20190430, Ch3 Angular Momentum

Monday, April 22, 2019 9:39 PM

- recap

erratum 勘误 $\otimes$ direct product 直积, not outer product

$|\psi\rangle \otimes \langle\phi| \longleftarrow$ outer product of state $|\psi\rangle$ and $|\phi\rangle$

HW. With $H = k\vec{S}_1 \cdot \vec{S}_2$ for two spin-$\frac{1}{2}$ electrons, check states $|S_1, S_2, S, m_S\rangle$ defined in lecture note are eigenstates, and find relevant eigenenergies. (here k is a constant).

- coupled density matrix.

two spin-$\frac{1}{2}$ particle

ρ_A for 1st one, ρ_B for 2nd one,

the overall density matrix for 2-particle system is

$$\rho = \rho_A \otimes \rho_B$$

example
$$\rho_A = |0\rangle\langle 0| \overset{\sigma_z \text{ basis}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho = \rho_A \otimes \rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_B = \frac{1}{2}\mathbb{1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho = \begin{pmatrix} 1 \times \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \times (\ ) \\ 0 \times (\ ) & 0 \times (\ ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in general, for $\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability P_i for the particle at state $|\psi_i\rangle$

for 2 particle, the overall state is described as

$$\rho = \sum_i P_i \, |\psi_i^1\rangle\langle\psi_i^1| \otimes |\phi_i^2\rangle\langle\phi_i^2|$$

$$= \sum_i P_i \left(\underbrace{|\psi_i^1\rangle \otimes |\phi_i^2\rangle}\right) \cdot \left(\langle\psi_i^1| \otimes \langle\phi_i^2|\right)$$

in general $\quad$ 2 particle state, separable, as a special case

$$\rho = \sum_i P_i |\psi_i\rangle \otimes \langle \psi_i|$$

a general two particle state.

example.

① pure state case.

particle 1 at state of $|0\rangle$

particle 2 at state of $|1\rangle$

density matrix $\rho = \rho_1 \otimes \rho_2 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$

the two particle state is $|0\rangle \otimes |1\rangle$

$$\Rightarrow \rho = (|0\rangle \otimes |1\rangle) \cdot (\langle 0| \otimes \langle 1|)$$

$$= |0\rangle\langle 0| \otimes |1\rangle\langle 1|$$

② for the two particle system, we have

$\frac{1}{3}$ of the chance two particles at $|\psi_1\rangle$

$\frac{1}{3}$ of the chance at $|\psi_2\rangle$

$\frac{1}{3}$ chance at $|\psi_3\rangle$

$$\Rightarrow \rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3|.$$

if $|\psi_1\rangle = |0\rangle|0\rangle$, $|\psi_2\rangle = |1\rangle|1\rangle$,

$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$

for this case $\rho \neq \sum_i P_i |\phi_i\rangle|\phi'_i\rangle \otimes \langle\phi_i|\langle\phi'_i|$

because of entanglement of $|\psi_3\rangle$.

- dynamics

recap for 1 particle.

$$\rho = \sum_i P_i |\psi_i(t)\rangle\langle\psi_i(t)|$$

mathematically

$$\rho = \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|$$

$$i\hbar \frac{\partial}{\partial t}|\psi_i(t)\rangle = H|\psi_i(t)\rangle$$

mathematically $|\psi\rangle\langle\phi| = |\psi\rangle \otimes \langle\phi|$

$$\dot{\rho} = \frac{d}{dt}\rho = \sum_i p_i \left(\frac{d}{dt}|\psi_i(t)\rangle\right)\langle\psi_i(t)| + \sum_i p_i |\psi_i(t)\rangle\left(\frac{d}{dt}\langle\psi_i(t)|\right)$$

$$= \sum_i p_i \frac{H}{i\hbar}|\psi_i(t)\rangle\langle\psi_i(t)| + \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|\frac{H}{-i\hbar}$$

$$= \frac{1}{i\hbar}\left(H \cdot \underbrace{\sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|}_{\rho} - \underbrace{\sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|}_{\rho} H\right)$$

$$\Rightarrow \dot{\rho} = \frac{1}{i\hbar}[H, \rho]$$

for multiparticle system, we have

ρ_{multi} , H_{multi}

$$\Rightarrow \dot{\rho}_{multi} = \frac{1}{i\hbar}[H_{multi}, \rho_{multi}]$$

- trace

$tr(\rho) = \sum_i \langle\psi_i|\rho|\psi_i\rangle$, $\{|\psi_i\rangle\}$ as a basis.

for multi particle, we need a $\{|\psi_i\rangle_{multi}\}$ as a basis

$$tr(\rho_{multi}) = \sum_i \langle\psi_i|_{multi}\, \rho_{multi}\, |\psi_i\rangle_{multi}.$$

example:

2 spin-$\frac{1}{2}$ particle, there are 4 states, for example

$|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$. form a basis.

- partial trace. for two particles, A and B

a partial trace on particle A is

$$tr_A(\rho) = \sum_i \left(\langle\psi_i|^A \otimes \mathbb{1}^B\right)\cdot\rho\cdot\left(|\psi_i\rangle^A \otimes \mathbb{1}^B\right)$$

example. $\rho = M_A \otimes M_B$, M_A, M_B are matrices.

for spin-$\frac{1}{2}$ particles.

$$tr_A(\rho) = (\langle 0|^A \otimes \mathbb{1}^B)\cdot \rho \cdot (|0\rangle^A \otimes \mathbb{1}^B)$$
$$+ (\langle 1|^A \otimes \mathbb{1}^B)\cdot \rho\, (|1\rangle^A \otimes \mathbb{1}^B)$$

$$(\langle 0|^A \otimes \mathbb{1}^B)\cdot M_A \otimes M_B \cdot (|0\rangle^A \otimes \mathbb{1}^B) = (\langle 0|\cdot M_A |0\rangle) \otimes \underline{(\mathbb{1}^B \cdot M_B \cdot \mathbb{1}^B)}$$
$$= \langle 0|M_A|0\rangle M_B$$

$$(\langle 1|^A \otimes \mathbb{1}^B)\cdot M_A \otimes M_B \cdot (|1\rangle^A \otimes \mathbb{1}^B)$$

what is $|0\rangle^A \otimes \mathbb{1}^B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$tr_A(\rho) = \langle 0|M_A|0\rangle M_B + \langle 1|M_A|1\rangle M_B$$
$$= (\langle 0|M_A|0\rangle + \langle 1|M_A|1\rangle)\, M_B$$
$$= \underset{\text{number}}{\underset{\uparrow}{tr(M_A)}}\ \underset{\text{matrix}}{\underset{\uparrow}{M_B}} \ \underline{\underline{\ M_A \text{ be physical}\ }}\ M_B$$

- partial trace and entangled state.

$\rho = M_A \otimes M_B$ — separable

$\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ — entangled state

$tr_A(\rho) = ?$

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle)(\langle 0|\langle 0| + \langle 1|\langle 1|)$$
$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$tr_A(\rho) = (\langle 0|\otimes\mathbb{1})\cdot\rho\cdot(|0\rangle\otimes\mathbb{1}) + (\langle 1|\otimes\mathbb{1})\cdot\rho\cdot(|1\rangle\otimes\mathbb{1})$$
$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}_B.$$

totally mixed state.

have no information what state particle B is.

consistent w/ observation

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$\downarrow$ measure particle A

A: "$|0\rangle$" $\swarrow$ $\qquad$ $\searrow$ A: "$|1\rangle$"

"$|00\rangle$"

$\downarrow \qquad\qquad \downarrow$

"$|0\rangle$" for B $\qquad$ "$|1\rangle$" for B.

$tr_A(\rho)$ means we discard (丢弃) information of A and what we left with B.