Chapter 3

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1 Coupled density matrix

For a two spin-1/2 particle, ρ_A for the first one, ρ_B for the second. For example,

$$\rho_A = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \rho_B = \frac{1}{2}I = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho = \rho_A \otimes \rho_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in general, for $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability p_i for the particle at state $|\psi_i\rangle$. For 2 particles, the overall state is described as

$$\begin{aligned} \rho &= \sum_i p_i |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes |\phi_i^{(2)}\rangle\langle\phi_i^{(2)}| & (1) \\ &= \sum_i p_i \left(|\psi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle\right) \cdot \left(\langle\psi_i^{(1)}| \otimes \langle\phi_i^{(2)}|\right) & (2) \end{aligned}$$

Box 1.1: Examples

Pure state case: particle 1 at $|0\rangle$, particle 2 at $|1\rangle$. Density matrix $\rho = |01\rangle\langle 01|$
Mixed state case: For a two particle system, we have $\frac{1}{3}$ of chance two particles at $|\psi_1\rangle$, $\frac{1}{3}$ of chance two particles at $|\psi_2\rangle$, and $\frac{1}{3}$ of chance two particles at $|\psi_3\rangle$. The density matrix

$$\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3|$$

2 Dynamics

For 1 particle, $\rho = \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|$, the Shrödinger's equation

$$i\hbar \frac{\partial}{\partial t}|\psi_i(t)\rangle = H|\psi_i(t)\rangle \tag{3}$$

so

$$\dot{\rho} = \frac{d}{dt}\rho = \sum_i p_i \left(\frac{d}{dt}\right) \tag{4}$$

for multi particle system, we have $\rho_{\text{multi}}, H_{\text{multi}}$

$$\Rightarrow \dot{\rho}_{\text{multi}} = \frac{1}{i\hbar}[H_{\text{multi}}, \rho_{\text{multi}}] \tag{5}$$

3 Trace

Under basis $\{|\psi_i\rangle\}$,

$$tr(\rho) = \sum_i \langle\psi_i|\rho|\psi_i\rangle \tag{6}$$

for multi particles, we need a $\{|\psi_i\rangle_{\text{multi}}\}$ as a basis,

4 Partial trace

Definition 4.1: Partial trace

Suppose we have a two particle system, then the partial trace on particle A is

$$tr_A(\rho) = \sum_i \left(\langle\psi_i|^A \otimes I^B\right) \cdot \rho \cdot \left(|\psi_i\rangle^A \otimes I^B\right) \tag{7}$$

The definition here is a bit complicated, we can refer to *Quantum Computation and Quantum Information, Michael A. Nielsen* and read the corresponding section. In that book, the author introduce the *reduced density operator* and the *partial trace* at the same time, and here is the definition in the book.

Definition 4.2: Reduced density operator and partial trace

Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A \equiv tr_B(\rho^{AB}) \tag{8}$$

where tr_B is a map of operators known as the *partial trace* over system B. The partial trace is defined by

$$\text{tr}_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) \equiv |a_1\rangle\langle a_2| \,\text{tr}\left(|b_1\rangle\langle b_2|\right) \tag{9}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B. The trace operation appearing on the right hand side is the usual trace operation for system B, so $tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$. We have defined the partial trace operation only on a special subclass of operators on AB; the specification is completed by requiring in addition to Equation (9) that the partial trace be linear in its input.

Specifically, suppose a quantum system is in the product state $\rho^{AB} = \rho \otimes \sigma$, where ρ is a density operator for system A, and σ is a density operator for system B. Then

$$\rho^A = \mathrm{tr}_B(\rho \otimes \sigma) = \rho\,\mathrm{tr}(\sigma) = \rho \tag{10}$$

4.1 Partial trace and entangled state

If we have $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, then what is $tr_A(\rho)$? First, we can calculate the density matrix as follows:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \tag{11}$$

$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \tag{12}$$

$$\Rightarrow tr_A(\rho) = (\langle 0| \otimes I)\cdot \tag{13}$$