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RESEARCH NOTE

Summation of the Born series for the normal modes of the Earth

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SUMMARY

We derive an explicit formula for the response of a laterally heterogeneous, self-gravitating, rotating, dissipative, and physically dispersive earth to an earthquake. The long-period spectrum at a single station is given by $u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}]^{-1} \cdot \mathbf{S}_0(\omega)$ where $\mathbf{R}_0$ and $\mathbf{S}_0(\omega)$ are the unperturbed source and receiver vectors and H denotes the Hermitian transpose. The quantities $\mathbf{V}(\omega)$, $\mathbf{W}$, and $\mathbf{T}$ are the potential energy, Coriolis, and relative kinetic energy matrices, and $\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}$ is the Lagrangian matrix whose eigenvalues are the perturbed complex eigenfrequencies. This simple result is obtained by formal summation of the Born series.

Key words: Born approximation, free oscillations, lateral heterogeneity, normal modes, synthetic seismograms.

1 INTRODUCTION

To calculate accurate synthetic long-period seismograms, it is necessary to account for the splitting and coupling of Earth's normal mode multiplets caused by rotation, deviations from spherical symmetry, and dissipation. A common approach to this problem has been to approximate the response by the first term in the Born series (Woodhouse 1983; Tanimoto 1984; Snieder & Romanowicz 1988; Romanowicz & Snieder 1988). However, this approximation is only valid for a relatively short time after an earthquake due to the presence of secular terms. So far there have been no attempts to compute higher order corrections to the first Born approximation. In this paper we show how to determine the complete response, taking all the higher order scattering interactions into account, by formal summation of the Born series. The final result is exact except for a linearized treatment of boundary topography perturbations.

2 SUMMATION OF THE BORN SERIES

We model the Earth as a self-gravitating, dissipative and physically dispersive continuum occupying a volume V with surface ∂V. Let $\boldsymbol{\Omega}$ denote the uniform angular velocity of rotation of this body about its centre of mass O, and let $\rho(\mathbf{r})$ denote the distribution of density within the body. We wish to determine the response of this earth model to a point source earthquake situated at $\mathbf{r}_q$. Let $\mathbf{M}(\omega)$ denote the Fourier transform of the moment tensor of this earthquake, and let $\mathbf{u}(\mathbf{r}, \omega)$ denote the Fourier transform of the resulting displacement field in the rotating frame. Our convention is that $\exp(-i\omega t)$ occurs in the Fourier integral when transforming to the frequency domain. To determine $\mathbf{u}(\mathbf{r}, \omega)$ we must solve the linearized equations (Dahlen 1972, 1977)

$$\mathscr{L}(\mathbf{r}, \omega)\mathbf{u}(\mathbf{r}, \omega) - \omega^2\rho(\mathbf{r})\mathbf{u}(\mathbf{r}, \omega) + 2i\omega\rho(\mathbf{r})\boldsymbol{\Omega} \times \mathbf{u}(\mathbf{r}, \omega)$$
$$= -\mathbf{M}(\omega) \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_q), \qquad \mathbf{r} \in V, \tag{2.1a}$$
$$\mathscr{B}(\mathbf{r}, \omega)\mathbf{u}(\mathbf{r}, \omega) = \hat{\mathbf{n}}(\mathbf{r}) \cdot [\boldsymbol{\Lambda}(\mathbf{r}, \omega) : \boldsymbol{\nabla}\mathbf{u}(\mathbf{r}, \omega)] = 0, \qquad \mathbf{r} \in \partial V. \tag{2.1b}$$

The quantity $\hat{\mathbf{n}}(\mathbf{r})$ is the unit outward normal on ∂V, and $\boldsymbol{\Lambda}(\mathbf{r}, \omega)$ is the fourth-order tensor that relates the incremental Piola–Kirchhoff stress to the displacement gradient. We account for dissipation and physical dispersion by allowing this 'elastic' tensor to be complex and frequency dependent; in order that the time domain response be real and causal, $\boldsymbol{\Lambda}(\mathbf{r}, \omega)$ must be analytic in the upper half of the complex ω plane, and it must satisfy $\boldsymbol{\Lambda}^*(\mathbf{r}, \omega) = \boldsymbol{\Lambda}(\mathbf{r}, \omega^*)$, where the asterisk denotes complex conjugation (Nowick & Berry 1972; O'Connell & Budiansky 1978). Both the elastic-gravitational restoring forces and the internal frictional forces are embodied in the integro-differential operator $\mathscr{L}(\mathbf{r}, \omega)$, given by

$$\mathscr{L}(\mathbf{r}, \omega)\mathbf{u}(\mathbf{r}, \omega) = \rho(\mathbf{r}) \int_V \rho(\mathbf{r}')\mathbf{g}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{u}(\mathbf{r}', \omega)\, d^3\mathbf{r}' + \rho(\mathbf{r})\mathbf{u}(\mathbf{r}, \omega) \cdot \nabla\nabla[\Phi(\mathbf{r}) + \Psi(\mathbf{r})] - \nabla \cdot [\mathbf{\Lambda}(\mathbf{r}, \omega) : \nabla\mathbf{u}(\mathbf{r}, \omega)], \qquad \mathbf{r} \in V. \tag{2.2}$$

Here

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{2.3}$$

is the initial gravitational potential and

$$\Psi(\mathbf{r}) = -\tfrac{1}{2}[\Omega^2 r^2 - (\mathbf{\Omega} \cdot \mathbf{r})^2] \tag{2.4}$$

is the centripetal potential. The self-gravitational kernel $\mathbf{g}(\mathbf{r}-\mathbf{r}')$ is given by

$$\mathbf{g}(\mathbf{r}-\mathbf{r}') = G\left[\frac{3(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{\mathbf{I}_3}{|\mathbf{r}-\mathbf{r}'|^3}\right], \tag{2.5}$$

where $\mathbf{I}_3$ is the 3×3 identity tensor.

Equations (2.1) are a non-Hermitian boundary value problem due to the complex nature of $\mathbf{\Lambda}(\mathbf{r}, \omega)$. Even if $\rho(\mathbf{r})$ and $\mathbf{\Lambda}(\mathbf{r}, \omega)$ were spherically symmetric, it would be difficult to solve this problem directly due to the presence of rotation and dissipation. But let us suppose we know how to solve for the displacement response $\mathbf{u}_0(\mathbf{r}, \omega)$ of a non-rotating, non-dissipative body occupying the same volume V as the rotating, dissipative body. This unperturbed earth model is characterized by a density distribution $\rho_0(\mathbf{r})$ and a real frequency-independent elastic tensor $\mathbf{\Lambda}_0(\mathbf{r})$. In practice, the unperturbed model will usually be spherical and isotropic, but is is unnecessary to assume that in what follows. The unperturbed displacement field is the solution to the Hermitian boundary value problem

$$\mathscr{L}_0(\mathbf{r})\mathbf{u}_0(\mathbf{r}, \omega) - \omega^2\rho_0(\mathbf{r})\mathbf{u}_0(\mathbf{r}, \omega) = -\mathbf{M}(\omega) \cdot \nabla\delta(\mathbf{r}-\mathbf{r}_q), \quad \mathbf{r} \in V, \tag{2.6a}$$

$$\mathscr{B}_0(\mathbf{r})\mathbf{u}_0(\mathbf{r}, \omega) = \hat{\mathbf{n}}(\mathbf{r}) \cdot [\mathbf{\Lambda}_0(\mathbf{r}) : \nabla\mathbf{u}_0(\mathbf{r}, \omega)] = 0, \qquad \mathbf{r} \in \partial V, \tag{2.6b}$$

where

$$\mathscr{L}_0(\mathbf{r})\mathbf{u}_0(\mathbf{r}, \omega) = \rho_0(\mathbf{r}) \int_V \rho_0(\mathbf{r}')\mathbf{g}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{u}_0(\mathbf{r}', \omega)\, d^3\mathbf{r}' + \rho_0(\mathbf{r})\mathbf{u}_0(\mathbf{r}, \omega) \cdot \nabla\nabla\Phi_0(\mathbf{r}) - \nabla \cdot [\mathbf{\Lambda}_0(\mathbf{r}) : \nabla\mathbf{u}_0(\mathbf{r}, \omega)], \qquad \mathbf{r} \in V. \tag{2.7}$$

The quantity

$$\Phi_0(\mathbf{r}) = -G \int_V \frac{\rho_0(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{2.8}$$

is the initial gravitational potential in the unperturbed earth model. The self-gravitational effect of any displaced density discontinuities is fully accounted for in equations (2.2) and (2.7); however, to avoid consideration of both stress continuity conditions and boundary perturbations, we shall temporarily assume that none of the physical properties $\rho(\mathbf{r})$, $\mathbf{\Lambda}(\mathbf{r}, \omega)$, $\rho_0(\mathbf{r})$ and $\mathbf{\Lambda}_0(\mathbf{r})$ have internal jump discontinuities.

We denote the real eigenfrequencies of the unperturbed earth model by $\pm\omega_{0k}$ and the associated eigenfunctions by $\mathbf{s}_{0k}(\mathbf{r})$. The response $\mathbf{u}_0(\mathbf{r}, \omega)$ can be written in terms of these unperturbed eigenfrequencies and eigenfunctions in the form (Gilbert 1970)

$$\mathbf{u}_0(\mathbf{r}, \omega) = \sum_k (\omega_{0k}^2 - \omega^2)^{-1}[\mathbf{M}(\omega) : \nabla\mathbf{s}_{0k}^*(\mathbf{r}_q)]\mathbf{s}_{0k}(\mathbf{r}). \tag{2.9}$$

The orthonormalization convention in (2.9) and elsewhere in this paper is

$$\int_V \rho_0(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r}) \cdot \mathbf{s}_{0l}(\mathbf{r})\, d^3\mathbf{r} = \delta_{kl}, \tag{2.10}$$

where δ_{kl} is the Kronecker delta symbol. The time-domain response corresponding to (2.9) is

$$\mathbf{u}_0(\mathbf{r}, t) = \sum_k \omega_{0k}^{-1}\mathbf{s}_{0k}(\mathbf{r}) \int_0^t [\mathbf{M}(\tau) : \nabla\mathbf{s}_{0k}^*(\mathbf{r}_q)] \sin\omega_{0k}(t-\tau)\, d\tau, \qquad t \geq 0. \tag{2.11}$$

Let us turn our attention to the perturbed problem again. We define the quantities $\mathbf{u}_S(\mathbf{r}, \omega)$, $\rho_S(\mathbf{r})$, $\mathbf{\Lambda}_S(\mathbf{r}, \omega)$ and $\Phi_S(\mathbf{r})$ by

$$\mathbf{u}(\mathbf{r}, \omega) = \mathbf{u}_0(\mathbf{r}, \omega) + \mathbf{u}_S(\mathbf{r}, \omega), \tag{2.12a}$$

$$\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + \rho_S(\mathbf{r}), \tag{2.12b}$$

$$\mathbf{\Lambda}(\mathbf{r}, \omega) = \mathbf{\Lambda}_0(\mathbf{r}) + \mathbf{\Lambda}_S(\mathbf{r}, \omega), \tag{2.12c}$$

$$\Phi(\mathbf{r}) = \Phi_0(\mathbf{r}) + \Phi_S(\mathbf{r}), \tag{2.12d}$$

so that

$$\Phi_S(\mathbf{r}) = -G \int_V \frac{\rho_S(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'. \tag{2.13}$$

The perturbations $\rho_S(\mathbf{r})$ and $\mathbf{\Lambda}_S(\mathbf{r}, \omega)$ can be regarded as a *continuous* distribution of scatterers embedded in the volume V, and $\mathbf{u}_S(\mathbf{r}, \omega)$ is the perturbation to the displacement field $\mathbf{u}_0(\mathbf{r}, \omega)$ due to the rotation $\mathbf{\Omega}$ and the presence of these scatterers. Upon inserting the decomposition (2.12) into (2.1) we find that the scattered field satisfies

$$\mathscr{L}_0(\mathbf{r})\mathbf{u}_S(\mathbf{r}, \omega) - \omega^2\rho_0(\mathbf{r})\mathbf{u}_S(\mathbf{r}, \omega) = \gamma_V(\mathbf{r}, \omega), \qquad \mathbf{r} \in V, \tag{2.14a}$$

$$\mathscr{B}_0(\mathbf{r})\mathbf{u}_S(\mathbf{r}, \omega) = \gamma_{\partial V}(\mathbf{r}, \omega), \qquad \mathbf{r} \in \partial V, \tag{2.14b}$$

where

$$\gamma_V(\mathbf{r}, \omega) = -[\mathscr{L}(\mathbf{r}, \omega) - \mathscr{L}_0(\mathbf{r})]\mathbf{u}(\mathbf{r}, \omega) + \omega^2\rho_S(\mathbf{r})\mathbf{u}(\mathbf{r}, \omega) - 2i\omega\rho(\mathbf{r})\mathbf{\Omega} \times \mathbf{u}(\mathbf{r}, \omega), \qquad \mathbf{r} \in V, \tag{2.15a}$$

$$\gamma_{\partial V}(\mathbf{r}, \omega) = -[\mathscr{B}(\mathbf{r}, \omega) - \mathscr{B}_0(\mathbf{r})]\mathbf{u}(\mathbf{r}, \omega) = -\hat{\mathbf{n}}(\mathbf{r}) \cdot [\mathbf{\Lambda}_S(\mathbf{r}, \omega) : \nabla\mathbf{u}(\mathbf{r}, \omega)], \qquad \mathbf{r} \in \partial V. \tag{2.15b}$$

The solution to the boundary value problem (2.14) for arbitrary $\gamma_V(\mathbf{r}, \omega)$ and $\gamma_{\partial V}(\mathbf{r}, \omega)$ is

$$\mathbf{u}_S(\mathbf{r}, \omega) = \int_V \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) \cdot \gamma_V(\mathbf{r}', \omega)\, d^3\mathbf{r}' + \int_{\partial V} \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) \cdot \gamma_{\partial V}(\mathbf{r}', \omega)\, d^2\mathbf{r}', \tag{2.16}$$

where $\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega)$ is the unperturbed 3×3 Green's tensor satisfying

$$\mathscr{L}_0(\mathbf{r})\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) - \omega^2\rho_0(\mathbf{r})\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{I}_3\delta(\mathbf{r}-\mathbf{r}'), \qquad \mathbf{r} \in V, \tag{2.17a}$$

$$\mathscr{B}_0(\mathbf{r})G_0(\mathbf{r},\mathbf{r}',\omega)=0, \qquad \mathbf{r}\in\partial V. \tag{2.17b}$$

The quantity $G_0(\mathbf{r},\mathbf{r}',\omega)$ can be written in terms of the unperturbed eigenfrequencies and eigenfunctions in the form

$$G_0(\mathbf{r},\mathbf{r}',\omega)=\sum_k(\omega_{0k}^2-\omega^2)^{-1}\mathbf{s}_{0k}(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r}'). \tag{2.18}$$

Upon inserting equations (2.15) and (2.18) into (2.16) and integrating the term involving $\Lambda(\mathbf{r},\omega)$ by parts, we obtain the result

$$\begin{aligned}\mathbf{u}_S(\mathbf{r},\omega)=&\sum_k(\omega_{0k}^2-\omega^2)^{-1}\mathbf{s}_{0k}(\mathbf{r})\int_V[\omega^2\rho_S(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}',\omega)\\&-2i\omega\rho(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\boldsymbol{\Omega}\times\mathbf{u}(\mathbf{r}',\omega)\\&-\nabla\mathbf{s}_{0k}^*(\mathbf{r}'):\Lambda_S(\mathbf{r}',\omega):\nabla\mathbf{u}(\mathbf{r}',\omega)\\&-\rho_S(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\nabla\nabla\Phi_0(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}',\omega)\\&-\rho(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\nabla\nabla\Phi_S(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}',\omega)\\&-\rho(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\nabla\nabla\Psi(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}',\omega)]\,d^3\mathbf{r}'\\&-\sum_k(\omega_{0k}^2-\omega^2)^{-1}\mathbf{s}_{0k}(\mathbf{r})\int_V\int_V[\rho_0(\mathbf{r}')\rho_S(\mathbf{r}'')\mathbf{s}_{0k}^*(\mathbf{r}')\\&\cdot\mathbf{g}(\mathbf{r}'-\mathbf{r}'')\cdot\mathbf{u}(\mathbf{r}'',\omega)\\&+\rho_S(\mathbf{r}')\rho_0(\mathbf{r}'')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\mathbf{g}(\mathbf{r}'-\mathbf{r}'')\cdot\mathbf{u}(\mathbf{r}'',\omega)\\&+\rho_S(\mathbf{r}')\rho_S(\mathbf{r}'')\mathbf{s}_{0k}^*(\mathbf{r}')\cdot\mathbf{g}(\mathbf{r}'-\mathbf{r}'')\cdot\mathbf{u}(\mathbf{r}'',\omega)]\,d^3\mathbf{r}'\,d^3\mathbf{r}''.\end{aligned} \tag{2.19}$$

Equation (2.19) is an integral equation for $\mathbf{u}_S(\mathbf{r},\omega)$ since the right side involves the full response $\mathbf{u}(\mathbf{r},\omega)=\mathbf{u}_0(\mathbf{r},\omega)+\mathbf{u}_S(\mathbf{r},\omega)$. In the jargon of quantum mechanics (2.19) is called the Lippmann–Schwinger equation (Schiff 1968; Boehm 1936).

To solve the Lippmann–Schwinger equation, we suppose the scattered wave field $\mathbf{u}_S(\mathbf{r},\omega)$ can be expressed as a Born series

$$\mathbf{u}_S(\mathbf{r},\omega)=\sum_{n=1}^{\infty}\mathbf{u}_n(\mathbf{r},\omega), \tag{2.20}$$

where each succeeding term is presumed to be one order higher in the quantities $\boldsymbol{\Omega}$, $\rho_S(\mathbf{r})$ and $\Lambda_S(\mathbf{r},\omega)$. Each higher order correction can be determined by solving (2.19) iteratively, starting with the expression (2.10) for the unperturbed solution $\mathbf{u}_0(\mathbf{r},\omega)$. The results of this iterative calculation are most conveniently expressed in a matrix notation. Let $\hat{\mathbf{v}}$ denote the polarization direction of the displacement sensor at $\mathbf{r}$, and define the unperturbed receiver and source column vectors $\mathbf{R}_0$ and $\mathbf{S}_0(\omega)$ by

$$R_{0k}=\hat{\mathbf{v}}\cdot\mathbf{s}_{0k}^*(\mathbf{r}), \tag{2.21a}$$

$$S_{0k}(\omega)=\mathbf{M}(\omega):\nabla\mathbf{s}_{0k}^*(\mathbf{r}_q). \tag{2.21b}$$

Let $\boldsymbol{\Sigma}_0$ denote the $\infty\times\infty$ matrix of unperturbed eigenfrequencies

$$\boldsymbol{\Sigma}_0=\mathrm{diag}\,(\cdots\omega_{0k}\cdots). \tag{2.22}$$

The unperturbed spectrum $u_0(\omega)=\hat{\mathbf{v}}\cdot\mathbf{u}_0(\mathbf{r},\omega)$ can be rewritten in the form

$$u_0(\omega)=\mathbf{R}_0^H\cdot(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\cdot\mathbf{S}_0(\omega), \tag{2.23}$$

where $\mathbf{I}_\infty$ is the $\infty\times\infty$ identity tensor and the superscript H denotes the Hermitian or complex conjugate transpose.

The first term in the Born series (2.20) is called the first Born approximation (Boehm 1936; Schiff 1968; Hudson & Heritage 1981). It is obtained by substituting $\mathbf{u}_0(\mathbf{r},\omega)$ for $\mathbf{u}(\mathbf{r},\omega)$ on the right-hand side of the Lippmann–Schwinger equation (2.19). In matrix notation, the first Born approximation $u_1(\omega)=\hat{\mathbf{v}}\cdot\mathbf{u}_1(\mathbf{r},\omega)$ is found to be

$$u_1(\omega)=\mathbf{R}_0^H\cdot\mathbf{K}(\omega)\cdot(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\cdot\mathbf{S}_0(\omega), \tag{2.24}$$

where

$$\begin{aligned}\mathbf{K}(\omega)=&(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\\&\cdot[\omega^2\mathbf{T}_S-\omega(\mathbf{W}+\tilde{\mathbf{W}})-\mathbf{V}_S(\omega)-\tilde{\mathbf{V}}_S(\omega)].\end{aligned} \tag{2.25}$$

The elements of the matrices $\mathbf{T}_S, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V}_S(\omega)$ and $\tilde{\mathbf{V}}_S(\omega)$ are defined by

$$T_{Skl}=\int_V\rho_S(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})\,d^3\mathbf{r}, \tag{2.26a}$$

$$W_{kl}=2i\int_V\rho_0(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\boldsymbol{\Omega}\times\mathbf{s}_{0l}(\mathbf{r})\,d^3\mathbf{r}, \tag{2.26b}$$

$$\tilde{W}_{kl}=2i\int_V\rho_S(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\boldsymbol{\Omega}\times\mathbf{s}_{0l}(\mathbf{r})\,d^3\mathbf{r}, \tag{2.26c}$$

$$\begin{aligned}V_{Skl}(\omega)=&\int_V[\nabla\mathbf{s}_{0k}^*(\mathbf{r}):\Lambda_S(\mathbf{r},\omega):\nabla\mathbf{s}_{0l}(\mathbf{r})\\&+\rho_S(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\nabla\nabla\Phi_0(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})\\&+\rho_0(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\nabla\nabla\Phi_S(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})\\&+\rho_0(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\nabla\nabla\Psi(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})]\,d^3\mathbf{r}\\&+\int_V\int_V[\rho_0(\mathbf{r})\rho_S(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r})\cdot\mathbf{g}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{s}_{0l}(\mathbf{r}')\\&+\rho_S(\mathbf{r})\rho_0(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r})\cdot\mathbf{g}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{s}_{0l}(\mathbf{r}')]\,d^3\mathbf{r}\,d^3\mathbf{r}',\end{aligned} \tag{2.26d}$$

$$\begin{aligned}\tilde{V}_{Skl}(\omega)=&\int_V[\rho_S(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\nabla\nabla\Phi_S(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})\\&+\rho_S(\mathbf{r})\mathbf{s}_{0k}^*(\mathbf{r})\cdot\nabla\nabla\Psi(\mathbf{r})\cdot\mathbf{s}_{0l}(\mathbf{r})]\,d^3\mathbf{r}\\&+\int_V\int_V[\rho_S(\mathbf{r})\rho_S(\mathbf{r}')\mathbf{s}_{0k}^*(\mathbf{r})\cdot\mathbf{g}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{s}_{0l}(\mathbf{r}')]\,d^3\mathbf{r}\,d^3\mathbf{r}'.\end{aligned} \tag{2.26e}$$

We have decomposed the Coriolis matrix and the perturbed part of the potential energy matrix into terms $\mathbf{W}$ and $\mathbf{V}_S(\omega)$ depending linearly on $\boldsymbol{\Omega}$ and $\rho_S(\mathbf{r})$, and other terms $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{V}}_S(\omega)$ whose dependence on these perturbations is quadratic.

The second Born approximation is obtained by substituting $\mathbf{u}_0(\mathbf{r},\omega)+\mathbf{u}_1(\mathbf{r},\omega)$ for $\mathbf{u}(\mathbf{r},\omega)$ on the right side of the Lippmann–Schwinger equation. In matrix notation, $u_2(\omega)=\hat{\mathbf{v}}\cdot\mathbf{u}_2(\omega)$ is given by

$$u_2(\omega)=\mathbf{R}_0^H\cdot\mathbf{K}(\omega)\cdot\mathbf{K}(\omega)\cdot(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\cdot\mathbf{S}_0(\omega). \tag{2.27}$$

Each succeeding term in the series incorporates an additional factor of $\mathbf{K}(\omega)$, so the total response $u(\omega)=\hat{\mathbf{v}}\cdot\mathbf{u}(\mathbf{r},\omega)$ is given by the sum

$$\begin{aligned}u(\omega)=&\mathbf{R}_0^H\cdot[\mathbf{I}_\infty+\mathbf{K}(\omega)+\mathbf{K}(\omega)\cdot\mathbf{K}(\omega)+\cdots]\\&\cdot(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\cdot\mathbf{S}_0(\omega).\end{aligned} \tag{2.28}$$

Formally, this infinite series can be summed to yield the result

$$u(\omega)=\mathbf{R}_0^H\cdot[\mathbf{I}_\infty-\mathbf{K}(\omega)]^{-1}\cdot(\boldsymbol{\Sigma}_0^2-\omega^2\mathbf{I}_\infty)^{-1}\cdot\mathbf{S}_0(\omega). \tag{2.29}$$

This expression can be rewritten, using the fact that $(\boldsymbol{\Sigma}_0^2 - \omega^2 \mathbf{I}_\infty)$ is a diagonal matrix, in the form

$$u(\omega) = \mathbf{R}_0^H \cdot [\boldsymbol{\Sigma}_0^2 + \mathbf{V}_S(\omega) + \tilde{\mathbf{V}}_S(\omega) + \omega(\mathbf{W} + \tilde{\mathbf{W}}) - \omega^2(\mathbf{I}_\infty + \mathbf{T}_S)]^{-1} \cdot \mathbf{S}_0(\omega). \tag{2.30}$$

The quantities $\boldsymbol{\Sigma}_0^2 + \mathbf{V}_S(\omega)$ and $\mathbf{I}_\infty + \mathbf{T}_S$ are the exact potential and relative kinetic energy matrices for the perturbed problem. To emphasize this, it is useful to relabel the corresponding unperturbed matrices $\boldsymbol{\Sigma}_0^2$ and $\mathbf{I}_\infty$ by

$$\mathbf{V}_0 = \boldsymbol{\Sigma}_0^2 \tag{2.31a}$$

$$\mathbf{T}_0 = \mathbf{I}_\infty. \tag{2.31b}$$

The unperturbed spectrum can be rewritten in this new notation as

$$u_0(\omega) = \mathbf{R}_0^H \cdot (\mathbf{V}_0 = \omega^2 \mathbf{T}_0)^{-1} \cdot \mathbf{S}_0(\omega), \tag{2.32}$$

whereas the perturbed spectrum is

$$u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V}_0 + \mathbf{V}_S(\omega) + \tilde{\mathbf{V}}_S(\omega) + \omega(\mathbf{W} + \tilde{\mathbf{W}}) - \omega^2(\mathbf{T}_0 + \mathbf{T}_S)]^{-1} \cdot \mathbf{S}_0(\omega). \tag{2.33}$$

Equation (2.33) describes the exact displacement field in an arbitrary dissipative and dispersive, rotating earth model. There is no restriction on the magnitude of the perturbations $\rho_S(\mathbf{r})$ and $\Lambda(\mathbf{r}, \omega)$; the only restrictions are that the perturbed and unperturbed earth model must both occupy the same volume V with surface ∂V, and all of the quantities $\rho(\mathbf{r})$, $\Lambda(\mathbf{r}, \omega)$, $\rho_0(\mathbf{r})$ and $\Lambda_0(\mathbf{r})$ must be smooth. Perturbations in the location of either the external boundary or any internal discontinuity surfaces cannot be treated exactly, rather they must be linearized. The details of this linearization procedure, including the complications that arise from internal fluid–solid discontinuities, are discussed by Woodhouse & Dahlen (1987). Terms that are ignored in linearizing the boundary perturbations are of the same order as the quadratic volumetric perturbations $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{V}}_S(\omega)$, and it is conventional to ignore these quantities also. This leads us, finally, to approximate the spectrum $u(\omega)$ by

$$u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]^{-1} \cdot \mathbf{S}_0(\omega), \tag{2.34}$$

where

$$\mathbf{V}(\omega) = \mathbf{V}_0 + \mathbf{V}_S(\omega) + \mathbf{V}_B(\omega), \tag{2.35a}$$

$$\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_S + \mathbf{T}_B. \tag{2.35b}$$

The quantities $\mathbf{V}_B(\omega)$ and $\mathbf{T}_B$ represent the contributions to the potential and relative kinetic energy matrices due to boundary perturbations. Both the Coriolis and relative kinetic energy matrices $\mathbf{W}$ and $\mathbf{T}$ are Hermitian: $\mathbf{W}^H = \mathbf{W}$ and $\mathbf{T}^H = \mathbf{T}$. Furthermore, $\mathbf{T}$ is always positive definite: $\mathbf{q}^H \cdot \mathbf{T} \cdot \mathbf{q} > 0$ for any non-zero column vector $\mathbf{q} \neq 0$. In general, the potential energy matrix $\mathbf{V}(\omega)$ is both non-Hermitian and frequency dependent, due to the effects of dissipation and dispersion. Complete expressions for the elements of $\mathbf{V}(\omega)$, $\mathbf{W}$, and $\mathbf{T}$ are given by Woodhouse (1980) and Mochizuki (1986), for the case of a spherical isotropic starting model.

Equation (2.34) is the main result of this paper. It is an explicit expression for the response of an arbitrary dissipative and dispersive, rotating earth model to an earthquake. The perturbed response (2.34) has the same form as the unperturbed response (2.32), with the diagonal matrix $(\mathbf{V}_0 - \omega^2 \mathbf{T}_0)^{-1}$ replaced by $[\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]^{-1}$. The quantity $\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}$ is the Lagrangian matrix that must be diagonalized to find the perturbed complex eigenfrequencies using a variational method (Park & Gilbert 1986). By Cramer's rule the inverse $[\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]^{-1}$ is given by

$$[\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]^{-1} = \frac{\operatorname{adj} [\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]}{\det [\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]}, \tag{2.36}$$

where $\operatorname{adj} [\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}]$ is the classical adjoint or transposed matrix of cofactors (Horn & Johnson 1985). The response $u(\omega)$ clearly exhibits poles at the perturbed complex eigenfrequencies, since these are the zeros of the secular equation

$$\det [\mathbf{V}(\omega) + \omega \mathbf{W} - \omega^2 \mathbf{T}] = 0. \tag{2.37}$$

Paradoxically, it is only after the summation has been performed that $u(\omega)$ is resonant at the correct perturbed eigenfrequencies. If the Born series (2.28) is truncated after any finite number of terms, the resulting partial sum is still resonant at the unperturbed eigenfrequencies; the nth Born approximation $\mathbf{u}_n(\mathbf{r}, \omega)$ is proportional to $(\boldsymbol{\Sigma}_0^2 - \omega^2 \mathbf{I}_\infty)^{-n-1}$, and this multiplicity of unperturbed poles leads to secular terms that grow like t^n in the time domain (Bender & Orzag 1978). Geller, Hara, & Tsuboi (1989) have shown how to remove this secularity to first order by introducing the first-order perturbations to the eigenfrequencies and eigenfunctions. Presumably, such a procedure could be extended to higher order by increasingly laborious manipulations of the higher order perturbations to the eigenfrequencies and eigenfunctions. Equation (2.34), which is remarkably simple, is valid to infinite order.

Inspection reveals that we have been able to sum the Born series because of the separate appearance of the source and receiver coordinates in (2.18). A similar method is applicable to any problem for which the unperturbed Green's function or tensor has this form. For example, in solid state physics a summation of a perturbation series is performed in order to find the Green's function for a system of interacting particles from the Green's function for a system of non-interacting particles (Jones & March 1973; Doniach & Sondheimer 1974).

3 COMMENT ON THE DERIVATION

The Born series in equation (2.28) only converges if $\|\mathbf{K}(\omega)\| < 1$, where $\|\cdot\|$ is an appropriate matrix norm. It is always possible to guarantee convergence by choosing ω to have a sufficiently large positive or negative imaginary part. However, the series diverges in the vicinity of the unperturbed eigenfrequencies $\omega = \pm\omega_{0k}$, because of the presence of the term $(\boldsymbol{\Sigma}_0^2 - \omega^2 \mathbf{T}_0)^{-1}$ in the definition (2.25) of $\mathbf{K}(\omega)$. Because of this divergence, any truncated version of equation (2.28) is undefined on the real ω axis, where we wish to know the spectrum. Equation (2.33) on the other hand is analytic over the entire ω plane except for singularities at the perturbed eigenfrequencies, where $\det [\mathbf{V}_0 + \mathbf{V}_S(\omega) + \tilde{\mathbf{V}}_S(\omega) + \omega(\mathbf{W} + \tilde{\mathbf{W}}) - \omega^2(\mathbf{T}_0 + \mathbf{T}_S)] = 0$. In fact, equation (2.33) is the analytic continuation of the Born series representation (Carrier, Krook & Pearson 1966). Both (2.33) and its generalization (2.34) are valid everywhere along the real ω axis, since dissipation gives each of the perturbed eigenfrequencies a small positive imaginary part.

The fact that (2.33) represents a formal summation of the Born series is appealing, since it shows that the response $u(\omega)$ includes all the higher order scattering interactions. However, the divergence of the truncated series for real ω suggest that the decomposition into unperturbed and perturbed problems may not be fundamental. In fact, it is easy to derive (2.33) more directly, as we now show. Our only assumption is that the perturbed displacement field can be expressed as a sum over the unperturbed eigenfunctions:

$$\mathbf{u}(\mathbf{r}, \omega) = \sum_l c_l(\omega)\mathbf{s}_{0l}(\mathbf{r}). \tag{3.1}$$

The coefficients $c_l(\omega)$ are the elements of a column vector $\mathbf{c}(\omega)$ that we wish to find. We substitute the expansion (3.1) into equation (2.1a), dot the result with an arbitrary conjugate eigenfunction $\mathbf{s}^*_{0k}(\mathbf{r})$, integrate over the volume V, and make use of the orthonormality (2.10). Upon integrating the term involving $\Lambda(\mathbf{r}, \omega)$ by parts and using the boundary condition (2.1b), we obtain

$$[\mathbf{V}_0 + \mathbf{V}_S(\omega) + \tilde{\mathbf{V}}_S(\omega) + \omega(\mathbf{W} + \tilde{\mathbf{W}}) - \omega^2(\mathbf{T}_0 + \mathbf{T}_S)] \cdot \mathbf{c}(\omega)$$
$$= \mathbf{S}_0(\omega). \tag{3.2}$$

Solving for $\mathbf{c}(\omega)$ and inserting the result in (3.1), we find that $u(\omega) = \hat{\boldsymbol{v}} \cdot \mathbf{u}(\mathbf{r}, \omega)$ is given by

$$u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V}_0 + \mathbf{V}_S(\omega) + \tilde{\mathbf{V}}_S(\omega) + \omega(\mathbf{W} + \tilde{\mathbf{W}})$$
$$- \omega^2(\mathbf{T}_0 + \mathbf{T}_S)]^{-1} \cdot \mathbf{S}_0(\omega), \tag{3.3}$$

which is the same as (2.33). Once again, linearizing boundary perturbations and ignoring the quadratic quantities $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{V}}_S(\omega)$ leads us to approximate the seismic spectrum by (2.34).

4 TIME DOMAIN RESPONSE

Equation (2.34) gives the frequency-domain response in terms of the unperturbed receiver and source vectors $\mathbf{R}_0$ and $\mathbf{S}_0(\omega)$. To determine the corresponding response in the time domain, it is necessary to recast the description in terms of the perturbed rather than the unperturbed normal modes. This is straightforward in the absence of dissipation and dispersion, as we show next.

4.1 Non-rotating elastic earth model

If the earth model is non-rotating and perfectly elastic, the response (2.34) simplifies to

$$u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V} - \omega^2\mathbf{T}]^{-1} \cdot \mathbf{S}_0(\omega), \tag{4.1}$$

where $\mathbf{V}$ is Hermitian, $\mathbf{V}^H = \mathbf{V}$, and frequency independent. In this case, there exists a congruent transform $\mathbf{Q}$ that simultaneously diagonalizes $\mathbf{V}$ and $\mathbf{T}$ (Horn & Johnson 1985):

$$\mathbf{Q}^H \cdot \mathbf{V} \cdot \mathbf{Q} = \mathbf{\Sigma}^2, \tag{4.2a}$$

$$\mathbf{Q}^H \cdot \mathbf{T} \cdot \mathbf{Q} = \mathbf{I}_\infty. \tag{4.2b}$$

The associated perturbed eigenfunctions are given in terms of the elements of $\mathbf{Q}$ by

$$\mathbf{s}_k(\mathbf{r}) = \sum_l Q_{lk}\mathbf{s}_{0l}(\mathbf{r}), \tag{4.3}$$

and $\mathbf{\Sigma}$ is the infinite diagonal matrix of perturbed real positive eigenfrequencies ω_k:

$$\mathbf{\Sigma} = \operatorname{diag}(\cdots \omega_k \cdots). \tag{4.4}$$

To find $\mathbf{\Sigma}$ and $\mathbf{Q}$ we solve the eigenvalue problem

$$\mathbf{V} \cdot \mathbf{Q} - \mathbf{T} \cdot \mathbf{Q} \cdot \mathbf{\Sigma}^2 = \mathbf{0}. \tag{4.5}$$

Equations (4.2) together imply that

$$(\mathbf{V} - \omega^2\mathbf{T})^{-1} = \mathbf{Q} \cdot (\mathbf{\Sigma}^2 - \omega^2\mathbf{I}_\infty)^{-1} \cdot \mathbf{Q}^H. \tag{4.6}$$

Thus if we define transformed source and receiver vectors $\mathbf{R}$ and $\mathbf{S}(\omega)$ by

$$\mathbf{R} = \mathbf{Q}^H \cdot \mathbf{R}_0, \tag{4.7a}$$

$$\mathbf{S}(\omega) = \mathbf{Q}^H \cdot \mathbf{S}_0(\omega), \tag{4.7b}$$

we can rewrite (4.1) in the form

$$u(\omega) = \mathbf{R}^H \cdot [\mathbf{\Sigma}^2 - \omega^2\mathbf{I}_\infty]^{-1} \cdot \mathbf{S}(\omega). \tag{4.8}$$

Expression (4.8) is of the same form as the representation (2.23) of $u_0(\omega)$, but with the unperturbed eigenfrequencies and eigenfunctions replaced by their perturbed counterparts, as expected. The perturbed source vector $\mathbf{S}(\omega)$ determines which combination of unperturbed modes is excited by the source, and the perturbed receiver vector $\mathbf{R}$ determines which combination of unperturbed modes is observed at the receiver. The corresponding time-domain response is the analogue of (2.11), namely (Gilbert 1970)

$$\mathbf{u}(\mathbf{r}, t) = \sum_k \omega_k^{-1}\mathbf{s}_k(\mathbf{r}) \int_0^t [\mathbf{M}(\tau) : \nabla\mathbf{s}^*_k(\mathbf{r}_q)]$$
$$\times \sin[\omega_k(t - \tau)]\, d\tau, \qquad t \geq 0. \tag{4.9}$$

The perturbed eigenfunctions $\mathbf{s}_k$ are orthonormal in the sense

$$\int_V \rho(\mathbf{r})\mathbf{s}^*_k \cdot \mathbf{s}_l(\mathbf{r})\, d^3\mathbf{r} = \delta_{kl}, \tag{4.10}$$

where the volume of integration is the perturbed shape of the Earth.

4.2 Rotating elastic earth model

If the earth model is rotating but still perfectly elastic, the theoretical spectrum is given by

$$u(\omega) = \mathbf{R}_0^H \cdot [\mathbf{V} + \omega\mathbf{W} - \omega^2\mathbf{T}]^{-1} \cdot \mathbf{S}_0(\omega), \tag{4.11}$$

where all of $\mathbf{V}, \mathbf{W}$, and $\mathbf{T}$ are Hermitian and frequency independent. Let $\mathbf{\Sigma}$ again denote the infinite diagonal matrix of perturbed real positive eigenfrequencies ω_k, and let $\mathbf{Q}_\pm$ be the $\infty \times \infty$ matrices whose columns are the eigenvectors associated with $\pm\mathbf{\Sigma}$:

$$\mathbf{V} \cdot \mathbf{Q}_\pm \pm \mathbf{W} \cdot \mathbf{Q}_\pm \cdot \mathbf{\Sigma} - \mathbf{T} \cdot \mathbf{Q}_\pm \cdot \mathbf{\Sigma}^2 = \mathbf{0}. \tag{4.12}$$

The perturbed eigenfunctions $\mathbf{s}_{\pm k}(\mathbf{r})$ associated with $\pm\omega_k$ are given in terms of the elements of $\mathbf{Q}_\pm$ by

$$\mathbf{s}_{\pm k}(\mathbf{r}) = \sum_l Q_{\pm lk}\mathbf{s}_{0l}(\mathbf{r}). \tag{4.13}$$

Eigenfunctions associated with negative and positive eigenfrequencies are simply related by $\mathbf{s}_{\mp k}(\mathbf{r}) = \mathbf{s}^*_{\pm k}(\mathbf{r})$.

The inverse $(\mathbf{V} + \omega\mathbf{W} - \omega^2\mathbf{T})^{-1}$ can be expressed in terms of $\mathbf{\Sigma}$ and $\mathbf{Q}_\pm$ in a manner analogous to (4.6) using the theory of lambda matrices (Lancaster 1966). The details are uninteresting, and we simply quote the results here. If the matrices $\mathbf{Q}_\pm$ are normalized such that

$$\tfrac{1}{2}[\mathbf{Q}^H_\pm \cdot \mathbf{T} \cdot \mathbf{Q}_\pm + \mathbf{\Sigma}^{-1} \cdot \mathbf{Q}^H_\pm \cdot \mathbf{V} \cdot \mathbf{Q}_\pm \cdot \mathbf{\Sigma}^{-1}] = \mathbf{I}_\infty, \tag{4.14}$$

then

$$(\mathbf{V} + \omega\mathbf{W} - \omega^2\mathbf{T})^{-1} = \tfrac{1}{2}\mathbf{Q}_+ \cdot \mathbf{\Sigma}^{-1} \cdot [\mathbf{\Sigma} - \omega\mathbf{I}_\infty]^{-1} \cdot \mathbf{Q}^H_+ + \tfrac{1}{2}\mathbf{Q}_- \cdot \mathbf{\Sigma}^{-1} \cdot [\mathbf{\Sigma} + \omega\mathbf{I}_\infty]^{-1} \cdot \mathbf{Q}^H_-. \tag{4.15}$$

Upon defining the transformed source and receiver vectors $\mathbf{R} = \mathbf{R}_+ = \mathbf{R}^*_-$ and $\mathbf{S}(\omega) = \mathbf{S}_+(\omega) = \mathbf{S}^*_-(\omega)$ by

$$\mathbf{R} = \mathbf{Q}^H_+ \cdot \mathbf{R}_0, \tag{4.16a}$$

$$\mathbf{S}(\omega) = \mathbf{Q}^H_+ \cdot \mathbf{S}_0(\omega), \tag{4.16b}$$

we can rewrite (4.11) in the form

$$u(\omega) = \frac{1}{2}\sum_k \omega_k^{-1}(\omega_k - \omega)^{-1} R^*_k S_k(\omega) + \frac{1}{2}\sum_k \omega_k^{-1}(\omega_k + \omega)^{-1} R_k S^*_k(\omega). \tag{4.17}$$

The associated time-domain response is given in terms of the perturbed eigenfrequencies and eigenfunctions by

$$\mathbf{u}(\mathbf{r}, t) = \mathcal{Re} \sum_k (i\omega_k)^{-1}\mathbf{s}_k(\mathbf{r}) \times \int_0^t [\mathbf{M}(\tau) : \nabla\mathbf{s}^*_k(\mathbf{r}_q)] \exp[i\omega_k(t - \tau)]\, d\tau, \qquad t \geq 0. \tag{4.18}$$

From (4.12) and (4.14) we deduce that the perturbed eigenfunctions $\mathbf{s}_k(\mathbf{r})$ are orthonormal in the sense

$$\int_V \rho(\mathbf{r})\mathbf{s}^*_k(\mathbf{r}) \cdot \mathbf{s}_l(\mathbf{r})\, d^3\mathbf{r} - 2i(\omega_k + \omega_l)^{-1} \times \int_V \rho(\mathbf{r})\mathbf{s}^*_k(\mathbf{r}) \cdot \mathbf{\Omega} \times \mathbf{s}_l(\mathbf{r})\, d^3\mathbf{r} = \delta_{kl}. \tag{4.19}$$

These results for the excitation of the normal modes of a rotating earth model are identical to those obtained using different methods by Dahlen (1978; 1980) and Wahr (1981).

5 NARROW BAND APPROXIMATION FOR A DISSIPATIVE EARTH MODEL

It is difficult to simplify (2.34) further in the case of a general frequency-dependent $\mathbf{V}(\omega)$. We can, however, easily obtain an approximate expression for $u(\omega)$ in the vicinity of a fiducial or reference frequency ω_r. To lowest order in $\delta\omega = \omega - \omega_r$ and the perturbations $\mathbf{V}_S, \mathbf{V}_B, \mathbf{W}, \mathbf{T}_S$ and $\mathbf{T}_B$, we can write

$$\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T} \approx 2\omega_r(\mathbf{V}' - \delta\omega\mathbf{I}_\infty), \tag{5.1}$$

where

$$2\omega_r\mathbf{V}' = (\mathbf{V}_0 - \omega_r^2\mathbf{I}_\infty) + \mathbf{V}_S(\omega_r) + \mathbf{V}_B(\omega_r) + \omega_r\mathbf{W} - \omega_r^2(\mathbf{T}_S + \mathbf{T}_B). \tag{5.2}$$

To the same order, the inverse $[\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}]^{-1}$ is given by

$$[\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}]^{-1} \approx (2\omega_r)^{-1}(\mathbf{V}' - \delta\omega\mathbf{I}_\infty)^{-1}. \tag{5.3}$$

Let

$$\mathbf{\Delta} = \operatorname{diag}(\cdots \delta\omega_k \cdots) \tag{5.4}$$

denote the diagonal matrix of complex eigenfrequency perturbations in the vicinity of ω_r, and let $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ be the associated matrices of right and left eigenvectors, so that

$$\mathbf{V}' \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{\Delta}, \tag{5.5a}$$

$$\mathbf{Q}^{-1} \cdot \mathbf{V}' = \mathbf{\Delta} \cdot \mathbf{Q}^{-1}. \tag{5.5b}$$

Equations (5.5) assume that $\mathbf{V}'$ is non-defective, in which case the eigenfrequency perturbations $\delta\omega_k$ are distinct. Associated with every perturbed eigenfrequency $\omega_r + \delta\omega_k$, there is both a perturbed eigenfunction $\boldsymbol{\sigma}_k(\mathbf{r})$ and a perturbed dual eigenfunction $\boldsymbol{\eta}_k(\mathbf{r})$ given by

$$\boldsymbol{\sigma}_k(\mathbf{r}) = \sum_l Q_{lk}\mathbf{s}_{0l}(\mathbf{r}), \tag{5.6a}$$

$$\boldsymbol{\eta}_k(\mathbf{r}) = \sum_l Q_{lk}^{-H}\mathbf{s}_{0l}(\mathbf{r}). \tag{5.6b}$$

These comprise dual orthonormal bases in the sense

$$\int_V \rho_0(\mathbf{r})\boldsymbol{\eta}^*_k(\mathbf{r}) \cdot \boldsymbol{\sigma}_l(\mathbf{r})\, d^3\mathbf{r} = \delta_{kl}. \tag{5.7}$$

The inverse $(\mathbf{V}' - \delta\omega\mathbf{I}_\infty)^{-1}$ can be written in terms of $\mathbf{\Delta}$ and the right and left eigenvectors in the form

$$(\mathbf{V}' - \delta\omega\mathbf{I}_\infty)^{-1} = \mathbf{Q} \cdot (\mathbf{\Delta} - \delta\omega\mathbf{I}_\infty)^{-1} \cdot \mathbf{Q}^{-1}. \tag{5.8}$$

Upon defining the transformed source and receiver vectors

$$\mathbf{R} = \mathbf{Q}^H \cdot \mathbf{R}_0, \tag{5.9a}$$

$$\mathbf{S}(\omega) = \mathbf{Q}^{-1} \cdot \mathbf{S}_0(\omega), \tag{5.9b}$$

we find that the spectrum in the vicinity of ω_r is

$$u(\omega_r + \delta\omega) \approx (2\omega_r)^{-1}\mathbf{R}^H \cdot (\mathbf{\Delta} - \delta\omega\mathbf{I}_\infty)^{-1} \cdot \mathbf{S}(\omega). \tag{5.10}$$

The corresponding time-domain response, obtained using the symmetry $u(-\omega) = u^*(\omega)$, is

$$\mathbf{u}(\mathbf{r}, t) \approx \mathcal{Re} \sum_k (i\omega_r)^{-1}\boldsymbol{\sigma}_k(\mathbf{r}) \int_0^t [\mathbf{M}(\tau) : \nabla\boldsymbol{\eta}^*_k(\mathbf{r}_q)] \times \exp[i(\omega_r + \delta\omega_k)(t - \tau)]\, d\tau, \qquad t \geq 0. \tag{5.11}$$

Equation (5.11) is equivalent to the expression employed by Park & Gilbert (1986). They show that linearization about a fiducial frequency is an acceptable approximation, at least for smooth models of the lateral heterogeneity of the Earth.

6 CONCLUSIONS

The frequency-domain response to an earthquake can, in principle, be determined by numerical inversion of the same matrix $\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}$ that must be diagonalized to find the perturbed eigenfrequencies. Direct implementation of such a method would be expensive, since an independent inversion must be performed at every frequency of interest. Iterative inversion techniques that take advantage of the simple dependence of $\mathbf{V}(\omega) + \omega\mathbf{W} - \omega^2\mathbf{T}$ on ω could, however, be developed. If the lateral heterogeneity of the Earth is rough enough to cause strong coupling among normal mode multiplets over a large range of frequencies, the approximation (5.11) might not be justifiable. Equation (2.34) could be used in that case to calculate accurate seismic spectra.

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