

Chapter 3: Angular Momentum

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1 Coupled density matrix

A two spin-1/2 particle system, ρ_A for the first one, ρ_B for the second. For example,

$$\rho_A = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_B = \frac{1}{2}I = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho = \rho_A \otimes \rho_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in general, for $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability p_i for the particle at state $|\psi_i\rangle$. For 2 particle system, the overall state is described as

$$\rho = \sum_i p_i |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes |\phi_i^{(2)}\rangle\langle\phi_i^{(2)}| \quad (1)$$

$$= \sum_i p_i \left(|\psi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle \right) \cdot \left(\langle\psi_i^{(1)}| \otimes \langle\phi_i^{(2)}| \right) \quad (2)$$

Box 1.1: Examples of coupled density matrix

Pure state case:

particle 1 at $|0\rangle$, particle 2 at $|1\rangle$. Density matrix $\rho = |01\rangle\langle 01|$

Mixed state case:

For a two particle system, we have $\frac{1}{3}$ of chance two particles at $|\psi_1\rangle$, $\frac{1}{3}$ of chance two particles at $|\psi_2\rangle$, and $\frac{1}{3}$ of chance two particles at $|\psi_3\rangle$. The density matrix

$$\rho = \frac{1}{3} |\psi_1\rangle\langle\psi_1| + \frac{1}{3} |\psi_2\rangle\langle\psi_2| + \frac{1}{3} |\psi_3\rangle\langle\psi_3|$$

if $|\psi_1\rangle = |00\rangle$, $|\psi_2\rangle = |11\rangle$, $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, for this case, because of entanglement of $|\psi_3\rangle$,

$$\rho \neq \sum_i p_i \left(|\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle \right) \cdot \left(\langle\phi_i^{(1)}| \otimes \langle\phi_i^{(2)}| \right) \quad (3)$$

2 Dynamics

For 1 particle, $\rho = \sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)|$, the Shrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\psi_i(t)\rangle = H |\psi_i(t)\rangle \quad (4)$$

so

$$\dot{\rho} = \frac{d}{dt} \rho = \sum_i p_i \left(\frac{d}{dt} |\psi_i(t)\rangle \right) \langle \psi_i(t)| + \sum_i p_i |\psi_i(t)\rangle \left(\frac{d}{dt} \langle \psi_i(t)| \right) \quad (5)$$

$$= \sum_i p_i \frac{H}{i\hbar} |\psi_i(t)\rangle \langle \psi_i(t)| + \sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)| \frac{H}{-i\hbar} \quad (6)$$

$$= \frac{1}{i\hbar} \left[H \cdot \left(\sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)| \right) - \left(\sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)| \right) \cdot H \right] \quad (7)$$

$$= \frac{1}{i\hbar} [H, \rho] \quad (8)$$

for multi particle system, we have $\rho_{\text{multi}}, H_{\text{multi}}$

$$\Rightarrow \dot{\rho}_{\text{multi}} = \frac{1}{i\hbar} [H_{\text{multi}}, \rho_{\text{multi}}] \quad (9)$$

3 Trace

Under basis $\{|\psi_i\rangle\}$,

$$\text{tr}(\rho) = \sum_i \langle \psi_i | \rho | \psi_i \rangle \quad (10)$$

for multi particles, we need a $\{|\psi_i\rangle_{\text{multi}}\}$ as a basis,

$$\text{tr}(\rho_{\text{multi}}) = \sum_i \langle \psi_i |_{\text{multi}} \rho_{\text{multi}} | \psi_i \rangle_{\text{multi}} \quad (11)$$

4 Partial trace

Definition 4.1: Partial trace for 2 particle system

Suppose we have a two particle system, then the partial trace on particle A is

$$\text{tr}_A(\rho) = \sum_i (\langle \psi_i |^A \otimes I^B) \cdot \rho \cdot (| \psi_i \rangle^A \otimes I^B) \quad (12)$$

Here, $|\psi_i\rangle$ is one of the basis of particle A, and ρ is the density matrix of the whole system. We can give an example below.

Box 4.1: An example of partial trace

$\rho = M_A \otimes M_B$, where M_A, M_B are matrixes. For spin-1/2 particles,

$$\begin{aligned} \text{tr}_A(\rho) &= \sum_i (\langle \psi_i |^A \otimes I^B) \cdot \rho \cdot (| \psi_i \rangle^A \otimes I^B) \\ &= (\langle 0 |^A \otimes I^B) \cdot \rho \cdot (| 0 \rangle^A \otimes I^B) + (\langle 1 |^A \otimes I^B) \cdot \rho \cdot (| 1 \rangle^A \otimes I^B) \end{aligned} \quad (13)$$

where $\rho = M_A \otimes M_B$, so

$$(\langle 0 |^A \otimes I^B) \cdot (M_A \otimes M_B) \cdot (| 0 \rangle^A \otimes I^B) = \langle 0 | M_A | 0 \rangle \otimes (I^B \cdot M_B \cdot I^B) \quad (14)$$

$$= \langle 0 | M_A | 0 \rangle M_B \quad (15)$$

we can easily get that

$$(\langle 1 |^A \otimes I^B) \cdot M_A \otimes M_B \cdot (| 1 \rangle^A \otimes I^B) = \langle 1 | M_A | 1 \rangle M_B \quad (16)$$

Here, $| 0 \rangle^A \otimes I^B$ means that

$$| 0 \rangle^A \otimes I^B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (17)$$

So

$$\text{tr}_A(\rho) = \langle 0 | M_A | 0 \rangle M_B + \langle 1 | M_A | 1 \rangle M_B \quad (18)$$

$$= (\langle 0 | M_A | 0 \rangle + \langle 1 | M_A | 1 \rangle) M_B \quad (19)$$

$$= \text{tr}(M_A) M_B \stackrel{M_A \text{ be physical}}{=} M_B \quad (20)$$

The definition here is a bit complicated, we can refer to *Quantum Computation and Quantum Information*, Michael A. Nielsen and read the corresponding section. In that book, the author introduce the *reduced density operator* and the *partial trace* at the same time. Here is the definition in the book.

Definition 4.2: Reduced density operator and partial trace

Suppose we have physical systems A and B , whose state is described by a density operator ρ^{AB} . The reduced density operator for system A is defined by

$$\rho^A \equiv \text{tr}_B(\rho^{AB}) \quad (21)$$

where tr_B is a map of operators known as the *partial trace* over system B . The partial trace is defined by

$$\text{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \text{tr}(|b_1\rangle\langle b_2|) \quad (22)$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A , and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B . The trace operation appearing on the right hand side is the usual trace operation for system B , so $tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$. We have defined the partial trace operation only on a special subclass of operators on AB ; the specification is completed by requiring in addition to Equation (22) that the partial trace be linear in its input.

It means that $tr_A(|a_1b_1\rangle\langle a_2b_2|) = |b_1\rangle\langle b_2| \cdot tr(|a_1\rangle\langle a_2|)$. Due to $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, we can always decompose ρ such that $\rho = \sum_{i,j,i',j'} c_{i,j,i',j'} |\phi_i^A \phi_j^B\rangle\langle\phi_{i'}^A \phi_{j'}^B|$. Specifically, suppose a quantum system is in the product state $\rho^{AB} = \rho \otimes \sigma$, where ρ is a density operator for system A , and σ is a density operator for system B . Then

$$\rho^A = tr_B(\rho \otimes \sigma) = \rho tr(\sigma) = \rho \quad (23)$$

We want to proof that the two definitions are equivalent. For a matrix $|a_1b_1\rangle\langle a_2b_2|$, by definition 4.1, we have

$$tr(|a_1b_1\rangle\langle a_2b_2|) = \sum_i \langle i|^A I^B \cdot |a_1b_1\rangle\langle a_2b_2| \cdot |i\rangle^A I^B \quad (24)$$

$$= \sum_i \langle i|^A a_1\rangle\langle a_2|i\rangle^A \otimes I^B |b_1\rangle\langle b_2| I^B \quad (25)$$

$$= |b_1\rangle\langle b_2| tr(|a_1\rangle\langle a_2|) \quad (26)$$

so these two definitions are equivalent.

4.1 Partial trace and entangled state

If we have $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, then what is $tr_A(\rho)$? First, we can calculate the density matrix as follows:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \quad (27)$$

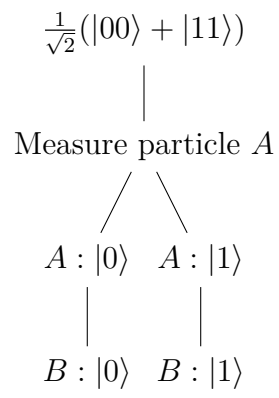
$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \quad (28)$$

by definition 4.1, we have

$$tr_A(\rho) = (\langle 0| \otimes I) \cdot \rho \cdot (|0\rangle \otimes I) + (\langle 1| \otimes I) \cdot \rho \cdot (|1\rangle \otimes I) \quad (29)$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I_B \quad (30)$$

which is a totally mixed state. It gives no information of particle B 's state.



$tr_A(\rho)$ means we discard information of A , and see what we left with B .