Chapter 3: Angular Momentum

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Attention: Quiz 4/30, two pieces of A4 double sided material, problems covering chapter 2. Basic calculas and linear algebra tutorials at 2 p.m. this Saturday.

1 Recap

Question: How to write the time evolution form of a state under the basis of $\{Y_{lm}\}$?

If we have $|\psi(0)\rangle = \sum_i c_i|\alpha_i\rangle \Rightarrow |\psi(t)\rangle = \sum_i c_i e^{-iE_n t/\hbar}|\alpha_i\rangle$, where $H|\alpha_i\rangle = E_i|\alpha_i\rangle$, then we also have

$$\langle\theta,\varphi|\psi(0)\rangle = \sum_{l,m} c_{l,m} Y_l^m \Rightarrow \langle\theta,\varphi|\psi(t)\rangle = \sum_{l,m} c_{l,m} Y_l^m e^{-iE_{l,m}t/\hbar} \tag{1.1}$$

Box 1.1: Orbital angular momentum

1. $L^2, L_x, L_y, L_z,\ [L^2, L_i] = 0,\ [L_k, L_l] = i\varepsilon_{klm}L_m$
2. Eigenfunctions for L^2 and L_i
3. Spherical harmonics $Y_l^m,\ l = 0, 1, 2, ...,\ m = \pm l, \pm(l-1), ...$

$$Y_l^m(\theta,\varphi) = \langle\theta,\varphi|l,m\rangle \propto e^{im\varphi} \tag{1.2}$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta,\ Y_1^{\pm1} = \sqrt{\frac{3}{8\pi}}\sin\theta\ e^{\pm i\varphi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\ Y_2^{\pm1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{\pm i\varphi},\ Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{\pm 2i\varphi}$$

Box 1.2: General properties of angular momentum

1. $J^2, J_x, J_y, J_z,\ [J^2, J_i] = 0,\ [J_k, J_l] = i\varepsilon_{klm}J_m$, eigenstate $|j,m\rangle$
2. $J_+, J_-,\ [J_z, J_+] = \hbar J_+,\ [J_z, J_-] = -\hbar J_-$

3. $J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle,\ J_z|j,m\rangle = m\hbar|j,m\rangle$, where j : integer or half integer, $m : -j, -j+1, ..., j-1, j$
4. $\begin{cases} J_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}|j,m+1\rangle,\ J_+|j,j\rangle = 0 \\ J_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}|j,m-1\rangle,\ J_-|j,-j\rangle = 0 \end{cases}$

Proof. Let $J_+|j,m\rangle = c|j,m+1\rangle$, we can evaluate J_-J_+ as:

$$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[j_x, J_y] = J^2 - J_z^2 - \hbar J_z$$

we have $\langle j,m|J_-J_+|j,m\rangle = |c|^2$, so

$$\Rightarrow |c|^2 = \hbar^2 j(j+1) - m^2\hbar^2 - m\hbar^2 = (j(j+1) - m(m+1))\,\hbar^2$$
$$\Rightarrow J_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}|j,m+1\rangle$$

Similarly, $J_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}|j,m-1\rangle$ □

2 Coupled spin-1/2 system

2.1 States

We have 2 particles, each with spin-1/2, use basis $\{|0\rangle, |1\rangle\}$ for each. Combinations: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, use direct product $\otimes$ to join the two qubits. For example,

$$|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix},\ |\psi_2\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \longmapsto |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{pmatrix} \tag{2.1}$$

For two qubits system, a general state could be expressed as $|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$.

2.2 Operators

Operator A and B for the first and the second particle, we have the operator $A \otimes B$:

$$(A \otimes B) \cdot (|\psi_1\rangle \otimes |\psi_2\rangle) = A|\psi_1\rangle \otimes B|\psi_2\rangle \tag{2.2}$$

In matrix form, we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{2.3}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \tag{2.4}$$

We can proof (**??**) mathematically. Here's some properties of outer products on operators.

Box 2.1: Properties of outer products on operators

1. $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$
2. $A \otimes B + C \otimes D \neq AC + BD$ (wrong dimensions)

Proof.

$$(C \otimes D)|\psi_1\rangle|\psi_2\rangle = C|\psi_1\rangle \otimes D|\psi_2\rangle \tag{2.5}$$

then apply $A \otimes B$ on it,

$$\begin{aligned}(A \otimes B) \cdot (C \otimes D) \cdot |\psi_1\rangle|\psi_2\rangle &= (A \otimes B) \cdot [C|\psi_1\rangle \otimes D|\psi_2\rangle] && (2.6)\\ &= (AC|\psi_1\rangle) \otimes (BD|\psi_2\rangle) && (2.7)\\ &= (AC \otimes BD) \cdot |\psi_1\rangle|\psi_2\rangle && (2.8)\end{aligned}$$

□

2.3 Entanglement, quantum logic gates

A separable state: $|\varphi\rangle = |a\rangle \otimes |b\rangle$, but if $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we cannot make it into the form $|a\rangle \otimes |b\rangle$, then the two particles are in entanglement state. Physically, we can say there's special correlation between the two particles. Entanglement is one special form of superposition. We can use some examples to clarify the entanglement and the non-entanglement state.

- Example(non-entanglement): $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Example(non-entanglement): $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$
- Example(entanglement):
  $|\psi\rangle = c\left(|00\rangle + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle\right) = c\left(|00\rangle + \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \neq |a\rangle \otimes |b\rangle$ is not so much entangled.

Some other important entanglement states

- Bell state $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$
- W-state $|w\rangle = \frac{1}{\sqrt{n}}(|10...0\rangle + |010...0\rangle + ... + |00...10\rangle + |00...01\rangle)$ is the permutations of the required state.

2.3.1 Properties of entangled states

(1) Measurement

Suppose we create a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then separate the two particles, after measure the state of the first particle(such as $|0\rangle$), then we know the state of the second one immediately, no matter how far they separated.

(2) Quantum correlation V.S. classical correlation

In this case, we need to distinguish the state $|\psi\rangle$ with classical case that suppose we have $2n$ pairs of particles in total, n of them are at state $|00\rangle$ and the rest are at state $|11\rangle$, then if we measure both the quantum case and the classical case, the result should be the same: half of the times we get $|00\rangle$, another half at $|11\rangle$.

Suppose we have $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we can do a *half flip*,

$$|0\rangle \longmapsto |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{2.9}$$

$$|1\rangle \longmapsto |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2.10}$$

then in the quantum case,

$$|\psi\rangle \longmapsto \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.11}$$

while in the classical case, it should be:

$$|00\rangle \longmapsto |++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \tag{2.12}$$

$$|11\rangle \longmapsto |++\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \tag{2.13}$$

If we repeat measurement along σ_z after the *half flip*, then we can easily find the difference.

2.3.2 Entangling operators

The Controlled-Not gate(CNOT) of two qubits

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.14}$$

the truth table of the CNOT gate is:

Before	After
$\|0\rangle \otimes \|0\rangle$	$\|0\rangle \otimes \|0\rangle$
$\|0\rangle \otimes \|1\rangle$	$\|0\rangle \otimes \|1\rangle$
$\|1\rangle \otimes \|0\rangle$	$\|1\rangle \otimes \|1\rangle$
$\|1\rangle \otimes \|1\rangle$	$\|1\rangle \otimes \|0\rangle$

Suppose the input state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.15}$$

after a CNOT gate, the output state should be:

$$|\psi_{\text{out}}\rangle = CNOT \cdot |\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \tag{2.16}$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.17}$$

which is an entangled state!

We need high fidelity to implement CNOT. For example, if we have 99.9% rate making CNOT to be successful, then we can only apply CNOT for roughly $O(1000)$ times.

2.3.3 How to construct a CNOT operation

Suppose we have $H_1 = \hbar\omega\sigma_z^A\sigma_x^B$, then the evolution $U = e^{-iH_1t/\hbar} = e^{-i\omega t\sigma_z^A\sigma_x^B}$

$$\Rightarrow U = \cos(\omega t)I - i\sin(\omega t)\sigma_z^A\sigma_x^B \tag{2.18}$$

we have

$$\begin{cases} H_0 = \hbar\omega_0\sigma_z^1 \\ H_2 = \hbar\omega_2\sigma_x^2 \end{cases} \tag{2.19}$$

then

$$\begin{cases} CNOT = e^{-i\frac{\pi}{4}}e^{-iH_0t_0}e^{-iH_2t_2}e^{-iH_1t_1} \\ t_0 = \frac{7\pi}{4\omega_0},\ t_2 = \frac{7\pi}{4\omega_2},\ t_1 = \frac{\pi}{4\omega_1} \end{cases} \tag{2.20}$$

we call this kind of implementation *pulse sequence.*