Quiz 2

Problem 1: 20 points

One dimensional infinity deep square well
For a particle with mass m in a potential

$$V(x) = \begin{cases} +\infty & x < -\frac{a}{2} \\ 0 & -\frac{a}{2} \le x \le \frac{a}{2} \\ +\infty & x > \frac{a}{2} \end{cases},$$

1. (5 pts) write out the eigen energy E_n
2. (5 pts) write out the eigen wave function in position space $\phi_n(x)$
3. (10 pts) with $\psi(x, t = 0) = \sqrt{\frac{1}{a}}\left(\cos\left(\frac{\pi x}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\right)$, derive and find $\psi(x, t = \frac{2ma^2}{\pi\hbar})$, hint: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

Solution.

(1) $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

(2) $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a} + \frac{n\pi}{2}\right)$

(3) When $t = 0$,

$$\begin{aligned} \psi(x,0) &= \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{a}\right) - \sqrt{\frac{1}{a}}\sin\left(\frac{2\pi x}{a}\right) && (1) \\ &= \sqrt{\frac{1}{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a} + \frac{\pi}{2}\right) + \sqrt{\frac{1}{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a} + \frac{2\pi}{2}\right) && (2) \\ &= \sqrt{\frac{1}{2}}\phi_1 + \sqrt{\frac{1}{2}}\phi_2 && (3) \end{aligned}$$

so

$$\psi(x,t) = \sqrt{\frac{1}{2}}e^{\frac{iE_1 t}{\hbar}}\phi_1 + \sqrt{\frac{1}{2}}e^{\frac{iE_2 t}{\hbar}}\phi_2 \qquad (4)$$

where $E_1 = \frac{\pi^2\hbar^2}{2ma^2},\ E_2 = \frac{4\pi^2\hbar^2}{2ma^2},\ t = \frac{2ma^2}{\pi\hbar}$, so

$$\psi(x,t) = -\sqrt{\frac{1}{2}}\phi_1 + \sqrt{\frac{1}{2}}\phi_2 \qquad (5)$$

□

Problem 2: 15 points

One dimensional Harmonic Oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

with

$$a = \frac{1}{\sqrt{\hbar\omega}} \left(i \frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2} m\omega^2} \hat{x} \right), \ a^\dagger = \frac{1}{\sqrt{\hbar\omega}} \left(-i \frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2} m\omega^2} \hat{x} \right)$$

1. (2 pts) write out $[a, a^\dagger]$
2. (1 pt) $a|n\rangle$
3. (1 pt) $a^\dagger|n\rangle$
4. (1 pt) E_n
5. (10 pts) derive and evaluate $a^3(a^\dagger)^2|n = 0\rangle$