Chapter 2

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1 Time evolution operator and Hamiltonian

Here, we talk about non-relativistic situation, and we think about time as a parameter, not an operator. The position representation $\langle x|\psi\rangle = \psi(x)$, adding the time evolution, we have

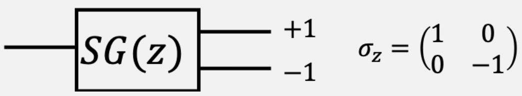
$$\langle x|\psi(t)\rangle = \psi(x,t)$$

First, we talk about 6 postulates of quantum mechanics.

Box 1.1: Postulates of Quantum Mechanics

Postulate 1. At any time t, the state of a physical system is defined by a ket $|\psi\rangle$, or *state* in a relevant Hilbert space H.

Postulate 2. The only possible result of measuring observable A is one of the eigenvalues of A



Aside:

1. If A is Hermitian, then the measurement gives a real number.
2. If A's spectrum is discrete, then we only see quantized result.

Postulate 3. Every measurable physical quantity A is described by a Hermitian operator.

Postulate 4. If $A|u_\alpha\rangle = a_\alpha|u_\alpha\rangle$, then for a system in $|\psi\rangle$, when we measure A, then the probability of getting a_α is $P(a_\alpha) = |\langle u_\alpha|\psi\rangle|^2$.
Aside: If we have degenerate a_α's $\{|u_{\alpha,1}\rangle, |u_{\alpha,2}\rangle, ...\}$ share the same eigenvalue, then $P(a_\alpha) = \sum_i |\langle u_{\alpha,i}|\psi\rangle|^2$
Example: $A = I$, all $a_\alpha = 1$

Postulate 5. If a measurement projects $|\psi\rangle$ into a new state $|u_\alpha\rangle$, then a physical new state should be $|u'_\alpha\rangle = \frac{|u_\alpha\rangle}{\sqrt{\langle u_\alpha|u_\alpha\rangle}}$, so that $\langle u'_\alpha|u'_\alpha\rangle = 1$.

Postulate 6. Between measurement the state vector $|\psi(t)\rangle$ evolves in time with time dependent Shrödinger's equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

here $\hat{H}$ is a Hamiltonian.

We let a displacement dt' on state $|\psi(t)\rangle$,

$$\Rightarrow U(dt')|\psi(t)\rangle = |\psi(t+dt')\rangle, \text{ where } UU^\dagger = 1 \tag{1}$$

It's similar to momentum, in that case, we have

$$\begin{cases} U(dt') = I - i\frac{\hat{H}}{\hbar}dt' \\ \hat{H} \text{ is Hermitian, called Hamiltonian} \end{cases}$$

so (1) could be evaluated as:

$$\text{LHS} = \left(I - i\frac{\hat{H}}{\hbar}dt'\right)\psi(x,t) = \psi(x,t) - i\frac{\hat{H}}{\hbar}dt'\psi(x,t) \tag{2}$$

$$\text{RHS} = \psi(x,t+dt') = \psi(x,t) + \left(\frac{\partial}{\partial t}\psi(x,t)\right)dt' \tag{3}$$

$$\Rightarrow \boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)} \tag{4}$$

which is Shrödinger's equation in position representation. In general, we have

$$\boxed{i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle} \tag{5}$$

H: Hamiltonian in analog to classical mechanics,

$$H = T + V, \quad \begin{cases} T = \frac{p^2}{2m} \text{ is kinetic energy} \\ V \text{ is potential energy} \end{cases} \tag{6}$$

and in quantum mechanics, we have

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + \hat{V}(x) \tag{7}$$

Here are some examples of Hamiltonians in different systems.

Box 1.2: Examples of Hamiltonians in different systems

1. A free particle $V = 0$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

2. Hydrogen atom

$$\hat{H} = \frac{\hat{p}_e^2}{2m_e} + \frac{\hat{p}_n^2}{2m_n} - \frac{e^2}{4\pi\varepsilon_0|\vec{r}_e - \vec{r}_n|}$$

3. A particle magnetic moment $\vec{\mu}$, in external magnetic field $\vec{B}$

$$\hat{H} = -\vec{\mu} \cdot \vec{B}$$