Quiz 2

Problem 1: 20 points

One dimensional infinity deep square well.
For a particle with mass m in a potential

$$V(x) = \begin{cases} +\infty & x < -\frac{a}{2} \\ 0 & -\frac{a}{2} \le x \le \frac{a}{2} \\ +\infty & x > \frac{a}{2} \end{cases},$$

1. (5 pts) write out the eigen energy E_n
2. (5 pts) write out the eigen wave function in position space $\phi_n(x)$
3. (10 pts) with $\psi(x, t = 0) = \sqrt{\frac{1}{a}}\left(\cos\left(\frac{\pi x}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\right)$, derive and find $\psi(x, t = \frac{2ma^2}{\pi\hbar})$, hint: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

Solution.

(1) $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

(2) $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a} + \frac{n\pi}{2}\right)$

(3) When $t = 0$,

$$\begin{aligned} \psi(x,0) &= \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{a}\right) - \sqrt{\frac{1}{a}}\sin\left(\frac{2\pi x}{a}\right) && (1) \\ &= \sqrt{\frac{1}{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a} + \frac{\pi}{2}\right) + \sqrt{\frac{1}{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a} + \frac{2\pi}{2}\right) && (2) \\ &= \sqrt{\frac{1}{2}}\phi_1 + \sqrt{\frac{1}{2}}\phi_2 && (3) \end{aligned}$$

so

$$\psi(x,t) = \sqrt{\frac{1}{2}}e^{\frac{iE_1 t}{\hbar}}\phi_1 + \sqrt{\frac{1}{2}}e^{\frac{iE_2 t}{\hbar}}\phi_2 \tag{4}$$

where $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$, $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$, $t = \frac{2ma^2}{\pi\hbar}$, so

$$\psi(x,t) = -\sqrt{\frac{1}{2}}\phi_1 + \sqrt{\frac{1}{2}}\phi_2 \tag{5}$$

□

Problem 2: 15 points

One dimensional Harmonic Oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

with

$$a = \frac{1}{\sqrt{\hbar\omega}}\left(i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\hat{x}\right),\ a^\dagger = \frac{1}{\sqrt{\hbar\omega}}\left(-i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\hat{x}\right)$$

1. (2 pts) write out $[a, a^\dagger]$
2. (1 pt) $a|n\rangle$
3. (1 pt) $a^\dagger|n\rangle$
4. (1 pt) E_n
5. (10 pts) derive and evaluate $a^3(a^\dagger)^2|n=0\rangle$

Solution.

(1) $[a, a^\dagger] = 1$

(2) $a|n\rangle = \sqrt{n}|n-1\rangle$

(3) $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

(4) $E_n = \hbar\omega(n + \frac{1}{2})$

(5) $a^3a^\dagger a^\dagger|0\rangle = a^3a^\dagger|1\rangle = a^3\sqrt{2}|2\rangle = 0$ □

Problem 3: 15 points

Dynamic of a spin-$\frac{1}{2}$ particle

$$H = \hbar\omega\sigma_x,\ |\psi, t=0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

1. (5 pts) find $U = e^{-\frac{iHt}{\hbar}}$
2. (5 pts) find $|\psi, t\rangle$
3. (5 pts) evaluate $\langle\psi, t|\sigma_z|\psi, t\rangle$ to find the average value of measuring along σ_z over time.

Solution.

(1) The time evolution operator

$$U(t) = e^{\frac{-iHt}{\hbar}} = e^{-i\omega t\sigma_x} = \cos(\omega t)I - i\sin(\omega t)\sigma_x \tag{6}$$

$$= \begin{pmatrix}\cos(\omega t) & -i\sin(\omega t)\\ -i\sin(\omega t) & \cos(\omega t)\end{pmatrix} \tag{7}$$

(2) We have $|\psi, 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$|\psi, t\rangle = U(t)|\psi, 0\rangle = \begin{pmatrix} \cos(\omega t) & -i\sin(\omega t) \\ -i\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -i\sin(\omega t) \end{pmatrix} \tag{8}$$

(3) We know $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so

$$\langle \psi, t|\sigma_z|\psi, t\rangle = \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} \tag{9}$$

$$= \cos^2\omega t - \sin^2\omega t = \cos(2\omega t) \tag{10}$$

$\square$