Chapter 3

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Quiz 4/30, two pieces of A4 double sided material, problems covering chapter 2. Basic calculas and linear algebra tutorials at 2 p.m. this Saturday.

If we have $|\psi(t=0)\rangle = \sum_i c_i|\alpha_i\rangle \mapsto |\psi(t)\rangle = \sum_i c_i e^{-iE_n t/\hbar}|\alpha_i\rangle$,

1 Coupled spin-1/2 system

1.1 States

We have 2 particles, each with spin-1/2, use basis $\{|0\rangle, |1\rangle\}$ for each. Combinations: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, use outer product $\otimes$ to join the two qubits. For example,

$$|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \; |\psi_2\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \longmapsto |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} a_1 a_2 \\ a_1 b_2 \\ b_1 a_2 \\ b_1 b_2 \end{pmatrix} \tag{1}$$

For two qubits system, a general state expressed as $|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$.

1.2 Operators

Operator A and B for the first and the second particle, we have the operator $A \otimes B$:

$$(A \otimes B) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = A|\psi_1\rangle \otimes B|\psi_2\rangle \tag{2}$$

In matrix form, we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{3}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \tag{4}$$

We can proof (2) mathematically. Here's some properties of outer products on operators.

Box 1.1: Properties of outer products on operators

(1) $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$
(2) $A \otimes B$

Proof. $(C \otimes D)|\psi_1\rangle|\psi_2\rangle$ $\square$

1.3 Entanglement, quantum logic gates

A separable state: $|\varphi\rangle = |a\rangle \otimes |b\rangle$, but if $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we cannot make it into the form $|a\rangle \otimes |b\rangle$, then the two particles are in entanglement state. Physically, we can say there's special correlation between the two particles. Entanglement is one special form of superposition. We can use some examples to clarify the entanglement and the non-entanglement state.

- Example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
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Some other important entanglement states

- Bell state $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$
- W-state $|w\rangle = \frac{1}{\sqrt{n}}(|10...0\rangle + |010...0\rangle + ... + |00...10\rangle + |00...01\rangle)$ is the permutations of the required state.

1.3.1 Properties of entangled states

(1) Measurement

Suppose we create a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then separate the two particles, after measure the state of the first particle(such as $|0\rangle$), then we know the state of the second one immediately, no matter how far them separated.

(2) Quantum correlation V.S. classical correlation

Suppose we have $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we can do a half flip,

$$|0\rangle \longmapsto |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|1\rangle \longmapsto |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

then

$$|\psi\rangle \longmapsto \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$$

1.3.2 Entangling operators

The Controlled-Not gate(CNOT) of two qubits

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

the truth table of the CNOT gate is:

Before	After
$\|0\rangle \otimes \|0\rangle$	$\|0\rangle \otimes \|0\rangle$
$\|0\rangle \otimes \|1\rangle$	$\|0\rangle \otimes \|1\rangle$
$\|1\rangle \otimes \|0\rangle$	$\|1\rangle \otimes \|1\rangle$
$\|1\rangle \otimes \|1\rangle$	$\|1\rangle \otimes \|0\rangle$

1.3.3 How to construct a CNOT gate

Suppose we have $H_1 = \hbar\omega\sigma_z^A\sigma_x^B$, then the evolution $U = e^{-iHt/\hbar}$