A Polynomial algorithm for k-cut

for a fixed k

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Problem Statement

k-CUT

The k-cut problem is to find a partition of an edge weighted graph into k nonempty components, such that the total edge weight between components is minimum.

NP-complete

k-CUT problem is NP-Complete.

Reduction

MAXIMUM CLIQUE $<$ k-CUT

MAXIMAL CLIQUE

The decision problem states that :

Given a graph $G = (V, E)$ and a positive integer M, does the largest complete subgraph in G contain exactly M vertices.

Greedy Approach

One natural greedy approach be calculating (S, T) min-cut recursively.
Given, $G = (V, E)$

```
for every subset set V' of V such that |V'| = k {
  w* = min(w*, k-cut(V', G))
}


k-cut(V', G) {
  for every v belongs to V' do {
    1. cut = minimum (S,T) cut
        here v belongs to S and V'\{v} is a subset of T
    2. G' is the graph formed after removing the part of cut
       containing v
    3. w* = min(cut + k-cut(V' - {v}, G'))
  }
}
```

Greedy Approach

One natural greedy approach be calculating (S,T) min-cut recursively.
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    3. w* = min(cut + k-cut(V' - {v}, G'))
  }
}
```

This **does not work** ! [Counter-example]

Naive Approach

This is a purely enumerative approach.

For all possible partitions $V_1, V_2, \ldots V_k$ of V return the minimum k-cut.

Time complexity of this method is $O(n^k)$.

Content

Statement

Main Theorem

Let C be a minimum k-cut in $G = (V, E)$ separating V into k components $V_1, V_2, ..., V_k$. Let C_i be the section of the cut separating V_i from $V \setminus V_i$. Let $w(A)$ be the sum of weights of edges in set A. The weight of the cut satisfies, $w(C) = \frac{1}{2}\sum_{i=1}^{k} w(C_i)$.

Here components are labeled such that :

$$w(C_1) \leq w(C_2) \leq ... \leq w(C_k)$$

Among all solutions choose the one that satisfies **Maximality Assumption**.

Definitions

- core set S and terminal set T
- minimum (S, T)-cut for core S and terminal set T
- maximal minimum (S, T)-cut [unique]

The main theorem is stated for an optimal cut C satisfying the **maximality assumption** and for $k \geq 4$. The case of $k = 3$ is treated separately later.

Content

Statement

Theorem 1

Let $k \geq 4$ and $|V_1| \geq k-2$ then there is a set S of $k-2$ vertices from V_1 and a set T containing one vertex in each $V_1, V_2, ...V_k$ such that C_1 is the maximal minimum (S, T)-cut.

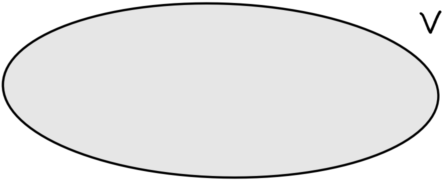
Theorem 1 deals with vertices where $k \geq 4$ and $|V_1| \geq k-2$. The proof of theorem 1 includes 3 parts.

Part 1

Part 1

The source set V_s, of any minimum (S, T)-cut, with S a core, $S \subset V_1$ and T a terminal set, is contained in V_1.

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



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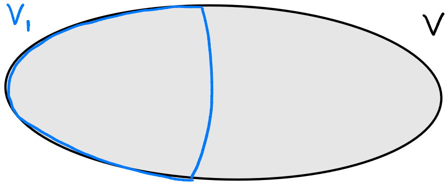

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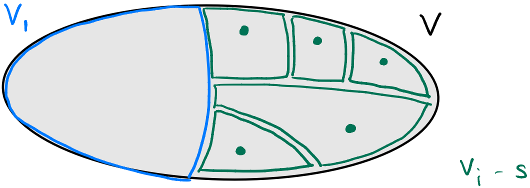

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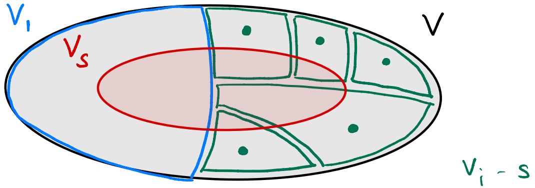

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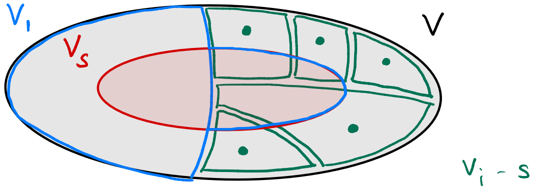
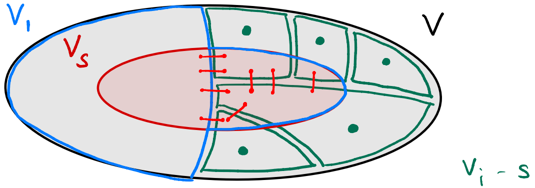


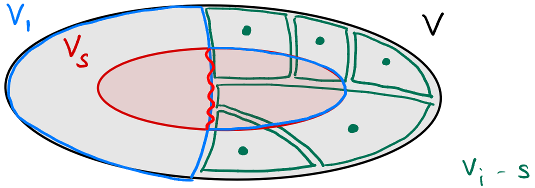
extend V_1 to $V_1 \cup V_S$

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



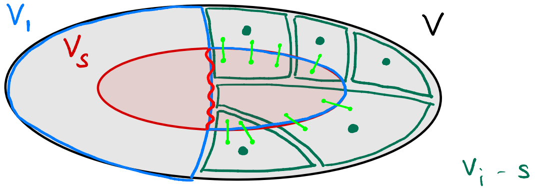
removed edges

$S \subset V_1$, V_s of (S,T)-mincut is inside V_1. $V_s \subset V_1$.



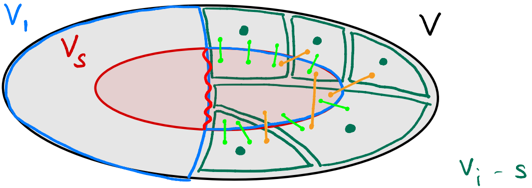
subset of removed edges

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



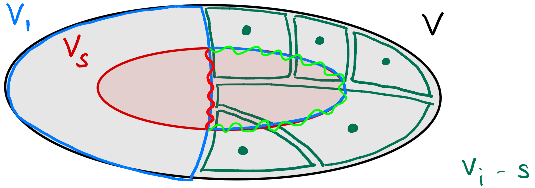
added edges

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



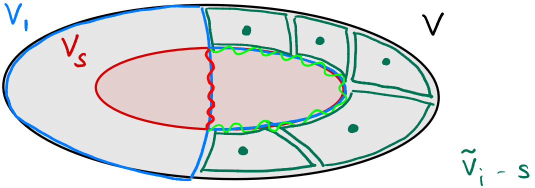
existing edges

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



superset of added edges

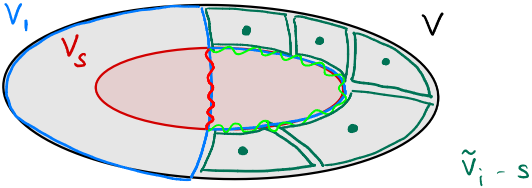
$S \subset V_1$, V_s of (S,T)-mincut is inside V_1. $V_s \subset V_1$.



$$\omega(\text{red cut}) \geq \omega(\text{green cut})$$

(V_s is min (S,T)-cut)

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



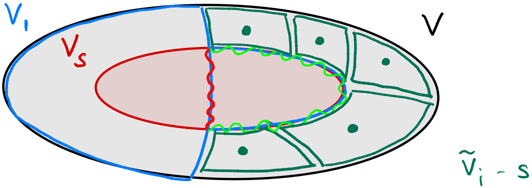
$$\omega(\sim) \geq \omega(\sim)$$

(V_S is min (S,T)-cut)

$$(V_1 \cup V_S,\ \underset{\tilde{V}_2}{\underset{\parallel}{V_2 \setminus V_S}},\ \underset{\tilde{V}_3}{\underset{\parallel}{V_3 \setminus V_S}}, \dots,\ \underset{\tilde{V}_k}{\underset{\parallel}{V_k \setminus V_S}})$$ is k-cut

so: $\omega(\sim) - \omega(\sim) \geqslant 0$

$S \subset V_1$, V_s of (S,T)-mincut is inside V_1. $V_s \subset V_1$.



$$w(\textcolor{red}{\sim}) \geq w(\textcolor{green}{\sim})$$

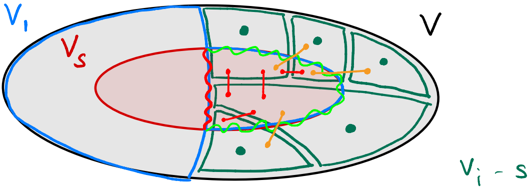
(V_s is min (S,T)-cut)

$(V_1 \cup V_s , V_2 \setminus V_s , V_3 \setminus V_s , \dots , V_k \setminus V_s)$ is k-cut

so: $w(\textcolor{green}{\sim}) - w(\textcolor{red}{\sim}) \geq 0$

$$\therefore \boxed{w(\textcolor{green}{\sim}) = w(\textcolor{red}{\sim})}$$

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



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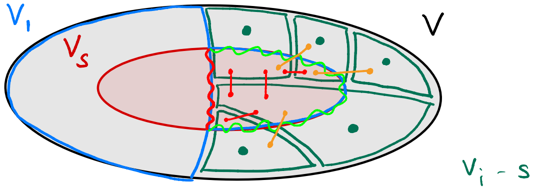
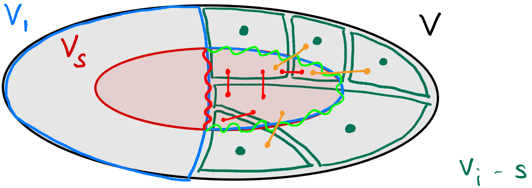


$\Delta_{cost} = 0 = w(\sim) - w(\sim) - w(\bullet) - w(\bullet)$

$S \subset V_1$, V_S of (S,T) - mincut is inside V_1. $V_S \subset V_1$.

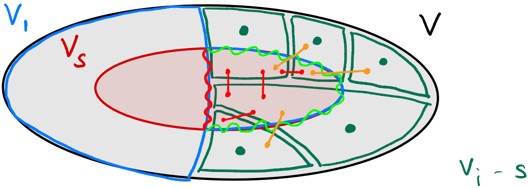


$$\Delta_{cost} = 0 = w(\sim) - w(\sim) - w(\bullet) - w(\bullet)$$

$$= - w(\bullet) - w(\bullet)$$

$S \subset V_1$, V_s of (S,T) - mincut

is inside V_1 . $V_s \subset V_1$.



$$\Delta_{cost} = 0 = \omega(\sim) - \omega(\sim) - \omega(\bullet) - \omega(\bullet)$$

$$= -\omega(\bullet) - \omega(\bullet)$$

so : $\omega(\bullet) = \omega(\bullet) = 0$

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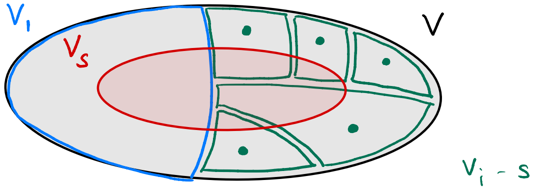
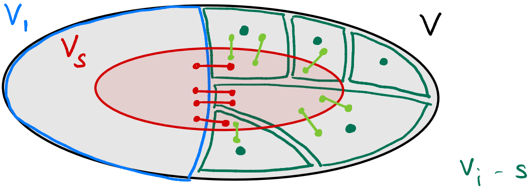


Does $w(V_i)$ change?

$S \subset V_1$, V_s of (S,T)-mincut is inside V_1. $V_s \subset V_1$.



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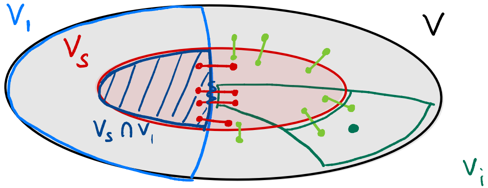

$V_S \cap V_1$ is a (S,T)-mincut

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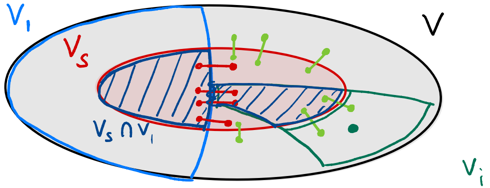
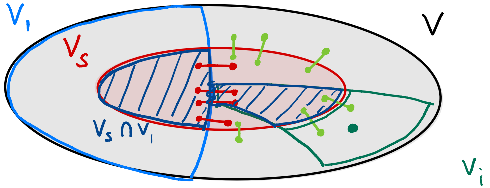


$V_s \cap V_1$ is a (S,T)-mincut

$V_s \cap (V_i \cup V_1)$ is a (S,T)-cut

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.

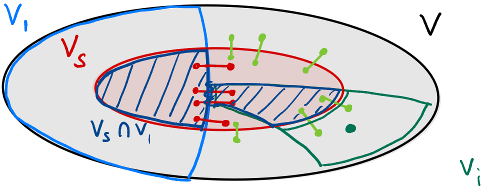


$V_S \cap V_1$ is a (S,T)-mincut

$V_S \cap (V_i \cup V_1)$ is a (S,T)-cut

$\therefore \quad w(\bullet v_i) \leq w(\bullet v_i) \quad \forall i > 1$

$S \subset V_1$, V_S of (S,T) - mincut is inside V_1. $V_S \subset V_1$.



$V_S \cap V_1$ is a (S,T) - mincut

$V_S \cap (V_i \cup V_1)$ is a (S,T) - cut

$\therefore \quad w(\bullet\, v_i) \leq w(\bullet\, v_i) \qquad \forall i > 1$

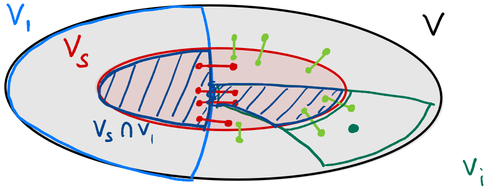
$\vdots$

$+$

$w(\sim) \leq w(\sim)$

$S \subset V_1$, V_s of (S,T) - mincut

is inside V_1 . $V_s \subset V_1$.



$V_s \cap V_1$ is a (s,T) - mincut

$V_s \cap (V_i \cup V_1)$ is a (s, T) - cut

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$w(\sim) = w(\sim)$

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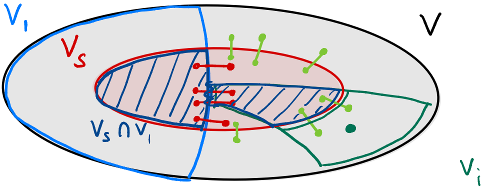


$V_s \cap V_1$ is a (S,T) - mincut

$V_s \cap (V_i \cup V_1)$ is a (S,T) - cut

$$\therefore \quad w(\bullet v_i) = w(\bullet v_i) \qquad \forall i > 1$$

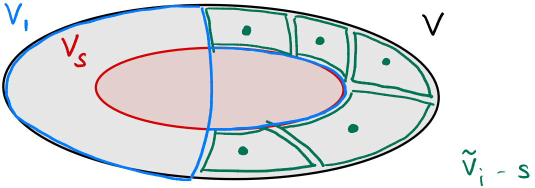
$$\vdots$$

$$+ \quad \overline{}$$

$$w(\sim) = w(\sim)$$

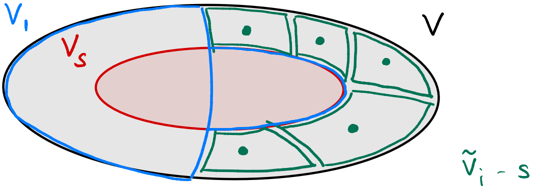
$$\therefore \quad w(\hat{V}_i) = w(V_i) - w(\bullet v_i) + w(\bullet v_i)$$
$$= w(V_i)$$

$S \subset V_1$, V_S of (S,T)-mincut is inside V_1. $V_S \subset V_1$.



$$\begin{array}{ccccccc} w(V_1 \cup V_S) & \leq & w(\tilde{V}_2) & \cdots & \leq & w(\tilde{V}_k) \\ \| & & \| & & & \| \\ w(V_1) & \leq & w(V_2) & \cdots & \leq & w(V_k) \end{array}$$

$S \subset V_1$, V_s of (S,T)-mincut is inside V_1. $V_s \subset V_1$.



$$
\begin{array}{ccccccc}
w(V_1 \cup V_s) & \leq & w(\tilde{V}_2) & \ldots & \leq & w(\tilde{V}_k) \\
\| & & \| & & & \| \\
w(V_1) & \leq & w(V_2) & \ldots & \leq & w(V_k)
\end{array}
$$

but $V_1 \subset (V_1 \cup V_s)$!

Part 2

Part 2

If a minimum (S, T)-cut is not identical to C_1, then this minimum (S, T)-cut has a weight strictly smaller than $w(C_1)$.

Part 3

Part 3

If every minimum (S, T)-cut has a weight strictly smaller than $w(C_1)$, then there exists a k-cut of smaller weight than C, hence a contradiction.

A k-cut ($k \geq 3$) with lower weight

A k-cut $(k \geq 3)$ with lower weight

Let $S = \{s_1, \ldots, s_{k-2}\}$ be the max wt core.

A k-cut $(k \geq 3)$ with lower weight

Let $S = \{s_1, \ldots, s_{k-2}\}$ be the max wt core.

some $s_0 \in V_1 - V_S$

let $\hat{S} = S \cup \{s_0\}$

A k-cut $(k \geqslant 3)$ with lower weight

Let $S = \{s_1, \ldots, s_{k-2}\}$ be the max wt core.

some $s_0 \in V_1 - V_S$

let $\hat{S} = S \cup \{s_0\}$

<u>Def</u>

$\underline{\text{Def}^n}$: $\widehat{S_i} = \hat{S} \setminus \{s_i\}$ $\quad i \in \{1, \ldots, k-2\}$

A k-cut ($k \geq 3$) with lower weight

Let $S = \{s_1, \dots, s_{k-2}\}$ be the max wt core.

some $s_0 \in V_1 - V_S$

let $\hat{S} = S \cup \{s_0\}$

<u>Def^n</u>: $\hat{S_i} = \hat{S} \setminus \{s_i\}$ $\quad i \in \{1, \dots, k-2\}$

Claim: $s_i \notin V_{\hat{S_i}}$

A k-cut $(k \geq 3)$ with lower weight

Claim: $s_i \notin V_{\hat{s}_i}$

FSOC $s_i \in V_{\hat{s}_i}$

$V_{\hat{s}_i}$ is a (S, T)-cut, so

$$\omega(V_{\hat{s}_i}) \geq \omega(V_S)$$

A k-cut ($k \geq 3$) with lower weight

Claim: $s_i \notin V_{\hat{S_i}}$

FSOC $s_i \in V_{\hat{S_i}}$

$V_{\hat{S_i}}$ is a (S, T)-cut, so

$$w(V_{\hat{S_i}}) \geq w(V_S)$$

but S is max wt. core, so

$$w(V_{\hat{S_i}}) \leq w(V_S)$$

A k-cut ($k \geq 3$) with lower weight

Claim: $s_i \notin V_{\hat{s_i}}$

FSOC $s_i \in V_{\hat{s_i}}$

$V_{\hat{s_i}}$ is a (S, T)-cut, so

$$w(V_{\hat{s_i}}) \geq w(V_S)$$

but S is max wt. core, so

$$w(V_{\hat{s_i}}) \leq w(V_S)$$

$$\therefore \quad w(V_{\hat{s_i}}) = w(V_S)$$

A k-cut ($k \geq 3$) with lower weight

Claim: $s_i \notin V_{\hat{s_i}}$

FSOC $s_i \in V_{\hat{s_i}}$

$V_{\hat{s_i}}$ is a (s, T)-cut, so

$$w(V_{\hat{s_i}}) \geq w(V_s)$$

but s is max wt. core, so

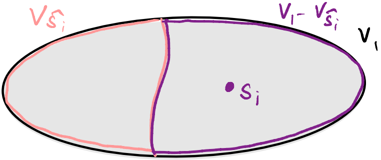
$$w(V_{\hat{s_i}}) \leq w(V_s)$$

$$\therefore \quad w(V_{\hat{s_i}}) = w(V_s)$$

but $s_0 \notin V_s$, so $V_s \cup V_{\hat{s_i}} \supsetneq V_s$ is a larger min (s, T) cut!

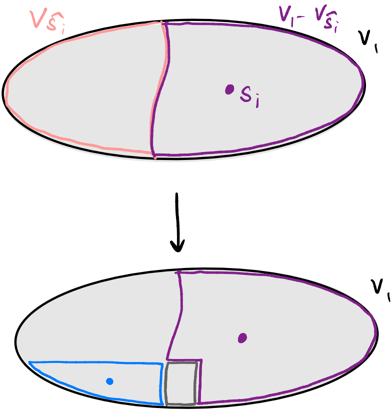
A k-cut ($k \geq 3$) with lower weight

Lemma: $s_i \notin V_{\hat{s_i}}$



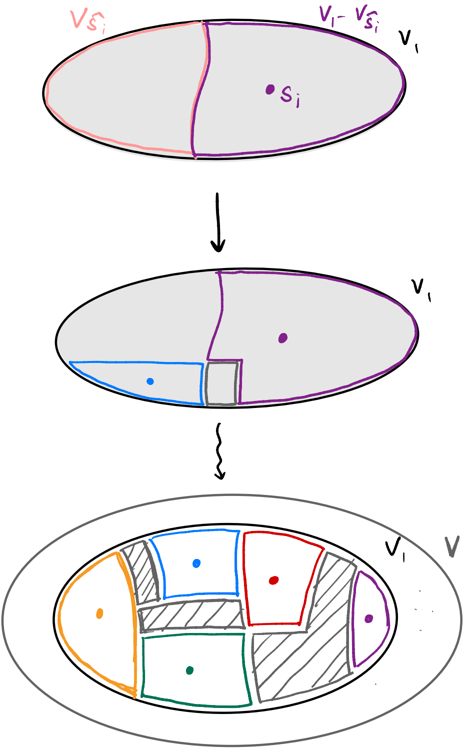
A k-cut ($k \geq 3$) with lower weight

Lemma: $s_i \notin V_{\hat{s_i}}$

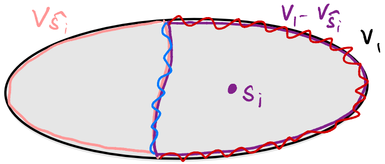


A k-cut ($k \geq 3$) with lower weight

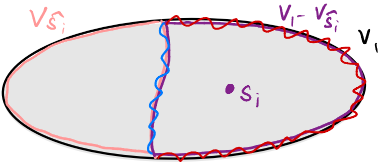
Lemma: $s_i \notin V_{\hat{s}_i}$



A k-cut ($k \geq 3$) with lower weight

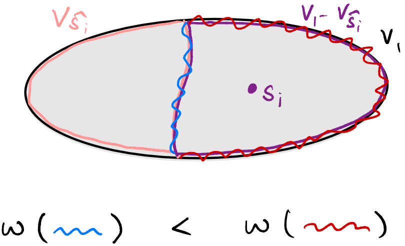


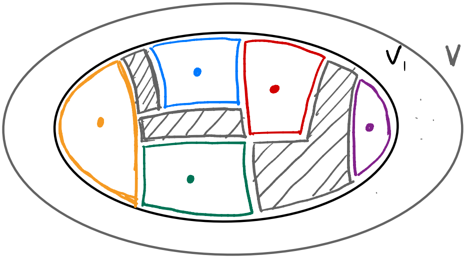
A k-cut ($k \geq 3$) with lower weight



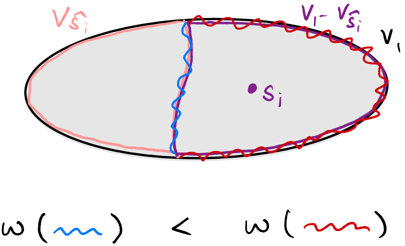
$w($ ~~ $) < w($ ~~ $)$

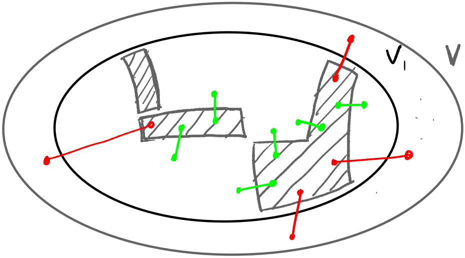
A k-cut ($k \geq 3$) with lower weight





A k-cut $(k \geq 3)$ with lower weight





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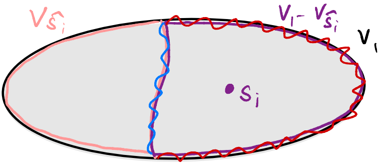
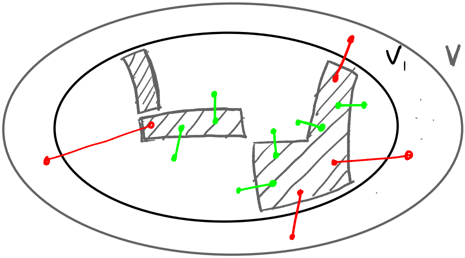


$w(\quad) < w(\quad)$



if $w(\cdot) \geq w(\cdot)$

remove from V_1

A k-cut ($k \geq 3$) with lower weight

$$w(C^*) < 2\,w(C_1)$$

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$$w(C) = \frac{1}{2} \sum_{i=1}^{k} w(C_i)$$

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A k-cut $(k \geq 3)$ with lower weight

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$$\geq \frac{k}{2}\, w(C_1)$$

$$\therefore \quad w(C^*) < \frac{4}{k}\, w(C)$$

Content

Statement

Theorem 2

Let $|V-1| \geq 2$, then there are two vertices in V_1, $S=\{s_1,s_2\}$ and a set $T=\{t_1,t_2\}$, containing one vertex in each of V_2, V_3 such that C_1 is the maximal minimum (S,T)-cut.

$k = 3$

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Let $S = \{s_1, s_2\}$ be max wt core

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Some $s_0 \in V_1 \setminus S$

$$\hat{S} = S \cup \{s_0\}$$

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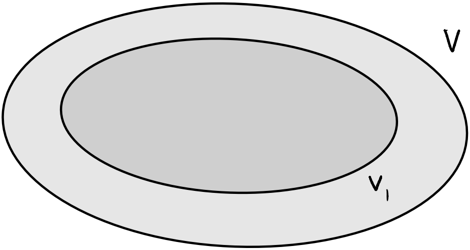

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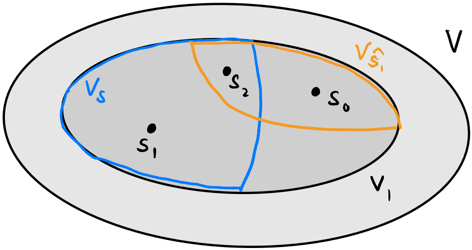

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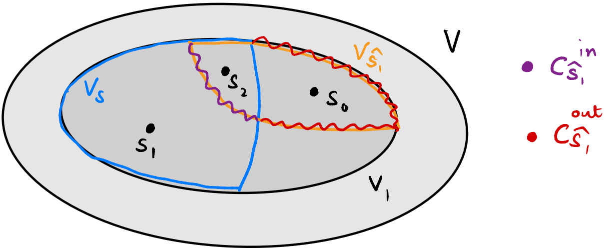

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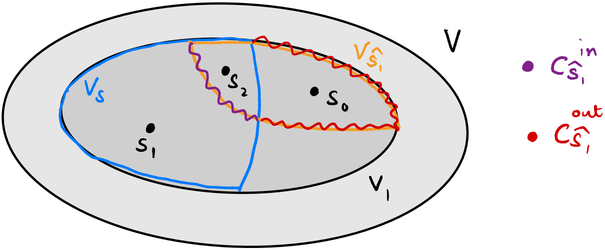


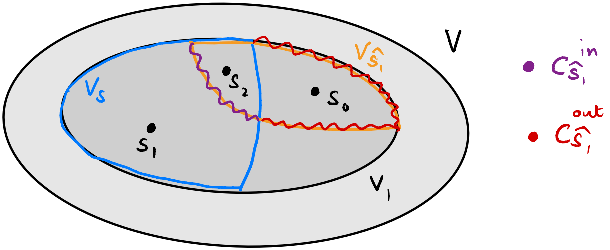
1. $\omega(\bullet) + \omega(c_{\hat{s}_1}^{in}) = W_1$
2. $\omega(\bullet) + \omega(c_{\hat{s}_1}^{out}) = W_2$

$k = 3$

Let $S = \{s_1, s_2\}$ be max wt core

Some $s_0 \in V_1 \setminus S$

$$\hat{S} = S \cup \{s_0\}$$



1. $w(\bullet) + w(c_{\hat{S}_1}^{in}) = W_1$
2. $w(\bullet) + w(c_{\hat{S}_1}^{out}) = W_2$

$+$

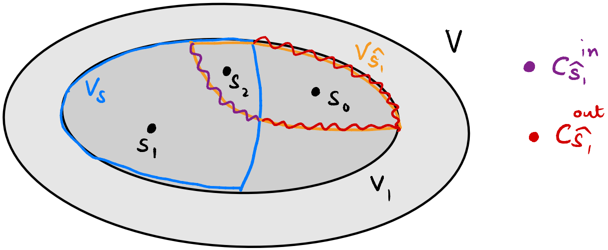
$$2w(\bullet) + w(c_{\hat{S}_1}^{in}) + w(c_{\hat{S}_1}^{out})$$

$k = 3$

Let $S = \{s_1, s_2\}$ be max wt core

Some $s_0 \in V_1 \setminus S$

$\hat{S} = S \cup \{s_0\}$



1. $w(\bullet) + w(c_{\hat{S}_1}^{in}) = W_1$

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$+$

$$2w(\bullet) + w(c_{\hat{S}_1}^{in}) + w(c_{\hat{S}_1}^{out})$$

$$\leq 2w(\bullet) + w(\bullet) \leq 3w(\bullet)$$

$$< 3w(c_1)$$

$k = 3$

Let $S = \{s_1, s_2\}$ be max wt core

Some $s_0 \in V_1 \setminus S$

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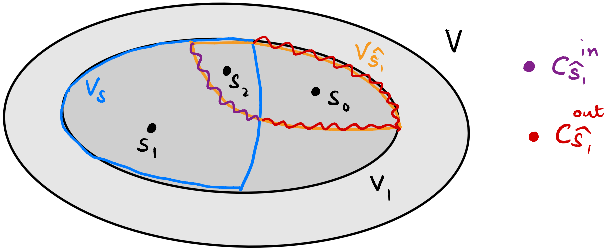

So W_1 or W_2 has cost strictly less than $3/2\, w(C_1)$

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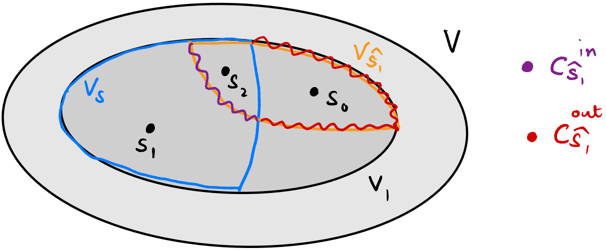

So W_1 or W_2 has cost strictly less than $3/2\, w(C_1)$

But $w(C) = \frac{1}{2} \sum_{i=1}^{3} w(C_i) \geq \frac{3}{2} w(C_1)$

Content

Case 1: $k = 2$

This is trivial!

This requires calculating max-flow $|V| - 1$ times.

Case 2: $k = 3$

3-cut Algorithm :

Phase 1: Consider all possible partitions in which $|V_1| = 1$. For each $v \in V$, let w_1 be the sum of edges adjacent to v. Let w_2 be the weight of the minimum 2-cut in $V - \{v\}$. If $w_1 + w_2 < w^*$, then $w^* = w_1 + w_2$ and the resulting 3 partition (V_1, V_2, V_3) is recorded.

Phase 3: Consider all possible vertices (s_1, s_2, t_1, t_2). Find the maximal minimum (s,t)-cut with source set V_s and sink set V_t. Find the minimum (t_1, t_2)-cut in the set V_t. Let sum of these cuts be w. If $w < w^*$, then $w * = w$ and record the corresponding partition.

Output: w^* and the optimal partition (V_1, V_2, V_3)

Case 3: $k > 3$

k-cut algorithm

Initialization: Let $w^* = \infty$

Phase 1: Consider all possible partitions in which $|V_1| = 1, 2, ..k - 3$. For each subset S of i vertices, let w_1 be the sum of the weights of the edges separating S from $V \setminus S$. Let w_2 be the optimal $(k-1)$-cut in $V \setminus S$. Let w_2 be the optimal $(k-1)$-cut in $V \setminus S$. If $w_1 + w_2 < w^*$ set $w^* = w$.

Case 3: $k > 3$

Phase 2: Consider all vertex subsets K of size $(k-2)+(k-1)=2k-3$. For all $\binom{|K|}{k-2}$ core set S and terminal sets $T = K - S$, find the maximal minimum (S,T)-cut. Let the value of such a cut be w_1. Find optimum $(k-1)$-cut in the corresponding sink set of value, say w_2. if $w_1 + w_2 < w^*$, then $w * = w_1 + w_2$.

Time complexity : $O\left(n^{k^2}\right)$

Thanks for Listening.

Bye Bye