

# Chapter 3: Angular Momentum

Yuquan Chen

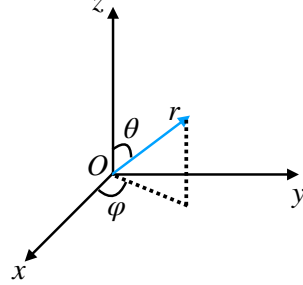
2019/04/16

We have discussed position and momentum operator before, now let's consider the rotation of a system, which leads to angular position and angular momentum. Before we dive in, there's some prerequisites we should know.

## 1 Some prerequisites

In a 3D system, we use  $\vec{r} = (x, y, z)$  to represent the coordinate, and the momentum is  $\vec{p} = (p_x, p_y, p_z)$ . In position representation, we have  $p_x \leftrightarrow -i\hbar \frac{\partial}{\partial x}$ ,  $p_y \leftrightarrow -i\hbar \frac{\partial}{\partial y}$ , and  $p_z \leftrightarrow -i\hbar \frac{\partial}{\partial z}$ , together we get  $\vec{p} = (-i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}) = -i\hbar \vec{\nabla}$ .

Now let's switch to spherical coordinate  $(r, \theta, \varphi)$ .



For a point in space, we use a ket  $|\psi\rangle$  to represent it,

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

now let's consider a rotation along  $\hat{z}$  direction. Here  $\varphi$  becomes  $\varphi + d\varphi$ , and  $r, \theta$  remain the same. We have

$$|x, y, z\rangle \xrightarrow{\text{rotation}} |x', y', z'\rangle \quad (2)$$

and the corresponding

$$(r, \theta, \varphi) \xrightarrow{\text{rotation}} (r, \theta, \varphi + d\varphi) \quad (3)$$

then we try to find out the expression of  $x', y', z'$  in terms of  $r, \theta, \varphi, d\varphi$

$$\begin{cases} x' = r \sin \theta \cos(\varphi + d\varphi) \simeq r \sin \theta \cos \varphi - r \sin \theta \sin \varphi d\varphi \\ y' = r \sin \theta \sin(\varphi + d\varphi) \simeq r \sin \theta \sin \varphi + r \sin \theta \cos \varphi d\varphi \\ z' = r \cos \theta = z \end{cases} \Rightarrow \begin{cases} x' = x - y d\varphi \\ y' = y + x d\varphi \\ z' = z \end{cases} \quad (4)$$

On a spin- $\frac{1}{2}$  system, we have Pauli operators  $\sigma_x, \sigma_y, \sigma_z$ . In the  $\sigma_z$  basis, we have

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5)$$

and

$$[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m, \quad \varepsilon_{klm} = \begin{cases} 1 & \text{if } k, l, m \text{ in order} \\ -1 & \text{if out of order} \end{cases} \quad (6)$$

define the spin operator,

$$\begin{cases} \vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad S_z = \frac{\hbar}{2}\sigma_z \\ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \end{cases} \quad (7)$$

here we should mention that  $[S_k, S_l] = i\hbar\varepsilon_{klm}S_m$  is generally true for angular momentum operators, including spin- $\frac{1}{2}$  operators.

$$\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{4}\hbar^2 I \quad (8)$$

notice that for each  $k$  in  $x, y, z$ , we have  $\sigma_k^2 = I$  so  $\vec{S}^2 \propto I$ , and we also have  $[\vec{S}^2, S_k] = 0$ . If we define unite length vector

$$\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \quad (9)$$

we have

$$\vec{\sigma} \cdot \vec{n} = \sigma_x \sin \theta \cos \varphi + \sigma_y \sin \theta \sin \varphi + \sigma_z \cos \theta \quad (10)$$

Noticed that if you try to use Mathematica to implement  $\vec{\sigma} \cdot \vec{n}$ , use the expression  $\vec{n} \cdot \vec{\sigma}$  instead. When we are dealing with matrix exponential, we use Taylor expansion

$$e^{i\phi\hat{\sigma}_x} = I + i\phi\sigma_x - \frac{\phi^2\sigma_x^2}{2!} + \frac{i\phi^3\sigma_x^3}{3!} - \dots \quad (11)$$

we use

$$\begin{cases} e^{\hat{A}} = I + \hat{A} + \frac{\hat{A}^2}{2!} + \dots + \frac{\hat{A}^n}{n!} + \dots \\ \sigma_x^2 = I \end{cases} \quad (12)$$

to get

$$e^{i\phi\hat{\sigma}_x} = \left( I - \frac{\phi^2}{2!}I + \frac{\phi^4}{4!}I - \dots \right) + i\sigma_x \left( \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots \right) \quad (13)$$

$$= \cos \phi I + i \sin \phi \sigma_x \quad (14)$$

## 2 Orbital angular momentum

Let's consider a small rotation:  $\varphi \rightarrow \varphi + d\varphi$ ,

$$\begin{cases} x' \simeq x - yd\varphi \\ y' \simeq y + xd\varphi \\ z' = z \end{cases} \quad (15)$$

if we have a state  $|\psi\rangle$  expressed in position coordinate  $\langle x, y, z|\psi\rangle = \psi(x, y, z)$  (we can also write it in spherical basis  $\psi(r, \theta, \varphi)$ , as we discussed above), then we apply a rotation to the state, we get from

$$\psi(r, \theta, \varphi) \rightarrow \psi(r, \theta, \varphi - d\varphi) \quad (16)$$

and we know

$$\psi(r, \theta, \varphi - d\varphi) \simeq \psi(r, \theta, \varphi) - d\varphi \cdot \frac{\partial}{\partial \varphi} \psi(r, \theta, \varphi) \quad (17)$$

we can express this small rotation as

$$I - \frac{\partial}{\partial \varphi} = I + \frac{1}{i\hbar} \hat{L}_z \quad (18)$$

here we can see the relationship:

$$\hat{L}_z \stackrel{\varphi \text{ coordinate}}{\longleftrightarrow} -i\hbar \frac{\partial}{\partial \varphi} \longleftrightarrow p_x \stackrel{x \text{ coordinate}}{\longleftrightarrow} -i\hbar \frac{\partial}{\partial x} \quad (19)$$

then we express  $\hat{L}_z$  in  $x, y, z$  coordinates,

$$\psi(x, y, z) \xrightarrow{I - \frac{\partial}{\partial \varphi}} \psi(x', y', z') \quad (20)$$

rotate axis along  $z$  by  $d\varphi$

$$\begin{cases} x' = x - yd\varphi \\ y' = y + xd\varphi \\ z' = z \end{cases} \xrightarrow{\text{want } -d\varphi} \begin{cases} x' = x + yd\varphi \\ y' = y - xd\varphi \\ z' = z \end{cases} \quad (21)$$

so

$$\psi(x', y', z') \simeq \psi(x + yd\varphi, y - xd\varphi, z) \quad (22)$$

$$= \psi(x, y, z) + yd\varphi \frac{\partial}{\partial x} \psi(x, y, z) - xd\varphi \frac{\partial}{\partial y} \psi(x, y, z) \quad (23)$$

we have

$$p_x \leftrightarrow -i\hbar \frac{\partial}{\partial x}, \quad p_y \leftrightarrow -i\hbar \frac{\partial}{\partial y} \quad (24)$$

so

$$\psi(x', y', z') = \psi(x, y, z) + y d\varphi \left( \frac{\hat{p}_x}{-i\hbar} \right) \psi(x, y, z) - x d\varphi \left( \frac{\hat{p}_y}{-i\hbar} \right) \psi(x, y, z) \quad (25)$$

$$= \left( I + \frac{1}{-i\hbar} d\varphi (yp_x - xp_y) \right) \psi(x, y, z) \quad (26)$$

$$= \left( I + \frac{1}{i\hbar} \hat{L}_z \right) \psi(x, y, z) \quad (27)$$

So  $\hat{L}_z$  expressed with  $x, y, z$  coordinate with respected operators should be

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \quad (28)$$

similarly, we can write out the  $\hat{L}_x$  and  $\hat{L}_y$ . In total,

$$\boxed{\begin{cases} \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{cases}} \quad (29)$$

which is the same as classical mechanics  $\vec{L} = \vec{r} \times \vec{p}$ , so we get an orbital angular momentum operator in analog to classical. For now, we've already find out the  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  represented in  $x, y, z$  coordinate. Let's try to write them in  $r, \theta, \varphi$  coordinate. We can replace the  $\hat{p}_i$  as  $-i\hbar \frac{\partial}{\partial i}$ , and use (1) to replace  $x, y, z$  to  $r, \theta, \varphi$ .

$$\begin{cases} L_x = i\hbar \left( \sin \varphi \frac{\partial}{\partial \varphi} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right) \\ L_y = i\hbar \left( -\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right) \\ L_z = -i\hbar \frac{\partial}{\partial \varphi} \end{cases} \quad (30)$$

After defined  $L_i$  in both  $x, y, z$  and  $r, \theta, \varphi$  coordinates, we can now explore some properties of orbital angular momentum.

$$1. [L_x, L_y] = i\hbar L_z$$

*Proof.* We already know

$$\begin{cases} [x, p_x] = i\hbar \\ [x, y] = [x, p_y] = [p_x, p_y] = 0 \end{cases} \quad (31)$$

so

$$[yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [yp_z, -xp_z] + [-zp_y, zp_x] + [-zp_y, -xp_z] \quad (32)$$

$$= [yp_z, zp_x] + [-zp_y, -xp_z] \quad (33)$$

$$= yp_x[p_z, z] + p_yx[z, p_z] \quad (34)$$

$$= i\hbar (xp_y - yp_x) = i\hbar L_z \quad (35)$$

□

Furthermore, we can easily proof that

$$[L_k, L_l] = i\hbar \varepsilon_{k,l,m} L_m, \quad \varepsilon_{klm} = \begin{cases} 1 & \text{if } k, l, m \text{ in order} \\ -1 & \text{if out of order} \end{cases} \quad (36)$$

2.  $[L^2, L_i] = 0$ , which is similar to  $[S^2, S_i] = 0$ .

*Proof.* We know that

$$L^2 = L_x^2 + L_y^2 + L_z^2, \quad [A^2, B] = A[A, B] + [A, B]A$$

so

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] \quad (37)$$

$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z \quad (38)$$

$$= -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z \quad (39)$$

$$= 0 \quad (40)$$

□

so in total,

### Box 2.1: Properties of orbital angular momentum

- 1.

$$[L_k, L_l] = i\hbar \varepsilon_{k,l,m} L_m, \quad \varepsilon_{klm} = \begin{cases} 1 & \text{if } k, l, m \text{ in order} \\ -1 & \text{if out of order} \end{cases}$$

- 2.

$$[L^2, L_i] = 0$$

Now let's consider the eigen equation with  $L^2$  and  $L_z$ . Recall that if  $\hat{A}, \hat{B}$  are Hermitian, and  $[\hat{A}, \hat{B}] = 0$ , then there exists  $\{|\psi\rangle\}$  giving

$$\begin{cases} \hat{A}|\psi\rangle = a|\psi\rangle \\ \hat{B}|\psi\rangle = b|\psi\rangle \end{cases}$$

or we can say  $\{|\psi\rangle\}$  is the mutual eigen state. Here we already have  $[L^2, L_z] = 0$ , let  $|y\rangle$  to be one of these states, so

$$\begin{cases} L_z|y\rangle = m\hbar|y\rangle \\ L^2|y\rangle = \beta\hbar^2|y\rangle \end{cases} \quad (41)$$

where  $m, \beta$  are numbers.  $|y\rangle$  needs to be normalized,  $\langle y|y\rangle = 1$

$$\beta\hbar^2 = \langle y|L^2|y\rangle = \langle y|L_x^2 + L_y^2 + L_z^2|y\rangle \geq \langle y|L_z^2|y\rangle = m^2\hbar^2 \quad (42)$$

so

$$\beta \geq m^2 \quad (43)$$

in spherical coordinate, as above we can express  $L^2, L_z$  all in just  $\theta, \varphi$  without  $r$ .  $L^2, L_z$  only relates to angular coordinates  $\{\theta, \varphi\}$  as the angular representation. The eigen function for  $L_z$  in coordinates  $\{\theta, \varphi\}$  is  $\langle \theta, \varphi | L_z | y \rangle = \lambda \langle \theta, \varphi | y \rangle$ . To make the expression more convenient (we will know the reason soon), let's set  $\lambda = m\hbar$ .

$$\langle \theta, \varphi | y \rangle = y(\theta, \varphi) \quad (44)$$

$$\langle \theta, \varphi | L_z | y \rangle = m\hbar \langle \theta, \varphi | y \rangle \quad (45)$$

$$= -i\hbar \frac{\partial}{\partial \varphi} \langle \theta, \varphi | y \rangle \quad (46)$$

then we can solve the equation

$$-i\hbar \frac{\partial}{\partial \varphi} y(\theta, \varphi) = m\hbar y(\theta, \varphi) \Rightarrow y(\theta, \varphi) \propto e^{im\varphi} \quad (47)$$

and  $\varphi \rightarrow \varphi + 2\pi$  should give the same wave function, so we have

$$e^{im\varphi} = e^{im(\varphi+2\pi)} \Rightarrow e^{i2\pi m} = 1 \quad (48)$$

so  $m$  should be an integer,  $m = 0, \pm 1, \pm 2, \dots$ , and now we know why we need to set  $\lambda = m\hbar$  before. Further, we have the eigen function for  $L^2$

$$\langle \theta, \varphi | L^2 | y \rangle = \beta \hbar^2 \langle \theta, \varphi | y \rangle \quad (49)$$

notice that the original eigen function is  $\langle \theta, \varphi | L^2 | y \rangle = \lambda' \langle \theta, \varphi | y \rangle$ , and the reason to let  $\lambda' = \beta \hbar^2$  is similar to the former one, we will see it soon. By using (30) to calculate  $L^2 = L_x^2 + L_y^2 + L_z^2$ , we get

$$\Rightarrow L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \quad (50)$$

$$\Rightarrow - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] y(\theta, \varphi) = \beta y(\theta, \varphi) \quad (51)$$

use (47) we can get  $\frac{\partial^2}{\partial \varphi^2} y(\theta, \varphi) = -m^2 y(\theta, \varphi)$ ,

$$\Rightarrow \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{m^2}{\sin^2 \theta} - \beta \right] y(\theta, \varphi) = 0 \quad (52)$$

There's a solution proportional to a spherical function  $P_l^m(\cos \theta)$  called associated Legendre polynomial. We need

$$\begin{cases} \beta = l(l+1), \quad l = 0, 1, 2, \dots \\ m = -l, -l+1, \dots, l-1, l \end{cases} \quad (53)$$

the full solution is called Spherical Harmonics function

$$Y_{lm} = y(\theta, \varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\varphi} \quad (54)$$

as a normalized solution, with  $\begin{cases} l = 0, 1, 2, \dots \\ m = -l, -l+1, \dots, l-1, l \end{cases}$  as quantum number (the solution  $Y_{lm}$  needs two numbers to specify, just like an ID). As we discussed above,  $Y_{lm}$  is normalized, which means

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin \theta Y_{l'm'}^* Y_{lm} d\theta = \delta_{ll'} \delta_{mm'} = \begin{cases} 1 & \text{if } l = l', m = m' \\ 0 & \text{otherwise} \end{cases} \quad (55)$$

From  $Y_{lm}$ , we know  $|y\rangle = |l, m\rangle$ .  $\{|l, m\rangle\}$  form a basis, which has  $2l+1$  dimension.

### 3 General properties of angular momentum

This section corresponds to section 3.5 of *Sakurai*. Similar to  $\vec{S}, \vec{L}$ , we use  $\vec{J}$  for general case, and the corresponding eigen state is  $|j, m\rangle$ , with

1.  $[J_k, J_l] = i\hbar \varepsilon_{klm} J_m$
2.  $[J^2, J_i] = 0$
3.  $J_z |j, m\rangle = m\hbar |j, m\rangle$
4.  $J^2 |j, m\rangle = \beta \hbar^2 |j, m\rangle$

and we will proof  $\beta = j(j+1)$  soon. Define ladder operator  $J_+, J_-$  as below

$$\begin{cases} J_+ = J_x + iJ_y \\ J_- = J_x - iJ_y \end{cases} \quad (56)$$

from the definition, we can get

$$J_- = J_+^\dagger \quad (57)$$

$$[J_z, J_+] = \hbar J_+ \quad (58)$$

$$[J_z, J_-] = -\hbar J_- \quad (59)$$

$$[J_+, J_-] = 2\hbar J_z \quad (60)$$

$$J_- J_+ = J^2 - J_z^2 - \hbar J_z \quad (61)$$

$$J_+ J_- = J^2 - J_z^2 + \hbar J_z \quad (62)$$

with  $J_z |j, m\rangle = m\hbar |j, m\rangle$ , try to find some properties.

$$J_z J_+ |j, m\rangle = ([J_z, J_+] + J_+ J_z) |j, m\rangle = (\hbar J_+ + J_+ J_z) |j, m\rangle = (m+1)\hbar J_+ |j, m\rangle \quad (63)$$

Let  $|\xi\rangle = J_+ |j, m\rangle$ , so there is  $J_z |\xi\rangle = (m+1)\hbar |\xi\rangle$ . We then know  $|\xi\rangle$  has something to do with  $|j, m+1\rangle$  (which means  $J_+$  increases quantum state  $|j, m\rangle$  by one unit),  $J_+ |j, m\rangle \rightarrow c |j, m+1\rangle$ . Similarly, we have  $J_- |j, m\rangle \rightarrow c' |j, m-1\rangle$ , and we try to find out  $c, c'$ .

1. Upper bound for  $|j, m\rangle = |j, M_{\text{up}}\rangle$ ,  $J_+|j, M_{\text{up}}\rangle = 0$

$$0 = J_- J_+ |j, M_{\text{up}}\rangle = (J^2 - J_z^2 - \hbar J_z) |j, M_{\text{up}}\rangle \quad (64)$$

$$= (\beta \hbar^2 - M_{\text{up}}^2 \hbar^2 - M_{\text{up}} \hbar^2) |j, M_{\text{up}}\rangle \quad (65)$$

$$\Rightarrow \beta = M_{\text{up}}(M_{\text{up}} + 1) \quad (66)$$

2. Lower bound of  $|j, m\rangle = |j, M_{\text{low}}\rangle$ ,  $J_-|j, M_{\text{low}}\rangle = 0$

$$0 = J_+ J_- |j, M_{\text{low}}\rangle \Rightarrow \beta = M_{\text{low}}(M_{\text{low}} - 1) \quad (67)$$

if we have  $j = M_{\text{up}} = -M_{\text{low}}$ , then  $\boxed{\beta = j(j+1)}$

$$\boxed{J^2 |j, m\rangle = j(j+1) \hbar^2 |j, m\rangle} \quad (68)$$

$$\boxed{J_z |j, m\rangle = m \hbar |j, m\rangle, m = -j, -j+1, \dots, j-1, j} \quad (69)$$

From  $[J^2, J_+] = 0$ ,  $[J^2, J_-] = 0$ ,

$$J^2 J_+ |j, m\rangle = J_+ J^2 |j, m\rangle = j(j+1) \hbar^2 J_+ |j, m\rangle \quad (70)$$

then we can calculate  $c, c'$ ,

$$\boxed{\begin{cases} J_+ |j, m\rangle = \hbar \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle \\ J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle \end{cases}} \quad (71)$$

We can find out some other properties.

$$[J^2, J_+] = 0, [J^2, J_-] = 0 \quad (72)$$

we can also say  $\{|j, m\rangle\}$  form a basis,  $\langle j', m' | j, m \rangle = \delta_{j'j} \delta_{m'm}$ .