Chapter 3: Angular Momentum

Yuquan Chen

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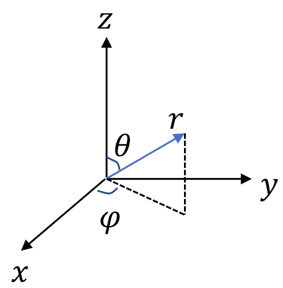
We have discussed position and momentum operator before, now let's consider the rotation of a system, which leads to angular position and angular momentum. Before we dive in, there's some prerequisites we should know.

1 Some prerequisites

In a 3D system, we use $\vec{r} = (x, y, z)$ to represent the coordinate, and the momentum is $\vec{p} = (p_x, p_y, p_z)$. In position representation, we have $p_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}$, $p_y \leftrightarrow -i\hbar\frac{\partial}{\partial y}$, and $p_z \leftrightarrow -i\hbar\frac{\partial}{\partial z}$, together we get $\vec{p} = (-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}) = -i\hbar\vec{\nabla}$.

Now let's switch to spherical coordinate (r, θ, φ).



For a point in space, we use a ket $|\psi\rangle$ to represent it,

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad \Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

now let's consider a rotation along $\hat{z}$ direction. Here φ becomes $\varphi + d\varphi$, and r, θ remain the same. We have

$$|x, y, z\rangle \xrightarrow{\text{rotation}} |x', y', z'\rangle \tag{1}$$

and the corresponding

$$(r, \theta, \varphi) \xrightarrow{\text{rotation}} (r, \theta, \varphi + d\varphi) \tag{2}$$

then we try to find out the expression of x', y', z' in terms of $r, \theta, \varphi, d\varphi$

$$\begin{cases} x' = r\sin\theta\cos(\varphi + d\varphi) \simeq r\sin\theta\cos\varphi - r\sin\theta\sin\varphi d\varphi \\ y' = r\sin\theta\sin(\varphi + d\varphi) \simeq r\sin\theta\sin\varphi + r\sin\theta\cos\varphi d\varphi \\ z' = r\cos\theta = z \end{cases} \Rightarrow \begin{cases} x' = x - y d\varphi \\ y' = y + x d\varphi \\ z' = z \end{cases} \tag{3}$$

On a spin-$\frac{1}{2}$ system, we have Pauli operators $\sigma_x, \sigma_y, \sigma_z$. In the σ_z basis, we have

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$