Homework 01

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Problem 1: 1.1

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

Proof.

$$
\begin{aligned}
&[AB, CD] = ABCD - CDAB \\
&= ABCD - CDAB + (CADB - CADB) + (ACDB - ACDB) + (ACBD - ACBD) \\
&= (-ACDB - ACBD) + (ACBD + ABCD) - (CDAB - CADB) + (CADB + ACDB) \\
&= -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB
\end{aligned}
$$

□

Problem 2: 1.4

Using the rules of bra-ket algebra, prove or evaluate the following:
(a) $tr(XY) = tr(YX)$, where X and Y are operators.
(b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.
(c) $exp[if(A)] =?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known.
(d) $\sum_{a'} \psi^*_{a'}(\boldsymbol{x'})\psi_{a'}(\boldsymbol{x''})$, where $\psi_{a'}(\boldsymbol{x'}) = \langle \boldsymbol{x'}|a'\rangle$.

Solution. Suppose we have a normalized orthogonal basis $\{|\alpha_i\rangle\} = \{|\alpha_1\rangle, ..., |\alpha_n\rangle\}$, all the operators in this problem will be represented under $\{|\alpha_i\rangle\}$.
(a) The ith element on the diagonal of the matrix XY is:

$$(XY)_i = X_{i1}Y_{1i} + X_{i2}Y_{2i} + ... + X_{in}Y_{ni} = \sum_{j=1}^{n} X_{ij}Y_{ji}$$

then

$$tr(XY) = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}Y_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} Y_{ji}X_{ij} = tr(YX)$$

(b) By definition, for any state $|\alpha\rangle$, we define $X^\dagger$ as:

$$(X|\alpha\rangle)^\dagger = \langle\alpha|X^\dagger$$

so

$$(XY|\alpha\rangle)^\dagger = (Y|\alpha\rangle)^\dagger X^\dagger = \langle\alpha|Y^\dagger X^\dagger$$

which means

$$(XY)^\dagger = Y^\dagger X^\dagger$$

(c) By another definition of function of operators (see *Quantum Computation and Quantum Information, Michael A. Nielsen, Anniversary edition* page 75), we first write

$$A = \sum_i a_i |i\rangle\langle i|$$

where $|i\rangle$ is the eigenvector of A, and a_i is the corresponding eigenvalue. Then by definition,

$$e^{if(A)} = \sum_i e^{ia_i}|i\rangle\langle i|$$

(d) We have $\psi_{a'}(\boldsymbol{x'}) = \langle\boldsymbol{x'}|a'\rangle$, $\psi^*_{a'}(\boldsymbol{x'}) = \langle a'|\boldsymbol{x'}\rangle$, $\psi_{a'}(\boldsymbol{x''}) = \langle\boldsymbol{x''}|a'\rangle$, so

$$\begin{aligned}
\sum_{a'} \psi^*_{a'}(\boldsymbol{x'})\psi_{a'}(\boldsymbol{x''}) &= \sum_{a'}\langle a'|\boldsymbol{x'}\rangle\langle\boldsymbol{x''}|a'\rangle \\
&= \sum_{a'}\langle\boldsymbol{x''}|a'\rangle\langle a'|\boldsymbol{x'}\rangle \\
&= \langle\boldsymbol{x''}|\left(\sum_{a'}|a'\rangle\langle a'|\right)|\boldsymbol{x'}\rangle \\
&= \langle\boldsymbol{x''}|\boldsymbol{x'}\rangle
\end{aligned}$$

□

Problem 3: 1.6

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

Solution. The condition is: $|i\rangle$ and $|j\rangle$ corresponding to the same eigenvalue.

Suppose $|i\rangle$ and $|j\rangle$ corresponding to the same eigenvalue α, we have

$$A|i\rangle = \alpha|i\rangle,\ A|j\rangle = \alpha|j\rangle$$

so

$$A(|i\rangle + |j\rangle) = \alpha|i\rangle + \alpha|j\rangle = \alpha(|i\rangle + |j\rangle),$$

then $(|i\rangle + |j\rangle)$ is also an eigenket of A. □

Problem 4: 1.14

A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

(a) Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?
(b) Give a physical example where all this is relevant.

Solution. Let the observable in the problem as A.
(a) Let λ as the eigenvalue, we have $A|\alpha\rangle = \lambda|\alpha\rangle \Rightarrow (A-\lambda I)|\alpha\rangle = 0$. So there is

$$det(A-\lambda I) = 0$$

we can get the solution $\lambda = -1, 1, 0$ and the corresponding eigenkets

$$|\alpha_1\rangle = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix},\ |\alpha_2\rangle = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix},\ |\alpha_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

So there's no degeneracy. Here, we can also use *Mathematica* to get the eigenkets and the corresponding eigenvalues.

```
In[∘]:= 1/√2 {{0, 1, 0}, {1, 0, 1}, {0, 1, 0}} // Eigenvalues
                                                 特征值
Out[∘]= {-1, 1, 0}

        Normalize /@ Eigenvectors[1/√2 {{0, 1, 0}, {1, 0, 1}, {0, 1, 0}}]
        正规化        特征向量
Out[∘]= {{1/2, -1/√2, 1/2}, {1/2, 1/√2, 1/2}, {-1/√2, 0, 1/√2}}
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(b) Due to A is Hermitian, we can regard A as a Hamiltonian, so the different eigenvalues of A represent different energy level of A, and these corresponding eigenvectors are the eigenstates. □

Problem 5: Additional

If operator U satisfies $UU^\dagger = I$ in a certain representation, show this is true for any other representations.

Proof. Suppose we have an orthogonal normalized basis $\{|\alpha_i\rangle\}$ and an operator $\hat{U}$, let U to be the corresponding matrix representation under basis $\{|\alpha_i\rangle\}$. So we have

$$U_{ij} = \langle\alpha_i|\hat{U}|\alpha_j\rangle$$

Let $\{|\alpha_i'\rangle\}$ be a new orthogonal normalized basis, so the new basis could be represented under the old basis $\{|\alpha_i\rangle\}$:

$$|\alpha_i'\rangle = \sum_j S_{ji}|\alpha_j\rangle$$

or we can simply write it as:

$$|\alpha_i'\rangle = S|\alpha_i\rangle$$

where S is a unitary operator under basis $\{|\alpha_i\rangle\}$.
Let U' to be the corresponding matrix representation of $\hat{U}$ under new basis $\{|\alpha_i'\rangle\}$, we have

$$U' = S^{-1}US = S^\dagger US$$

then

$$\begin{aligned}
U'U'^\dagger &= S^\dagger US(S^\dagger US)^\dagger \\
&= S^\dagger US \cdot (US)^\dagger S \\
&= S^\dagger US \cdot S^\dagger U^\dagger S \\
&= S^\dagger UU^\dagger S = S^\dagger S = I
\end{aligned}$$

□