

作业 07

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题 1

With

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

find eigenvalue and eigenstates of $\vec{\sigma} \cdot \vec{n}$

解. 直接用 Mathematica 求解如下

```
In[ ]:=  $\vec{\sigma}$  = Table[PauliMatrix[ $\vec{i}$ ], { $\vec{i}$ , 3}];
           |表格 |泡利自旋矩阵

In[ ]:=  $\vec{n}$  = {Sin[ $\theta$ ] Cos[ $\varphi$ ], Sin[ $\theta$ ] Sin[ $\varphi$ ], Cos[ $\theta$ ]};
           |正弦 |余弦 |正弦 |正弦 |余弦

In[ ]:= Grid[
           |格子
           Insert[
           |插入
           Transpose[{Eigenvalues[ $\vec{n} \cdot \vec{\sigma}$ ],
           |转置 |特征值
           TraditionalForm /@ Assuming[{ $\theta$ ,  $\varphi$ ] ∈ Reals, Normalize /@ Simplify[
           |传统格式 |假定 |实数域 |正规化 |化简
           Eigenvectors[ $\vec{n} \cdot \vec{\sigma}$ ]]]], {"Eigenvalue", "Eigenvector"}, 1],
           |特征向量
           Frame → All]
           |边框 |全部
```

Eigenvalue	Eigenvector
-1	$\left\{ \frac{\tan\left(\frac{\theta}{2}\right) (-\cos(\varphi) + i \sin(\varphi))}{\sqrt{1 + \left (i \sin(\varphi) - \cos(\varphi)) \tan\left(\frac{\theta}{2}\right) \right ^2}}, \frac{1}{\sqrt{1 + \left (i \sin(\varphi) - \cos(\varphi)) \tan\left(\frac{\theta}{2}\right) \right ^2}} \right\}$
1	$\left\{ \frac{\cot\left(\frac{\theta}{2}\right) (\cos(\varphi) - i \sin(\varphi))}{\sqrt{1 + \left \cot\left(\frac{\theta}{2}\right) (\cos(\varphi) - i \sin(\varphi)) \right ^2}}, \frac{1}{\sqrt{1 + \left \cot\left(\frac{\theta}{2}\right) (\cos(\varphi) - i \sin(\varphi)) \right ^2}} \right\}$

□

题 2

With $\psi(\theta, \varphi) = \frac{1}{\sqrt{3}} (\sqrt{2}Y_1^0(\theta, \varphi) + Y_1^1(\theta, \varphi))$, without integrations, find $\langle L^2 \rangle, \langle L_z \rangle$

解. 我们已知

$$\begin{aligned} L_z Y_{lm} &= m\hbar Y_{lm} \\ L^2 Y_{lm} &= l(l+1)\hbar^2 Y_{lm} \end{aligned}$$

所以

$$L^2|\psi\rangle = L^2 \left(\sqrt{\frac{2}{3}}Y_1^0 + \sqrt{\frac{1}{3}}Y_1^1 \right) = \sqrt{\frac{2}{3}}2\hbar^2 Y_1^0 + \sqrt{\frac{1}{3}}2\hbar^2 Y_1^1 \quad (1)$$

$$L_z|\psi\rangle = \sqrt{\frac{1}{3}}\hbar Y_1^1 \quad (2)$$

故

$$\langle L^2 \rangle = 2\hbar^2 \quad (3)$$

$$\langle L_z \rangle = \frac{\hbar}{3} \quad (4)$$

□

题 3

With $\phi(t=0) = \psi(\theta, \varphi)$ as above, $H = \frac{\vec{L}^2}{2mR^2}$, find $\phi(t=T)$

解. 因为

$$L^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle \quad (5)$$

我们可以把算符 L^2 展开为

$$L^2 = \sum l(l+1)|l, m\rangle\langle l, m| \quad (6)$$

同样可以把演化算符 $e^{-\frac{i}{\hbar}Ht}$ 做如下展开:

$$\begin{aligned} e^{-\frac{i}{\hbar}Ht} &= e^{-\frac{it}{2m\hbar R^2}L^2} \\ &= \sum e^{-\frac{it}{2m\hbar R^2}l(l+1)}|l, m\rangle\langle l, m| \end{aligned} \quad (7)$$

所以

$$\begin{aligned}
\phi(t) &= e^{-\frac{i}{\hbar} H t} \phi(0) \\
&= \left(\sum e^{-\frac{i t}{2 m \hbar R^2} l(l+1)} |l, m\rangle \langle l, m| \right) \left(\sqrt{\frac{2}{3}} |1, 0\rangle + \sqrt{\frac{1}{3}} |1, 1\rangle \right) \\
&= e^{-\frac{i t}{m R^2 \hbar}} \left(\sqrt{\frac{2}{3}} Y_1^0 + \sqrt{\frac{1}{3}} Y_1^1 \right)
\end{aligned} \tag{8}$$

□

题 4: Uncertainty principle

With $J_z|j, m\rangle = m\hbar|j, m\rangle$, $\vec{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$,

$$\langle \Delta A \rangle = \sqrt{\langle j, m | A^2 | j, m \rangle - (\langle j, m | A | j, m \rangle)^2}$$

find $\langle \Delta J_x \rangle \langle \Delta J_y \rangle$ and $\langle [J_x, J_y] \rangle$, check $\langle \Delta J_x \rangle \langle \Delta J_y \rangle \geq \frac{1}{2} |\langle [J_x, J_y] \rangle|$.

When $\langle \Delta J_x \rangle \langle \Delta J_y \rangle = \frac{1}{2} |\langle [J_x, J_y] \rangle|$, what's the requirement of m ?

解. 我们先证明在 J_z 的本征态 $|j, m\rangle$ 下, $\langle J_x \rangle = \langle J_y \rangle = 0$ 。因为

$$[J_y, J_z] = i\hbar J_x \tag{9}$$

两边在 $|j, m\rangle$ 态下求平均值,

$$\text{LHS} = \langle j, m | J_y J_z - J_z J_y | j, m \rangle = m\hbar \langle j, m | J_y | j, m \rangle - m\hbar \langle j, m | J_y | j, m \rangle = 0 \tag{10}$$

$$\text{RHS} = i\hbar \langle j, m | J_x | j, m \rangle = i\hbar \langle J_x \rangle \tag{11}$$

故有 $\langle J_x \rangle = 0$, 同理可得 $\langle J_y \rangle = 0$ 。由对称性, 可知 $\langle \Delta J_x \rangle = \langle \Delta J_y \rangle$, 又有

$$\langle \Delta J_x \rangle^2 = \langle J_x^2 \rangle - \langle J_x \rangle^2 = \langle J_x^2 \rangle \tag{12}$$

则

$$\begin{aligned}
\langle \Delta J_x \rangle^2 &= \langle \Delta J_y \rangle^2 = \frac{1}{2} (\langle \Delta J_x \rangle^2 + \langle \Delta J_y \rangle^2) \\
&= \frac{1}{2} (\langle J_x^2 \rangle + \langle J_y^2 \rangle) = \frac{1}{2} \langle J_x^2 + J_y^2 \rangle \\
&= \frac{1}{2} \langle J^2 - J_z^2 \rangle = \frac{1}{2} (\langle J^2 \rangle - \langle J_z^2 \rangle) \\
&= \frac{1}{2} [j(j+1) - m^2] \hbar^2
\end{aligned} \tag{13}$$

我们已知 $[J_x, J_y] = i\hbar J_z$, 则 $|\langle [J_x, J_y] \rangle| = |i\hbar \langle J_z \rangle| = |m| \hbar^2$,

$$\langle \Delta J_x \rangle \langle \Delta J_y \rangle = \langle \Delta J_x \rangle^2 = \frac{1}{2} [j(j+1) - m^2] \hbar^2 \quad (14)$$

$$\geq \frac{1}{2} |m| \hbar^2 \quad (15)$$

当 $m = \pm j$ 时取等号。 □

题 5

Write out $J_- J_+$ as a matrix in the basis of J_z , with $j = 1$, and find the eigenvalue and eigenstates of $J_- J_+$

解.

$$J_- J_+ = (J_x + iJ_y)(J_x - iJ_y) \quad (16)$$

$$\begin{aligned} &= J_x^2 - i[J_x, J_y] + J_y^2 \\ &= J_x^2 + \hbar J_z + J_y^2 \\ &= J^2 - J_z^2 + \hbar J_z \end{aligned} \quad (17)$$

当 $j = 1$ 时, $m = \{-1, 0, 1\}$, 故 J_z 的基 $|j, m\rangle$ 有 3 个, 为 $\{|1, -1\rangle, |1, 0\rangle, |1, 1\rangle\}$, 由关系

$$\begin{cases} J_z |j, m\rangle = m\hbar |j, m\rangle \\ J_z^2 |j, m\rangle = m^2 \hbar^2 |j, m\rangle \\ J^2 |j, m\rangle = j(j+1) \hbar^2 |j, m\rangle \end{cases}$$

可以得到 $J_- J_+$ 的矩阵表示为

$$J_- J_+ = 2\hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (18)$$

本征值和对应的本征态可以写为:

本征値	本征态
0	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$2\hbar^2$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$2\hbar^2$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

□