作业 06

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题 1

Prove $e^{-i\omega t\hat{\sigma}_x} = cos\omega tI - isin\omega t\hat{\sigma}_x$

解. 因为

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} + (-1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{1}$$

所以

$$\begin{aligned} e^{-i\omega t\hat{\sigma}_x} &= e^{-i\omega t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} + e^{i\omega t} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \\ &= cos\omega tI - isin\omega t\hat{\sigma}_x \end{aligned} \tag{2}$$

也可以用上课时说的 $\begin{cases} e^A = I + A + \frac{A^2}{2!} + ... \\ \sigma_x^2 = I \end{cases}$ 来化简，这两种方式等效。 □

题 2

Detuned Rabi flopping for a spin-1/2 particle with energy spacing ω_0 apply an oscillating magnetic with frequency $\omega_0 + \delta$, and Rabi rate Ω, so we have $H(t) = \hbar\frac{\omega_0}{2}\sigma_z + \hbar\Omega\sigma_x cos((\omega + \delta)t)$

(1) Choose a proper transformation and apply rotating wave approximation to make $H(t)$ time-independent, so that $H_{int} = -\hbar\frac{\delta}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x$

(2) Solve for eigenvalue λ_+, λ_- and eigenstate $|\psi_+\rangle, |\psi_-\rangle$ for H_{int} in the basis of σ_z $\{|0\rangle, |1\rangle\}$

(3) With $|\psi(t=0)\rangle = |\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ solve for the overlap between $|\psi_0\rangle$ and $|\psi(t)\rangle$, defined as $|\langle\psi_0|\psi(t)\rangle|^2$. Hint: use $|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle$, and $H = \lambda_+|\psi_+\rangle\langle\psi_+| +$

$\lambda_-|\psi_-\rangle\langle\psi_-|$. We can assume Ω is real for simplicity.

解.

（1）暂时没弄懂课堂上这步的化简

$t \sim \mu s \quad 10^{-6}s$

$(\omega_0+\omega)t \to$ large

Rotating wave approximation $e^{i(\omega_0+\omega)t} \to 0$

(RWA) $e^{i(\omega-\omega_0)t}$ survives

let $\omega=\omega_0$, and apply RWA

$\Rightarrow H_{1,int} \triangleq \frac{\hbar\Omega}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|) = \frac{\hbar\Omega}{2}\sigma_x$

体系的哈密顿量

$$\begin{aligned} H(t) &= \hbar\frac{\omega_0}{2}\sigma_z + \hbar\Omega\sigma_x cos((\omega+\delta)t) \\ &= \hbar\frac{\omega_0+\delta}{2}\sigma_z + \left(\hbar\Omega\sigma_x cos((\omega+\delta)t) - \hbar\frac{\delta}{2}\sigma_z\right) \end{aligned} \tag{3}$$

令 $H_0 = \hbar\frac{\omega_0+\delta}{2}\sigma_z$，$H_1 = \hbar\Omega\sigma_x cos((\omega+\delta)t) - \hbar\frac{\delta}{2}\sigma_z$，则

$$H_{int} = e^{\frac{i}{\hbar}H_0 t} H_1 e^{-\frac{i}{\hbar}H_0 t} \tag{4}$$

（2）现在已知 $H_{int} = -\hbar\frac{\delta}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x$，我们求解它的本征值和本征态：

```
In[∘]:= Hint = -ℏ δ/2 PauliMatrix[3] + ℏ Ω/2 PauliMatrix[1];
                      泡利自旋矩阵               泡利自旋矩阵

Grid[Insert[Transpose[{Eigenvalues[Hint], Normalize /@ Eigenvectors[Hint]}],
格子  插入    转置         特征值               正规化       特征向量
  {"Eigenvalue", "Eigenvector"}, 1], Frame → All]
                                     边框    全部
```

Out[∘]=

Eigenvalue	Eigenvector
$-\frac{1}{2}\sqrt{\delta^2+\Omega^2}\,\hbar$	$\left\{-\frac{\delta+\sqrt{\delta^2+\Omega^2}}{\Omega\sqrt{1+\mathrm{Abs}\left[\frac{\delta+\sqrt{\delta^2+\Omega^2}}{\Omega}\right]^2}}, \frac{1}{\sqrt{1+\mathrm{Abs}\left[\frac{\delta+\sqrt{\delta^2+\Omega^2}}{\Omega}\right]^2}}\right\}$
$\frac{1}{2}\sqrt{\delta^2+\Omega^2}\,\hbar$	$\left\{-\frac{\delta-\sqrt{\delta^2+\Omega^2}}{\Omega\sqrt{1+\mathrm{Abs}\left[\frac{\delta-\sqrt{\delta^2+\Omega^2}}{\Omega}\right]^2}}, \frac{1}{\sqrt{1+\mathrm{Abs}\left[\frac{\delta-\sqrt{\delta^2+\Omega^2}}{\Omega}\right]^2}}\right\}$

（3）已知 $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}\lambda_+ t}|\psi_+\rangle\langle\psi_+| + e^{-\frac{i}{\hbar}\lambda_- t}|\psi_-\rangle\langle\psi_-| \tag{5}$$

所以

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = e^{-\frac{i}{\hbar}\lambda_+ t}|\psi_+\rangle\langle\psi_+|0\rangle + e^{-\frac{i}{\hbar}\lambda_- t}|\psi_-\rangle\langle\psi_-|0\rangle \tag{6}$$

左乘 $\langle\psi(0)|$，

$$\begin{aligned}\langle\psi(0)|\psi(t)\rangle &= e^{-\frac{i}{\hbar}\lambda_+ t}\langle 0|\psi_+\rangle\langle\psi_+|0\rangle + e^{-\frac{i}{\hbar}\lambda_- t}\langle 0|\psi_-\rangle\langle\psi_-|0\rangle \\ &= e^{-\frac{i}{\hbar}\lambda_+ t}|\langle 0|\psi_+\rangle|^2 + e^{-\frac{i}{\hbar}\lambda_- t}|\langle 0|\psi_-\rangle|^2\end{aligned} \tag{7}$$

令 $\{\delta,\Omega\}\in\mathbb{R}$，可以求出 overlap

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \frac{2\delta^2+\Omega^2\cos\left(t\sqrt{\delta^2+\Omega^2}\right)+\Omega^2}{2\left(\delta^2+\Omega^2\right)} \tag{8}$$

□

题 3

Proof $Tr(\rho^2)=1$ correspond to $\rho=|\psi\rangle\langle\psi|$, a pure state.

解. 假设 $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ 为一般的混态，我们求解它的 $Tr(\rho^2)$。

$$\rho^2=\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)\left(\sum_j p_j|\psi_j\rangle\langle\psi_j|\right) \tag{9}$$

选取一组正交归一基 $\{|n\rangle\}$，每个态 $|\psi_i\rangle$ 可以在这组基下展开为

$$|\psi_i\rangle=\sum_n a_{in}|n\rangle \tag{10}$$

则

$$\rho=\sum_{i,n,n'} p_i a_{in}a^*_{in'}|n\rangle\langle n'| \tag{11}$$

ρ^2 可以展开为

$$\begin{aligned}\rho^2 &= \left(\sum_{i,n,n'} p_i a_{in}a^*_{in'}|n\rangle\langle n'|\right)\left(\sum_{j,m,m'} p_j a_{jm}a^*_{jkm'}|m\rangle\langle m'|\right) \\ &= \sum_{i,n,n',j,m,m'} p_ip_j a_{in}a^*_{in'}a_{jm}a^*_{jm'}|n\rangle\langle n'|m\rangle\langle m'| \\ &= \sum_{i,j,n,m,m'} p_ip_j a_{in}a^*_{im}a_{jm}a^*_{jm'}|n\rangle\langle m'|\end{aligned} \tag{12}$$

则

$$
\begin{aligned}
Tr(\rho^2) &= \sum_{i,j,k,n,m,m'} p_i p_j a_{in} a_{im}^* a_{jm} a_{jm'}^* \langle k|n\rangle\langle m'|k\rangle \\
&= \sum_{i,j,k,m} p_i p_j \left(a_{ik} a_{jk}^*\right)\left(a_{im}^* a_{jm}\right) \\
&= \sum_{ij} p_i p_j \langle\psi_j|\psi_i\rangle\langle\psi_i|\psi_j\rangle \\
&= \sum_{ij} p_i p_j |\langle\psi_i|\psi_j\rangle|^2 \qquad (13) \\
&\leq \sum_{ij} p_i p_j = 1 \qquad (14)
\end{aligned}
$$

当且仅当对 $\forall i, j, |\psi_i\rangle = |\psi_j\rangle$ 时等号成立。此时

$$
\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = |\psi\rangle\langle\psi| \qquad (15)
$$

为纯态的密度矩阵。 $\square$