

Quiz 2

Problem 1: 20 points

One dimensional infinity deep square well.

For a particle with mass m in a potential

$$V(x) = \begin{cases} +\infty & x < -\frac{a}{2} \\ 0 & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ +\infty & x > \frac{a}{2} \end{cases},$$

1. (5 pts) write out the eigen energy E_n
2. (5 pts) write out the eigen wave function in position space $\phi_n(x)$
3. (10 pts) with $\psi(x, t = 0) = \sqrt{\frac{1}{a}} (\cos(\frac{\pi x}{a}) - \sin(\frac{2\pi x}{a}))$, derive and find $\psi(x, t = \frac{2ma^2}{\pi\hbar})$, hint: $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

Solution.

$$(1) E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$(2) \phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a} + \frac{n\pi}{2}\right)$$

(3) When $t = 0$,

$$\psi(x, 0) = \sqrt{\frac{1}{a}} \cos\left(\frac{\pi x}{a}\right) - \sqrt{\frac{1}{a}} \sin\left(\frac{2\pi x}{a}\right) \quad (1)$$

$$= \sqrt{\frac{1}{2}} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a} + \frac{\pi}{2}\right) + \sqrt{\frac{1}{2}} \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a} + \frac{2\pi}{2}\right) \quad (2)$$

$$= \sqrt{\frac{1}{2}} \phi_1 + \sqrt{\frac{1}{2}} \phi_2 \quad (3)$$

so

$$\psi(x, t) = \sqrt{\frac{1}{2}} e^{\frac{iE_1 t}{\hbar}} \phi_1 + \sqrt{\frac{1}{2}} e^{\frac{iE_2 t}{\hbar}} \phi_2 \quad (4)$$

where $E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$, $E_2 = \frac{4\pi^2 \hbar^2}{2ma^2}$, $t = \frac{2ma^2}{\pi\hbar}$, so

$$\psi(x, t) = -\sqrt{\frac{1}{2}} \phi_1 + \sqrt{\frac{1}{2}} \phi_2 \quad (5)$$

□

Problem 2: 15 points

One dimensional Harmonic Oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

with

$$a = \frac{1}{\sqrt{\hbar\omega}} \left(i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\hat{x} \right), \quad a^\dagger = \frac{1}{\sqrt{\hbar\omega}} \left(-i\frac{\hat{p}}{\sqrt{2m}} + \sqrt{\frac{1}{2}m\omega^2}\hat{x} \right)$$

1. (2 pts) write out $[a, a^\dagger]$
2. (1 pt) $a|n\rangle$
3. (1 pt) $a^\dagger|n\rangle$
4. (1 pt) E_n
5. (10 pts) derive and evaluate $a^3(a^\dagger)^2|n=0\rangle$

Solution.

- (1) $[a, a^\dagger] = 1$
- (2) $a|n\rangle = \sqrt{n}|n-1\rangle$
- (3) $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$
- (4) $E_n = \hbar\omega(n + \frac{1}{2})$
- (5) $a^3a^\dagger a^\dagger|0\rangle = a^3a^\dagger|1\rangle = a^3\sqrt{2}|2\rangle = 0$

□

Problem 3: 15 points

Dynamic of a spin- $\frac{1}{2}$ particle

$$H = \hbar\omega\sigma_x, \quad |\psi, t=0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1. (5 pts) find $U = e^{-\frac{iHt}{\hbar}}$
2. (5 pts) find $|\psi, t\rangle$
3. (5 pts) evaluate $\langle\psi, t|\sigma_z|\psi, t\rangle$ to find the average value of measuring along σ_z over time.