

Chapter 3

Yuquan Chen

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1 Coupled density matrix

For a two spin-1/2 particle, ρ_A for the first one, ρ_B for the second. For example,

$$\rho_A = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_B = \frac{1}{2}I = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho = \rho_A \otimes \rho_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in general, for $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ of one particle, the physical meaning is there is a statistics, that there is a probability p_i for the particle at state $|\psi_i\rangle$. For 2 particles, the overall state is described as

$$\rho = \sum_i p_i |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes |\phi_i^{(2)}\rangle\langle\phi_i^{(2)}| \quad (1)$$

$$= \sum_i p_i \left(|\psi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle \right) \cdot \left(\langle\psi_i^{(1)}| \otimes \langle\phi_i^{(2)}| \right) \quad (2)$$

Box 1.1: Examples

Pure state case: particle 1 at $|0\rangle$, particle 2 at $|1\rangle$. Density matrix $\rho = |01\rangle\langle 01|$

Mixed state case: For a two particle system, we have $\frac{1}{3}$ of chance two particles at $|\psi_1\rangle$, $\frac{1}{3}$ of chance two particles at $|\psi_2\rangle$, and $\frac{1}{3}$ of chance two particles at $|\psi_3\rangle$. The density matrix

$$\rho = \frac{1}{3} |\psi_1\rangle\langle\psi_1| + \frac{1}{3} |\psi_2\rangle\langle\psi_2| + \frac{1}{3} |\psi_3\rangle\langle\psi_3|$$

2 Dynamics

For 1 particle, $\rho = \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)|$, the Shrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\psi_i(t)\rangle = H |\psi_i(t)\rangle \quad (3)$$

so

$$\dot{\rho} = \frac{d}{dt}\rho = \sum_i p_i \left(\frac{d}{dt} \right) \quad (4)$$

for multi particle system, we have $\rho_{\text{multi}}, H_{\text{multi}}$

$$\Rightarrow \dot{\rho}_{\text{multi}} = \frac{1}{i\hbar} [H_{\text{multi}}, \rho_{\text{multi}}] \quad (5)$$

3 Trace

Under basis $\{|\psi_i\rangle\}$,

$$\text{tr}(\rho) = \sum_i \langle \psi_i | \rho | \psi_i \rangle \quad (6)$$

for multi particles, we need a $\{|\psi_i\rangle_{\text{multi}}\}$ as a basis,

4 Partial trace

Definition 4.1: Partial trace

Suppose we have a two particle system, then the partial trace on particle A is

$$\text{tr}_A(\rho) = \sum_i (\langle \psi_i |^A \otimes I^B) \cdot \rho \cdot (|\psi_i\rangle^A \otimes I^B) \quad (7)$$

4.1 Partial trace and entangled state

If we have $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, then what is $\text{tr}_A(\rho)$? First, we can calculate the density matrix as follows:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \quad (8)$$

$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \quad (9)$$

$$\Rightarrow \text{tr}_A(\rho) = (\langle 0| \otimes I) \quad (10)$$