Chapter 3

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1 Coupled density matrix

For a two spin-1/2 particle, ρ_A for the first one, ρ_B for the second. For example,

$$\rho_A = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \rho_B = \frac{1}{2}I = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho = \rho_A \otimes \rho_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in general, for $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability p_i for the particle at state $|\psi_i\rangle$. For 2 particles, the overall state is described as

$$\rho = \sum_i p_i|\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes |\phi_i^{(2)}\rangle\langle\phi_i^{(2)}| \tag{1}$$

$$= \sum_i p_i \left(|\psi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle\right) \cdot \left(\langle\psi_i^{(1)}| \otimes \langle\phi_i^{(2)}|\right) \tag{2}$$

> **Box 1.1: Examples**
>
> **Pure state case:** particle 1 at $|0\rangle$, particle 2 at $|1\rangle$. Density matrix $\rho = |01\rangle\langle 01|$
> **Mixed state case:** For a two particle system, we have $\frac{1}{3}$ of chance two particles at $|\psi_1\rangle$, $\frac{1}{3}$ of chance two particles at $|\psi_2\rangle$, and $\frac{1}{3}$ of chance two particles at $|\psi_3\rangle$. The density matrix
> $$\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3|$$

2 Dynamics

For 1 particle, $\rho = \sum_i p_i|\psi_i(t)\rangle\langle\psi_i(t)|$, the Shrödinger's equation

$$i\hbar\frac{\partial}{\partial t}|\psi_i(t)\rangle = H|\psi_i(t)\rangle \tag{3}$$

so

$$\dot{\rho} = \frac{d}{dt}\rho = \sum_i p_i \left(\frac{d}{dt}\right) \tag{4}$$

for multi particle system, we have $\rho_{\text{multi}}, H_{\text{multi}}$

$$\Rightarrow \dot{\rho}_{\text{multi}} = \frac{1}{i\hbar}[H_{\text{multi}}, \rho_{\text{multi}}] \tag{5}$$

3   Trace

Under basis $\{|\psi_i\rangle\}$,

$$tr(\rho) = \sum_i \langle\psi_i|\rho|\psi_i\rangle \tag{6}$$

for multi particles, we need a $\{|\psi_i\rangle_{\text{multi}}\}$ as a basis,

4   Partial trace

Definition 4.1: Partial trace

Suppose we have a two particle system, then the partial trace on particle A is

$$tr_A(\rho) = \sum_i \left(\langle\psi_i|^A \otimes I^B\right) \cdot \rho \cdot \left(|\psi_i\rangle^A \otimes I^B\right) \tag{7}$$