3. Prove Beta - Binomial conjugation

$X \sim Bin(n, p)$

$f(x|p) = C_x^n \, p^x (1-p)^{n-x} \propto \exp\{x \ln p + (n-x) \ln(1-p)\}$

prior is beta distribution $\pi \sim Beta(\alpha, \beta)$

$\downarrow$

$\pi(p) \propto p^{\alpha-1}(1-p)^{\beta-1} = \exp\{(\alpha-1)\ln p + (\beta-1)\ln(1-p)\}$

$\pi(p|x) = \pi(p) f(x|p)$

$\propto \exp\{x \ln p + (n-x)\ln(1-p)\} \cdot \exp\{(\alpha-1)\ln p + (\beta-1)\ln(1-p)\}$

$= \exp\{(x+\alpha-1)\ln p + (n-x+\beta-1)\ln(1-p)\}$

$= p^{\alpha+x-1}(1-p)^{n+\beta-x-1}$

$\pi(p|x) \sim Beta(\alpha+x, n+\beta-x)$

4. Prove Gamma - Poisson conjugation

$X \sim Exp(\lambda)$

$f(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!} = \frac{1}{x!} \exp\{-\lambda + x \ln \lambda\}$

prior is gamma distribution $\pi \sim gamma(\alpha, \beta)$

$\pi(\lambda) \propto \lambda^{\alpha-1} e^{-\beta\lambda} = \exp\{-\beta\lambda + (\alpha-1) \ln \lambda\}$

$\pi(\lambda|x) = f(x|\lambda)\pi(\lambda)$

$\propto \exp\{-\lambda + x \ln \lambda\} \cdot \exp\{-\beta\lambda + (\alpha-1) \ln \lambda\}$

$= \lambda^{x+\alpha-1} e^{-(\beta+1)\lambda}$

$\pi(\lambda|X) \sim gamma(x+\alpha, \beta+1)$