestep.

We acknowledge that this is a simplification. In practice, Q-values are learned estimates of the long-term, discounted sum of future rewards and are subject to approximation errors, especially when function approximators like deep neural networks are used. They are not equivalent to the true, immediate utility gain. However, if the Q-values are accurate representations of long-term value attained by the agent, the fairness improvement will still be correct, even though it will not be realized within the single time step.

Additionally, this kind of assumption is standard for the theoretical analysis of mechanisms built on top of reinforcement learning. Its purpose is to isolate and analyze the properties of the GIFF framework itself, disentangled from the separate and complex issue of Q-value estimation error. By assuming perfect Q-values as inputs, we can prove that the GIFF mechanism for translating utility into fairness is principled—that its surrogate is a valid lower bound and that its tuning parameter β behaves predictably. The strong performance of GIFF in our diverse empirical evaluations, which use Q-values learned in complex and stochastic environments (or which appear from black-box sources), suggests that these desirable theoretical properties are robust enough to hold in practice even when this assumption is relaxed.

H BROADER IMPACT STATEMENT

Methods to approach fairness often come at a cost to some participants' utility. Resource allocation using GIFF has the potential to affect real-world stakeholders. However, users should exercise caution, as reliance on inaccurate learned Q-values or poor fairness metrics may propagate existing biases in the data , and misapplication in high-stakes environments could lead to unforeseen inequitable outcomes.

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