EPFL

Numerical Analysis and Computational Mathematics

Fall Semester 2019 - Section CSE

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Nonlinear equations: Bisection and Newton methods

Exercise I (MATLAB)

a) Write a MATLAB function implementing the bisection method (in a file called `bisection.m`) by following the layout found in the file `bisection_template.m`. The layout of the function reads:

```
function [xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax)
% BISECTION Find a zero of a nonlinear scalar function inside an interval.
%   XVECT=BISECTION(FUN,A,B,TOL,NMAX) finds a zero of the continuous
%   function FUN in the interval [A,B] using the bisection method and returns
%   a vector XVECT containing the successive approximations of the zero (iterates).
%   FUN accepts real scalar input x and returns a real scalar value;
%   FUN can also be an inline object.
%   TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
%   If the search fails an error message is displayed.
%
%   [XVECT,ESTERRVECT,RESVECT,NIT]=BISECTION(FUN,...) also returns the vector
%   ESTERRVECT of error estimators for each iterate, the vector RESVECT of residual
%   evaluations for each iterate, and NIT the number of iterations.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

return
```

where `fun` is an inline function defined outside `bisection.m`. Note that inside `bisection.m` function, it is possible to evaluate `fun` at a point `x` using the command `fx = fun(x)`. (*Hint*: follow the pseudocodes in Algorithm 2.1 and Algorithm 2.2 in the notes).

Verify the implementation of the bisection method by approximating the zero $\alpha \in (a, b)$ of the function $f(x) = \sin(2x) - 1 + x$, for $a = -1$ and $b = 3$, with a graphical visualization; set the tolerance for the stopping criterion $tol = 10^{-1}$ and the maximum number of iterations $n_{max} = 100$. As an example, write a MATLAB script `bisection_example.m`, with the following MATLAB commands:


```
fun = @(x) sin(2*x) - 1 + x; % inline function
a = -1; b = 3;
xv = linspace(a,b,2001); % xv=[a:(b-a)/2000:b]
plot(xv, fun(xv)); grid on; % plot of the function f
tol = 1e-4; nmax = 100;
[xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax);
x_nit = xvect(end); % final iterate (approximated zero); NOTE: length(xvect)=nit+1
fx_nit = resvect(end); % final residual fun(x nit); NOTE: length(resvect)=nit+1
hold on
plot(xvect,resvect,'*k',x_nit,fx_nit,'or'); % iterates in black, the final in red
```

b) Repeat for values of $tol = 10^{-2}, 10^{-3}$ and 10^{-4} to verify that decreasing tolerance results in a convergent sequence of approximated solutions (*hint*: use the magnifying glass tool to zoom in close to the zero).

c) **(Optional)** Improve the function `bisection.m` to make it more robust. If $f(a)$ and $f(b)$ have the same sign, an error message should be displayed. If one of the endpoints of the interval (a, b) given in the input is a zero of the function, it should be returned as the solution without performing any iteration.

Exercise II (Theoretical and MATLAB)

Consider the problem to find the zero $\alpha \in (0, 2)$ of the function $f(x) = (1 - x)\sin(4x) + 1/6$.

a) Plot the function $f(x)$ in MATLAB for $x \in [0, 2]$. Does it satisfy the conditions required to apply the bisection method ? Motivate the answer.

b) Estimate the minimum number of iterations n_{min} required by the bisection method to reach an approximation $x^{(n_{min})}$ of the zero α yielding the error $e^{(n_{min})} := |x^{(n_{min})} - \alpha| < \epsilon$, with $\epsilon = 10^{-6}$.

c) Following point b), use the `bisection.m` function to find the first 20 iterates of the bisection method applied to the function $f(x)$ for $x \in [0, 2]$. What are the residuals $r^{(n)} := f(x^{(n)})$ corresponding to the iterates $n = 19$ and $n = 20$? (*Hint*: $n \geq 0$.)

d) We say that the residuals $r^{(n)} := f(x^{(n)})$ converge monotonically with n if $|r^{(n)}| \geq |r^{(n+1)}|$ for all $n \geq 0$. In general, this property is desirable for an iterative method because the solution after taking another iteration can never be worse than the previous one. Is the convergence of the residuals monotonic for the function $f(x)$ considered here? (*Hint*: following point c), plot in semilogarithmic scale the absolute residuals $|r^{(n)}|$ vs. n for $n \geq 0$.)

e) An iterative method converges *linearly* (convergence of order 1) if:

$$\lim_{n\to\infty} \frac{|x^{(n+1)} - \alpha|}{|x^{(n)} - \alpha|} = \mu, \qquad \text{for some } 0 < \mu < 1, \tag{1}$$

where μ is the asymptotic convergence factor. Let us assume that after 20 iterations the method has found a solution that is very close to the zero so that $\alpha \simeq x^{(20)}$. Plot the sequence $a_n := |x^{(n+1)} - x^{(20)}|/|x^{(n)} - x^{(20)}|$ based on the 18 first iterates $x^{(n)}$ for the function above. Is it possible to verify the order of convergence from this graph by using definition (1)?

f) Consider instead the sequence of error estimators $\tilde{e}^{(n)} := |I^{(n+1)}| = (b-a)/2^{n+1}$ (for $0 \leq n \leq 18$). Does it converge linearly, i.e.:

$$\lim_{n\to\infty} \frac{|\tilde{e}^{(n+1)}|}{|\tilde{e}^{(n)}|} = \nu, \qquad \text{for some } 0 < \nu < 1?$$

What does this tell you about the convergence of the method? (*Hint*: plot the sequence $\tilde{a}_n := \tilde{e}^{(n+1)}/\tilde{e}^{(n)}$ vs. n for $0 \leq n \leq 18$.)

Exercise III (Theoretical and MATLAB)

Let us consider the function $f(x) = x^3 - 2x - 5$ in the interval $(1, 3)$. Since $f(x)$ is a polynomial, it is continuous and continuously differentiable. We can observe that:

$$f(1) = -6 < 0 \quad \text{and} \quad f(3) = 16 > 0,$$

thus, it has at least one zero $\alpha \in (1, 3)$.

a) Verify the uniqueness of the zero $\alpha \in (1, 3)$.

b) Write the Newton scheme for the function $f(x)$.

c) Compute the first three Newton iterations starting from the initial guess $x^{(0)} = 1.5$.