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Nievinski, F. G. (2009), "*Humidity conversion*," Appendix IV in "Ray-tracing Options to Mitigate the Neutral Atmosphere Delay in GPS." M.Sc.E. thesis, Department of Geodesy and Geomatics Engineering Technical Report No. 262, University of New Brunswick, Fredericton, New Brunswick, Canada, 232 pp. <http://hdl.handle.net/1882/1050>

# Appendix IV

# Humidity conversion

There are many ways to express humidity values. For example, the refractivity formulation eq. (2.3) or eq. (2.28) asks for partial pressure of water vapor; radiosondes usually measure relative humidity [Dabberdt et al., 2002] but are required to report dew point temperature, following international convention [Brettle and Galvin, 2003]; in the Canadian regional NWM [Côté et al., 1998] as well as in the CIRA86aQ_UoG climatology [Kirchengast et al., 1999], specific humidity is preferred. The present section aims at serving as a handy primer on converting among different yet equivalent forms of expressing humidity.[1]

Tables IV.1 and IV.2 summarize the expressions to convert from any humidity variable to any other (in a total of $6^2$ combinations), using partial pressure of water vapor $P_w$ as an intermediary variable; pressure $P$ and temperature $T$ are assumed available; details can be found in the derivations that follow in the rest of the present section.

Let us start considering a volume $V$ of humid air at temperature $T$ and total pressure $P$. Its constituents are mass $m_d$ of dry air and mass $m_w$ of water vapor. Its total mass $m$ is therefore:

$$m = m_d + m_w. \qquad \text{(IV.1)}$$

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[1]Useful general references are Guyot [1998]; Wallace and Hobbs [2006]; Glickman [2000]; Curry [2002a,b].

Table IV.1: Conversion *to* partial pressure of water vapor.

| Humidity variable | Input | Output | Expression |
|---|---|---|---|
| Partial pressure of water vapor | $P, T, P_{\mathrm{w}}$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = P_{\mathrm{w}}$ |
| Specific humidity | $P, T, q$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = qP/(\varepsilon + (1-\varepsilon)q)$ |
| Mixing ratio | $P, T, r$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = rP/(r + \varepsilon)$ |
| Relative humidity | $P, T, RH$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = (RH/100)P_{\mathrm{w}}^{\mathrm{s}}, P_{\mathrm{w}}^{\mathrm{s}} = \mathrm{P}_{\mathrm{w}}^{\mathrm{s}}(T)$ |
| Dew point temperature | $P, T, T_{\mathrm{d}}$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = \mathrm{P}_{\mathrm{w}}^{\mathrm{s}}(T = T_{\mathrm{d}})$ |
| Virtual temperature | $P, T, T_{\mathrm{v}}$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = P(1 - T/T_{\mathrm{v}})/(1 - \varepsilon)$ |

Table IV.2: Conversion *from* partial pressure of water vapor.

| Humidity variable | Input | Output | Expression |
|---|---|---|---|
| Partial pressure of water vapor | $P, T, P_{\mathrm{w}}$ | $P_{\mathrm{w}}$ | $P_{\mathrm{w}} = P_{\mathrm{w}}$ |
| Specific humidity | $P, T, P_{\mathrm{w}}$ | $q$ | $q = P_{\mathrm{w}}\varepsilon/(P_{\mathrm{w}}\varepsilon + P_{\mathrm{d}}), P_{\mathrm{d}} = P - P_{\mathrm{w}}$ |
| Mixing ratio | $P, T, P_{\mathrm{w}}$ | $r$ | $r = (P_{\mathrm{w}}/P_{\mathrm{d}})\varepsilon, P_{\mathrm{d}} = P - P_{\mathrm{w}}$ |
| Relative humidity | $P, T, P_{\mathrm{w}}$ | $RH$ | $RH = 100P_{\mathrm{w}}/P_{\mathrm{w}}^{\mathrm{s}}, P_{\mathrm{w}}^{\mathrm{s}} = \mathrm{P}_{\mathrm{w}}^{\mathrm{s}}(T)$ |
| Dew point temperature | $P, T, P_{\mathrm{w}}$ | $T_{\mathrm{d}}$ | $T_{\mathrm{d}} = (\mathrm{P}_{\mathrm{w}}^{\mathrm{s}})^{-1}(P_{\mathrm{w}}^{\mathrm{s}} = P_{\mathrm{w}})$ |
| Virtual temperature | $P, T, P_{\mathrm{w}}$ | $T_{\mathrm{v}}$ | $T_{\mathrm{v}} = T/(1 - (P_{\mathrm{w}}/P)(1 - \varepsilon))$ |

*Specific humidity q is defined as:*

$$q \equiv \frac{m_{\mathrm{w}}}{m}. \tag{IV.2}$$

*Mixing ratio r is defined as*

$$r \equiv \frac{m_{\mathrm{w}}}{m_{\mathrm{d}}}. \tag{IV.3}$$

The two are obviously closely related:

$$q = \frac{r}{r+1}, \qquad\qquad r = \frac{1}{1-q}.$$

To have a sense for the magnitude of such variables, we quote Guyot [1998], who states that "In the atmosphere, the magnitude of $r$ is a few grams per kilogram in the middle latitudes, but in the tropics it can reach approximately $20\,\mathrm{g/kg}$."

The relationship between *partial pressure of water vapor $P_{\mathrm{w}}$* and either $q$ or $r$ requires a bit or theory. The partial pressure of a particular gas in a mixture of gases is the pressure

that such a gas would exert if it were to occupy alone the volume occupied by the whole mixture, at the same temperature. Assuming that the constituent gases do not interact chemically, *Dalton's law of partial pressures* states that

$$P = P_d + P_w. \tag{IV.4}$$

Each gas constituent is assumed to obey the *state equation of ideal gases*,

$$P_d = \rho_d T R_d, \tag{IV.5}$$

$$P_w = \rho_w T R_w, \tag{IV.6}$$

where the same temperature $T$ is shared by the two gases, following the assumption that they are in equilibrium. $\rho_d$ and $\rho_w$ are the *gases densities* ($\rho_w$ is sometimes called *absolute humidity* or *volumetric humidity*),

$$\rho_d = m_d/V, \tag{IV.7}$$

$$\rho_w = m_w/V; \tag{IV.8}$$

$R_d$, $R_w$ are, respectively, the specific *gas constants* for dry gases and water vapor:

$$R_d \equiv R^*/M_d,$$

$$R_w \equiv R^*/M_w,$$

defined in terms of the *universal gas constant*

$$R^* \equiv 8.316963 \times 10^3 \, \text{J/kmol} \times \text{K},$$

and the *molar masses* of dry air and water vapor,

$$M_d \equiv 28.9644\,\mathrm{kg/kmol},$$

$$M_w \equiv 18.0152\,\mathrm{kg/kmol}.$$

(The numerical values are taken from Glickman [2000].) With that, the mixing ratio $r$ can be rewritten as

$$
\begin{aligned}
r &= \frac{m_w}{m_d} \\
&= \frac{\rho_w V}{\rho_d V} \\
&= \frac{\rho_w}{\rho_d} \\
&= \frac{P_w/(T R_w)}{P_d/(T R_d)} \\
&= \frac{P_w/R_w}{P_d/R_d} \\
&= \frac{P_w M_w/R^*}{P_d M_d/R^*} \\
&= \frac{P_w M_w}{P_d M_d} \\
&= \frac{P_w}{P_d}\varepsilon,
\end{aligned}
\tag{IV.9}
$$

where

$$\varepsilon \equiv \frac{M_w}{M_d}. \tag{IV.10}$$

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Similarly, specific humidity $q$ can be expressed as

$$
\begin{aligned}
q &= \frac{m_\mathrm{w}}{m_\mathrm{w} + m_\mathrm{d}} \\
&= \frac{\rho_\mathrm{w} V}{\rho_\mathrm{w} V + \rho_\mathrm{d} V} \\
&= \frac{\rho_\mathrm{w}}{\rho_\mathrm{w} + \rho_\mathrm{d}} \\
&= \frac{P_\mathrm{w}/(T R_\mathrm{w})}{P_\mathrm{w}/(T R_\mathrm{w}) + P_\mathrm{d}/(T R_\mathrm{d})} \\
&= \frac{P_\mathrm{w}/R_\mathrm{w}}{P_\mathrm{w}/R_\mathrm{w} + P_\mathrm{d}/R_\mathrm{d}} \qquad\qquad \text{(IV.11)} \\
&= \frac{P_\mathrm{w} M_\mathrm{w}/R^*}{P_\mathrm{w} M_\mathrm{w}/R^* + P_\mathrm{d} M_\mathrm{d}/R^*} \\
&= \frac{P_\mathrm{w} M_\mathrm{w}}{P_\mathrm{w} M_\mathrm{w} + P_\mathrm{d} M_\mathrm{d}} \\
&= \frac{P_\mathrm{w} M_\mathrm{w}}{P_\mathrm{w} M_\mathrm{w} + P_\mathrm{d} M_\mathrm{d}} \frac{1/M_\mathrm{d}}{1/M_\mathrm{d}} \\
&= \frac{P_\mathrm{w} \varepsilon}{P_\mathrm{w} \varepsilon + P_\mathrm{d}}.
\end{aligned}
$$

The inverse relationships require some care in their derivation: we shall never try to simplify expressions dividing them by $q$ or $P_\mathrm{w}$, because those quantities might assume the zero value:

$$
\begin{aligned}
r &= \frac{P_\mathrm{w}}{P_\mathrm{d}} \varepsilon \\
r &= \frac{P_\mathrm{w}}{P - P_\mathrm{w}} \varepsilon \\
r - r P_\mathrm{w} - P_\mathrm{w} \varepsilon &= 0 \\
-P_\mathrm{w}(r + \varepsilon) &= -r P \\
P_\mathrm{w} &= \frac{r P}{r + \varepsilon}
\end{aligned}
$$

and

$$q = \frac{P_{\mathrm{w}}\varepsilon}{P_{\mathrm{w}}\varepsilon + P_{\mathrm{d}}}$$

$$q = \frac{P_{\mathrm{w}}\varepsilon}{P_{\mathrm{w}}\varepsilon + P - P_{\mathrm{w}}}$$

$$qP_{\mathrm{w}}\varepsilon + qP - qP_{\mathrm{w}} - P_{\mathrm{w}}\varepsilon = 0$$

$$P_{\mathrm{w}}(q\varepsilon - q - \varepsilon) = -qP$$

$$P_{\mathrm{w}} = \frac{-qP}{q\varepsilon - q - \varepsilon}$$

$$P_{\mathrm{w}} = \frac{qP}{\varepsilon + q - q\varepsilon}$$

$$P_{\mathrm{w}} = \frac{qP}{\varepsilon + (1 - \varepsilon)q}.$$

In the remaining conversions below, we will need the so-called *saturation water vapor pressure* $P_{\mathrm{w}}^{\mathrm{s}}$. It is important to realize that $P_{\mathrm{w}}^{\mathrm{s}}$ itself is *not* a way of expressing humidity values, just an auxiliary quantity involved in the conversion. It is defined as the partial pressure at which water vapor is in equilibrium with a flat surface of liquid water, at a given temperature (equilibrium meaning that evaporation and condensation are both occurring — at the same rate, though). There are numerous formulas in use for $P_{\mathrm{w}}^{\mathrm{s}}$. Murphy and Koop [2005] provide a recent throughly review and comparison of both old and new formulations against experimental data, from which we quote:

> All of the commonly used parametrizations for the vapor pressure of supercooled water are extrapolations that were not originally intended for use below the freezing point. In addition, the World Meteorological Organization definition of the vapor pressure of supercooled water contains an easily overlooked typographical error. Recent data on the molar heat capacity of supercooled water are used [by us] to derive its vapor pressure.

We therefore adopt their formulation:

$$\ln(P_w^s) \approx + 54.842763$$
$$- 6763.22/T$$
$$- 4.210\ln(T)$$
$$+ 0.000367T$$
$$+ \tanh\left(0.0415(T - 218.8)\right)$$
$$\times \left(53.878 - 1331.22/T - 9.44523\ln(T) + 0.014025T\right),$$

(IV.12)

valid for $123\,\text{K} < T < 332\,\text{K}$. Below we denote $P_w^s = \text{P}_w^s(T)$ the function that returns the value of $P_w^s$ at a particular value of $T$. We also need the inverse function, which we denote $T = (\text{P}_w^s)^{-1}(P_w^s)$, returning the value of $T$ at a particular value of $P_w^s$.[2]

With $P_w^s$ at hand, we define *relative humidity RH* as the ratio

$$RH \equiv 100\frac{P_w}{P_w^s},$$

(IV.13)

and *dew point temperature* $T_d$ as the temperature to which the air must be lowered, at constant pressure, for the water vapor to saturate:

$$T_d \mid P_w = \text{P}_w^s(T = T_d).$$

(IV.14)

To have a sense for such variables, we quote Wallace and Hobbs [2006, p. 83]:

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[2]Equation (IV.12) cannot be inverted analytically, so we resort to the Newton–Raphson numerical method; a good approximate solution upon which to start and improve iteratively is given by Bolton [1980, eq. (10)]:

$$x = \ln\left((P_w^s/1000)/0.6112\right)/17.67,$$
$$T = 273.15 + 243.5x/(1 - x).$$

At the earth's surface, the pressure typically varies by only a few percent from place to place and from time to time. Therefore, the dew point [temperature] is a good indicator of the moisture content of the air. In warm, humid weather the dew point is also a convenient indicator of the level of human discomfort. (...) In contrast to the dew point, relative humidity depends as much upon the temperature of the air as upon its moisture content.

Finally, *virtual temperature $T_v$* is defined such that

$$T_v \mid P = \rho T_v R_d, \tag{IV.15}$$

where $P = P_d + P_w$ and $\rho = \rho_d + \rho_w$ refer to the moist air as a whole, even though eq. (IV.15) employs the constant of dry gases $R_d$, as explained by Wallace and Hobbs [2006, p. 66]:

Moist air has a lower apparent molecular weight than dry air. Therefore, the gas constant for 1 kg of moist air is larger than that for 1 kg of dry air. However, rather than use a gas constant for moist air, the exact value of which would depend on the amount of water vapor in the air (which varies considerably), it is more convenient to retain the gas constant for dry air and use a fictitious temperature (called the *virtual temperature*) in the ideal gas equation.

To express $T_v$ in terms of $P_w$, first we re-write $\rho$ as

$$
\begin{aligned}
\rho &= P_d/(TR_d) + P_w/(TR_w) \\
&= P_d M_d/(TR^*) + P_w M_w/(TR^*) \\
&= \frac{P_d M_d + P_w M_w}{TR^*} \\
&= \frac{P_d M_d + P_w M_w}{TR^*} \frac{M_d}{M_d} \frac{P}{P} \\
&= \frac{P M_d}{TR^*}\left(1 - (P_w/P)(1-\varepsilon)\right),
\end{aligned} \tag{IV.16}
$$

then we isolate $T_v$ and substitute $\rho$, obtaining

$$
\begin{aligned}
T_v &= P/(\rho R_d) \\
&= P/(\rho R^*/M_d) \\
&= PM_d/(\rho R^*) \\
&= PM_d \left( R^* \frac{PM_d}{TR^*} \left(1 - (P_w/P)(1-\varepsilon)\right) \right) \\
&= \frac{T}{1 - (P_w/P)(1-\varepsilon)}.
\end{aligned}
\tag{IV.17}
$$