Numerical Analysis and Computational Mathematics

Fall Semester 2019 - Section CSE

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Solutions - Nonlinear equations: Bisection and Newton methods

Solution I (MATLAB)

a) We consider the following implementation of the MATLAB function `bisection.m`:

Listing 1: Possible implementation of the bisection method.

```
function [xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax)
% BISECTION Find a zero of a nonlinear scalar function inside an interval.
%   XVECT=BISECTION(FUN,A,B,TOL,NMAX) finds a zero of the continuous
%   function FUN in the interval [A,B] using the bisection method and returns
%   a vector XVECT containing the successive approximations of the zero (iterates).
%   FUN accepts real scalar input x and returns a real scalar value;
%   FUN can also be an inline object.
%   TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
%   If the search fails an error message is displayed.
%
%   [XVECT,ESTERRVECT,RESVECT,NIT]=BISECTION(FUN,...) also returns the vector
%   ESTERRVECT of error estimators for each iterate, the vector RESVECT of residual
%   evaluations for each iterate, and NIT the number of iterations.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

if a ≥ b
   error(' b must be greater than a (b > a)');
end

% evaluate f at the endpoints
fa = fun(a);
fb = fun(b);
if sign(fa) * sign(fb) > 0
   error(' The sign of FUN at the extrema of the interval must be different');
end

if fa == 0 % a is the solution
```

```
    xvect = a; fx = 0; esterr = 0; nit  = 0;
    resvect = fx; esterrvect = esterr;
    return
elseif fb == 0 % b is the solution
    xvect = b; fx = 0; esterr = 0; nit  = 0;
    resvect = fx; esterrvect = esterr;
    return
end

nit = 0;
xvect = []; resvect = []; esterrvect = [];

% initial approximate solution
x = (a + b) / 2;
% initial error estimator is the half of the length of the interval
esterr  = (b - a) / 2;
fx = fun(x);
xvect = x;
resvect = fx;
esterrvect = esterr;

% loop until convergence or maximum number of iterations reached
while esterr ≥ tol && nit < nmax

    if fx == 0 % we found the solution
      return;
    end
    if sign(fx) * sign(fa) < 0     % alpha is in (a,x)
      b = x;
    elseif sign(fx) * sign(fb) < 0 % alpha is in (x,b)
      a = x;
    else
      error('Algorithm not operating correctly');
    end
    % calculate mid-point of updated interval
    x = (a + b) / 2;
    % the error estimator is now half of the previous one
    esterr = esterr / 2;
    fx = fun(x);
    xvect = [xvect, x];
    resvect = [resvect, fx];
    esterrvect = [esterrvect, esterr];
    nit = nit + 1;

end

if esterrvect(end) > tol
    warning(['bisection stopped without converging to the desired tolerance ',...
             'because the maximum number of iterations was reached']);
end

return
```

We solve the nonlinear equation $f(x)$ by considering the following commands, for example taking $tol = 10^{-1}$ as stopping criterion:

```
fun = @(x) sin(2*x) - 1 + x;
a = -1; b = 3; tol = 1e-4; nmax = 100;
[xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax);
x_nit = xvect( nit+1 ) % last iterate (approximated zero)
nit
% x_nit =
%
%     0.3125
%
% nit =
%
%     5
```

We observe that the bisection method converges in $n = 5$ iterations to the approximated zero $x^{(5)} = 0.3125$.

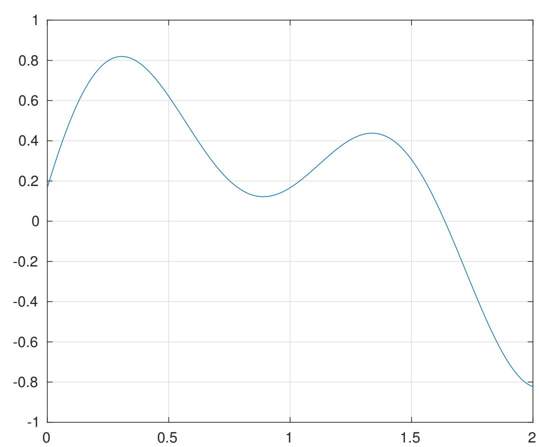
b) If we decrease the tolerance to the suggested values ($tol = [10^{-2}, 10^{-3}, 10^{-4}]$), we obtain the following approximated solutions $[x^{(8)} = 0.3516, x^{(11)} = 0.3525, x^{(15)} = 0.3522]$, where the numbers in bracket represent again the iterations needed by the algorithm to converge.

c) Note that in Listing 1 we made the algorithm more robust by adding several checks on the input arguments and on some corner cases, and by displaying warning/error messages that can help the user if something goes wrong.

Solution II (Theoretical and MATLAB)

a) We consider the following MATLAB commands to plot the function $f(x)$:

```
fun = @(x) (1-x) .* sin(4*x) + 1/6;
a = 0; b = 2;
xv = linspace(a,b,1001); % xv=[a:(b-a)/1000:b]
plot( xv, fun(xv) ); grid on
```



The conditions that need to be satisfied in order to apply the bisection method are the continuity of $f(x)$ in $[a,b]$ and $f(a)f(b) < 0$ (i.e. f must change its sign in (a,b)). In our example, these conditions are satisfied since $f(x) \in C^\infty([0,2])$ and $f(0) = 1/6 > 0$ and $f(2) = -\sin(8) + 1/6 < 0$. The conditions imply that there exists at least one zero in the interval $(0,2)$; by referring to the previous plot, we deduce that the zero $\alpha \in (0,2)$ is also unique.

b) The error estimator (error bound) at the nth iteration of the bisection method is $\tilde{e}^{(n)} = (b-a)/2^{n+1}$, for which:

$$e^{(n)} = |x^{(n)} - \alpha| \leq \tilde{e}^{(n)} = \frac{b-a}{2^{n+1}}, \quad n = 0, 1, \dots, \infty.$$

Thus the stopping criterion based on the error estimator implies that n needs to be at least big enough such that $\tilde{e}^{(n)} < \epsilon$. Taking the logarithm on both sides is permitted since both sides are strictly positive; therefore:

$$n_{min} > \log_2\left(\frac{b-a}{\epsilon}\right) - 1 = \frac{\log(b-a) - \log(\epsilon)}{\log 2} - 1.$$

By inserting the values in the above relation, we obtain the condition $n_{min} > 19.9316$, so $n_{min} = 20$ iterations will suffice.

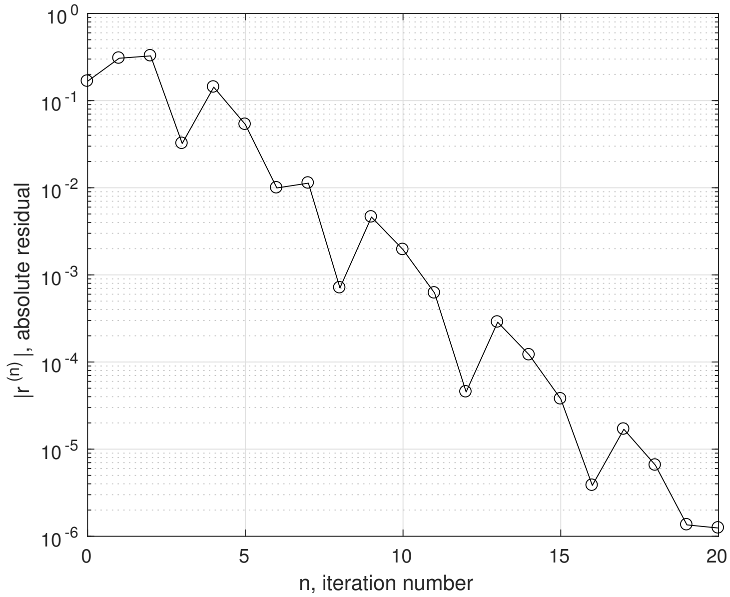
c) We consider the following MATLAB commands:

```
tol = 1e-6; nmax = 100;
[xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax);
% NOTE: the first entry of the vector xvect contains the initial guess of
% the zero obtained for n=0, i.e. the mid point of the interval [a,b]
resvect( 19 + 1 )
resvect( 20 + 1 )
% ans =
%
%   -1.3514e-06
%
% ans =
%
%    1.2434e-06
```

Therefore, we have $r^{(19)} = -1.3514 \cdot 10^{-6}$ and $r^{(20)} = 1.2434 \cdot 10^{-6}$.

d) The residuals of the bisection method are not monotonically converging, neither are the errors. We verify this property for the function $f(x)$ by plotting the absolute residuals $|r^{(n)}|$ vs. n with the following MATLAB commands (in continuation to point c)):

```
tol = 1e-6; nmax = 100;
[xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax);
% NOTE: the first entry of the vector xvect contains the initial guess of
% the zero obtained for n=0, i.e. the mid point of the interval [a,b]
resvect( 19 + 1 )
resvect( 20 + 1 )
nvect = 0 : nit;
resvect_abs = abs( resvect );
semilogy( nvect, resvect_abs, '-ok' ); grid on
```



From the plot (in semilogarithmic scale for the sake of clarity), we conclude that the convergence of the residuals is not monotonic.

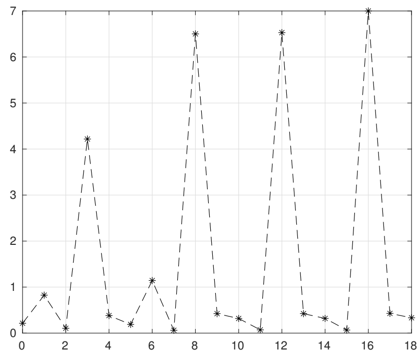
e) We consider the following MATLAB commands to obtain the plot representing the sequence a_n vs. the number of iterations n.

```
alpha = xvect( 20 + 1 ); % approximation of exact zero (alpha)
nv = [ 0 : 18 ];
nv_ind = nv + 1; % Matlab indexes start from 1
err_n_plus_1 = abs( xvect( nv_ind + 1 ) - alpha );
err_n = abs( xvect( nv_ind ) - alpha );
a_n = err_n_plus_1 ./ err_n;
plot( nv, a_n, '--k*' ); grid on
```

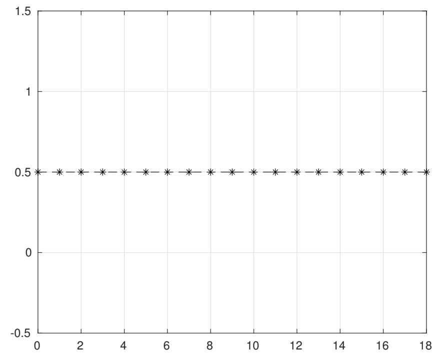
As with the residuals, the convergence of the error $e^{(n)} = |x^{(n)} - \alpha|$ is not monotonic in n in this case (see also point f), Subfigure (a)); therefore, a coefficient μ representing the asymptotic convergence factor cannot be deduced from the sequence a_n vs. n and we cannot infer that the convergence is linear. However, note that this does not mean that the method does not converge.

f) Since $\tilde{e}^{(n)} = (b-a)/2^{n+1}$ and $\tilde{e}^{(n+1)} = (b-a)/2^{n+2}$, it follows that $\tilde{a}_n := \tilde{e}^{(n+1)}/\tilde{e}^{(n)} = \nu = 1/2$ for all $n \geq 0$ and the sequence of the error estimators (bounds) $\left\{\tilde{e}^{(n)}\right\}_{n=1}^{\infty}$ converges linearly by definition (the order of convergence is 1). We obtain the plot of the sequence $\tilde{a}_n$ vs. n in Subfigure (b) by considering the following commands, from which we graphically verify that $\nu = 1/2 < 1$:

```
esterr_n_plus_1 = ( b - a ) ./ 2.^( nv + 1 );
esterr_n = ( b - a ) ./ 2.^( nv );
a_tilde_n = esterr_n_plus_1 ./ esterr_n;
plot( nv, a_tilde_n, '--k*' ); grid on
```



(a) a_n vs. n



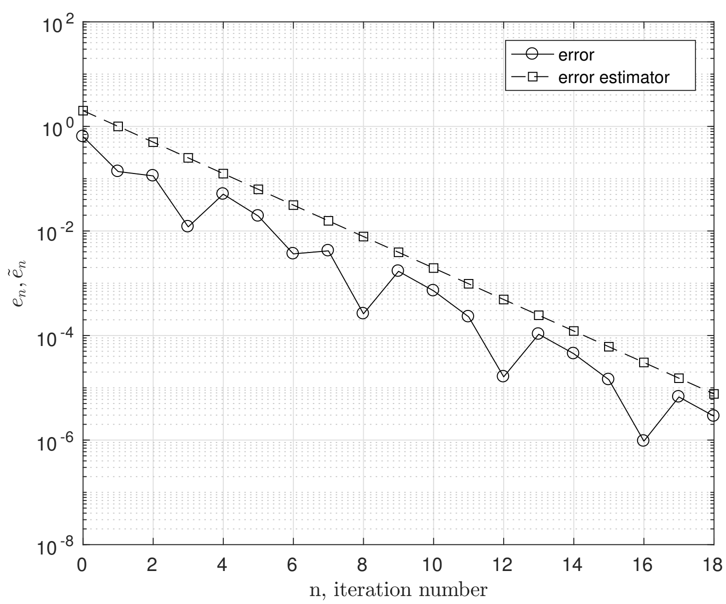
(b) $\tilde{a}_n$ vs. n

Even if we cannot establish the convergence order of the bisection method according to the definition (1) (i.e. for the error $e^{(n)} = |x^{(n)} - \alpha|$), we observe that the error estimators (bounds) $\left\{\tilde{e}^{(n)}\right\}_{n=1}^{\infty}$ represent a dominating sequence of the errors $\left\{e^{(n)}\right\}_{n=1}^{\infty}$, i.e.:

$$e^{(n)} = |x^{(n)} - \alpha| \leq \tilde{e}^{(n)}, \quad n = 0, 1, \ldots, \infty.$$

In this case, the behavior of the error resembles the one of the estimator, which is convergent of order 1. We verify this by plotting the errors $e^{(n)}$ and error estimators $\tilde{e}^{(n)}$ vs. n with the following commands:

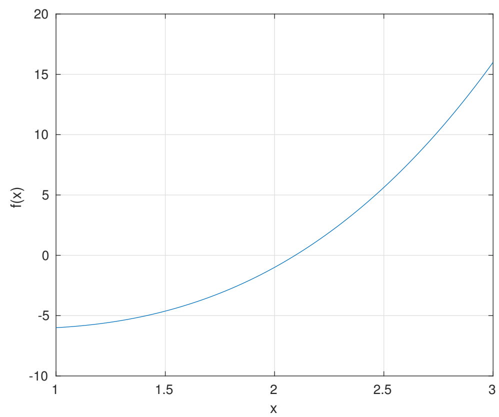
```
semilogy( nv, err_n, '-ko', nv, esterr_n, '--ks' ); grid on
```



We remark again that the method does not have order 1 because the error is not monotonically decreasing.

Solution III (Theoretical and MATLAB)

a) Since $f(x) \in C^0([1,3])$ and $f(1)f(3) < 0$, then there exists at least one zero $\alpha \in (1,3)$. By studying the function $f(x)$ in $(1,3)$ we deduce that the zero α is also unique since $f(1) < 0$ and the function is strictly increasing; indeed, the first derivative of $f(x)$ is strictly positive in the interval, being $f'(x) = 3x^2 - 2 > 0$ for all $x \in [1,3]$.



b) The Newton method reads:

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})} = x^{(n)} - \frac{(x^{(n)})^3 - 2x^{(n)} - 5}{3(x^{(n)})^2 - 2} \quad \text{for } n = 0, 1, 2, \dots,$$

until a stopping criterion is satisfied, provided that $f'(x^{(n)}) \neq 0$ for all $n = 0, 1, 2, \dots$.

c) We use MATLAB to determine the first three iterates; we consider the following commands:

```
fun = @(x) x.^3 - 2*x - 5;
dfun = @(x) 3*x.^2 - 2;
newton_iterate = @(xn) xn - fun( xn ) / dfun( xn );
x0 = 1.5;
x1 = newton_iterate( x0 )
x2 = newton_iterate( x1 )
x3 = newton_iterate( x2 )
% x1 =
%
%     2.4737
%
% x2 =
%
%     2.1564
%
% x3 =
%
%     2.0966
```

We deduce that the first three iterates, starting from $x^{(0)} = 1.5$, are $x^{(1)} = 2.4737$, $x^{(2)} = 2.1564$ and $x^{(3)} = 2.0966$. Note that $\alpha \simeq 2.0946$.