Chapter 3

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1 Coupled density matrix

For a two spin-1/2 particle, ρ_A for the first one, ρ_B for the second. For example,

$$\rho_A = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \rho_B = \frac{1}{2}I = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in general, for $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability p_i for the particle at state $|\psi_i\rangle$. For 2 particles, the overall state is described as

$$\rho = \sum_{i} p_i |\psi_i^{(1)}\rangle \langle \psi_i^{(1)}| \otimes |\phi_i^{(2)}\rangle \langle \phi_i^{(2)}| \tag{1}$$

$$= \sum_{i} p_{i} \left(|\psi_{i}^{(1)}\rangle \otimes |\phi_{i}^{(2)}\rangle \right) \cdot \left(\langle \psi_{i}^{(1)}| \otimes \langle \phi_{i}^{(2)}| \right)$$
 (2)

Box 1.1: Examples

Pure state case: particle 1 at $|0\rangle$, particle 2 at $|1\rangle$. Density matrix $\rho = |01\rangle\langle 01|$ **Mixed state case:** For a two particles system, we have $\frac{1}{3}$ of chance two particles at $|\psi_1\rangle$, $\frac{1}{3}$ of chance two particles at $|\psi_2\rangle$, and $\frac{1}{3}$ of chance two particles at $|\psi_3\rangle$. The density matrix

$$\rho = \frac{1}{3} |\psi_1\rangle \langle \psi_1| + \frac{1}{3} |\psi_2\rangle \langle \psi_2| + \frac{1}{3} |\psi_3\rangle \langle \psi_3|$$

2 Dynamics

For 1 particle, $\rho = \sum_{i} p_{i} |\psi_{i}(t)\rangle \langle \psi_{i}(t)|$, the Shrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\psi_i(t)\rangle = H|\psi_i(t)\rangle$$
 (3)

$$\dot{\rho} = \frac{d}{dt}\rho = \sum_{i} p_i \left(\frac{d}{dt}\right) \tag{4}$$

for multi particle system, we have $\rho_{\text{multi}}, H_{\text{multi}}$

$$\Rightarrow \dot{\rho}_{\text{multi}} = \frac{1}{i\hbar} [H_{\text{multi}}, \rho_{\text{multi}}] \tag{5}$$

3 Trace

Under basis $\{|\psi_i\rangle\}$,

$$tr(\rho) = \sum_{i} \langle \psi_i | \rho | \psi_i \rangle \tag{6}$$

for multi particles, we need a $\{|\psi_i\rangle_{\text{multi}}\}$ as a basis,

4 Partial trace

Definition 4.1: Partial trace

Suppose we have a two particle system, then the partial trace on particle A is

$$tr_A(\rho) = \sum_i \left(\langle \psi_i | A \otimes I^B \right) \cdot \rho \cdot \left(|\psi_i \rangle^A \otimes I^B \right) \tag{7}$$

The definition here is a bit complicated, we can refer to *Quantum Computation and Quantum Information*, *Michael A. Nielsen* and read the corresponding section. In that book, the author introduce the *reduced density operator* and the *partial trace* at the same time, and here is the definition in the book.

Definition 4.2: Reduced density operator and partial trace

Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB} . The reduced density operator for system A is defined by

$$\rho^A \equiv tr_B(\rho^{AB}) \tag{8}$$

where tr_B is a map of operators known as the partial trace over system B. The partial trace is defined by

$$\operatorname{tr}_{B}(|a_{1}\rangle\langle a_{2}|\otimes|b_{1}\rangle\langle b_{2}|) \equiv |a_{1}\rangle\langle a_{2}|\operatorname{tr}(|b_{1}\rangle\langle b_{2}|) \tag{9}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B. The trace operation appearing on the right hand side is the usual trace operation for system B, so $tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$. We have defined the partial trace operation only on a special subclass of operators

on AB; the specification is completed by requiring in addition to Equation (9) that the partial trace be linear in its input.

4.1 Partial trace and entangled state

If we have $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, then what is $tr_A(\rho)$? First, we can calculate the density matrix as follows:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \tag{10}$$

$$= \frac{1}{2}(|00\rangle\langle00| + |00\rangle\langle11| + |11\rangle\langle00| + |11\rangle\langle11|) \tag{11}$$

$$\Rightarrow tr_A(\rho) = (\langle 0 | \otimes I) \cdot \tag{12}$$