Numerical Analysis and Computational Mathematics

Fall Semester 2019 - Section CSE

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Linear systems: direct & iterative methods

Exercise I (MATLAB)
Let us consider the linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$, $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$ for $n \geq 1$. We are interested in computing the solution $\mathbf{x}$ by means of the LU factorization method.

a) Let us consider the following two matrices:

$$A_1 = \begin{bmatrix} 3 & -2 & 0 & & & & \\ -1 & 3 & -2 & 0 & & & \\ 0 & -1 & 3 & -2 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -1 & 3 & -2 \\ & & & & 0 & -1 & 3 \end{bmatrix}, \qquad A_2 \ : \ (A_2)_{ij} = \frac{1}{i+j-1}, \quad i,j = 1,\dots n.$$

The matrix A_2 is called the Hilbert matrix of dimension n. Generate the matrices A_1 and A_2 for $n = 4$ in MATLAB (for A_2 use the MATLAB command `hilb(n)`), and determine if there exists a unique LU factorization of the matrices without performing the pivoting technique. (*Hint*: if necessary, use the MATLAB function `eig` to compute the eigenvalues of a matrix).

b) Now set $n = 9$ and compute by means of the LU factorization method the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the linear systems $A_1\mathbf{x}_1 = \mathbf{b}_1$ and $A_2\mathbf{x}_2 = \mathbf{b}_2$, where $\mathbf{x}_1 = \mathbf{x}_2 = (1, 1, \dots, 1)^T$, respectively. (*Hint*: use the the MATLAB functions `lu`, `forward_substitutions.m` and `backward_substitutions.m`). Compute the relative errors $e_{\mathrm{rel},i} := \frac{\mathbf{x}_i - \widehat{\mathbf{x}}_i}{\mathbf{x}_i}$, where $\widehat{\mathbf{x}}_i$ are the approximate solutions of the linear systems by LU factorization with $i = 1, 2$. Similarly, compute the relative residuals $r_{\mathrm{rel},i} := \frac{\mathbf{r}_i}{\mathbf{b}_i}$, where $\mathbf{r}_i := \mathbf{b}_i - A_i\widehat{\mathbf{x}}_i$, and the condition numbers $K_2(A_i)$ for $i = 1, 2$. Comment and motivate the results obtained. (*Hint*: use the MATLAB function `cond` to compute the condition number of the matrices and assume that the perturbation on the matrices are negligible).

c) Repeat point b) for $n = 4, 5, \dots, 13$. Plot the relative errors, the relative residuals, and the condition numbers vs. n (use the semilogarithmic scale) and comment the results obtained. What are the relative errors obtained for the two linear systems when $n = 13$? Can we consider the approximate solutions $\widehat{\mathbf{x}}_i$ satisfactory?

Exercise II (MATLAB)
Let us consider the Jacobi and Gauss-Seidel iterative methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$.

a) Let us consider the non singular matrices:

$$A_1 = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1.65 & -1 \\ 0 & 1 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 5 & -3 & -2 \\ -3 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix},$$

$$A_3 \in \mathbb{R}^{n_3\times n_3}, \quad A_3 = \begin{bmatrix} 4 & -1 & 0 & & & & \\ -1 & 4 & -1 & 0 & & & \\ 0 & -1 & 4 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & 0 & -1 & 4 & -1 \\ & & & & 0 & -1 & 4 \end{bmatrix}, \quad \text{with } n_3 = 100;$$

we observe that the diagonal elements of the previous matrices are nonzero, i.e. $(A_p)_{ii} \neq 0$ for $i = 1, \ldots, n_p$, with $p = 1, 2$, and 3. Are the Jacobi and Gauss-Seidel methods convergent for any choice of the initial solution $\mathbf{x}^{(0)}$ for the matrices A_1, A_2, and A_3? Motivate the answer. (*Hint*: proceed by inspecting the matrices A_p without assembling the iteration matrices of the Jacobi and Gauss-Seidel methods, then, if not possible to establish a priori the convergence properties of the methods, compute the spectral radii of the iteration matrices; in all the cases, use the latter approach for verification. To extract a lower or upper triangular matrix from a given matrix, use the MATLAB functions `tril` and `triu`, respectively).

b) Write the MATLAB functions `jacobi.m` and `gauss_seidel.m` which implement the Jacobi and Gauss-Seidel methods for solving the generic linear system $A\mathbf{x} = \mathbf{b}$. The functions should read as inputs the matrix A, the vector $\mathbf{b}$, the initial solution $\mathbf{x}^{(0)}$, the maximum number of iterations allowed k_{max}, and the tolerance *tol* on the stopping criterion, which we assume to be based on the norm of the residual (i.e. we require that $r^{(k)} = \mathbf{r}^{(k)} = \left\|A\mathbf{x}^{(k)} - \mathbf{b}\right\| < tol$). The outputs should be the vector $\mathbf{x}^{(k)}$, the corresponding number of iterations k, and the norm of the residual $r^{(k)} = \left\|\mathbf{r}^{(k)}\right\|$. Use the functions `jacobi_template.m` and `gauss_seidel_template.m` as templates:

```
function [ x, k, res ] = jacobi( A, b, x0, tol, kmax )
% JACOBI solve the linear system A x = b by means of the
% Jacobi iterative method; diagonal elements of A must be nonzero.
% Stopping criterion based on the residual.
%  [ x, k, res ] = jacobi( A, b, x0, tol, kmax )
%  Inputs:  A    = matrix (square matrix)
%           b    = vector (right hand side of the linear system)
%           x0   = initial solution (colum vector)
%           tol  = tolerence for the stopping driterion based on residual
%           kmax = maximum number of iterations
%  Outputs: x    = solution vector (column vector)
%           k    = number of iterations at convergence
%           res  = value of the norm of the residual at convergence
%

return
```

```
function [ x, k, res ] = gauss_seidel( A, b, x0, tol, kmax )
% GAUSS_SEIDEL solve the linear system A x = b by means
% of the Gauss-Seidel iterative method; diagonal elements of A
% must be nonzero. Stopping criterion based on the residual.
%  [ x, k, res ] = gauss_seidel( A, b, x0, tol, kmax )
%  Inputs:  A    = matrix (square matrix)
%           b    = vector (right hand side of the linear system)
%           x0   = initial solution (colum vector)
%           tol  = tolerence for the stopping driterion based on residual
%           kmax = maximum number of iterations
%  Outputs: x    = solution vector (column vector)
%           k    = number of iterations at convergence
%           res  = value of the norm of the residual at convergence
%

return
```

c) If possible, solve the linear systems $A_p\mathbf{x}_p = \mathbf{b}_p$ associated to the matrices A_p, with $p = 1,2,3$, defined at point a) by means of the Jacobi and Gauss-Seidel methods implemented in the corresponding MATLAB functions at point b). Assume the exact solutions $\mathbf{x}_p = (1,1,\ldots,1)^T \in \mathbb{R}^{n_p}$. Report the norm of the errors ($e_p^{(k_p)} = \left\|\mathbf{e}_p^{(k_p)}\right\| = \left\|\mathbf{x}_p - \mathbf{x}_p^{(k_p)}\right\|$), the number of iterations k_p necessary to converge to the prescribed tolerance $tol = 10^{-6}$, and the norm of the residual $r_p^{(k_p)} = \left\|\mathbf{r}_p^{(k_p)}\right\|$; set `kmax` $= 1000$ and the initial solutions $\mathbf{x}_p^{(0)} = (0,0,\ldots,0)^T \in \mathbb{R}^{n_p}$ for all $p = 1,2,3$. Discuss the results obtained for the Jacobi and Gauss-Seidel methods in relation to point a).

d) Let us now consider the family of matrices depending on a parameter $\gamma \in \mathbb{R}$ such that:

$$A_4 = \begin{bmatrix} 8 & \gamma & -2 & -1 \\ -2 & 2 & -\gamma & -3 \\ -1 & -2 & 18 & -18 \\ -1 & -3 & -7 & 25 \end{bmatrix}, \quad \text{with } \gamma \in [-10, 25].$$

We remark that the matrix A_4 is non singular for $\gamma \in [-10, 25]$. Graphically study the convergence properties of the Jacobi and Gauss-Seidel methods for the solution of the linear systems associated to the matrix A_4 for $\gamma \in [-10, 25]$. Which of the two methods would you choose for $\gamma = 0$, 9, and 15? Motivate the answer. (*Hint*: compute the spectral radii associated to the iteration matrices of the Jacobi and Gauss-Seidel methods for a "sufficiently" large set of parameters $\gamma \in [-10, 25]$).