作业 07

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题 1

With

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \ \ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

find eigenvalue and eigenstates of $\vec{\sigma} \cdot \vec{n}$

解. 直接用 Mathematica 求解如下

```
In[∘]:= σ⃗ = Table[PauliMatrix[i], {i, 3}];
            表格   泡利自旋矩阵

In[∘]:= n⃗ = {Sin[θ] Cos[φ], Sin[θ] Sin[φ], Cos[θ]};
             正弦   余弦    正弦   正弦    余弦

In[∘]:= Grid[
        格子
         Insert[
         插入
          Transpose[{Eigenvalues[n⃗.σ⃗],
          转置       特征值
            TraditionalForm /@ Assuming[{θ, φ} ∈ Reals, Normalize /@ Simplify[
            传统格式            假定       实数域   正规化      化简
                Eigenvectors[n⃗.σ⃗]]]}], {"Eigenvalue", "Eigenvector"}, 1],
                特征向量
         Frame → All]
         边框    全部
```

Out[∘]=

Eigenvalue	Eigenvector
-1	$\left\{\frac{\tan\left(\frac{\theta}{2}\right)\left(-\cos(\varphi)+i\sin(\varphi)\right)}{\sqrt{1+\left\lvert\left(i\sin(\varphi)-\cos(\varphi)\right)\tan\left(\frac{\theta}{2}\right)\right\rvert^2}}, \frac{1}{\sqrt{1+\left\lvert\left(i\sin(\varphi)-\cos(\varphi)\right)\tan\left(\frac{\theta}{2}\right)\right\rvert^2}}\right\}$
1	$\left\{\frac{\cot\left(\frac{\theta}{2}\right)\left(\cos(\varphi)-i\sin(\varphi)\right)}{\sqrt{1+\left\lvert\cot\left(\frac{\theta}{2}\right)\left(\cos(\varphi)-i\sin(\varphi)\right)\right\rvert^2}}, \frac{1}{\sqrt{1+\left\lvert\cot\left(\frac{\theta}{2}\right)\left(\cos(\varphi)-i\sin(\varphi)\right)\right\rvert^2}}\right\}$

$\square$

题 2

With $\psi(\theta,\varphi)=\frac{1}{\sqrt{3}}\left(\sqrt{2}Y_1^0(\theta,\varphi)+Y_1^1(\theta,\varphi)\right)$, without integrations, find $\langle L^2\rangle, \langle L_z\rangle$

解. 我们已知

$$L_z Y_{lm} = m\hbar Y_{lm}$$
$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$$

所以

$$L^2|\psi\rangle = L^2\left(\sqrt{\frac{2}{3}}Y_1^0+\sqrt{\frac{1}{3}}Y_1^1\right)=\sqrt{\frac{2}{3}}2\hbar^2Y_1^0+\sqrt{\frac{1}{3}}2\hbar^2Y_1^1 \tag{1}$$
$$L_z|\psi\rangle = \sqrt{\frac{1}{3}}\hbar Y_1^1 \tag{2}$$

故

$$\langle L^2\rangle = 2\hbar^2 \tag{3}$$
$$\langle L_z\rangle = \frac{\hbar}{3} \tag{4}$$

$\square$

题 3

With $\phi(t=0)=\psi(\theta,\varphi)$ as above, $H=\frac{\vec{L}^2}{2mR^2}$, find $\phi(t=T)$

解. 因为

$$L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle \tag{5}$$

我们可以把算符 L^2 展开为

$$L^2=\sum l(l+1)|l,m\rangle\langle l,m| \tag{6}$$

同样可以把演化算符 $e^{-\frac{i}{\hbar}Ht}$ 做如下展开：

$$\begin{aligned} e^{-\frac{i}{\hbar}Ht} &= e^{-\frac{it}{2m\hbar R^2}L^2} \\ &= \sum e^{-\frac{it}{2m\hbar R^2}l(l+1)}|l,m\rangle\langle l,m| \end{aligned} \tag{7}$$

所以

$$
\begin{aligned}
\phi(t) &= e^{-\frac{i}{\hbar}Ht}\phi(0) \\
&= \left(\sum e^{-\frac{it}{2m\hbar R^2}l(l+1)}|l,m\rangle\langle l,m|\right)\left(\sqrt{\frac{2}{3}}|1,0\rangle + \sqrt{\frac{1}{3}}|1,1\rangle\right) \\
&= e^{-\frac{it}{mR^2\hbar}}\left(\sqrt{\frac{2}{3}}Y_1^0 + \sqrt{\frac{1}{3}}Y_1^1\right)
\end{aligned} \tag{8}
$$

□

题 4: Uncertainty principle

With $J_z|j,m\rangle = m\hbar|j,m\rangle$, $\vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$,

$$\langle\Delta A\rangle = \sqrt{\langle j,m|A^2|j,m\rangle - (\langle j,m|A|j,m\rangle)^2}$$

find $\langle\Delta J_x\rangle\langle\Delta J_y\rangle$ and $\langle[J_x,J_y]\rangle$, check $\langle\Delta J_x\rangle\langle\Delta J_y\rangle \geq \frac{1}{2}|\langle[J_x,J_y]\rangle|$.

When $\langle\Delta J_x\rangle\langle\Delta J_y\rangle = \frac{1}{2}|\langle[J_x,J_y]\rangle|$, what's the requirement of m?

解. 我们先证明在 J_z 的本征态 $|j,m\rangle$ 下，$\langle J_x\rangle = \langle J_y\rangle = 0$。因为

$$[J_y, J_z] = i\hbar J_x \tag{9}$$

两边在 $|j,m\rangle$ 态下求平均值，

$$\text{LHS} = \langle j,m|J_yJ_z - J_zJ_y|j,m\rangle = m\hbar\langle j,m|J_y|j,m\rangle - m\hbar\langle j,m|J_y|j,m\rangle = 0 \tag{10}$$

$$\text{RHS} = i\hbar\langle j,m|J_x|j,m\rangle = i\hbar\langle J_x\rangle \tag{11}$$

故有 $\langle J_x\rangle = 0$，同理可得 $\langle J_y\rangle = 0$。由对称性，可知 $\langle\Delta J_x\rangle = \langle\Delta J_y\rangle$，又有

$$\langle\Delta J_x\rangle^2 = \langle J_x^2\rangle - \langle J_x\rangle^2 = \langle J_x^2\rangle \tag{12}$$

则

$$
\begin{aligned}
\langle\Delta J_x\rangle^2 = \langle\Delta J_y\rangle^2 &= \frac{1}{2}\left(\langle\Delta J_x\rangle^2 + \langle\Delta J_y\rangle^2\right) \\
&= \frac{1}{2}\left(\langle J_x^2\rangle + \langle J_y^2\rangle\right) = \frac{1}{2}\langle J_x^2 + J_y^2\rangle \\
&= \frac{1}{2}\langle J^2 - J_z^2\rangle = \frac{1}{2}(\langle J^2\rangle - \langle J_z^2\rangle) \\
&= \frac{1}{2}\left[j(j+1) - m^2\right]\hbar^2
\end{aligned} \tag{13}
$$

我们已知 $[J_x, J_y] = i\hbar J_z$，则 $|\langle [J_x, J_y]\rangle| = |i\hbar\langle J_z\rangle| = |m|\,\hbar^2$，

$$\langle \Delta J_x\rangle\langle \Delta J_y\rangle = \langle \Delta J_x\rangle^2 = \frac{1}{2}\left[j(j+1) - m^2\right]\hbar^2 \tag{14}$$

$$\geq \frac{1}{2}\,|m|\,\hbar^2 \tag{15}$$

当 $m = \pm j$ 时取等号。 □

题 5

Write out J_-J_+ as a matrix in the basis of J_z, with $j = 1$, and find the eigenvalue and eigenstates of J_-J_+

解.

$$\begin{aligned} J_-J_+ &= (J_x + iJ_y)(J_x - iJ_y) \qquad (16)\\ &= J_x^2 - i[J_x, J_y] + J_y^2 \\ &= J_x^2 + \hbar J_z + J_y^2 \\ &= J^2 - J_z^2 + \hbar J_z \qquad (17) \end{aligned}$$

当 $j = 1$ 时，$m = \{-1, 0, 1\}$，故 J_z 的基 $|j, m\rangle$ 有 3 个，为 $\{|1, -1\rangle, |1, 0\rangle, |1, 1\rangle\}$，由关系

$$\begin{cases} J_z|j, m\rangle = m\hbar|j, m\rangle \\ J_z^2|j, m\rangle = m^2\hbar^2|j, m\rangle \\ J^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle \end{cases}$$

可以得到 J_-J_+ 的矩阵表示为

$$J_-J_+ = 2\hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

本征值和对应的本征态可以写为:

本征值	本征态
0	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$2\hbar^2$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$2\hbar^2$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

□