Numerical Analysis and Computational Mathematics

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Nonlinear equations: Newton method

Exercise I (MATLAB)

With the help of the file `newton_template.m`, implement the Newton method in a MATLAB function, say `newton` (saved inside the file `newton.m`). The layout of the function is the following:

```
function [xvect, resvect, nit] = newton( fun, dfun, x0, tol, nmax )
% NEWTON Find a zero of a nonlinear scalar function.
%   [XVECT] = NEWTON(FUN,DFUN,X0,TOL,NMAX) finds a zero of the differentiable
%   function FUN using the Newton method and returns a vector XVECT containing
%   the successive approximations of the zero (iterates). DFUN is the derivative of FUN.
%   FUN and DFUN accept real scalar input x and return a real scalar value;
%   FUN and DFUN can also be inline objects. X0 is the initial guess.
%   TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
%   The stopping criterion based on the difference of successive iterates is used.
%   If the search fails a warning message is displayed.
%
%   [XVECT,RESVECT,NIT] = NEWTON(FUN,DFUN,X0,TOL,NMAX) also returns the vector
%   RESVECT of residual evaluations for each iterate, and NIT the number of iterations.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

return
```

As a stopping criterion for the Newton method, check if the difference of successive iterates at the step n is smaller than the prescribed tolerance tol, i.e. $|x^{(n)} - x^{(n-1)}| < tol$, with a limit on the maximum number of iterations n_{max} ($n \leq n_{max}$).

a) Use the function `newton` to find the zero $\alpha = 0$ of the nonlinear function $f(x) = \sin(2x) + x$ in $(-1, 1)$ starting from $x^{(0)} = 0.7$ with the tolerance $tol = 10^{-5}$ and $n_{max} = 50$. How many iterations are required for the convergence of the Newton method, say n_c, and how large is the error $e^{(n_c)} = |x^{(n_c)} - \alpha|$?

b) What is the expected convergence order of the Newton method to the zero α for the function $f(x)$ of point a)? Motivate the answer based on the theoretical convergence results.

c) Plot in semi-logarithmic scale the errors $e^{(n)} = |x^{(n)} - \alpha|$ vs. the iteration number n, with $n = 0, \ldots, n_{(max)}$, for the function $f(x)$ given at point a); set $x^{(0)} = 0.7, n_{(max)} = 6$ and $tol = 10^{-12}$. By comparing the result with the plot of $b_n = 2^{-n}$ for $n = 0, \ldots, n_{max}$, what can we deduce about the convergence order of the Newton method applied to the function $f(x)$?

d) Modify the function `newton` to implement the stopping criterion based on the residual, i.e. $|r^{(n)}| = |f(x^{(n)})| < tol$ and save it in a different file `newton_residual.m`. Apply it to find the unique zero $\alpha = 0$ of the function $f(x) = \exp(\beta x) - 1$, with $\beta = 10^{-3}, 1, 10^3$. Set the initial value $x^{(0)} = 0.1$, the tolerance $tol = 10^{-7}$ and $n_{max} = 150$. Compare the values of the (absolute) residual $|r^{(n_c)}|$ and error $e^{(n_c)}$ at convergence for the different values of β. Is the stopping criterion based on the residual satisfactory for these values of β? Why?

Exercise II (MATLAB)

Consider the Newton method for finding the zero $\alpha = 0$ of the function $f(x) = (\sin(x))^m$ in the interval $(-\pi/2, \pi/2)$ starting from the initial value $x^{(0)} = \pi/6$ for different $m = 1, 2, 3, \ldots$.

a) What are the expected convergence orders of the Newton method to the zero α for the function $f(x)$ with $m = 1, 2$, and 3? Motivate the answer based on the theoretical convergence results.

b) Use the function `newton` to find the zero α for $m = 1, 2$, and 3. Set $tol = 10^{-8}$ and use the stopping criterion based on the difference of successive iterates and maximum number of iteration $n_{max} = 50$. How many iterations are required to converge for the prescribed tolerance?

c) Similarly to Exercise 1 point c), plot the errors obtained for $m = 1, 2$, and $m = 3$ vs. the iteration number $n = 0, \ldots, n_{max}$ (set $n_{max} = 5$). Motivate the results obtained in relation to point a).

d) How should the Newton method be modified if the function $f(x)$ has a zero α of multiplicity $m > 1$, i.e. such that (for $f(x) \in C^m(I_\alpha)$, with I_α a neighborhood of α):

$$f(\alpha) = f'(\alpha) = \ldots = f^{m-1}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0?$$

Implement the modified Newton method in a function `newton_modified` taking as additional input the multiplicity m of the zero α.

e) What are the expected convergence orders of the (properly) modified Newton method applied to the function $f(x)$ in the cases $m = 1, 2$, and 3? Repeat the point c) by using the function `newton_modified` and comment the results obtained.

Exercise III (MATLAB)

Consider the polynomial $p(x) = -3x^3/8 + 5x^2/4 + x/2 - 1$ of order 3.

a) Plot the polynomial $p(x)$ in the interval $x \in (-1, 3)$. Plot in the same figure the tangent lines to the curve $(x, p(x))$ at $x = 0$ and $x = 2$ (*hint*: the tangent line at the point $(x_0, p(x_0))$ is given by $y(x) = p(x_0) + p'(x_0)(x - x_0)$).

b) Following point a), use the Newton method to find the zero $\alpha \in (0, 2)$ starting from the initial value $x^{(0)} = 10^{-3}$. Set $tol = 10^{-8}$ and $n_{max} = 20$. Does the method converge? To which value? In how many iterations?

c) Repeat the point b) by setting $x^{(0)} = 10^{-3}$ and $x^{(0)} = 0$ (*hint*: check the value of the iterates in `xvect`). Motivate the results with the help of the plot obtained at point a).

Exercise IV (MATLAB)

With the help of the file `newtonsys_template.m`, complete the implementation of the Newton method for systems of nonlinear equations in a MATLAB function `newtonsys` (saved inside the file `newtonsys.m`). The layout of the function is the following:

```
function [x, res, nit] = newtonsys( F, J, x0, tol, nmax )
% NEWTONSYS Find the zeros of a system of nonlinear equations.
%   [X] = NEWTONSYS(F,J,X0,TOL,NMAX) find the zero X of the
%   continuous and differentiable system of functions F nearest to X0 using the
%   Newton method. J is a function which takes X and returns the Jacobian matrix.
%   X0 is a column vector; F returns a column vector and J a square matrix.
%   The stopping criterion is based on the difference (norm) of successive
%   iterates.
%   If the search fails a warning message is displayed.
%
%   [X,RES,NITER] = NEWTONSYS(F,J,X0,TOL,NMAX) returns the value of the
%   residual RES in X and the number of iterations NITER required for computing X.
%   Note: only the final iterate is stored in X; similarly for RES.
%

return
```

As stopping criterion for the Newton method, consider the test on the increment of successive iterates $\mathbf{x}^{(n)}$ at the iteration step n, i.e. $\left\|\mathbf{x}^{(n)} - \mathbf{x}^{(n-1)}\right\|_2 < tol$ for a prescribed tolerance tol, with a limit on the maximum number of iterations n_{max} ($n < n_{max}$) (*hint*: use the MATLAB commands `norm` to compute the norm of a vector and `\` to solve systems of linear equations).
Use the function `newtonsys` to find the zero $\boldsymbol{\alpha} \in \mathbb{R}^d$ of the system of nonlinear equations $\boldsymbol{F}(\mathbf{x}) = \mathbf{0}$, with $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$, $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$, where $d = 2$ and:

$$\boldsymbol{F}(\mathbf{x}) = \begin{bmatrix} e^{x_1^2 + x_2^2} - 1 \\ e^{x_1^2 - x_2^2} - 1 \end{bmatrix}.$$

Find the zero $\boldsymbol{\alpha} = (0, 0)^T$ by setting $tol = 10^{-5}$ and $n_{max} = 100$ for the choices of the initial datum $\mathbf{x}^{(0)} = (1.5, -2)^T$ and $\mathbf{x}^{(0)} = (4, 4)^T$. Report the number of iterations required for the convergence of the method and discuss the choice of $\mathbf{x}^{(0)}$.