# Chapter 3: Angular Momentum

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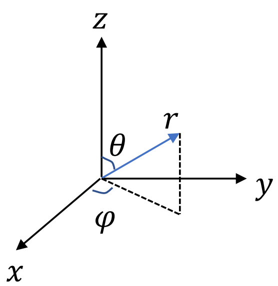
We have discussed position and momentum operator before, now let's consider the rotation of a system, which leads to angular position and angular momentum. Before we dive in, there's some prerequisites we should know.

## 1 Some prerequisites

In a 3D system, we use $\vec{r} = (x, y, z)$ to represent the coordinate, and the momentum is $\vec{p} = (p_x, p_y, p_z)$. In position representation, we have $p_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}$, $p_y \leftrightarrow -i\hbar\frac{\partial}{\partial y}$, and $p_z \leftrightarrow -i\hbar\frac{\partial}{\partial z}$, together we get $\vec{p} = (-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}) = -i\hbar\vec{\nabla}$.

Now let's switch to spherical coordinate $(r, \theta, \varphi)$.




For a point in space, we use a ket $|\psi\rangle$ to represent it,

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \tag{1}$$

now let's consider a rotation along $\hat{z}$ direction. Here $\varphi$ becomes $\varphi + d\varphi$, and $r, \theta$ remain the same. We have

$$|x, y, z\rangle \xrightarrow{\text{rotation}} |x', y', z'\rangle \tag{2}$$

and the corresponding

$$(r, \theta, \varphi) \xrightarrow{\text{rotation}} (r, \theta, \varphi + d\varphi) \tag{3}$$

then we try to find out the expression of $x', y', z'$ in terms of $r, \theta, \varphi, d\varphi$

$$\begin{cases} x' = r\sin\theta\cos(\varphi + d\varphi) \simeq r\sin\theta\cos\varphi - r\sin\theta\sin\varphi d\varphi \\ y' = r\sin\theta\sin(\varphi + d\varphi) \simeq r\sin\theta\sin\varphi + r\sin\theta\cos\varphi d\varphi \\ z' = r\cos\theta = z \end{cases} \Rightarrow \begin{cases} x' = x - yd\varphi \\ y' = y + xd\varphi \\ z' = z \end{cases} \tag{4}$$

On a spin-$\frac{1}{2}$ system, we have Pauli operators $\sigma_x, \sigma_y, \sigma_z$. In the $\sigma_z$ basis, we have

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and

$$[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m, \quad \varepsilon_{klm} = \begin{cases} 1 & \text{if } k, l, m \text{ in order} \\ -1 & \text{if out of order} \end{cases} \tag{6}$$

define the spin operator,

$$\begin{cases} \vec{S} = \frac{\hbar}{2}\vec{\sigma}, \ S_z = \frac{\hbar}{2}\sigma_z \\ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \end{cases} \tag{7}$$

here we should mention that $\boxed{[S_k, S_l] = i\hbar\varepsilon_{klm}S_m}$ is generally true for angular momentum operators, including spin-$\frac{1}{2}$ operators.

$$\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) = \frac{3}{4}\hbar^2 I \tag{8}$$

notice that for each $k$ in $x, y, z$, we have $\boxed{\sigma_k^2 = I}$ so $\vec{S}^2 \propto I$, and we also have $\boxed{[\vec{S}^2, S_k] = 0}$. If we define unite length vector

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{9}$$

we have

$$\vec{\sigma} \cdot \vec{n} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta \tag{10}$$

Noticed that if you try to use Mathematica to implement $\vec{\sigma} \cdot \vec{n}$, use the expression $\vec{n} \cdot \vec{\sigma}$ instead. When we are dealing with matrix exponential, we use Taylor expansion

$$e^{i\phi\hat{\sigma}_x} = I + i\phi\sigma_x - \frac{\phi^2\sigma_x^2}{2!} + \frac{i\phi^3\sigma_x^3}{3!} - ... \tag{11}$$

we use

$$\begin{cases} e^{\hat{A}} = I + \hat{A} + \frac{\hat{A}^2}{2!} + ... + \frac{\hat{A}^n}{n!} + ... \\ \sigma_x^2 = I \end{cases} \tag{12}$$

to get

$$e^{i\phi\hat{\sigma}_x} = \left(I - \frac{\phi^2}{2!}I + \frac{\phi^4}{4!}I - ...\right) + i\sigma_x\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - ...\right) \tag{13}$$

$$= \cos\phi I + i\sin\phi\sigma_x \tag{14}$$

## 2 Orbital angular momentum

Let's consider a small rotation: $\varphi \to \varphi + d\varphi$,

$$\begin{cases} x' \simeq x - yd\varphi \\ y' \simeq y + xd\varphi \\ z' = z \end{cases} \tag{15}$$

if we have a state $|\psi\rangle$ expressed in position coordinate $\langle x, y, z|\psi\rangle = \psi(x, y, z)$ (we can also write it in spherical basis $\psi(r, \theta, \varphi)$, as we discussed above), then we apply a rotation to the state, we get from

$$\psi(r, \theta, \varphi) \to \psi(r, \theta, \varphi - d\varphi) \tag{16}$$

and we know

$$\psi(r, \theta, \varphi - d\varphi) \simeq \psi(r, \theta, \varphi) - d\varphi \cdot \frac{\partial}{\partial \varphi}\psi(r, \theta, \varphi) \tag{17}$$

we can express this small rotation as

$$I - \frac{\partial}{\partial \varphi} = I + \frac{1}{i\hbar}\hat{L}_z \tag{18}$$

here we can see the relationship:

$$\hat{L}_z \xlongequal{\varphi \text{ coordinate}} -i\hbar\frac{\partial}{\partial \varphi} \longleftrightarrow p_x \xlongequal{x \text{ coordinate}} -i\hbar\frac{\partial}{\partial x} \tag{19}$$

then we express $\hat{L}_z$ in $x, y, z$ coordinates,

$$\psi(x, y, z) \xrightarrow{I - \frac{\partial}{\partial \varphi}} \psi(x', y', z') \tag{20}$$

rotate axis along $z$ by $d\varphi$

$$\begin{cases} x' = x - yd\varphi \\ y' = y + xd\varphi \\ z' = z \end{cases} \xrightarrow{\text{want } -d\varphi} \begin{cases} x' = x + yd\varphi \\ y' = y - xd\varphi \\ z' = z \end{cases} \tag{21}$$

so

$$\psi(x', y', z') \simeq \psi(x + yd\varphi, y - xd\varphi, z) \tag{22}$$

$$= \psi(x, y, z) + yd\varphi\frac{\partial}{\partial x}\psi(x, y, z) - xd\varphi\frac{\partial}{\partial y}\psi(x, y, z) \tag{23}$$

we have

$$p_x \leftrightarrow -i\hbar\frac{\partial}{\partial x}, \; p_y \leftrightarrow -i\hbar\frac{\partial}{\partial y} \tag{24}$$

so

$$\psi(x', y', z') = \psi(x,y,z) + yd\varphi\left(\frac{\hat{p}_x}{-i\hbar}\right)\psi(x,y,z) - xd\varphi\left(\frac{\hat{p}_y}{-i\hbar}\right)\psi(x,y,z) \tag{25}$$

$$= \left(I + \frac{1}{-i\hbar}d\varphi(yp_x - xp_y)\right)\psi(x,y,z) \tag{26}$$

$$= \left(I + \frac{1}{i\hbar}\hat{L}_z\right)\psi(x,y,z) \tag{27}$$

So $\hat{L}_z$ expressed with $x, y, z$ coordinate with respected operators should be

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{28}$$

similarly, we can write out the $\hat{L}_x$ and $\hat{L}_y$. In total,

$$\boxed{\begin{cases} \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{cases}} \tag{29}$$

which is the same as classical mechanics $\vec{L} = \vec{r}\times\vec{p}$, so we get an orbital angular momentum operator in analog to classical. For now, we've already find out the $\hat{L}_x, \hat{L}_y, \hat{L}_z$ represented in $x, y, z$ coordinate. Let's try to write them in $r, \theta, \varphi$ coordinate. We can replace the $\hat{p}_i$ as $-i\hbar\frac{\partial}{\partial i}$, and use (1) to replace $x, y, z$ to $r, \theta, \varphi$.

$$\begin{cases} L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\varphi} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \\ L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\ L_z = -i\hbar\frac{\partial}{\partial\varphi} \end{cases} \tag{30}$$

After defined $L_i$ in both $x, y, z$ and $r, \theta, \varphi$ coordinates, we can now explore some properties of orbital angular momentum.

1. $[L_x, L_y] = i\hbar L_z$

   *Proof.* We already know

$$\begin{cases} [x, p_x] = i\hbar \\ [x, y] = [x, p_y] = [p_x, p_y] = 0 \end{cases} \tag{31}$$

   so

$$[yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [yp_z, -xp_z] + [-zp_y, zp_x] + [-zp_y, -xp_z] \tag{32}$$

$$= [yp_z, zp_x] + [-zp_y, -xp_z] \tag{33}$$

$$= yp_x[p_z, z] + p_y x[z, p_z] \tag{34}$$

$$= i\hbar\,(xp_y - yp_x) = i\hbar L_z \tag{35}$$

   □

Furthermore, we can easily proof that

$$\boxed{[L_k, L_l] = i\hbar\varepsilon_{k,l,m} L_m,\ \ \varepsilon_{klm} = \begin{cases} 1 & \text{if } k,l,m \text{ in order} \\ -1 & \text{if out of order} \end{cases}} \tag{36}$$

2. $\boxed{[L^2, L_i] = 0}$, which is similar to $[S^2, S_i] = 0$.

   *Proof.* We know that

$$L^2 = L_x^2 + L_y^2 + L_z^2,\ \ [A^2, B] = A[A,B] + [A,B]A$$

   so

$$\begin{aligned} [L^2, L_x] &= [L_x^2 + L_y^2 + L_z^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x] & (37)\\ &= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z & (38)\\ &= -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z & (39)\\ &= 0 & (40) \end{aligned}$$

   □

so in total,

**Box 2.1: Properties of orbital angular momentum**

1. 

$$[L_k, L_l] = i\hbar\varepsilon_{k,l,m} L_m,\ \ \varepsilon_{klm} = \begin{cases} 1 & \text{if } k,l,m \text{ in order} \\ -1 & \text{if out of order} \end{cases}$$

2. 

$$[L^2, L_i] = 0$$

Now let's consider the eigen equation with $L^2$ and $L_z$. Recall that if $\hat{A}, \hat{B}$ are Hermitian, and $[\hat{A}, \hat{B}] = 0$, then there exists $\{|\psi\rangle\}$ giving

$$\begin{cases} \hat{A}|\psi\rangle = a|\psi\rangle \\ \hat{B}|\psi\rangle = b|\psi\rangle \end{cases}$$

or we can say $\{|\psi\rangle\}$ is the mutual eigen state. Here we already have $[L^2, L_z] = 0$, let $|y\rangle$ to be one of these states, so

$$\begin{cases} L_z|y\rangle = m\hbar|y\rangle \\ L^2|y\rangle = \beta\hbar^2|y\rangle \end{cases} \tag{41}$$

where $m, \beta$ are numbers. $|y\rangle$ needs to be normalized, $\langle y|y\rangle = 1$

$$\beta\hbar^2 = \langle y|L^2|y\rangle = \langle y|L_x^2 + L_y^2 + L_z^2|y\rangle \geq \langle y|L_z^2|y\rangle = m^2\hbar^2 \tag{42}$$

so

$$\beta \geq m^2 \tag{43}$$

in spherical coordinate, as above we can express $L^2, L_z$ all in just $\theta, \varphi$ without $r$. $L^2, L_z$ only relates to angular coordinates $\{\theta, \varphi\}$ as the angular representation. The eigen function for $L_z$ in coordinates $\{\theta, \varphi\}$ is $\langle \theta, \varphi | L_z | y \rangle = \lambda \langle \theta, \varphi | y \rangle$. To make the expression more convenient (we will know the reason soon), let's set $\lambda = m\hbar$.

$$\begin{aligned}
\langle \theta, \varphi | y \rangle &= y(\theta, \varphi) &(44)\\
\langle \theta, \varphi | L_z | y \rangle &= m\hbar \langle \theta, \varphi | y \rangle &(45)\\
&= -i\hbar \frac{\partial}{\partial \varphi} \langle \theta, \varphi | y \rangle &(46)
\end{aligned}$$

then we can solve the equation

$$-i\hbar \frac{\partial}{\partial \varphi} y(\theta, \varphi) = m\hbar y(\theta, \varphi) \Rightarrow y(\theta, \varphi) \propto e^{im\varphi} \tag{47}$$

and $\varphi \to \varphi + 2\pi$ should give the same wave function, so we have

$$e^{im\varphi} = e^{im(\varphi + 2\pi)} \Rightarrow e^{i2\pi m} = 1 \tag{48}$$

so $m$ should be an integer, $m = 0, \pm 1, \pm 2, ...$, and now we know why we need to set $\lambda = m\hbar$ before. Further, we have the eigen function for $L^2$

$$\langle \theta, \varphi | L^2 | y \rangle = \beta \hbar^2 \langle \theta, \varphi | y \rangle \tag{49}$$

notice that the original eigen function is $\langle \theta, \varphi | L^2 | y \rangle = \lambda' \langle \theta, \varphi | y \rangle$, and the reason to let $\lambda' = \beta \hbar^2$ is similar to the former one, we will see it soon.