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Modelling of coupled normal modes of the Earth: the spectral method

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SUMMARY

Modelling of coupled free oscillations or seismograms in an earth with small-scale lateral heterogeneities (a few hundred kilometres) is presently impossible without strong approximations, such as taking into account the coupling effect of the neighbouring modes only. Even within this assumption, first-order perturbation theory is generally insufficient, and variational theory must be used, leading to numerically heavy diagonalizations. An alternative method is presented in this paper. The first characteristic of this method is the use of higher order perturbation theory, which expresses the aspherical normal modes as a power series of perturbations. This perturbation theory generalizes the classical perturbation theory, in order to take into account density heterogeneities and secular terms by a renormalization technique. We show that from the third order on, the aspherical normal modes are computed with an accuracy a hundred times better than normal mode observations usually permit. The second characteristic is the use of a generalization of the spectral method in the tensor (elastic) case. Classically, interaction terms are treated as matrix products and require computations increasing as ℓmax^4, where ℓmax is the maximum angular order of the modelled modes, when coupling is fully taken into account for an earth model with small-scale heterogeneities. We show that such terms can be computed with a backward and forward Legendre transformation, for which computations increase only as ℓmax^3. This method is thus faster by an order of ℓmax than the variational approach. It is promising for the study of fully coupled modes and seismic waves in a realistic earth including small-scale lateral heterogeneities associated with narrow tectonic provinces such as in mid-oceanic ridges, subduction zones and continental margins.

Key words: normal modes, perturbation theory, spectral method.

1 INTRODUCTION

For the past five years, seismological studies have helped to increase our knowledge of large-scale, aspherical structure of the Earth. The existence of lateral variations of structure was first observed in the early 1960s (Ness, Harrison & Schlicher 1961; Benioff, Press & Smith 1961), and the first 3-D maps of the Earth's mantle were obtained twenty years later. These studies have used surface wave dispersion (Nakanishi & Anderson 1984), waveform modelling (Woodhouse & Dziewonski 1984) or arrival times (Dziewonski 1984). Increasingly sophisticated models, with additional parameters such as anisotropy (Nataf, Nakanishi & Anderson 1984, 1986; Montagner 1985, 1986) or anelasticity (Romanowicz 1989) are now available. The scale of the present models, developed into spherical harmonics, reaches an angular order of 12–15 (Montagner & Tanimoto 1989). This first generation of tomographic models has confirmed the relation of tectonic features at the surface and deep structure in the first 200 km of the Earth. But their limited resolution is unable to resolve the most interesting smaller features such as hotspots, subduction zones, the convection pattern under mid-oceanic ridges and more generally the convective structure of the mantle, and, moreover, it is not clear whether the seismic inversion for large-scale structure is not biased by such smaller scale features (Snieder & Neele 1989; Lognonné & Romanowicz 1990). As a matter of fact, the scale length of these features is always less than a thousand kilometres and even if the number of data might be sufficient for their resolution, the solution of the direct problem of propagation of seismic waves in this highly aspherical earth is one of the most important restrictions for the development of a second generation of tomographic models.

The direct problem of normal modes, for an elastic spherical earth, was solved at the beginning of the century (Love 1911). The solution consists of an infinite number of degenerate states of vibration. Aspherical structure destroys the symmetries, and introduces coupling between free oscillations. A first approach to solve for the normal modes of an aspherical earth is to use degenerate perturbation theory, which considers only isolated multiplets (Backus & Gilbert 1961; Madariaga 1971, 1972; Dahlen 1974; Woodhouse & Girnius 1982), or quasi-degenerate perturbation theory, which takes into account coupling within a super-multiplet (Luh 1973a, b; Woodhouse 1983; Dahlen 1987). Another approach used is variational or Galerkin theory (Morris *et al.* 1987; Park & Gilbert 1986). Nevertheless, in all cases a great number of aspherical parameters must be included: anelasticity, which breaks the Hermitian symmetry of the problem (Dahlen 1981; Park & Gilbert 1986), anisotropy (Montagner & Nataf 1986; Mochizuki 1986; Tanimoto 1986; Romanowicz & Snieder 1988), or pre-stress (Valette 1986). All these difficulties lead to a large number of computations and forbid the use of variational or Galerkin methods for the modelling of seismograms in an aspherical earth with small-scale heterogeneities of both stiffness tensor and density. Different approximations are usually made to reduce the computations, such as assuming that only neighbouring multiplets are coupled (Park 1987), or using asymptotic approximations (Jordan 1978; Dahlen 1979; Romanowicz 1987). Especially, in the latter case, one is led to assume that the asphericity is weak and that the scale of the heterogeneities is large.

In this paper, in order to overcome these restrictions and to perform modelling in a general aspherical earth, we propose a formalism based on higher order perturbation theory for the modelling of eigenfunctions, which can include fundamental and overtone, spheroidal and toroidal mode coupling without leading to a numerically heavy diagonalization of very large matrices. This method differs from other methods first by the way of computing the interaction matrices or the product of such matrices by a vector. We use a generalization of a method called 'the spectral method' (Orszag & Gotlieb 1977) to the case of tensorial (elastic) fields. This method is based, for its optimization, on a systematic use of group theory, spatial and time symmetries. Depending on the kind of coupling, the interaction of eigenfunctions will be computed in the spectral space of spherical eigenmodes (for Coriolis coupling for example), or in the spatial space for anelastic and anisotropic heterogeneities. Thus, all coupling operations due to heterogeneities of density and elasticity, classically treated as matrix multiplications, are faster by an order of ℓmax, ℓmax being the maximum angular order of the modelled waves. We do not need Clebsch–Gordan coefficients or explicit development of the asphericity into spherical harmonics. A second major difference is in the way of handling the effect of lateral heterogeneities in density. These heterogeneities are either neglected (Morris *et al.* 1987; Tsuboi & Geller 1989), or modelled with first-order perturbation theory (Woodhouse 1983; Dahlen 1987), or require a Cholesky decomposition of very large matrices (Masters, Park & Gilbert 1983; Park 1985). Here, we prove that the effect of lateral heterogeneities of density can be solved analytically with a renormalization technique, which incorporates the density into the elasto-dynamic operator. The Cholesky decomposition is thus no longer necessary, if one uses the variational method, or, if one uses higher order perturbation theory, the number of computations is reduced by half. Indeed, the last difference is the use of higher order perturbation theory. Up to now, expressions of eigenfrequency and eigenmode perturbations were either given only up to first order (Tanimoto 1984; Park 1987; Romanowicz 1987), without taking into account secular terms, or up to second order with a first-order secular equation (Dahlen 1987), but without taking into account the perturbation of the inner-product produced by the rotation and density lateral heterogeneities. We illustrate the importance of the secular equation and generalize to any nth order the results obtained by Dahlen (1987) to first order. We thus give exact expressions of the nth-order perturbation of eigenfrequency and eigenmode, and use it to compute the eigenmodes and eigenfrequencies.

In what follows, we first recall, for the simple case of a spherical, non-rotating, elastic, isotropic earth, (SNREI), the formalism of generalized spherical harmonics introduced by Phinney & Burridge (1973) and describe the symmetries which leave this earth model invariant. In the second part, we give the solution for the vibrations of a laterally heterogeneous spherical earth, starting from an SNREI model and using the higher orders of perturbation theory. Exact expressions up to third-order perturbations are thus given, by may be easily generalized to any higher orders. An analysis of the error of perturbation theory is also given, which illustrates the accuracy improvement of the higher order perturbation with respect to the first-order one. The numerical problem of the computation of splitting matrices for models with any scale of lateral heterogeneities is solved in Section 4 and the tensorial transformation method presented. We then give the expression of any nth-order perturbation of eigenmode and eigenfrequency with or without the help of the density renormalization technique. We then examine the accuracy of higher order perturbation theory for two examples of spheroidal modes, for which the coupling effects are significantly different, and compare both the eigenfrequency and eigenmode of the aspherical singlets with those obtained with the variational method.

2 THE SPECTRAL PROBLEM OF AN ASPHERICAL ROTATING EARTH

Let us begin by recalling the formulation of the spectral problem for the normal modes of a rotating, laterally heterogeneous and self-graviting earth, which leads to finding the eigenfunction $\mathbf{v}$, in bracket notation $|\mathbf{v}\rangle$, and the eigenfrequency ω, as a solution of the generalized eigenproblem (Valette 1986; Woodhouse & Dalhen 1978):

$$-\omega^2 |\mathbf{v}\rangle + \omega \mathbf{B}_0 |\mathbf{v}\rangle + \mathbf{A} |\mathbf{v}\rangle = \mathbf{0}, \tag{1}$$

where $\mathbf{B}_0$ is a self-adjoint operator associated to the Coriolis force, such that

$$\mathbf{B}_0 |\mathbf{v}\rangle = 2\mathrm{i}(\boldsymbol{\Omega} \times \mathbf{v}), \tag{2}$$

and where the self-adjoint operator $\mathbf{A}$ is defined, in the case

of an hydrostatic pre-stressed earth, as (Valette 1986)

$$\langle \mathbf{u}|\,\mathbf{A}\,|\mathbf{v}\rangle = \int_V \{d^{ijkl}D_i v_j D_k u_l^* + \rho S\{\mathbf{v}\cdot\mathbf{g}\,\mathrm{div}(\mathbf{u}^*) - \mathbf{u}^*\cdot\mathbf{Grad}(\mathbf{v}\cdot\mathbf{g})\}\,dV - \int_E \frac{\mathbf{Grad}[\psi(\mathbf{v})]\cdot\mathbf{Grad}[\psi(\mathbf{u}^*)]}{4\pi G}\,dV - \int_{\delta\Sigma}\rho S\{\mathbf{v}\cdot\mathbf{n}[\mathbf{u}^*]\cdot\mathbf{Grad}_\Sigma(p_0)\}\,d\Sigma. \quad (3)$$

where, in the solid part of the model

$$d^{ijkl} = c^{ijkl} - pg^{ij}g^{kl} + pg^{ik}g^{jl} + pg^{il}g^{jk},$$

and in the fluid part of the model

$$d^{ijkl} = \gamma p g^{ij}g^{kl}.$$

E denotes the whole universe, $\delta\Sigma$ the surfaces of discontinuities of the earth model, $\mathbf{n}$ the normal to these surfaces, ρ the density, $\mathbf{g}$ the aspherical gravity field, ψ the gravity redistribution potential, G the universal gravity constant, Ω the earth's rotation vector, p the istropic pre-stress, γ the adiabatic index of the fluid in the outer-core, g^{ij} the metric tensor and D_i is the Eulerian derivative. $S\{.\}$ is the symmetric part of a bilinear form b(u, v) and is defined as

$$S\{b(\mathbf{u},\mathbf{v})\} = 1/2\{b(\mathbf{u},\mathbf{v}) + b^*(\mathbf{v},\mathbf{u})\}.$$

As noted by Dahlen & Smith (1975), two normal modes, $\mathbf{u}$ and $\mathbf{v}$ are bi-orthogonal with respect to the inner product:

$$\langle \mathbf{u}|\mathbf{v}\rangle = \int_V \rho\,dV[\mathbf{u}^*\cdot\mathbf{v} + \mathbf{A}(\mathbf{v})\cdot\mathbf{u}^*/\omega_u\omega_v] = \langle \mathbf{u}|\mathbf{v}\rangle + \langle \mathbf{u}|\,\mathbf{A}\,|\mathbf{v}\rangle/\omega_n\omega_v, \quad (4)$$

where ω_u and ω_v are the eigenfrequencies of $\mathbf{u}$ and $\mathbf{v}$ and where the bracket product $\langle \mathbf{u}|\mathbf{v}\rangle$ is defined as

$$\langle \mathbf{u}|\mathbf{v}\rangle = \int_V \rho\,dV\,\mathbf{u}^*\cdot\mathbf{v}.$$

The goal of this paper is to present a method, based on higher order perturbation theory, which will allow us to solve, in a fast and accurate way, the normal mode equation (1) for any laterally heterogeneous earth model, including lateral variations in density and stiffness tensor. This method can be used of course for a more complicated earth model, including for example a pre-stress deviatoric part. We note however that some controversy still remains in the effect of such pre-stress on the expression of the operator $\mathbf{A}$ [e.g. between Biot's formulation of Dahlen (1972), Woodhouse & Dahlen (1978), Valette (1986) and Love's formulation of Geller (1988)].

3 THE SNREI MODEL: A BASIS FOR ALL ASPHERICAL MODES

3.1 Recall: SNREI normal modes

Any aspherical normal mode can be expressed in terms of its displacement at each point of the volume V of the aspherical earth model. However, we choose to express it in the basis of the normal modes of a spherical, non rotating, elastic and isotropic earth model (SNREI). These modes, depending on three discrete indexes n, ℓ, m, respectively associated to the spherical coordinates r, θ, ϕ, are solutions of the SNREI normal mode equation

$$-\omega^2|\mathbf{u}\rangle + \mathbf{A_0}|\mathbf{u}\rangle = \mathbf{0}, \quad (5)$$

where $\mathbf{A_0}$ is the elastodynamic operator of the spherical model. As rotation is not taken into account in the SNREI model, the SNREI normal modes ${}_n\mathbf{u}_\ell^m$, noted as $|{}_n\mathbf{u}_\ell^m\rangle$ in bracket notation, are orthogonal for the inner-product

$$\langle {}_n\mathbf{u}_\ell^m|{}_{n'}\mathbf{u}_{\ell'}^{m'}\rangle_0 = \int_{V_0} dV\rho_0\,{}_n\mathbf{u}_\ell^{m*}\cdot{}_{n'}\mathbf{u}_{\ell'}^{m'} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}, \quad (6)$$

where the index $_0$ in the bracket product means that the integration is done with the spherical density ρ_0 and for the spherical volume V_0.

In a spherical Cartesian basis, these solutions may be expressed with spherical harmonics and for example, each displacement field of a free mode nℓm is given, in operator notation, by

$$\mathbf{u}_\ell^m(r,\theta,\phi) = \mathbf{D}Y_\ell^m(\theta,\phi)$$

where the index n has been omitted, as it will be in what follows. The operator $\mathbf{D}$ is given by

$$\mathbf{D}Y_\ell^m(\theta,\phi) = U(r)Y_\ell^m(\theta,\phi)\mathbf{e}_r + V(r)\nabla_1 Y_\ell^m(\theta,\phi) - W(r)\mathbf{e}_r\times\nabla_1 Y_\ell^m(\theta,\phi). \quad (7)$$

where U, V, W are functions of r which depend on n and ℓ, the radial and angular orders of the mode respectively, $Y_\ell^m(\theta,\phi)$ is the fully normalized spherical harmonic, ∇_1 the gradient operator on the unit sphere, $\mathbf{e}_r$ the radial basis vector of the spherical Cartesian basis, $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, and θ and ϕ are the colatitude and longitude respectively.

Following Phinney & Burridge (1973), we express the solution in terms of the canonical spherical basis, where contravariant components of displacement are given by

$$u^\pm = 1/\sqrt{2}\,(\mp u_\theta - iu_\phi), \qquad u^0 = u_r, \quad (8)$$

and new unitary basis vectors are

$$\mathbf{e}_\pm = 1/\sqrt{2}\,(\mp\mathbf{e}_\theta + i\mathbf{e}_\phi), \qquad \mathbf{e}_0 = \mathbf{e}_r. \quad (9)$$

In this basis, using the operator $\mathbf{D}$, the displacement field $\mathbf{u}$ associated with a free mode nℓm is thus

$$\mathbf{u}_\ell^m(r,\theta,\phi) = \mathbf{D}Y_\ell^m(\theta,\phi) = \gamma_\ell U(r)Y_\ell^{0m}(\theta,\phi)\mathbf{e}_0 + \gamma_\ell\Omega^0{}_\ell[V(r)+iW(r)]Y_\ell^{+1m}(\theta,\phi)\mathbf{e}_+ + \gamma_\ell\Omega^0{}_\ell[V(r)-iW(r)]Y_\ell^{-1m}(\theta,\phi)\mathbf{e}_-, \quad (10)$$

where γ_ℓ and $\Omega_\ell{}^N$ are given by

$$\gamma_\ell = \sqrt{(\ell(\ell+1))/2}, \qquad \Omega_\ell{}^N = \sqrt{(\ell+N)(\ell-N+1)/2}. \quad (11)$$

U, V, W are the same functions as in (2) and $Y_\ell^{Nm}(\theta,\phi)$ is the generalized spherical harmonic defined in Phinney & Burridge (1973). The displacement field $\mathbf{u}$ may be written with the operator components

$$\mathbf{u}_\ell^m(r,\theta,\phi) = u_\ell{}^{\alpha m}(r,\theta,\phi)\mathbf{e}_\alpha = D_\ell{}^\alpha(r)Y_\ell{}^{\alpha m}(\theta,\phi)\mathbf{e}_\alpha,$$

using summation convention on indice, $\mathbf{u}$ denoting the displacement eigenfunction of a mode nℓm, $D_\ell{}^\alpha$ the associated operator component of displacement, which is

independent of m in the basis of generalized spherical harmonics. For spheroidal modes, we thus have

$$u_\ell^{\pm m}(r, \theta, \phi) = \gamma_\ell \Omega^0{}_\ell V(r) Y_\ell^{\pm 1m}(\theta, \phi),$$
$$u_\ell^{0m}(r, \theta, \phi) = \gamma_\ell U(r) Y_\ell^{m}(\theta, \phi), \quad (12)$$

and for toroidal modes,

$$u_\ell^{\pm m}(r, \theta, \phi) = \pm i\gamma_\ell \Omega^0{}_\ell W(r) Y_\ell^{\pm 1m}(\theta, \phi),$$
$$u_\ell^{0m}(r, \theta, \phi) = 0. \quad (13)$$

3.2 Expression of the aspherical modes in the SNREI normal mode basis

To ensure the completeness of the basis of spherical normal modes which will help us to express the aspherical modes, we shall map the volume V of the aspherical model, with elliptical of more complicated topography, into the volume V_0 of the spherical earth. This can be done by introducing some set of local mappings between the two volumes V and V_0 and leads to a new expression of (3) with all integration done within the volume of the spherical earth (Valette 1987; Woodhouse & Dahlen 1978). With this restriction, it is thus possible to express any displacement field $|\mathbf{v}\rangle$, defined in V_0, in the basis of the SNREI normal mode eigenfunctions, $|\mathbf{k}_0\rangle$ (where $\mathbf{k}_0$ is ${}_n\mathbf{u}_\ell^m$) as

$$|\mathbf{v}\rangle = \sum_{\mathbf{k}_0} |\mathbf{k}_0\rangle \langle \mathbf{k}_0 | \mathbf{v}\rangle_0, \quad (14)$$

where $\langle \mathbf{k}_0 | \mathbf{v}\rangle_0$ may be complex and is obtained with the spherical inner-product (6). In principle, the summation over $\mathbf{k}_0$ includes not only all seismic modes (e.g. modes with a period less than 55 mn), but also the sub-seismic and secular ones. However, it is practically limited to the few nearest frequency modes, this restriction being motivated either by the relatively fast decrease of coupling between modes along dispersion branches, or by selection rules.

It is now possible to define the perturbation of the operator **A** in such a way that

$$\langle \mathbf{u}| \mathbf{A} |\mathbf{v}\rangle = \langle \mathbf{u}| \mathbf{A_0} |\mathbf{v}\rangle_0 + \langle \mathbf{u}| \delta\mathbf{A} |\mathbf{v}\rangle_0 \quad (14bis)$$

where $\delta\mathbf{A}$ will now include two kinds of perturbations. The first one is related to the volumic mapping between the volume V of the aspherical earth and the volume V_0 of the spherical reference earth and can be, in practice, expressed only by its first-order approximation (Woodhouse & Dahlen 1978; Woodhouse 1980; Valette 1987). On the other hand, the second one can be exactly expressed and is related to the volumic lateral heterogeneities of the stiffness tensor, or to the asphericity of the discontinuities, located in the 'geocentric' position (i.e. after the mapping between V and V_0). We shall now assume that all integrations are done for the volume and density V_0, ρ_0 and thus forget the index $_0$ for the bracket products.

3.3 Unbroken symmetries in the SNREI model and coupling control in the aspherical model

Many methods can be used to control the coupling between SNREI modes induced by lateral heterogeneities, that is, to obtain *a priori* information on the projection of a given aspherical mode $|\mathbf{v}\rangle$ on the SNREI normal mode basis. One of the most powerful methods is to make use of all the unbroken symmetries of the SNREI model.

The connection between group theory, symmetries and earth's normal modes was first presented by Phinney & Burridge (1973), and later by Chao (1981) and appears from time to time in other papers. In this part, we will recall some symmetry properties of spherical harmonics and normal modes of a spherical earth, which will be used later for the computation of the normal modes of a general, aspherical earth. Each of these symmetries, which lead to the degeneracy of the SNREI eigenvibrations, can be associated with an operator, which, applied to the spherical model, leaves it unchanged. We will briefly recall these operators for the three most important symmetries, that is, for the spatial symmetries around the centre of the earth and around a plane containing the rotation axis, and for the time symmetry due to the pure elastic behaviour of this model. More details can be found in Appendix 1.

The first symmetry is the spatial symmetry **P** about the centre of the earth or parity symmetry. At each point, it transforms a vector into an opposite vector located at the antipodal point, which simply gives

$$\mathbf{Pu}(\mathbf{x}) = -\mathbf{u}(-\mathbf{x}).$$

In the SNREI case, we recall in Appendix 1, that the spheroidal modes of order ℓ have the same parity as ℓ:

$$\mathbf{Pu}_\ell = (-1)^\ell \mathbf{u}_\ell, \quad (15a)$$

and that the toroidal modes of order ℓ have the opposite parity to ℓ:

$$\mathbf{Pu}_\ell = (-1)^{\ell+1} \mathbf{u}_\ell. \quad (15b)$$

This symmetry is only broken by lateral variations in structure which are antisymmetrical with respect to **P**. It is not broken, in particular, by rotation or ellipticity of the earth.

The second spatial symmetry, unbroken in the SNREI case, is the orthogonal symmetry **S** about the meridian plane $\phi = 0$, or around any plane containing the earth's rotation axis. Its operator is such that

$$\mathbf{Su}_r(\theta, \phi) = \mathbf{u}_r(\theta, 2\pi - \phi), \qquad \mathbf{Su}_\theta(\theta, \phi) = \mathbf{u}_\theta(\theta, 2\pi - \phi),$$
$$\mathbf{Su}_\phi(\theta, \phi) = -\mathbf{u}_\phi(\theta, 2\pi - \phi),$$

that is in canonical coordinates,

$$S^\alpha(\mathbf{u}) = (-1)^\alpha u^{-\alpha}(\theta, 2\pi - \phi).$$

This symmetry separates the SNREI modes into **S** symmetrical ones, [cos(mϕ) modes], and antisymmetrical ones, [sin(mϕ) modes]. It is broken by rotation and by the **S** antisymmetric part of lateral heterogeneities.

The last symmetry **T** acts on the time variable and is just the complex conjugation

$$\mathbf{Tu} = \mathbf{u}^*.$$

This symmetry is unbroken for the SNREI model, which does not include anelasticity, and separates the SNREI modes into real and purely imaginary ones. More generally, if anelasticity is introduced, as shown by Dahlen (1981), the **T** operator maps the space of permissible eigenfunctions into its dual space. This **T** time symmetry is broken by anelasticity, but also by rotation.

Table 1. Structure of the eight symmetry subspaces associated with the eigenspace of the operators P, T, S. In the elastic case, only the four subspaces with $\eta_T = 1$ are necessary. (a) Structure of the real eigenfunctions. (b) Structure of the imaginary eigenfunctions.

(a)

Subset 1

$\beta^m{}_{+\ell} = \frac{1}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ even

$\zeta^m{}_{+\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ odd

$\eta_P, \eta_T, \eta_S = +1, +1, +1$

$\zeta^m{}_{+\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ odd

$\beta^m{}_{+\ell} = \frac{1}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ even

$\eta_P, \eta_T, \eta_S = +1, +1, -1$

Subset 2

$\beta^m{}_{+\ell} = \frac{1}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ odd

$\zeta^m{}_{+\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ even

$\eta_P, \eta_T, \eta_S = -1, +1, +1$

$\zeta^m{}_{+\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ even

$\beta^m{}_{+\ell} = \frac{1}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ odd

$\eta_P, \eta_T, \eta_S = -1, +1, -1$

(b)

Subset 3

$\zeta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ even

$\beta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ odd

$\eta_P, \eta_T, \eta_S = +1, -1, +1$

$\beta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ odd

$\zeta^m{}_{-\ell} = \frac{I}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ even

$\eta_P, \eta_T, \eta_S = +1, -1, -1$

Subset 4

$\zeta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ odd

$\beta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ even

$\eta_P, \eta_T, \eta_S = -1, -1, +1$

$\beta^m{}_{-\ell} = \frac{i}{\sqrt{2}}(u_\ell{}^m + (-1)^m u_\ell{}^{-m})$

Toroidal with ℓ even

$\zeta^m{}_{-\ell} = \frac{1}{\sqrt{2}}(u_\ell{}^m - (-1)^m u_\ell{}^{-m})$

Spheroidal with ℓ odd

$\eta_P, \eta_T, \eta_S = -1, -1, -1$

We have thus redefined three symmetry operators, which all commute with the SNREI operator $\mathbf{A_0}$, which implies that the SNREI eigenfunctions are their eigenfunctions as well. Note that, as the SNREI basis is complete, in terms of quantum mechanics the set of operators **P**, **T**, **S** and $\mathbf{A_0}$ constitutes a complete set of commuting operators (Cohen-Tannoudji, Diu & Laloë 1980). Thus, the space of displacement field over V_0, as well as the decomposition (14), may be expressed as a direct summation of the 2^3 subspaces associated with the different eigenvalues for the three operators P, T, S, $(\eta_P, \eta_S, \eta_T) = (\pm 1, \pm 1, \pm 1)$, and all elements of this basis, eigenmodes of the SNREI model, will thus be uniquely defined by the set of six indices $\mathbf{k}_0 = (\ell, n, \eta_P, |m|, \eta_S, n_T)$, where n is the radial order of the mode. (See Table 1 for a summary.) The coupling control of the interaction terms can now be done by decomposing the perturbation operator $\delta\mathbf{A}$ into its **P**, **S** and **T** symmetrical and antisymmetrical parts.

4 PERTURBATION THEORY STARTING FROM AN SNREI MODEL

4.1 Higher order perturbation series

A direct way to solve the aspherical eigenproblem (1) is to use a variational or Galerkin method to calculate the hybrid singlet eigenfrequencies and eigenfunctions in the basis of SNREI normal modes (Park & Gilbert 1986; Morris *et al.* 1987). The computations are, however, time consuming, especially with full interaction matrices for aspherical structure with a scale comparable to the wavelength of the normal modes. Another alternative is to use perturbation theory in various forms (Woodhouse 1983; Tanimoto 1984; Dahlen 1987; Snieder & Romatowicz 1988). Let us briefly recall this method starting from equation (1), which, as the brackets are now defined for the spherical density, can be rewritten as

$$-\omega^2\mathbf{K}\,|\mathbf{v}\rangle + \omega\mathbf{B}\,|\mathbf{v}\rangle + \mathbf{A}\,|\mathbf{v}\rangle = \mathbf{0}, \tag{16}$$

where the elasto-dynamic operator **A** is defined in (2), **K** and **B** are

$$\mathbf{K} = \rho/\rho_0\mathbf{1}, \qquad \mathbf{B} = \rho/\rho_0\mathbf{B_0},$$

and ω_v and $|\mathbf{v}\rangle$ are the eigenfrequency and eigenmode of the aspherical normal modes.

Let us expand the eigenmode $|\mathbf{v}\rangle$ and eigenfrequency ω_v in terms of a power series of a small parameter ϵ related for example to the perturbation $\delta\mathbf{A}$ as defined in (14bis). We shall define

$$\begin{aligned}\omega_v &= \omega_v{}^{(0)} + \delta_1\omega_v + \delta_2\omega_v + \ldots + \delta_n\omega_v + \ldots,\\ |\mathbf{v}\rangle &= |0, \mathbf{v}\rangle + |1, \mathbf{v}\rangle + |2, \mathbf{v}\rangle + \ldots + |n, \mathbf{v}\rangle + \ldots,\end{aligned} \tag{17}$$

where the eigenfrequencies $\delta_n\omega_v$ and the eigenmode $|n, \mathbf{v}\rangle$ are of order ϵ^n. Here, $\omega_v{}^{(0)}$ and $|0, \mathbf{v}\rangle$ are the eigenfrequency and eigenmode for the SNREI model with elastodynamic operator $\mathbf{A_0}$. We define in the same way, in terms of powers of ϵ, the squared eigenfrequency:

$$\begin{aligned}\omega_v^{2(n)} = \omega^2{}_v{}^{(0)} &+ \delta_1\omega^2{}_v + \delta_2\omega^2{}_v + \delta_3\omega^2{}_v + \ldots\\ &+ \delta_n\omega^2{}_v + \ldots,\end{aligned}$$

where

$$\begin{aligned}\omega_v^{(0)2} &= \omega^2{}_v{}^{(0)},\\ \delta_1\omega^2{}_v &= 2\omega_v^{(0)}\delta_1\omega_v,\\ \delta_2\omega^2{}_v &= \delta_1\omega_v{}^2 + 2\omega_v^{(0)}\delta_2\omega_v, \qquad \text{etc.}\end{aligned} \tag{18}$$

Expanding equation (16) into powers of ϵ yields the following power series:

$$-\omega_v{}^2\mathbf{K}\,|\mathbf{v}\rangle + \omega_v\mathbf{B}\,|\mathbf{v}\rangle - \mathbf{A}\,|\mathbf{v}\rangle = \mathbf{0} + \sum_{n=1}^{n=\infty}\mathbf{f}_n, \tag{19}$$

with

$$\mathbf{f}_n = (\mathbf{A_0} - \omega_v^{(0)2}\mathbf{1})\,|n, \mathbf{v}\rangle + |n-1, r, \mathbf{v}\rangle - \delta_n\omega^2{}_v\,|0, \mathbf{v}\rangle, \tag{20}$$

where $|n-1, r, \mathbf{v}\rangle$ can be expressed in terms of perturbations to the eigenfrequencies and eigenmodes up to order $n-1$. The first three such terms are, for example,

$$\begin{aligned}|0, r, \mathbf{v}\rangle &= \delta\mathbf{H}\,|0, \mathbf{v}\rangle,\\ |1, r, \mathbf{v}\rangle &= \delta\mathbf{H}\,|1, \mathbf{v}\rangle - \delta_1\omega^2{}_v\left[|1, \mathbf{v}\rangle + \left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}\right]|0, \mathbf{v}\rangle\right],\\ |2, r, \mathbf{v}\rangle &= \delta\mathbf{H}\,|2, \mathbf{v}\rangle - \delta_1\omega^2{}_v\left[|2, \mathbf{v}\rangle + \left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}\right]|1, \mathbf{v}\rangle\right]\\ &\quad - \delta_2\omega^2{}_v\left[|1, \mathbf{v}\rangle + \left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}\right]|0, \mathbf{v}\rangle\right]\\ &\quad + \left[\delta_2\omega_v - \frac{\delta_2\omega^2{}_v}{2\omega_v^{(0)}}\right]\mathbf{B}\,|0, \mathbf{v}\rangle,\end{aligned} \tag{21}$$

where

$$\delta\mathbf{H} = \delta\mathbf{A} + \omega_v^{(0)}\mathbf{B} - \omega_v^{(0)2}\delta\mathbf{K}, \qquad \delta\mathbf{K} = \mathbf{K} - \mathbf{1}.$$

More generally, the nth-order 'residual' term $|n, r, \mathbf{v}\rangle$ is given by

$$|n, r, \mathbf{v}\rangle = \delta\mathbf{H}\,|n, \mathbf{v}\rangle - \sum_{p=1}^{p=n} \delta_p\omega^2{}_v\left[|n-p+1, \mathbf{v}\rangle - \left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}\right]|n-p, \mathbf{v}\rangle\right] + \left[\delta_p\omega_v - \frac{\delta_p\omega^2{}_v}{2\omega_v^{(0)}}\right]\mathbf{B}\,|n-p, \mathbf{v}\rangle.$$

Since each term of order ϵ_n in equation (19) is equal to zero, we obtain the following relations

$$(\omega_v^{(0)2}\mathbf{1} - \mathbf{A}_0)\,|n, \mathbf{v}\rangle = |n-1, r, \mathbf{v}\rangle - \delta_n\omega^2{}_n\,|0, \mathbf{v}\rangle. \tag{22}$$

Multiplying the left side of this relation by $\langle \mathbf{v}, 0|$ yields an expression for the nth-order perturbation in the eigenfrequency, in terms of approximations up to order $n-1$:

$$\delta_n\omega^2{}_v = \langle \mathbf{v}, 0 \mid n-1, r, \mathbf{v}\rangle. \tag{23}$$

The first-order frequency perturbation $\delta_1\omega^2{}_v$ and the zeroth-order perturbation $|0, \mathbf{v}\rangle$ are solution of the eigenproblem obtained by projecting the 'residual' $\mathbf{f}_1$ on the subspace S_v associated to the SNREI multiplet with eigenfrequency equal to $\omega_v^{(0)}$:

$$\mathcal{P}\mathbf{f}_1 = \mathcal{P}\delta\mathbf{H}\,|0, \mathbf{v}\rangle - \delta_1\omega^2{}_v\,|0, \mathbf{v}\rangle = \mathbf{0}, \tag{24}$$

where $\mathcal{P}$ is the projector into the subspace S_v. If the eigenmodes $|0, \mathbf{v}\rangle$ are normalized, the eigenfrequencies are thus given by

$$\delta_1\omega^2{}_v = \langle \mathbf{v}, 0|\,\delta\mathbf{H}\,|0, \mathbf{v}\rangle.$$

Note that equation (24) is the definition of the isolated multiplet perturbation theory (*cf.* Messiah 1962; Madariaga 1971).

In order to calculate the nth approximation $|n, \mathbf{v}\rangle$ to the eigenvector, we can write, using relation (14),

$$|n, \mathbf{v}\rangle = \sum_{\mathbf{k}_0} |\mathbf{k}_0\rangle\langle \mathbf{k}_0 \mid n, \mathbf{v}\rangle$$

and thus we need to obtain the projection of $|n, \mathbf{v}\rangle$ onto the eigenmodes of the SNREI model. Equation (22) gives us only the projection on modes that do not belong to multiplet S_v, that is modes $|\mathbf{k}_0\rangle$ such that $\omega_{k_0}{}^{(0)} \neq \omega_v{}^{(0)}$. For these modes, from (22)

$$\langle \mathbf{k}_0 \mid n, \mathbf{v}\rangle = \frac{1}{\omega_v^{2(0)} - \omega_{k_0}^{2(0)}}\langle \mathbf{k}_0 \mid n-1, r, \mathbf{v}\rangle. \tag{25}$$

Within the multiplet S_v, we need to calculate the projection differently. As noted by Dahlen (1989, personnal communication), or Landau & Lifschitz (1965), Lyons (1985), the perturbation $|n, \mathbf{v}\rangle$, even to first order, is orthogonal to this subspace only if S_v is non-degenerate and if the inner product is not perturbed, that is if the angular order is zero and if no rotation and density lateral heterogeneities are taken into account. To obtain the expression of all nth-order projections on S_v, we make use of the following two relations, verified by any aspherical modes $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ of S_v:

$$\langle \mathbf{u}|\,-\omega_v{}^2\mathbf{K} + \omega_v\mathbf{B} + \mathbf{A}\,|\mathbf{v}\rangle = 0, \tag{26}$$

$$\langle \mathbf{u}|\,\mathbf{K} - 1(\omega_v + \omega_u)\mathbf{B}\,|\mathbf{v}\rangle = 1/2\,\langle \mathbf{u} \mid \mathbf{v}\rangle = \delta_{uv}. \tag{27}$$

Relation (26) stems directly from equation (16), and relation (27) is obtained from the definition of the inner product $\langle \mathbf{u} \mid \mathbf{v}\rangle$ and from equation (16). We then expand relation (25) and (26) into powers of ϵ, and obtain, after some algebra, the projection of the perturbation $|n, \mathbf{v}\rangle$ on the elements of S_v

$$\langle \mathbf{u}, 0 \mid n, \mathbf{v}\rangle = \frac{1}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathbf{A}^n_{uv} - \frac{\delta_1\omega^2{}_v}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathbf{B}^n_{uv}, \tag{28}$$

where $\mathbf{u} \neq \mathbf{v}$ and where $\mathbf{A}^n_{uv}$ and $\mathbf{B}^n_{uv}$ are given in Appendix 2. Note that we have used the zero-order ket $|0, \mathbf{u}\rangle$ of the multiplet S_v and not the SNREI modes $|\mathbf{u}_0\rangle$ of this multiplet to compute the projection of $|n, \mathbf{v}\rangle$. The last unknown is now the projection of the perturbation $|n, \mathbf{v}\rangle$ onto the unperturbed singlet $|0, \mathbf{v}\rangle$ itself. For this, we use expressions (2.8) and (2.9) of Appendix 2 and obtain

$$\langle \mathbf{v}, 0 \mid n, \mathbf{v}\rangle = -1/2\,\mathbf{B}^n_{vv}, \tag{29}$$

where we nave assumed that the phase of the eigenmode, which is unconstrained, is such that $\langle \mathbf{v}, 0 \mid \mathbf{v}\rangle$ is real, that is $\langle \mathbf{v}, 0 \mid n, \mathbf{v}\rangle$ is real for all order n. Hence, using (25), (28), and (29), we obtain

$$|n, \mathbf{v}\rangle = \sum_{\mathbf{k}_0 \notin S_v} \frac{1}{\omega_v^{2(0)} - \omega_{k_0}^{2(0)}}|\mathbf{k}_0\rangle\langle \mathbf{k}_0 \mid n-1, r, \mathbf{v}\rangle - |0, \mathbf{v}\rangle 1/2\,\mathbf{B}^n_{vv} + \sum_{\mathbf{u}\in S_v,\,\mathbf{u}\neq\mathbf{v}} |0, \mathbf{u}\rangle\left(\frac{1}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathbf{A}^n_{uv} - \frac{\delta_1\omega^2{}_v}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathbf{B}^n_{uv}\right). \tag{30}$$

Note that the first part of this equation can be also written with the ket $|0, \mathbf{k}\rangle$, solution of (24) for all multiplets, instead of the ket $|\mathbf{k}_0\rangle$, as both $|0, \mathbf{k}\rangle$ and $|\mathbf{k}_0\rangle$ are a base of the subspace S_k. Expressions (23) and (30) enable us to obtain the nth approximation to eigenfrequency and eigenfunction when we know approximations up to order $(n-1)$. We note that in the previous derivation, we have assumed that $\delta\mathbf{A}$ and $\mathbf{B}$ can be expanded using the same small coefficient ϵ. However, it is possible to obtain similar perturbation series directly in the product space (Valette 1989b), where eigenmodes are expressed in terms of the generalized ket

$$|\mathbf{v}\rangle = \begin{pmatrix} 1/\omega_v\mathbf{A}^{1/2}\mathbf{v} \\ \mathbf{v} \end{pmatrix}.$$

This approach is more similar to the classical Rayleigh Schroedinger perturbation of the Hamiltonian (see Appendix 4). Note also that the second part of (30), i.e. the summation over all singlets of S_v is generally omitted in the previously cited studies, and is necessary for the coherency of (30) with results obtained with other perturbation theories, such as the Brillouin–Wigner perturbation theory (Ziman 1969).

4.2 Secular equation

As seen in the previous section, all nth-order perturbations $|n, \mathbf{v}\rangle$ are, in general, not orthogonal to the subspace S_v, their projections on S_v being expressed using relations (28) and (29). However, expression (28) contains terms such as

$$\frac{1}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u},$$

where modes $\mathbf{u}$ and $\mathbf{v}$ are two different singlets of the same multiplet S_v. Such terms appear to be very unstable unless singlets are regularly split. In a few cases, the terms A^n_{uv} and B^n_{uv} cancel, as for the perturbations due to rotation and ellipticity, for which the eigenfunctions are simply related to $Y_\ell{}^m$ harmonic functions and the first-order frequencies are expressed in terms of the azimuthal order m:

$$\delta_1\omega^2 = 2\omega^{(0)2}(a + mb + m^2c),$$

where b and a, c are linearly related to the rotation velocity Ω and to the ellipticity parameter ϵ_e respectively (Dahlen & Sailor 1979). Equation (30) generalizes the results of Dahlen & Sailor's paper, especially for the eigenmode perturbations. However, equation (30) is not suitable for the eigenmode perturbations due to lateral variations, especially to those due to smooth models, for which some of the $2\ell + 1$ singlets frequencies $\delta_1\omega_v$ are grouped into doublets (Dahlen & Hensen 1985), or are very close to one another.

We thus propose to perform modifications to the classical perturbation theory in order to force the first nth-order perturbations to be orthogonal to the subspace S_v, which amounts to cancelling the first A^n_{uv} and B^n_{uv} terms of equation (28). Looking at the expression of (26) developed into powers of ϵ, we see in the Appendix 2 that an operator $\delta\mathcal{H}$ can be defined by

$$\delta\mathcal{H} = \mathcal{P}^*(\delta\mathbf{H} + \delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H} + \delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H} + \ldots)\mathcal{P},$$

whose nth-order terms appears in the expression of $A_{uv}{}^n$. Here, * denotes the adjoint operator and $\mathbf{G}_v$ is defined as

$$\mathbf{G}_v = \sum_{k_0 \notin S_v} \frac{1}{\omega^{2(0)}_v - \omega^{2(0)}_{k_0}} |\mathbf{k}_0\rangle\langle\mathbf{k}_0|.$$

In the same way, looking at the power series of expression (27), another operator $\mathcal{N}$ can be defined, such that

$$\mathcal{N} = \mathcal{P}^*(\mathbf{1} + \mathbf{G}_v\delta\mathbf{H} + \mathbf{G}_v\delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H} + \ldots)^* \times \left(\mathbf{1} + \delta\mathbf{K} - \frac{\mathbf{B}}{2\omega^{(0)}_v}\right) \times (\mathbf{1} + \mathbf{G}_v\delta\mathbf{H} + \mathbf{G}_v\delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H} + \ldots)\mathcal{P}.$$

Instead of using a strength coefficient ϵ related to $\delta\mathbf{H}$, we will thus develop (26) and (27) with help of a coupling coefficient related to $\delta\mathcal{H}$. The first-order terms of (26) and (27) thus lead to the following two equations, which must be verified by two different zeroth-order singlets $|0, \mathbf{v}\rangle$ and $|0, \mathbf{u}\rangle$ of the multiplet S_v:

$$\langle\mathbf{u}, 0|\,\delta\mathcal{H}\,|0, \mathbf{v}\rangle = 0, \qquad \langle\mathbf{u}, 0|\,\mathcal{N}\,|0, \mathbf{v}\rangle = 0, \qquad \text{for } \mathbf{u} \neq \mathbf{v}.$$

This can be done if the zeroth-order kets $|0, \mathbf{v}\rangle$ verify the following eigenproblem, which will be called the 'secular equation':

$$\delta\mathcal{H}\,|0, \mathbf{v}\rangle = \delta_1\omega^2{}_v\mathcal{N}\,|0, \mathbf{v}\rangle. \tag{31}$$

This equation appears obviously as a generalization of equation (24) defining the order zero of 'classical' perturbation theory

$$\mathcal{P}\delta\mathbf{H}\mathcal{P}\,|0, \mathbf{v}\rangle = \delta\omega^2{}_v\,|0, \mathbf{v}\rangle,$$

and also as a generalization of the equation obtained by Landau & Lifschitz (1965), Park (1987) or Dahlen (1987)

$$\mathcal{P}(\delta\mathbf{H} + \delta\mathbf{H}\mathbf{G}_v\delta\mathbf{H})\mathcal{P}\,|0, \mathbf{v}\rangle = \delta\omega^2{}_v\,|0, \mathbf{v}\rangle.$$

It is now necessary to perform modifications to the power series of equation (16). We do that by redefining the terms (20) of this power series as

$$\mathbf{f}_n = (\mathbf{A}_0 - \omega^{(0)2}_v\mathbf{1})\,|n, \mathbf{v}\rangle + |n-1, r, \mathbf{v}\rangle - \delta_n\omega^2{}_v\mathcal{N}\,|0, \mathbf{v}\rangle$$

where we have redefined the 'residual' part $|n, r, \mathbf{v}\rangle$ as

$$|0, r, \mathbf{v}\rangle = [(\mathbf{1} - \mathcal{P})\delta\mathbf{H} + \delta\mathcal{H}]\,|0, \mathbf{v}\rangle,$$

for order zero, and as

$$|n, r, \mathbf{v}\rangle \Rightarrow |n, r, \mathbf{v}\rangle - \mathcal{P}\delta\mathbf{H}[\mathbf{G}_v\delta\mathbf{H}]^n\,|0, \mathbf{v}\rangle - \delta_n\omega^2{}_v(\mathbf{1} - \mathcal{N})\,|0, \mathbf{v}\rangle,$$

for nth order. These new definitions lead to the same expression for the projection of the peturbation $|n, \mathbf{v}\rangle$ onto other multiplets than S_v as in (25) and to the same expression for the perturbation of the eigenfrequency as in (23).

In the same manner, as in the previous section, we now obtain, after some algebra, new expressions for the projections of the ket $|n, \mathbf{v}\rangle$ onto the singlets $|0, \mathbf{v}\rangle$, first for $\mathbf{u} = \mathbf{v}$:

$$\langle\mathbf{v}, 0 \mid n, \mathbf{v}\rangle = -1/2\mathcal{B}^n_{vv},$$

and then for $\mathbf{u} \neq \mathbf{v}$,

$$\langle\mathbf{v}, 0 \mid n, \mathbf{v}\rangle = \frac{1}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathcal{A}^n_{uv} - \frac{\delta_1\omega^2{}_v}{\delta_1\omega^2{}_v - \delta_1\omega^2{}_u}\mathcal{B}^n_{uv}, \tag{32}$$

where $\mathcal{A}^{1,2}_{uv}$ and $\mathcal{B}^{1,2}_{uv}$ are now defined as

$$\mathcal{A}^1{}_{uv} = 0, \qquad \mathcal{B}^1{}_{uv} = 0,$$

$$\mathcal{A}^2{}_{uv} = \delta_1\omega_v\delta_1\omega_u\langle\mathbf{u}, 0|\frac{\mathbf{B}}{2\omega^{(0)}_v}|0, \mathbf{v}\rangle,$$

$$\mathcal{B}^2{}_{uv} = \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega^{(0)}_v}\langle\mathbf{u}, 0|\frac{\mathbf{B}}{2\omega^{(0)}_v}|0, \mathbf{v}\rangle,$$

$$\mathcal{A}^3{}_{uv} = \left\{-\delta_1\omega^2{}_v\delta_1\omega^2{}_u\left[\langle\mathbf{u}, 1| + \langle\mathbf{u}, 0|\left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega^{(0)}_v}\right]\right] \times \mathbf{G}_v\left[\left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega^{(0)}_v}\right]|0, \mathbf{v}\rangle + |1, \mathbf{v}\rangle\right]\right\} + \left\{\left(\delta_1\omega_v\delta_2\omega_u + \delta_1\omega_u\delta_2\omega_v - \delta_1\omega_u\delta_1\omega_v \times \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega^{(0)}_v}\right)\langle\mathbf{u}, 0|\frac{\mathbf{B}}{2\omega^{(0)}_v}|0, \mathbf{v}\rangle\right\} + \delta_1\omega_v\delta_1\omega_u\left(\langle\mathbf{u}, 1|\frac{\mathbf{B}}{2\omega^{(0)}_v}|0, \mathbf{v}\rangle + \langle\mathbf{u}, 0|\frac{\mathbf{B}}{2\omega^{(0)}_v}|1, \mathbf{v}\rangle\right),$$

$$\mathscr{B}^3_{uv} = \left\{ -(\delta_1\omega^2{}_v + \delta_1\omega^2{}_u)\left[\langle \mathbf{u}, 1| + \langle \mathbf{u}, 0| \left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}} \right] \right] \right.$$
$$\left. \times \mathbf{G}_v \left[\left[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}} \right] |0, \mathbf{v}\rangle + |1, \mathbf{v}\rangle \right] \right\}$$
$$+ \frac{\delta_1\omega_u + \delta_2\omega_v}{2\omega_v^{(0)}} \left(\langle \mathbf{u}, 1| \frac{\mathbf{B}}{2\omega_v^{(0)}} |0, \mathbf{v}\rangle + \langle \mathbf{u}, 0| \frac{\mathbf{B}}{2\omega_v^{(0)}} |0, \mathbf{v}\rangle \right)$$
$$+ \left(\frac{(\delta_1\omega_u + \delta_1\omega_v)^2}{4\omega_v^{(0)2}} - \frac{\delta_2\omega_u + \delta_2\omega_v}{2\omega_v^{(0)}} \right) \langle \mathbf{u}, 0| \frac{\mathbf{B}}{2\omega_v^{(0)}} |0, \mathbf{v}\rangle. \quad (33)$$

We thus have the same expression as before for any nth-order perturbation in eigenfrequency and eigenmode associated to singlet $|\mathbf{v}\rangle$, noting that the $\delta_1\omega^2$ used in relation (32) are now the eigenfrequencies obtained by the secular equation. However, the main difference is that the secular equation leads to a first-order perturbation always orthogonal to the starting subspace S_v. It remains valid to second order, for which the non-orthogonality of the eigenmode perturbation is only related to the rotation, whose effect is linear and not quadratic in the frequency. This effect can thus be well modelled using relation (32). It can nevertheless be easily shown that, if the perturbation series is done in the product space (see Appendix 4) or if the rotation is neglected, the second-order perturbation is orthogonal to the subspace S_v. However, no such simplification is possible for the third-order perturbation, which always has a non-orthogonal component, but, as we shall see in the last section of this paper, it appears in practice to be very small and well expressed by relation (32).

Taking as zero-order ket the solution to the secular equation greatly simplifies the perturbation procedure. However, we shall now describe another modification to the until now described method, in order to reduce the effect induced by the perturbation of the inner-product, due to density and rotation.

4.3 Density renormalization

A renormalization procedure was previously proposed by Park (1985), in order to take into account the inner product perturbation due to density lateral variations and to physical dispersion (omitting however the perturbation due to rotation). However, we shall now prove that this procedure, generally done numerically by using a Cholesky decomposition, can be performed analytically for the density terms. In this case, instead of looking for the hybrid multiplet $|\mathbf{v}\rangle$, we search for the renormalized ket, defined as

$$|\mathbf{v}_r\rangle = \mathbf{K}^{1/2} |\mathbf{v}\rangle$$

with the same eigenfrequency. This renormalization, which is possible whenever the density ρ is non zero inside V is sufficient to obtain a standard eigenproblem from relation (16), which can be written as

$$-\omega^2 |\mathbf{v}_r\rangle + \omega \mathbf{B}_0 |\mathbf{v}_r\rangle + \mathbf{A}_r |\mathbf{v}_r\rangle = \mathbf{0}, \quad (34)$$

where, after renormalization, the Coriolis operator $\mathbf{B}_0$ is the same as in the spherical earth, and thus has selection rules which allow the coupling of a mode of angular order ℓ only with modes of angular order $\ell \pm 1$, and where $\mathbf{A}_r$, formally given by

$$\mathbf{A}_r = \mathbf{K}^{-1/2} \mathbf{A} \mathbf{K}^{-1/2},$$

can be expressed analytically. (See Appendix 3 for details and the expression of $\mathbf{A}_r$.) Note that the renormalization yields a different parametrization of the Hamiltonian, closer to parametrizations used for example in ray or asymptotic theory. The new aspherical parameters are thus $\rho_0/\rho\ C^{ijklr}$ and $\mathrm{Ln}(\rho/\rho_0)$, the expression of the former being related to the squared S and P velocities, and one looks for the renormalized eigenmodes, projected on the renormalized SNREI normal mode basis. The inner product (4) between two normal modes $\mathbf{u}$ and $\mathbf{v}$ is now transformed into

$$\langle \underline{\mathbf{u}} \,|\, \mathbf{v} \rangle = \langle \mathbf{u}| \mathbf{K} - \mathbf{B}/(\omega_u + \omega_v) |\mathbf{v}\rangle$$
$$= \langle \mathbf{u}_r| \mathbf{1} - \mathbf{B}_0/(\omega_u + \omega_v) |\mathbf{v}_r\rangle.$$

If any Cholesky decomposition is needed for the 'new' inner product, as for the perturbation series described in the relations (23) and (30), or for a variational procedure, it can be done around a fiducial frequency using the sparse operator

$$\mathbf{1} - \frac{\mathbf{B}}{2\omega_u^{(0)}},$$

which greatly reduces the number of required operations. A quick comparison of the results obtained with the density renormalization can be done with those obtained without renormalization, using for example Born linearization of the kinetic term (Tanimoto 1984). Making a development of the square root of the operator $\mathbf{K}^{1/2} \simeq \mathbf{I} + 1/2\delta\mathbf{K}$, we have to first order,

$$\delta\mathbf{A}_r = \mathbf{A}_r - \mathbf{A}_0 = \mathbf{K}^{-1/2}(\mathbf{A}_0 + \delta\mathbf{A})\mathbf{K}^{-1/2} - \mathbf{A}_0$$
$$= \delta\mathbf{A} - 1/2(\delta\mathbf{K}\mathbf{A}_0 + \mathbf{A}_0\delta\mathbf{K}).$$

We thus obtain the following first-order expression for the eigenfrequency:

$$2\omega_v^{(0)}\delta_1\omega_v = \langle \mathbf{v}, 0| \delta\mathbf{A} - \omega_k^{(0)2}\delta\mathbf{K} |0, \mathbf{v}\rangle,$$

and for the part of eigenmode orthogonal to S_v:

$$(\mathbf{1} - \mathscr{P}) |1, \mathbf{v}\rangle = \sum_{\mathbf{k}_0} \frac{1}{\omega_v^{(0)2} - \omega_{k_0}^{(0)2}} |\mathbf{k}_0\rangle$$
$$\times \langle \mathbf{k}_0| \delta\mathbf{A} - 1/2(\omega_v^{(0)2} + \omega_{k_0}^{(0)2})\delta\mathbf{K} |0, \mathbf{v}\rangle,$$

which can be written

$$(\mathbf{1} - \mathscr{P}) |1, \mathbf{v}\rangle = 1/2 \sum_{\mathbf{k}_0} |\mathbf{k}_0\rangle \langle \mathbf{k}_0| \delta\mathbf{K} |0, \mathbf{v}\rangle$$
$$+ \sum_{\mathbf{k}_0} \frac{1}{\omega_v^{(0)2} - \omega_{k_0}^{(0)2}} |\mathbf{k}_0\rangle \langle \mathbf{k}_0| \delta\mathbf{A} - \omega_v^{(0)2}\delta\mathbf{K} |0, \mathbf{v}\rangle. \quad (35)$$

The first part is due to the mass renormalization (i.e. the multiplication by $\mathbf{K}^{1/2}$) and the second part is exactly the first classical Born approximation. Similar comparison can be done for the part of $|1, \mathbf{v}\rangle$ projected into S_v. Note that, as each new order implies the computation of terms like $\delta\mathbf{K} |n-1, \mathbf{v}\rangle$ as well as terms like $\delta\mathbf{H} |n-1, \mathbf{v}\rangle$, the renormalization reduces the number of operations required

by two, and that for strongly coupled multiplets, for which, even to higher order, degenerate perturbation theory gives poor results, the quasi-degenerate perturbation with a renormalization must be used, giving an exact expression of the projection of the Hamiltonian into the subspace of these nearly resonant multiplets. In this case, no fiducial frequency needs to be used for the kinetic terms.

4.4 Perturbation theory and Rayleigh quotient

In order to compare the solutions obtained with different orders of perturbation theory, and those obtained with the variational method, it is necessary to have a measure of the error associated with an eigenmode computation. Most variational methods are based on a determination of the eigenfrequency using the Rayleigh quotient α defined, for an eigenvector $|\mathbf{v}\rangle$ of an operator $\mathbf{A}$, as

$$\alpha = \langle \mathbf{v}| \mathbf{A} |\mathbf{v}\rangle / \langle \mathbf{v} | \mathbf{v}\rangle. \tag{36}$$

Thus any comparison between variational and nth-order perturbation theory must compare the difference between the Rayleigh quotient obtained with variational methods and that obtained with the quotient for an nth-order approximation of eigenfunction $|\mathbf{v}_n\rangle$. We prove in Appendix 4 that this Rayleigh quotient is not related to the nth-order frequency approximation, but to its $(n+1)$th-order approximation.

Around the Rayleigh quotient frequency α, an estimate of the error bar of the eigenfrequency can be obtained using the Krylow–Weinstein theorem (Dautray & Lions 1984), which is recalled in Appendix 4.

For perturbation theory, we prove in the same appendix that, in the non-rotating quadratic case, an estimate of the error is given by

$$\sqrt{\beta^2 - \alpha^2} = 1/(2\omega_v) \times \sqrt{\langle \mathbf{v}, \mathbf{r}, \mathbf{n} \,|\, \mathbf{r}, \mathbf{n}, \mathbf{v}\rangle / \langle \mathbf{v}_n \,|\, \mathbf{v}_n\rangle - \langle \mathbf{v}_n \,|\, \mathbf{n}, \mathbf{r}, \mathbf{v}\rangle^2 / \langle \mathbf{v}_n \,|\, \mathbf{v}_n\rangle^2}. \tag{37a}$$

We note that this error bar is the norm of the projection of the remaining ket $|\mathbf{n}, \mathbf{r}, \mathbf{k}\rangle$ on the orthogonal subspace of $|\mathbf{v}_n\rangle$. In the rotating case, a similar estimation can be done in the product space, and the error is greater and given by the square root of

$$\sqrt{\beta^2 - \alpha^2} = 1/(2\omega_v) \times \sqrt{2\langle \mathbf{v}, \mathbf{r}, \mathbf{n} \,|\, \mathbf{n}, \mathbf{r}, \mathbf{k}\rangle / \langle \mathbf{v}_n \,|\, \mathbf{v}_n\rangle - \langle \mathbf{v}_n \,|\, \mathbf{n}, \mathbf{r}, \mathbf{v}\rangle^2 / \langle \mathbf{v}_n \,|\, \mathbf{v}_n\rangle^2}. \tag{37b}$$

5 THE TRANSFORMATION METHOD

5.1 Introduction

To compute the eigenfunctions with the expressions of Section 4, let us note that all higher order perturbations are expressed in terms of the projection on the reference basis of kets such as $\delta\mathbf{H}|\mathbf{v}\rangle$ and $\mathbf{B_0}|\mathbf{v}\rangle$. If the computation of $B_0|\mathbf{v}\rangle$ is straightforward, that of $\delta\mathbf{H}|\mathbf{v}\rangle$ is more time consuming, especially if the operator $\delta\mathbf{H}$ includes small-scale asphericities. For example, let us take a model with small-scale heterogeneities, which would be represented by a set of spherical harmonics of angular order smaller than ℓmax, and let us assume that one wants to compute all modes with an angular order smaller than ℓmax. If one wants to take into account all coupling effects, the SNREI basis necessary to the expression of the aspherical modes must include all SNREI modes with an angular order smaller than 2ℓmax, i.e. a basis with a dimension growing as ℓmax^2. All the computations of interaction term such as $\delta\mathbf{H}|\mathbf{u}\rangle$, in fact associated with convolution in the spectral space, will thus require a number of computations increasing as ℓmax^4 for small scales of lateral heterogeneities, for which the interaction matrices are full. Finally, the diagonalization of all ℓmax^2 singlets, either for the perturbation method or for the variational method, requires a number of computations growing as ℓmax^6. This high number of computations forbids a fully coupled computation of interaction terms, except for models with large-scale lateral heterogeneities where the interaction matrices are in fact band diagonal and may be expressed by using Clebsch–Gordan coefficients.

As an alternative to these spectral convolutions, we propose to use a generalization of the 'Spectral Method' or 'Transformation Method' to the tensorial elastic case, in which the number of computations only increases as ℓmax^3, leading to an number of computations growing only as ℓmax^5 for the eigenproblem. The transformation method, presently used in seismology for Fourier transformations (Kosloff & Baysal 1982), was first proposed in 1972 by Orzsag for atmospheric modelling with classical harmonics formalism, where it has been intensively developed. As the present algorithms cannot be applied to the tensorial case, we have thus developed a generalization of the transformation method using the generalized spherical harmonics. We now describe this method only for density and elastic coupling matrices, the generalization to other spectral interactions such as boundaries, gravity and Coriolis being easy to do. Let us take, for example, the computation of the elastic term of expression (3) for a field $|\mathbf{v}\rangle$ given by

$$|\mathbf{v}\rangle = \sum_{\mathbf{k_0}} |\mathbf{k_0}\rangle \langle \mathbf{k_0} \,|\, \mathbf{v}\rangle. \tag{38}$$

The projection of this elastic term on the reference basis of normal modes $|\mathbf{k}_0\rangle = |\ell, \mathrm{m}\rangle$ may be expressed as

$$\langle \ell, \mathrm{m}| \mathbf{E} |\mathbf{v}\rangle = \int_{V_0} [\mathrm{d}^{\alpha\beta\gamma\delta} \mathrm{D}_\alpha \mathrm{v}_\beta] \mathrm{D}_\gamma \mathrm{u}_\delta^{\mathrm{lm}^*} \, \mathrm{dV}, \tag{39}$$

which may be written in the canonical basis of Phinney & Burridge, taking expression (38) of $|\mathbf{v}\rangle$, where $\mathbf{k}$ is replaced by its indexes ℓ, m

$$\langle \ell_0, \mathrm{m}_0| \mathbf{E} |\mathbf{v}\rangle = \sum_{\alpha\beta\gamma\delta} \int_{V_0} \mathrm{dV} \, \mathrm{E}_{\ell_0}{}^{\gamma\delta^*}(\mathrm{r}) \, \mathrm{E}_\ell{}^{\alpha\beta}(\mathrm{r}) \times \mathrm{Y}_{\ell_0}{}^{\mathrm{N}_0\mathrm{m}_0^*}(\theta, \phi) \mathrm{d}_{\alpha\beta\gamma\delta}(\theta, \phi, \mathrm{r}) \times \sum_{\ell,\mathrm{m}} \langle \ell, \mathrm{m} |\mathbf{v}\rangle \mathrm{Y}_\ell{}^{\mathrm{Nm}}(\theta, \phi), \tag{40}$$

where $\mathrm{N}_0 = \gamma + \delta$, $\mathrm{N} = \alpha + \beta$ and $\mathrm{E}_\ell{}^{\alpha\beta}(\mathrm{r})$ is the radially dependent part of the reference normal modes strain tensor. If we define the kernel

$$\mathrm{X}_{\ell\ell_0}{}^{\mathrm{NN}_0}(\theta, \phi) = \sum_{\alpha\beta\gamma\delta} \int \mathrm{r}^2 \mathrm{dr} \, \mathrm{d}_{\alpha\beta\gamma\delta}(\theta, \phi, \mathrm{r}) \mathrm{E}_{\ell_0}{}^{\gamma\delta^*}(\mathrm{r}) \mathrm{E}_\ell{}^{\alpha\beta}(\mathrm{r}),$$

limiting the summation with the condition $N = \alpha + \beta$, $N_0 = \gamma + \delta$, we finally have

$$\langle \ell_0, m_0 | E | v \rangle = \sum_{NN_0} \int_\Sigma d\Sigma \, Y_{\ell_0}^{N_0 m_0 *}(\theta, \phi) \sum_\ell X_{\ell \ell_0}^{N N_0}(\theta, \phi) \times \sum_m \langle \ell, m | v \rangle Y_\ell^{Nm}(\theta, \phi). \quad (41)$$

The other terms and kernels are, for example, in Snieder & Romanowicz (1988). We see that the computation of such terms as in relation (40) or (41) consists essentially in the inverse and direct Legendre transformations of the tensorial field, whose Legendre coefficients are given by $\langle \ell, m | v \rangle$. These direct and inverse Legendre transformations may generally be performed before the radial integration, with expression (40), or after radial integration with the use of the kernal relation (41). Let us now describe how to perform these Legendre transformations.

5.2 A Gauss numerical integration for Legendre transformations

Gauss integration is a well-known method for the numerical integration of a polynomial of finite degree on the compact interval $[-1, +1]$. We recall the most important properties of this integration (Press *et al.* 1986): there is one and only one set of N points μ_i and weighting coefficients w_i such that

$$\int_{-1}^{+1} R(z)\, dz = \sum_{i=1}^{i=N} R(\mu_i) w_i, \quad (42)$$

for every polynomial R(z) of degree less than $2N - 1$. The points μ_i are the ith roots of the Legendre polynomial of order N, $P_N(\mu)$, and the weight w_i is given by

$$w_i = 2/((1 - \mu_i^2) P_N'(\mu_i)^2),$$

where P'_N is the derivative of P_N. Note the important symmetries of the Gauss points and weights:

$$\mu_{N-i} = -\mu_{i+1}, \qquad w_{N-i} = w_{i+1}. \quad (43)$$

Let us use Gauss points to perform the integration of generalized Legendre functions. All expessions which occur in direct or inverse transformations are of the form

$$\int_{-1}^{+1} P_{\ell_1}^{nm}(z) P_{\ell_2}^{nm}(z)\, dz. \quad (44)$$

The integration along longitude (or Fourier transformation) and the properties of the spherical canonical matrix $g_{\alpha\beta}$ cancels the terms with different n or m values. Expressing the generalized Legendre functions in the form (Vilenkin 1968)

$$P_\ell^{nm}(z) = \left(\frac{1+z}{1-z}\right)^{(m+n)/2} (1 - z)^j f(z, \ell - j),$$

where $j = \mathrm{Max}(m, N)$ and $f(z, \ell - j)$ is a polynomial of order $\ell - j$, we see that the product in relation (44) may be expressed as

$$P_{\ell_1}^{nm}(z) P_{\ell_2}^{nm}(z) = \left(\frac{1+z}{1-z}\right)^{m+n} (1 - z)^{2j} f(z, \ell_1 - j) f(z, \ell_2 - j).$$

As 2j is greater than $m + N$, this is a polynomial of order $\ell_1 + \ell_2$ and a Gauss numerical integration is thus possible for a set of N points greater than $(\ell_1 + \ell_2 + 1)/2$. Furthermore, as there are no Gauss points at the poles of the Legendre generalized functions, no singularity appears in the computation of the value of any field at the Gauss points. Let us insist upon the condition for the validity of the computation of the interaction terms. Let us assume that lateral heterogeneities of the earth model are of angular order smax, and the field $|v\rangle$ of angular order ℓmax. After interaction with asphericity, the resulting field has its maximum angular order equal to ℓmax + smax. Thus the numerical Legendre transformation up to order ℓmax will require integration of polynomials up to degree (2*ℓmax + smax) and the number of Gauss points (number of latitudes in the spatial collocation grid) needed will be ℓmax + (smax/2) + 1. Similar aliasing rules may be obtained for the Fourier points, using classical aliasing rules of the Fourier transformation. If these aliasing conditions are verified, the fields $|v\rangle$, $\mathbf{A}|v\rangle$ and $\mathbf{B}|v\rangle$ may be expressed either in the spectral space, in terms of their components $\langle \ell, m | v \rangle$, or in the spatial space on a collocation grid, in terms of the values $v(\mu_i, \phi_i)$. This equivalence will be true for the computation of all coupling operation and is the basis of the transformaion method. For Coriolis and ellipticity perturbations, the computations will be done in the spectral space, using low-order Clebsch–Gordan coefficients, but all other computations will be done on a collocation spatial grid, where they become simple tensor contractions for the elastic, density and gravity interactions.

5.3 Generalized discrete Legendre transformation

Using Gauss numerical integration and the bijection between the spectral and spatial space, we define the following forward and backward generalized Legendre transformation for a tensorial field U of order N.

Backward Legendre transformation

$$U^n(\mu_i, \phi_j) = \sum_{m=-M}^{m=+M} \exp(-im\phi_j) \sum_{\ell=|m|}^{\ell=\ell\mathrm{max}} U_\ell^{nm} P_\ell^{nm}(\mu_i). \quad (45)$$

Forward Legendre transformation

$$U_\ell^{nm} = \sum_{i=1}^{i=nmax} w_i P_\ell^{nm}(\mu_i) \frac{1}{M} \sum_j \exp(im\phi_j) U^n(\mu_i, \phi_j). \quad (46)$$

We give in Appendix 5 an optimized algorithm for the two transformations, more suitable for the decomposition of the eigenspace in terms of subspaces related to the operators **P**, **S**, **T**. Depending on the kind of field, the transformations will be two to eight times faster, respectively, for a complex spherical field or for a real hemispheric field. This algorithm generalizes the classical optimization used in atmospheric modelling (Butel 1984), taking into account the differences introduced by the tensorial cases, and the final expressions are given in Table 2. For this algorithm, only the spectral terms with positive N and M values are necessary. If one wants to use this method with relation (40), using N_z layers for the vertical integration, the number of required computations for the spectral method grows as $\ell\mathrm{max}^3 * N_z$ for the direct or inverse transformation of the field U (including

Table 2. Direct and inverse generalized Legendre transformations, as defined in Section 5. The U_l^{nm} are the Legendre coefficients of the field U^n defined at Gauss points μ_i and Fourier points ϕ_i. ω_i are the Gauss weights. A and B depend on the properties of the field U with respect to the symmetry operators P and T.

Direct Transform

$$U_\ell^{nm} = \sum_{i=nmax/2}^{i=nmax} w_i A_\ell^{nm}(\mu_i) \frac{1}{M} \sum_j \exp(im\phi_j) \text{ Real}(U^n(\mu_i,\phi_j)) + \sum_{i=nmax/2}^{i=nmax} w_i B_\ell^{nm}(\mu_i) \frac{1}{M} \sum_j \exp(im\phi_j) \text{ Imag}(U^n(\mu_i,\phi_j)).$$

Inverse Transform

$$\text{Real}(U^n(\mu_i,\phi_j)) = \sum_{m=-M}^{m=+M} \exp(-im\phi_j) \sum_{l=|m|}^{\ell=\ell max} U_\ell^{nm} \; 1/2 \; A_\ell^{nm}(\mu_i).$$

$$\text{Imag}(U^n(\mu_i,\phi_j)) = \sum_{m=-M}^{m=+M} \exp(-im\phi_j) \sum_{l=|m|}^{\ell=\ell max} U_\ell^{nm} \; 1/2 \; B_\ell^{nm}(\mu_i).$$

the displacement and the strain field) with field and asphericities up to order ℓmax, which is to be compared to the number of computations used with classical matrix products, which grows as $\ell max^4 * Nmax^2$, where Nmax is the number of fundamentals and overtones used in the truncated reference basis. If, on the other hand, relation (41) is used, the number of required computations is now growing as $Mmax^2$ ℓmax^3 $\ell og_2(\ell max)$ with the spectral method, the optimization being due to the fast Fourier transformation, and $Nmax^2$ ℓmax^4 with a classical matrix product. Note however that great simplifications are obtained if one takes into account the fact that the eigenfunctions $E_\ell^{\alpha\beta}(r)$ and the kernal $X_{\ell\ell_0}^{NN_0}(\theta, \phi)$ have slow variations with respect to ℓ and ℓ_0. Thus, in many cases, the use of the spectral method is faster than performing, in a classical manner, the matrix products found in the perturbation series procedure. The formalism and method described here can be conveniently applied to quantify the degree of coupling of target modes with a variety of neighbouring modes and thus to test the convergence of perturbation theory used in conjunction with presently available earth models. In what follows, we present several examples of such calculations.

6 NUMERICAL EXAMPLES

6.1 Description of the SNREI basis and of the laterally heterogeneous model used

We illustrate the accuracy of the higher order perturbation theory by taking a few examples concerning normal modes along the fundamental spheroidal branch, the first one at low frequency, around modes ${}_0S_{25}/{}_0S_{26}$ (i.e. with a period of 300 s), the second one at higher frequency, around mode ${}_0S_{43}$ (i.e. with a period of 200 s). In order to obtain most of the coupling effect, we have taken into account the coupling effect due to the 10 nearest spheroidal and toroidal fundamentals in the first case, and due to the 10 nearest spheroidal fundamentals in the second case. As we focus this paper on the coupling produced by stiffness and density lateral heterogeneities of the Earth, we have neglected rotation and ellipticity coupling effects. The anelasticity was neglected in the computation of the coupling effects. This assumption may of course produce an overestimation of coupling strength between modes of different branches, for which the Q ratio is very different (Park 1986; Lognonné 1989), and a more realistic modelling of aspherical normal modes including this effect will be the subject of a future paper (Lognonné, in preparation).

We used the global M84 + L02.56 model (Woodhouse & Dziewonski 1984; Dziewonski 1984), with weak and smooth spatial variations. We obtain perturbations in κ, μ and λ using the scaling relations

$$\delta\rho/\rho = 0.4\,\delta\beta/\beta, \quad \delta\alpha/\alpha = 0.8\,\delta\beta/\beta,$$

where ρ, α and β are respectively density, P velocity and S velocity, which suggest heterogeneities correlated with temperature variations. Note that for such smooth structure, coupling with overtones is forbidden by selection rules.

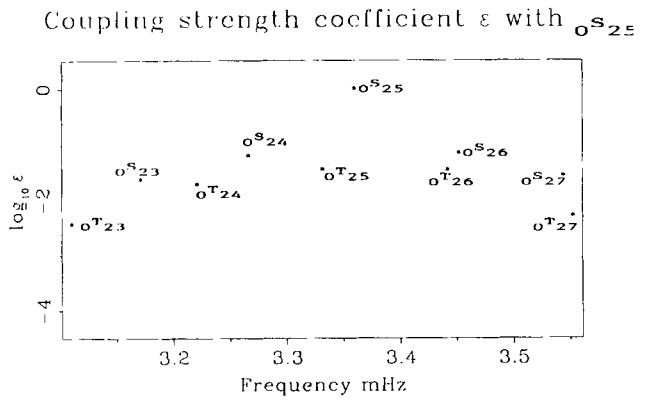
6.2 Coupling diagrams and sensitivity

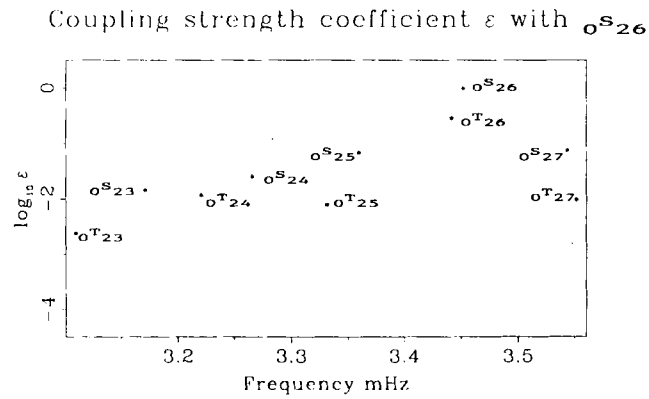
On Figs 1(a)–(c), respectively for the modes ${}_0S_{25}$, ${}_0S_{26}$ and ${}_0S_{43}$, we show the coupling strength parameters between these modes and their nearest neighbour fundamental modes, defined as (Park 1987)

$$\epsilon_{vk} = \frac{\| \langle \mathbf{k}, 0 | \, \delta\mathbf{H} \, | 0, \mathbf{v} \rangle \|}{\omega_v^{(0)} - \omega_k^{(0)}}$$

and related to the norm of the block terms of the operator $\mathbf{G}_v\delta\mathbf{H}$. For the structure M84 + L02.56 and for the very long-period modes ${}_0S_{25}$–${}_0S_{26}$, we see that most of the coupling parameters are very small, generally less than 0.1. Note however that the coupling parameter between spheroidal and toroidal can be stronger, especially in this frequency range between modes of same angular order l. The coupling between ${}_0S_{25}$–${}_0T_{25}$, ${}_0S_{26}$–${}_0T_{26}$ is due for its main part to the asymmetrical part of degree 1 of the upper mantle at depths around 400 km (Park 1985), however the coupling for these two modes shows big differences: while ${}_0S_{25}$–${}_0T_{25}$ are weakly coupled, with a coupling coefficient ϵ less than 0.1, ${}_0S_{26}$–${}_0T_{26}$ are strongly coupled, giving a coupling strength coefficient of the order of 0.25, and although this coefficient seems to be small, we will see in the next section that it leads to quasi-degeneracy, which, if not taken into account, can affect the convergence rate of perturbation theory. At higher frequency and for the mode ${}_0S_{43}$, the coupling along the dispersion branch is larger, leading to coupling parameters with the $l \pm 1$ modes larger than 0.1.

As the frequency variation is nearly linear along a dispersion curve, the decreasing of the coupling strength for the target mode is such that the strength coefficient is approximately divided by nine every three angular orders along the same branch. For the smooth model M84 + L0256, 90 per cent of the coupling effects is thus modelled using only the three nearest modes on each side.





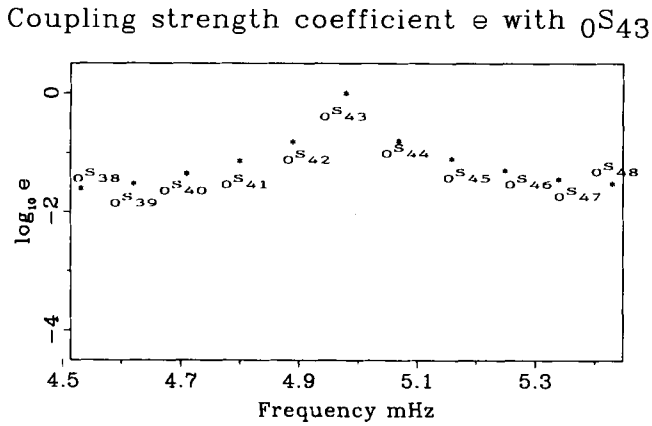


Figure 1. Coupling strength for spheroidal fundamentals $_0S_{25}$, $_0S_{26}$ and $_0S_{43}$. We observe a slow decrease of the strength of coupling when one moves away from the target mode on the dispersion branch, and this coupling becomes nine times smaller for an angular shift of $\Delta\ell = 3$. Note that $_0S_{26}$ is strongly coupled with $_0T_{26}$ and that the coupling along the branch increases with frequency.

6.3 Performance of 'classical' perturbation theory

The first test we performed, was to check the convergence of the perturbation series in the computation of the eigenfrequency and Krylow–Weinstein error. The theory was used up to third order without any secular renormalization, i.e. using 'classical' Rayleigh–Schroedinger perturbation theory. The eigenfrequencies of the singlets of the three modes $_0S_{25}$, $_0S_{26}$ and $_0S_{43}$ were computed, increasing amplitude of lateral variations from 0 (i.e. the spherical model PREM) to 5 (i.e. an earth with lateral heterogeneities five times greater than those of M84 + L02.56).

Let us first show the result for the weakly coupled mode $_0S_{25}$. For this mode, we see in Fig. 2 that the most important effect of higher order perturbation theory is to separate singlets that are mostly associated into doublets in the isolated mode case (Dahlen & Henson 1985). The coupling between toroidal and spheroidal modes weakly removes this quasi-degeneracy, as does the introduction of rotation, for which the Coriolis force breaks the **S** symmetry even to first order. Other non-linear effects may be observed in the second and third frequency approximations, such as a few branch crossings, for an asphericity three or five times greater than that of M84 + L02.56. We describe in Figs 3(a)

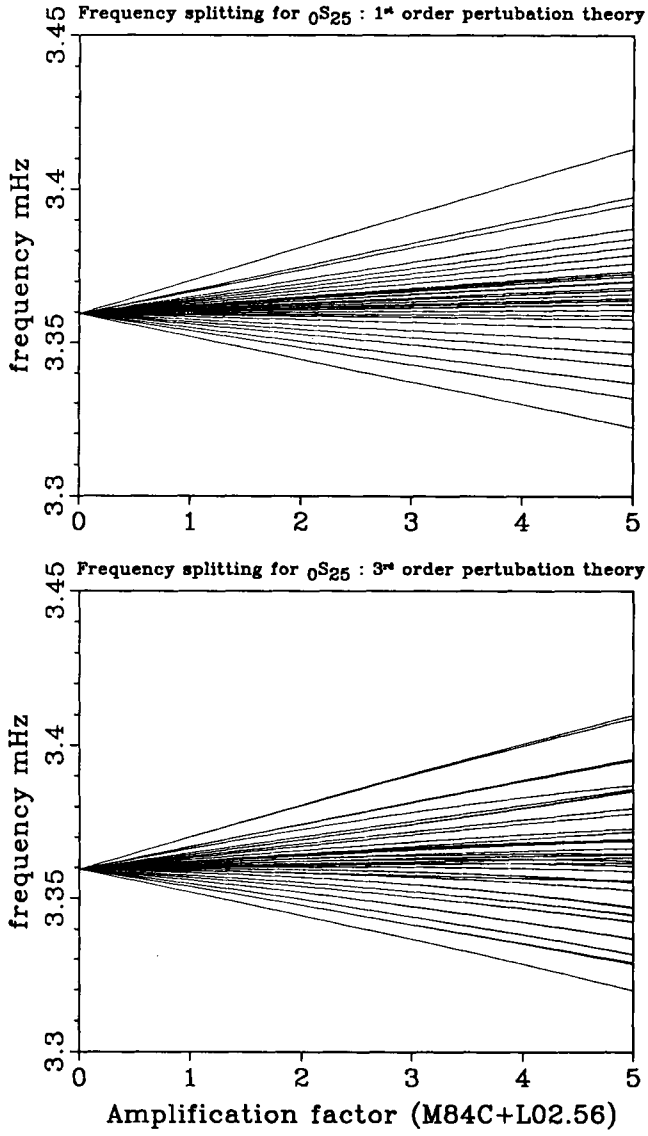
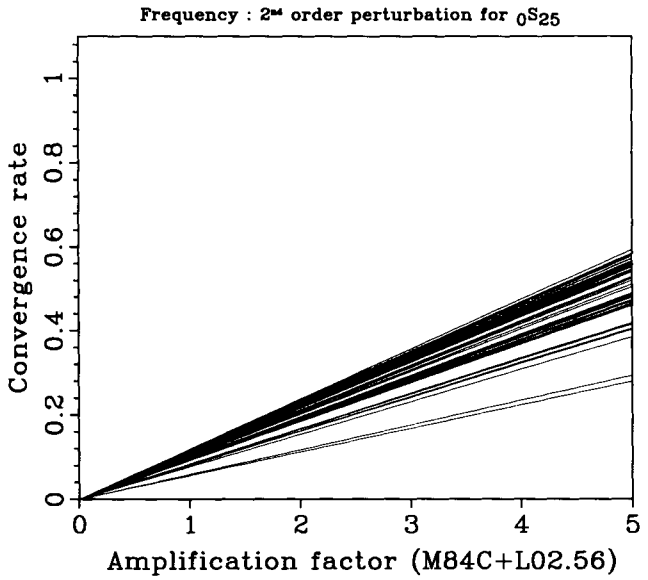


Figure 2. Frequency dispersion curves for the 51 singlets obtained with a first- and third-order approximation. Each line is associated with an aspherical singlet. Most of the curves corresponding to first-order perturbation in frequency contain in reality two quasi-degenerate singlets. Nevertheless, this degeneracy is removed by higher order perturbations (Fig. 5b).



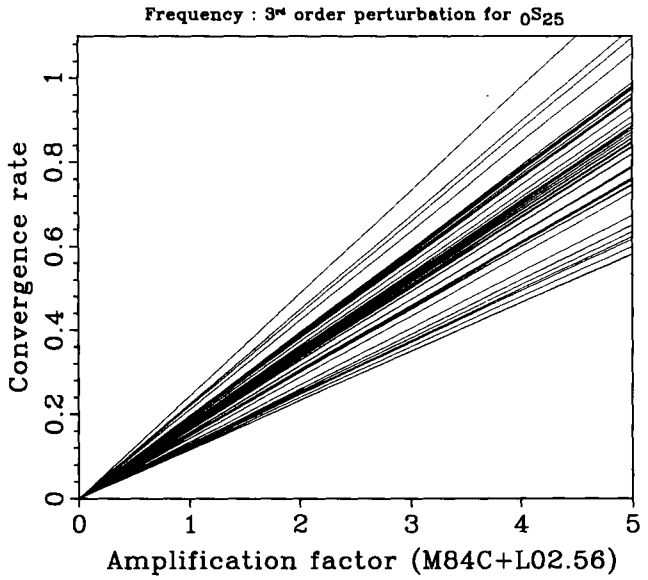


Figure 3. Convergence rate of the frequency defined as the ratio of the nth-order Krylow–Weinstein frequency error over the (n − 1)th one. For the weakly coupled mode $_0S_{25}$, the convergence of the perturbation series is extremely fast.

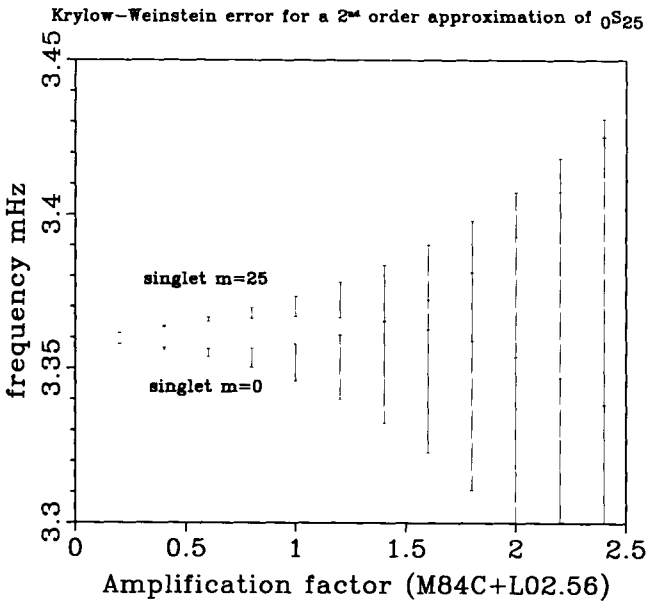
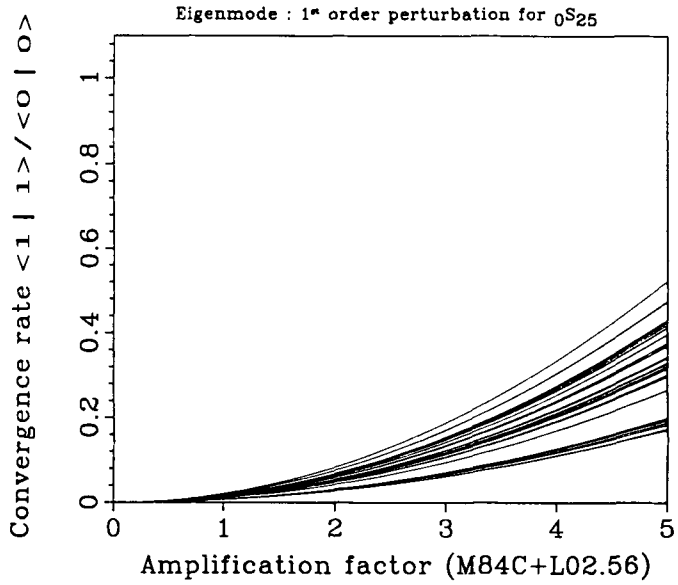


Figure 4. The Krylow–Weinstein error-bars associated with a second-order approximation of the eigenfrequencies of two singlets of $_0S_{25}$. In the aspherical case, the singlet order m is not related to the Fourier component m, but was taken here as the order in singlet frequency after first-order perturbation. Note that the 'classical' perturbation theory cannot separate singlets for asphericity twice larger than M84 + L0256. After fourth order however, the error is reduced to a few per thousand for M84 + L0256.

and (b) the convergence rate of the perturbation series, defined as the ratio of Krylow–Weinstein error bar of the nth approximation over that of the (n − 1)th one. We see that every new step of the perturbation series reduces the Krylow–Weinstein error by an order of 10. For weak heterogeneities, this leads to a separation of the different singlets within the error bar, but for heterogeneities two or three times bigger than L0256 + M84, this separation disappears (Fig. 4).

The relative success of perturbation theory for the mode $_0S_{25}$ is also confirmed for the convergence of the eigenmode perturbations, which is fast enough to change, after each new iteration, the energy of the hybrid ket $|\mathbf{v}_n\rangle$ by a few per cent only (Fig. 5a–c). We note however that the convergence rate is more efficient for the first perturbation than for the next one. Finally, the part of the energy of $_0S_{25}$ which belongs to other multiplets is very small and less than 2 per cent. The escape of energy from the starting multiplet is illustrated by the norm of the projection on the starting subspace of the nth approximation of all $2\ell l + 1$ singlets, and is plotted in Fig. 6 versus its eigenfrequency for the second-order approximation. This can allow us to make a first comparison between the solution obtained with higher order perturbation theory and that obtained with the variational method. We see that from the second order on, the accuracy stops to improve, the results obtained with the third order (Fig. 7) being very close to those of Fig. 6. However, the differences between the eigenfrequency approximations and those computed by the variational method are beyond the frequency resolution of the longest measurement of the earth's normal modes (Fig. 7), and the difference between projection values is also small, of the order of 10^{-3}. This mode is thus relatively well modelled with the 'classical' perturbation theory, for model M84 + L02.56, but apprarently not for models with larger heterogeneities.

On the other hand, for other, more coupled modes, the accuracy of 'classical' degenerate perturbation theory decreases dramatically even for M84 + L0256. It leads, for example for the mode $_0S_{26}$, to divergence in the estimation of the eigenfrequency for heterogeneities from 1 to 2.8 times larger than those of the Earth (Fig. 8). The same failure can be observed in the computation of the eigenmodes, and we see that for many singlets, from second order on, the series slowly improves the eigenfrequency and the eigenmode compared to the variational method. The solutions obtained



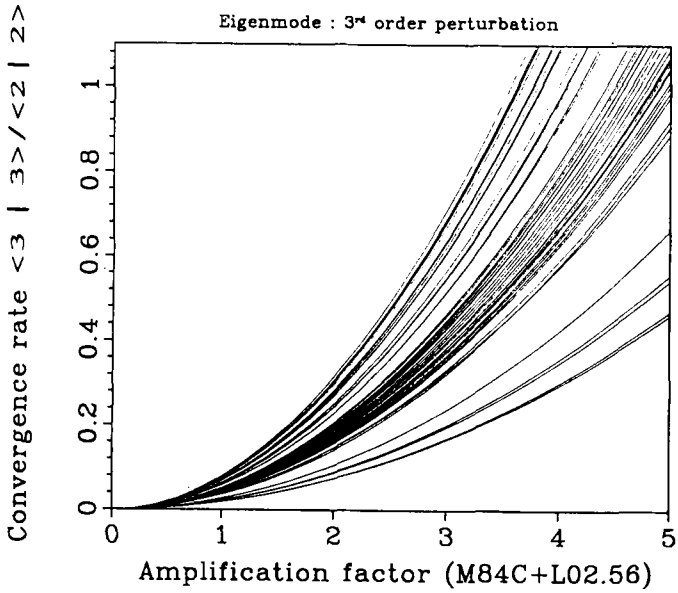


Figure 5. The convergence rate for eigenmodes defined as the ratio of the norm of the nth ket perturbation to that of the (n − 1)th ket perturbation for mode $_0S_{25}$. Just as for the eigenfrequency, the convergence of the eigenmode is extremely fast, each new order having its norm more than 20 times smaller than the previous one.

thus stay too far from those obtained with the variational method (Fig. 9). A more accurate comparison with the eigenmodes obtained using the variational method can be done by computing the orthogonal part of the approximated eigenmode with respect to the variational singlet. This orthogonal part can thus be directly related to the eigenmode error. Doing this for the three first approximations of one singlet of $_0S_{43}$ (Fig. 10), we see that the perturbation of the multiplet structure is, for this range of frequency, stronger than the appearance of components on the neighbouring SNREI multiplets. The classical perturbation theory is unable to model this effect, even with the last

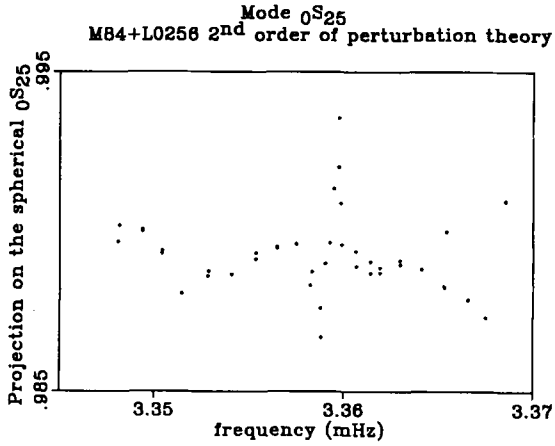


Figure 6. The eigenfrequencies and the remaining part of amplitude in the target multiplet for mode $_0S_{25}$ after the second order of the 'classical' perturbation theory. Up to this order, the evolution is slow and no notable difference with this approximation appears.

terms of equation (30). This failure appears already to first order, which explains that the higher orders do not contribute to any improvement, as it does for $_0S_{25}$ if the lateral heterogeneities are increased. The use of the secular equation (31), which will better constrain the three first-order perturbations, and in particular, all coupling effects within the multiplet, is necessary for all modes more sensitive to lateral heterogeneities than to rotation and ellipticity. We shall now present the results using this formalism.

6.4 Secular equation and renormalization

Figure 11 illustrates the power of both secular and density renormalization. The trace at the back shows, for one of the 87 aspherical singlets of $_0S_{43}$, the absolute value of all components on the 952 SNREI singlets belonging to modes

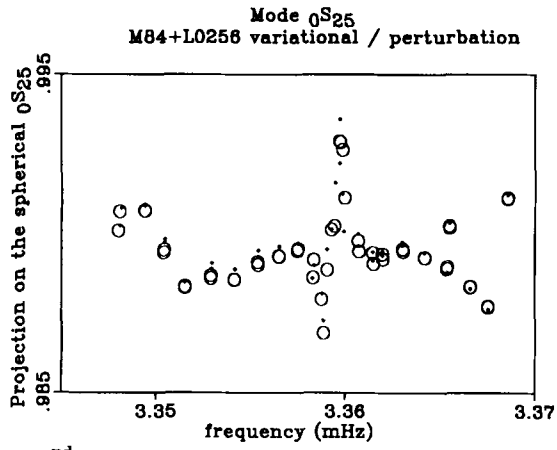


Figure 7. Comparison between the third-order approximation for the mode $_0S_{25}$ and the variational solution. Most of the singlets of $_0S_{25}$ are very close to the variational solution. Note that the difference between obtained eigenfrequencies is smaller than the resolution of the spectrum calculated using a 15 day long time series.

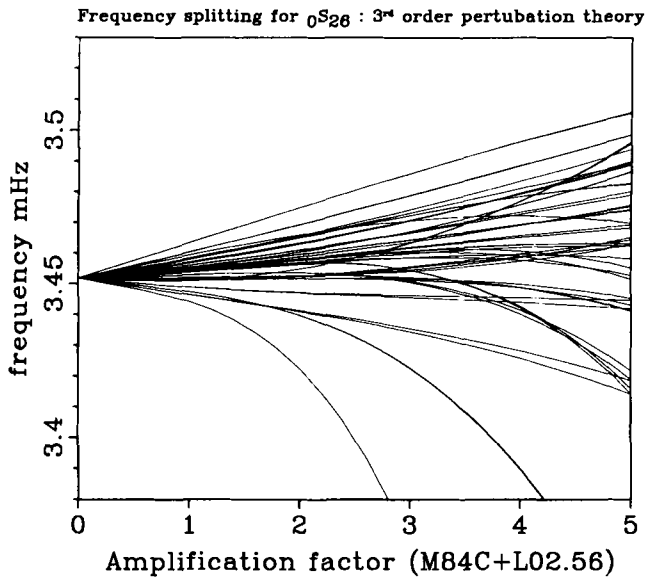


Figure 8. The same as Fig. 2(b) but for the more strongly coupled mode $_0S_{26}$. A great number of dispersion curve crossings are observed and lead to divergence of perturbation theory, due to a high $_0T_{26}$ character of some singlets. The mean frequency is strongly affected by the higher order perturbation.

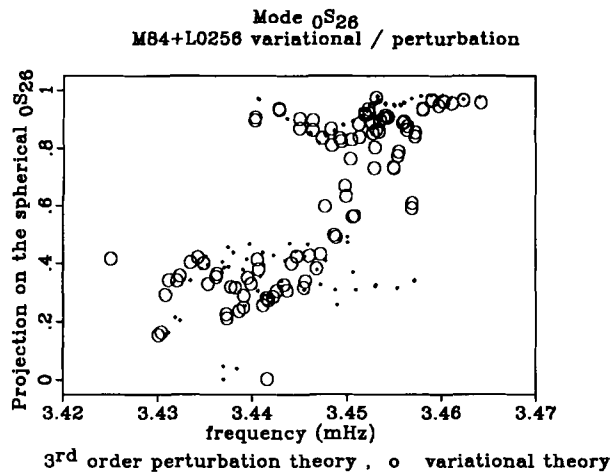


Figure 9. Comparison between the third-order perturbation for the $_0T_{26}$ and $_0S_{26}$ and the variational solution. The theory used here is always the 'classical' higher order. Most singlets of $_0T_{26}$–$_0S_{26}$ are very far from the variational solution.

$_0S_{38}$–$_0S_{48}$. The three first traces show the same thing, but for the orthogonal residual of three zero order approximations of this singlet obtained by different perturbation theories. From front to back, the third trace was obtained with the 'classical' perturbation theory, i.e. by solving the isolated multiplet case using the Born approximation for the density term. Although one would expect to find the largest residuals in the neighbouring modes $_0S_{38}$–$_0S_{42}$ and $_0S_{44}$–$_0S_{48}$, we see that they are still located in the multiplet $_0S_{43}$. The second trace shows the improvement obtained by using the density renormalization, in which the effect of lateral heterogeneities of density is better modelled, which leads to a more accurate modelling of the rearranging of the multiplet structure, but one sees that it is still insufficient. Finally, the first trace of Fig. 11 shows the residual after taking for zero order ket the solution of the first-order secular equation (32). We see that the projection of the residual on the target multiplet $_0S_{43}$ is no larger than the residual in the neighbouring modes $_0S_{38}$–$_0S_{42}$ and $_0S_{44}$–$_0S_{48}$. These coupling terms can thus be treated in an efficient way by higher order terms of perturbation theory.

Figure 12 shows the residual between the variational method and the solution obtained with higher order perturbation theory. Note that for any nth-order approximation, the zero-order one was taken as solution of the nth-order secular equation. We see that for all nth-order approximations, the largest residual still remains for the SNREI starting multiplet, here $_0S_{43}$. It is thus reduced in the next step by the (n + 1)th-order secular equation, while the residual in the neighbouring modes is well reduced by

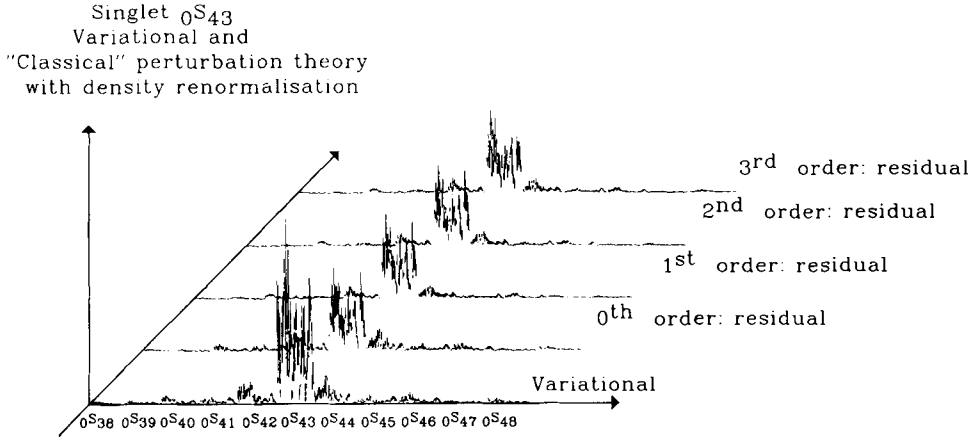


Figure 10. Comparison between the four first-order approximations obtained with the 'classical' perturbation theory (from front to back, the four last traces) and the solution obtained with the variational method (the first trace). The absolute value of the 952 components in the SNREI basis $_0S_{38}$–$_0S_{48}$ are shown for the variational singlet, or for the perturbation theory approximations. The residuals are defined as the orthogonal part of each approximation with respect to the variational singlet. Up to zero order, a big residual stays in the component associated to $_0S_{43}$, not reduced by the higher order approximations. Coupling terms with the neighbouring modes are thus very poorly modelled by the non-secular perturbation theory.

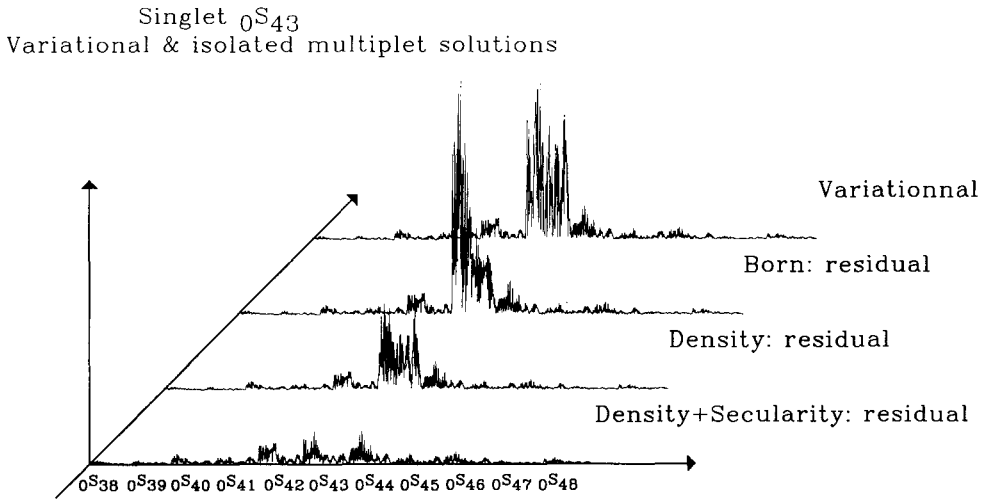


Figure 11. Comparison between three zero-order approximations obtained with various perturbation theories (from front to back, the three first traces) and the solution obtained with the variational method (the last trace). The residuals are defined as for Fig. 10, and the absolute value of their components in the SNREI basis $_0S_{38}$–$_0S_{48}$ is plotted. The third trace is the residual obtained using the isolated multiplet Born approximation, the second one using the zero-order perturbation theory with a density renormalization, and first one using the first-order secular equation, after density renormalization. The largest residual in the third trace is in the $_0S_{43}$ components, due to the non-orthogonal perturbations induced by the coupling of this mode with the neighbouring modes. It is first reduced by the density renormalization, and then by the secular equation. Finally, in the first trace, the residual is of order as that in the neighbouring modes $_0S_{42}$ and $_0S_{44}$.

the (n + 1)th-order eigenmode perturbation. Note that the direction cosine between these approximations and the variational solution are for the zero, first, second and third order equal to 0.9678, 0.1016, 0.9989, 1.0006, respectively. The same accuracy is obtained for the eigenfrequency estimation, and results are shown in Fig. 13. We see that both singlet splitting effects and multiplet mean frequency shift are determined, with a third-order approximation, within a relative error of 10^{-3} per thousand, that is an error a hundred times smaller than the measurement error of the Earth's eigenfrequencies.

The same improvements are obtained for the determination of both eigenfrequency and eigenmodes of multiplet $_0S_{26}$. Nevertheless, it is necessary to use here quasi-degenerate perturbation theory, i.e. to start from a supermultiplet including both $_0S_{26}$ and $_0T_{26}$ modes. All coupling terms with the other neighbouring toroidals and spheroidals are thus treated by the higher order perturbations. Already to first order, the improvement compared to the previous results of Fig. (9) is spectacular, as shown in Fig. (14), where now notable differences between the two solutions are distinguishable, and the higher orders allow us thus to obtain an accuracy even greater than for the case of $_0S_{43}$.

Seismograms can of course be computed using these eigenmodes and eigenfrequencies and expressed as a

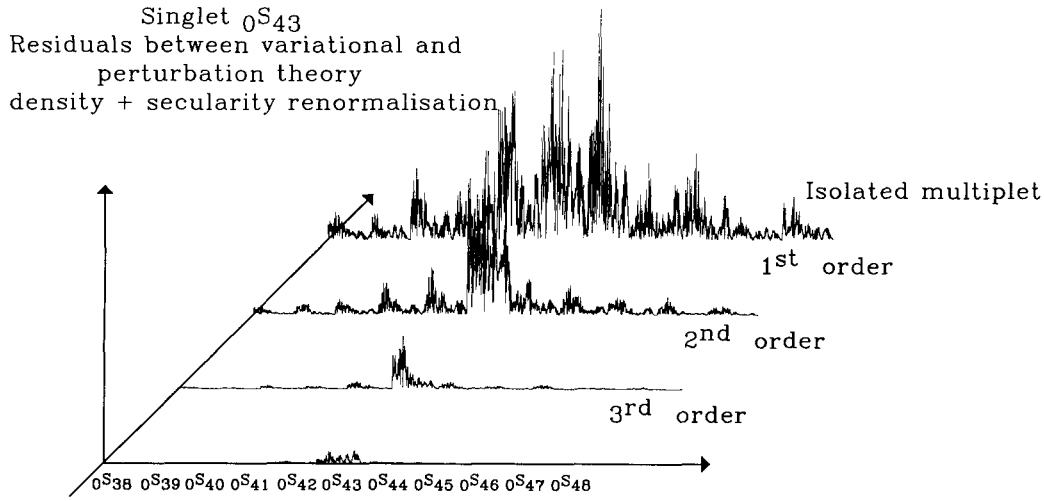


Figure 12. Convergence of the eigenmode to the variational solution for perturbations up to third order, obtained using the secular equation and density renormalization. The last trace is the first one of Fig. 11, where the corresponding variational solution was shown. For each new order, coupling effects within the neighbouring modes are well modelled by higher order perturbations. Residuals decrease regularly and the approximate eigenmode becomes more and more parallel to the variational one. Direction cosines for the approximations are respectively 0.9678, 1.016, 0.9989, 1.0006, which shows that already to third order, the eigenmode is modelled within a relative precision of 10^{-3} and less.

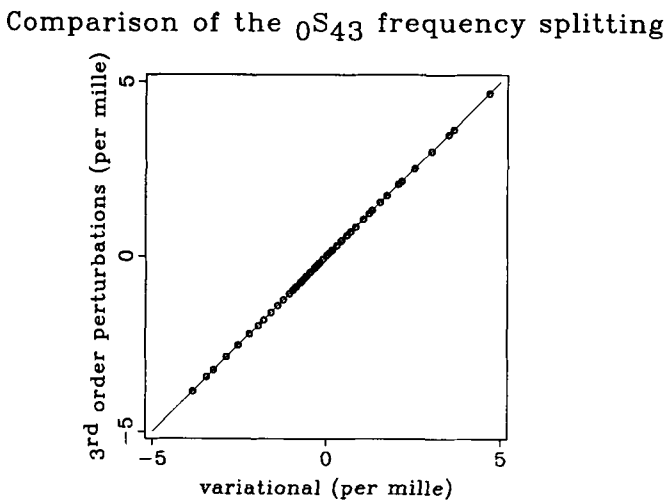


Figure 13. Comparison between the eigenfrequencies of the 87 singlets of $_0S_{43}$ obtained with the variational method and those obtained with the third-order perturbation theory in per million for model M84. No differences bigger than 10^{-3} per thousand are observable in both splitting and shifting effects.

summation of spherical normal modes, oscillating with the spherical frequency $\omega_v^{(0)}$, and having their amplitude slowly modulated with time by the effect of lateral heterogeneities (Dahlen 1987). These modulations, which lead to both frequency splitting or shifting and amplitude anomalies, can thus be modelled using higher order perturbation theory within an relative error smaller than 10^{-3}, smaller than the usual observation error for example, and likely smaller too than the effect due to physical dispersion or due to the non-quadratic Coriolis effect. We thus think that the higher order theory is a fast and sufficient accurate way for solving the normal mode equation of a laterally heterogeneous earth, or for computing seismograms for all realistic earth models. Note however that the accuracy of the eigenmodes can be insufficient if seismograms are computed without removing the 'fast' spherical oscillations with frequency

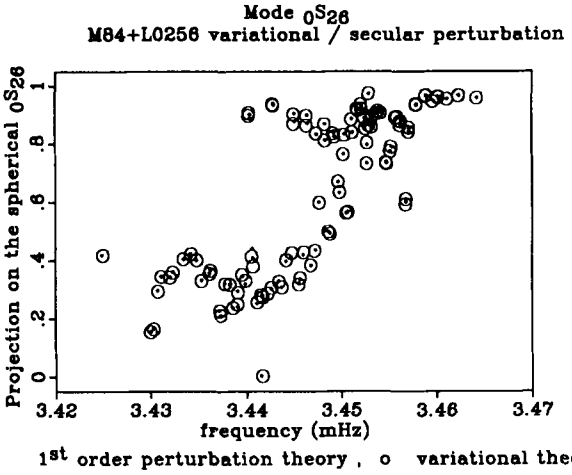


Figure 14. Comparison between the first-order perturbation for the supermultiplet $_0T_{26}$–$_0S_{26}$ and the variational solution. Theory used here is the quasi-degenerate perturbation theory with secular equation. Most singlets of $_0T_{26}$–$_0S_{26}$ have now converged to the variational solution.

Table 3. Comparison of the CPU times required in the variational method and in the higher order perturbation method on the CONVEX C1. The spectral method was not used in the computation of the matrix products, which were performed classically (matrix size: 952). The second comparison is representative of the required CPU time if one wants to take into account coupling effects of the ten nearest modes. CONVEX optimized assembly routines were used for EISPACK path TRED2/TQL2.

Computation of all singlets of the $_0S_{38}$–$_0S_{48}$ eigenproblem •	
Numerical Path	**CPU (Convex C1)**
Perturbation: 3rd order (Analytical)	308 s
Variational: RSG path (Cholesky)	1754 s
Variational: TRED2/TQL2 path (Analytical)	736 s

• With analytical or numerical density renormalisatio

Computation of all singlets of $_0S_{43}$ coupled with $_0S_{38}$–$_0S_{48}$ •	
Numerical Path	**CPU (Convex C1)**
Perturbation: 3rd order	28 s
Variational: TRED1/BISECT/TINVIT/TRBAK1 path	192 s
Variational: TRED2/TQL2 path (Mean time per multiplet)	66 s

• After density renormalisatio

$\omega_v^{(0)}$, as it happens if one sums directly the singlet's decaying sinusoids in the frequency domain.

From a computational point of view, the higher order perturbation theory leads to computer codes easily implemented on vectorized computers (Cray or Cray-like) or on highly parallelized computers (as CM-2 connection machine), which are from two to six times faster than eispack routines (see Table 3 for an accurate comparison). Note in particular that this method is six times faster than the variational method if one wants to compute a set of fundamental modes coupled with a given number of neighbouring modes, that is for block-band diagonal interaction matrices.

6.5 Coupling for a model with sharp structure

We want to conclude by testing the perturbation theory for a model with lateral heterogeneities sharper than those of M84, for example for the shear velocity model of M & T (Montagner & Tanimoto 1989), for which the coupling between fundamental and overtones is no longer forbidden, the maximum angular order s being now s = 15 (Fig. 15). The target modes $_0S_{25}$ and $_0S_{26}$ will now be coupled with all

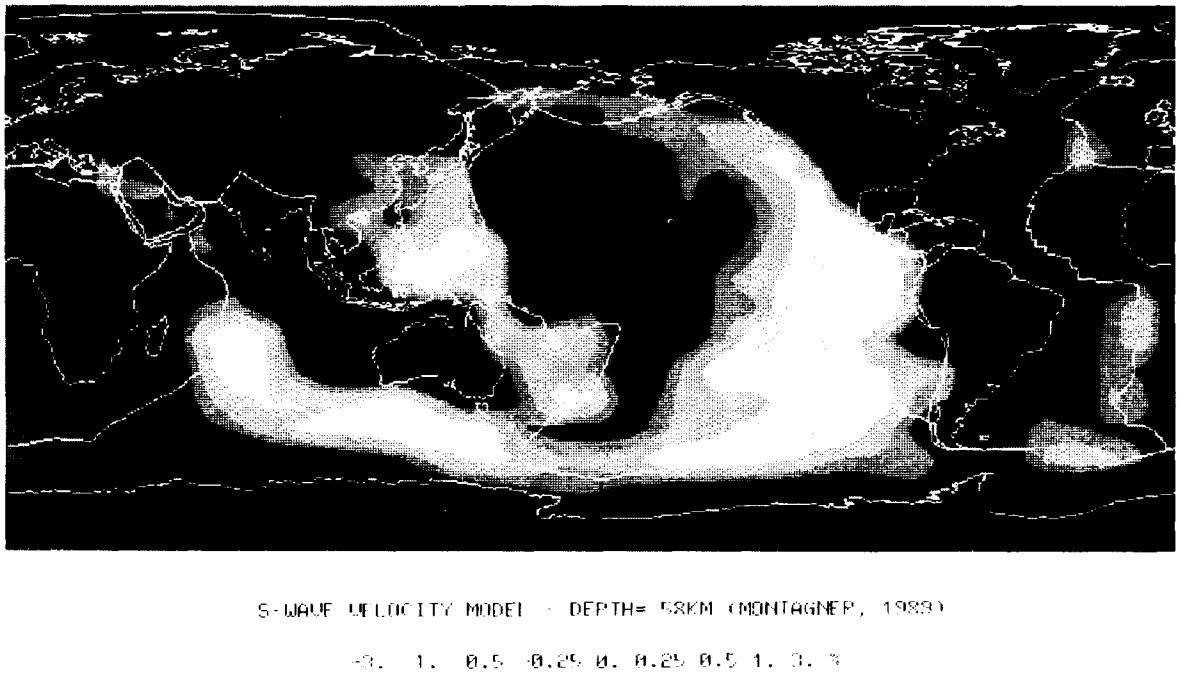


Figure 15. Map of shear velocity lateral heterogeneities at a depth of 58 km for the model of Montagner & Tanimoto (1990). Extremal values reach ±3 per cent of the spherical mean shear velocity. Note the very good correlation with surface tectonics and the well-determined odd angular orders of lateral heterogeneities.

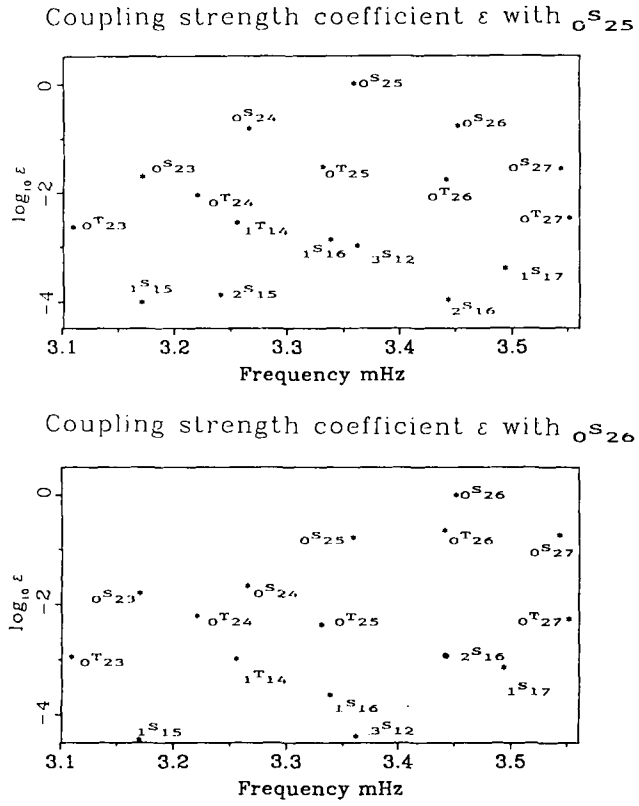


Figure 16. Coupling strength diagrams for the spheroidal fundamentals $_0S_{25}$ and $_0S_{26}$ for the M & T model. The coupling between modes of different polarities increased compared to Fig. 1(a) and (b). The selection rules still do not forbid coupling with overtones, which appear to have a coupling strength parameter ϵ between 10^{-2} and 10^{-3}.

fundamentals modes, and with overtones $_1S_{15}$, $_1S_{16}$, $_1S_{17}$, $_2S_{15}$, $_2S_{16}$, $_3S_{12}$ and $_1T_{14}$ (Fig. 16). These modes are weakly coupled with the target mode $_0S_{25}$, with a coupling strength parameter ϵ between 10^{-2} and 10^{-3} for the more strongly coupled overtones. A second big difference with the coupling diagrams for model M84C + L02.56 is that coupling between modes of opposite polarities is stronger. This is due to the odd part of lateral heterogeneities, which is stronger in the M & T model than in the W & D model.

The coupling for these modes increases and the energy lost is three times greater than for the W & D model. Nevertheless, the comparisons we have performed show that perturbation theory always gives a good accuracy with respect to variational theory. We have plotted in Fig. 17 the frequency splitting and the escape of energy from target multiplet $_0S_{25}$ induced by the M & T model. Crosses are the spherical frequencies of $_0S_{25}$ and $_3S_{12}$. The coupling pattern shows extrema of eigenmode coupling. Two are related to the most split singlets. Their eigenfrequency is thus relatively distant from the other singlet frequencies, which leads to a decrease of the effective coupling effects. As expected, another minimum of eigenmode coupling is observed for singlets which are weakly split, and thus close to the central frequency of $_0S_{25}$. On the contrary, we observe a maximum of eigenmode coupling, associated with a high escape of energy from the target mode, close to the central frequency of $_3S_{12}$, which can be explained as a resonance effect with this mode. Even with relatively small coupling, the overtones may thus affect the coupling pattern of fundamental modes. Nevertheless, we think that this coupling is still underestimated using M & T model, which does not have very small-scale lateral heterogeneities, and think that the effect of all small features of the Earth such as

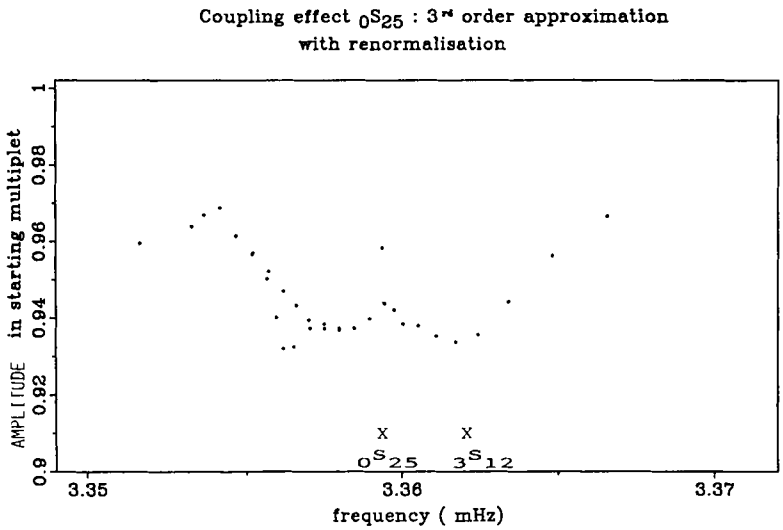


Figure 17. Frequency splitting and remaining part of amplitude in the target mode $_0S_{25}$ for the M & T model. The 'non-isolated' character of this mode increases by an order of 3 compared to the W & D model. Note the minimum of eigenmode coupling for singlets near the mean frequency of $_0S_{25}$ (left cross) and the maximum of coupling due to a resonance effect with overtone $_3S_{12}$.

trenches, ridges or even hotspots (Neele *et al.* 1989) will increase the coupling strength of overtones. For example, unpublished results show that the coupling coefficient between $_0S_{25}$ and $_3S_{12}$ due to lateral heterogeneities of a Aleutian trench model can be, for this trench alone, as large as that of M & T model. However, even for such sharp models, the higher order perturbation theory is still sufficient to compute realistic and accurate normal modes and seismograms.

7 CONCLUSIONS

The method presented in this paper gives a very convenient and fast way of computing multiplets for an earth model with sharp or smooth lateral heterogeneities. Its first advantage is due to the use of higher order perturbation theory, which leads to a faster way to compute normal modes than with the variational method. Non-quadratic effects due to rotation, which lead to a doubling of the dimension of the eigenproblem for the variational method can be easily taken into account without any noticeable increase of the computation time. This theory allows us to compute modes along a given dispersion curve, with the same number of neighbouring coupled modes for each of them, and without neglecting coupling with the other dispersion branches. A future paper (Lognonné, in preparation) will generalize the present theory in order to introduce anelasticity and physical dispersion (Lognonné 1989), which will avoid the overestimation of coupling effects between modes with different Q ratios using any perturbation theory starting from an elastic reference earth model.

Its second advantage is due to the use of the spectral method, which requires no Glebsch–Gordan coefficients. However, as the interaction matrices were needed in this paper in order to perform the comparison with the variational method, and as only a few neighbouring modes were taken into account in the coupling terms, we have used the spectral method only for the computation of the interaction matrices, and performed thus all matrix products classically. Nevertheless, this method will reduce the number of computations even more if used directly in the computation of terms such as $\mathbf{A}|\mathbf{u}\rangle$. In this case, the number of computations needed by the higher order perturbation theory will grow as $\ell\max^5$, while the number of computation for the variational method grows as $\ell\max^6$. This opens new prospects for fully coupled mode computations, particularly in order to model the effect of sharp tectonic features, crustal and interface variations more precisely and also in the inversion of normal modes or long-period surface waves.

More generally, the discrete Legendre transformation based on generalized Legendre functions and on symmetry subspace decomposition formalizes the optimization of spectral methods used in atmospheric modelling. Such methods will be helpful in solving global tensorial field problems, which appear in the fields of Earth magnetism or magnetohydrodynamics and in the field of global circulations.

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Appendix 1: Operator P, S and T and the S.N.R.E.I modes

We briefly recall the property of the S.N.R.E.I modes with respect to the three symmetry operators **P**, **S** and **T**, summarized in Table 1. For the operator **P**, we easily find, starting from its definition in Part 3.3 that, for spheroidals modes, we have for the horizontal ($\alpha=\pm1$) displacement components:

$$\begin{aligned}
\mathbf{P}^{\alpha}(\mathbf{u}_\ell{}^m) &= \gamma_\ell\, \Omega^0_\ell\, V(r)\, Y_\ell{}^{-\alpha m}(\pi-\theta,\phi+\pi)\\
&= \gamma_\ell\, \Omega^0_\ell\, V(r)\, (-1)^m\, P_\ell{}^{-\alpha m}(-\mu)\, \exp(-im\phi)\\
&= \gamma_\ell\, \Omega^0_\ell\, V(r)\, (-1)^\ell\, P_\ell{}^{\alpha m}(\mu)\, \exp(-im\phi)\\
&= (-1)^\ell\, u_\ell{}^{\alpha m},
\end{aligned}$$

and for the vertical ones:

$$\begin{aligned}
\mathbf{P}^{0}(\mathbf{u}_\ell{}^m) &= \gamma_\ell\, U(r)\, Y_\ell{}^{0m}(\pi-\theta,\phi+\pi)\\
&= \gamma_\ell\, U(r)\, (-1)^m\, P_\ell{}^{0m}(-\mu)\, \exp(-im\phi)\\
&= \gamma_\ell\, U(r)\, (-1)^\ell\, P_\ell{}^{0m}(\mu)\, \exp(-im\phi)\\
&= (-1)^\ell\, u_\ell{}^{0m},
\end{aligned}$$

which means that

$$\mathbf{Pu}=(-1)^\ell\, \mathbf{u}.$$

For toroidals, we have in the same way, for the horizontal components :

$$\begin{aligned}
\mathbf{P}^{\alpha}(\mathbf{u}_\ell{}^m) &= -\alpha\, i\, \gamma_\ell\, \Omega^0_\ell\, W(r)\, Y_\ell{}^{-\alpha m}(\pi-\theta,\phi+\pi)\\
&= -\alpha\, i\, \gamma_\ell\, \Omega^0_\ell\, W(r)\, (-1)^m\, P_\ell{}^{-\alpha m}(-\mu)\, \exp(-im\phi)\\
&= -\alpha\, i\, \gamma_\ell\, \Omega^0_\ell\, W(r)\, (-1)^\ell\, P_\ell{}^{\alpha m}(\mu)\, \exp(-im\phi)\\
&= (-1)^{\ell+1}\, u_l{}^{\alpha m}.
\end{aligned}$$

which means that

$$\mathbf{Pu} = (-1)^{\ell+1}\, \mathbf{u}.$$

For the operator **T**, we find easily, for the spheroidal displacement, for example for the horizon-

tal displacement

$$T^{\alpha}(\mathbf{u}_\ell^m) = (-1)^{\alpha}\, \gamma_\ell\, \Omega^0_\ell\, V(r)\, Y_\ell^{-\alpha m^*}(\theta,\phi)$$
$$= \gamma_\ell\, \Omega^0_\ell\, V(r)\, (-1)^{\alpha}\, P_\ell^{-\alpha m}(\mu)\, \exp(im\phi)$$
$$= \gamma_\ell\, \Omega^0_\ell\, V(r)\, (-1)^{m}\, P_\ell^{\alpha-m}(\mu)\, \exp(im\phi)$$
$$= (-1)^m\, \mathbf{u}_\ell^{\alpha-m}.$$

Doing this on the same way for the vertical component, we finally find that

$$\mathbf{T}\mathbf{u}_\ell^m = (-1)^m\, \mathbf{u}_\ell^{-m}.$$

In the same way, we find for toroidal modes that

$$\mathbf{T}\mathbf{u}_\ell^m = (-1)^m\, \mathbf{u}_\ell^{-m}.$$

The free modes **u** are not eigenfunctions of T. The eigenfunctions are easy to find and are simply:

$$\beta_{+\ell}^m = 1/\sqrt{2}\,(\,\mathbf{u}_\ell^m + (-1)^m\,\mathbf{u}_\ell^{-m})$$
$$\zeta_{+\ell}^m = i/\sqrt{2}\,(\,\mathbf{u}_\ell^m - (-1)^m\,\mathbf{u}_\ell^{-m})\,,$$

for eigenvalue +1,

$$\beta_{-\ell}^m = 1/\sqrt{2}\,(\,\mathbf{u}_\ell^m - (-1)^m\,\mathbf{u}_\ell^{-m})$$
$$\zeta_{-\ell}^m = i/\sqrt{2}\,(\,\mathbf{u}_\ell^m + (-1)^m\,\mathbf{u}_\ell^{-m})\,,$$

for eigenvalue -1.

Finally for the operator S, we find easily for the β_+ and ζ_+ functions defined above:

$$S^{\alpha}(\beta_{\pm\ell}^m) = \frac{1}{\sqrt{2}}(-1)^{\alpha}(\,\mathbf{u}_\ell^{-\alpha m}(r,\theta,-\phi) \pm (-1)^m\,\mathbf{u}_\ell^{-\alpha-m}(r,\theta,-\phi))$$
$$= 1/\sqrt{2}\,(-1)^{\alpha}\,(\,s_\ell^{-\alpha m}(r)\,P_\ell^{-\alpha m}(\theta)\,\exp(im\phi) \pm (-1)^m\,s_\ell^{-\alpha-m}(r)\,P_\ell^{-\alpha-m}(\theta)\,\exp(-im\phi))$$
$$= 1/\sqrt{2}\,(\,(-1)^m\,s_\ell^{-\alpha m}(r)\,Y_\ell^{\alpha-m}(\theta,\phi) \pm s_\ell^{-\alpha-m}(r)\,Y_\ell^{-\alpha-m}(\theta,\phi))$$

and thus that for spheroidal modes:

$$s_\ell^{-\alpha m} = s_\ell^{\alpha-m} \text{ and } s_\ell^{-\alpha-m} = s_\ell^{\alpha m}$$

so that

$$\mathbf{S}\beta_\pm = \pm\,\beta_\pm$$

and that for toroidals:

$$s_\ell^{-\alpha m} = -\,s_\ell^{\alpha-m} \text{ and } s_\ell^{-\alpha-m} = -\,s_\ell^{\alpha m}$$

so that

$$\mathbf{S}\beta_\pm = \mp\,\beta_\pm$$

We find the same results for $\zeta_\pm$ i.e. that

$\mathbf{S}\zeta_\pm = \mp\,\zeta_\pm$, for toroidal modes

$\mathbf{S}\zeta_\pm = \pm\,\zeta_\pm$, for spheroidal modes.

Appendix 2: Perturbation of the singlets

In order to determine the projection of all n^{th} order perturbations, let us recall that the $2\ell+1$ aspherical singlets with the same $\omega_v^{(0)}$ frequency verify the two relations (2.1) and (2.2):

$$\langle \mathbf{u}|\, -\omega_v^2\,\mathbf{K} + \omega_v\,\mathbf{B} + \mathbf{A}|\mathbf{v}\rangle = 0, \tag{2.1}$$

$$\langle \mathbf{u}|\mathbf{K} - 1/(\omega_v + \omega_u)\,\mathbf{B}|\mathbf{v}\rangle = 2\,\underline{\langle \mathbf{u}|\mathbf{v}\rangle} = \delta_{\mathbf{uv}}. \tag{2.2}$$

Let us substitute the bra and ket $\langle \mathbf{u}|$ and $|\mathbf{v}\rangle$ by the power series (17) of section 4.1 and expand (2.1) into a power series of ϵ, noting that, for two different singlets **v** and **u**, and since $\omega_u^{(0)}$ and $\omega_v^{(0)}$ are equal

$$-\omega_v{}^2\mathbf{K} + \omega_v\mathbf{B} + \mathbf{A} = \mathbf{A}_0 - \omega_v^{2(0)}\mathbf{I} + \delta\mathbf{H} - (\omega_v{}^2 - \omega_v{}^{(0)})(\mathbf{K} - \frac{1}{\omega_v+\omega_u}\mathbf{B}) + \frac{(\omega_v-\omega_v{}^{(0)})(\omega_u-\omega_u{}^{(0)})}{\omega_v+\omega_u}\mathbf{B}.$$

If (2.2) is verified to the n^{th} order,

$$\langle \mathbf{u}|\mathbf{K} - 1/(\omega_v + \omega_u)\,\mathbf{B}|\mathbf{v}\rangle = 0 + o(\epsilon^n),$$

expression (2.1) can thus be written as

$$\langle \mathbf{u}|\, -\omega_v{}^2\,\mathbf{K} + \omega_v\mathbf{B} + \mathbf{A}|\mathbf{v}\rangle = \langle \mathbf{u}|\mathbf{A}_0 - \omega_v^{(0)2}\mathbf{I} + \delta\mathbf{H}|\mathbf{v}\rangle + \frac{(\omega_v-\omega_v{}^{(0)})(\omega_u-\omega_v{}^{(0)})}{\omega_v+\omega_u}\langle \mathbf{u}|\mathbf{B}|\mathbf{v}\rangle + o(\epsilon^{n+1}).$$

We then have, remembering that $|0,v\rangle$ and $|0,u\rangle$ are eigenvectors of $\mathbf{A}_0$:

$$\begin{aligned}\langle u| - \omega_v^2 \mathbf{K} + \omega_v \mathbf{B} + \mathbf{A}|v\rangle = 0 \\ &+ \langle u,0|\delta H|0,v\rangle \\ &+ \langle u,1|\mathscr{P}\delta H|0,v\rangle + \langle u,0|\delta H\mathscr{P}|1,v\rangle + A^1_{uv} \\ &+ \ldots \\ &+ \langle u,n|\mathscr{P}\delta H|0,v\rangle + \langle u,0|\delta H\mathscr{P}|n,v\rangle + A^n_{uv} \\ &+ \ldots,\end{aligned} \tag{2.3}$$

where $\mathscr{P}$ is the projector into the subspace S_v and $\mathscr{I}-\mathscr{P}$ the projector into its orthogonal, and where

$$A^1_{uv} = \langle u,1|\mathbf{A}_0-\omega_v^{(0)2}|1,v\rangle + \langle u,1|(\mathscr{I}-\mathscr{P})\delta H|0,v\rangle + \langle u,0|\delta H(\mathscr{I}-\mathscr{P})|1,v\rangle,$$

$$\begin{aligned}A^2_{uv} = {} & \langle u,1|\delta H|1,v\rangle + \langle u,2|\mathbf{A}_0-\omega_v^{(0)2}|1,v\rangle + \langle u,1|\mathbf{A}_0-\omega_v^{(0)2}|2,v\rangle \\ & + \langle u,2|(\mathscr{I}-\mathscr{P})\delta H|0,v\rangle + \langle u,0|\delta H(\mathscr{I}-\mathscr{P})|2,v\rangle + \delta_1\omega_v\,\delta_1\omega_u\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle,\end{aligned}$$

$$\begin{aligned}A^3_{uv} = {} & \langle u,2|\delta H|1,v\rangle + \langle u,1|\delta H|2,v\rangle + \langle u,3|\mathbf{A}_0-\omega_v^{(0)2}|1,v\rangle + \langle u,1|\mathbf{A}_0-\omega_v^{(0)2}|3,v\rangle \\ & + \langle u,2|\mathbf{A}_0-\omega_v^{(0)2}|2,v\rangle + \langle u,3|(\mathscr{I}-\mathscr{P})\delta H|0,v\rangle + \langle u,0|\delta H(\mathscr{I}-\mathscr{P})|3,v\rangle \\ & + (\delta_1\omega_v\delta_2\omega_u+\delta_1\omega_u\,\delta_2\omega_v-\delta_1\omega_u\delta_1\omega_v\frac{\delta_1\omega_u+\delta_1\omega_v}{2\omega_v^{(0)}})\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle \\ & + \delta_1\omega_v\delta_1\omega_u\,(\langle u,1|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle+\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|1,v\rangle).\end{aligned} \tag{2.4}$$

Using relation (21) of section 4.1 and the following property of the first order perturbation (from (23),(24) of the same section)

$$\langle u,0|\delta H|n,v\rangle + \langle u,1|\mathbf{A}_0-\omega_v^{(0)2}|n,v\rangle = \langle u,0|\delta H\mathscr{P}|n,v\rangle,$$

and defining the operator $\mathbf{G}_v$ as

$$\mathbf{G}_v = \sum_{k\neq v}\frac{1}{\omega_v^{(0)}-\omega_k^{(0)}}|0,k\rangle\langle k,0|,$$

we obtain, after some algebra, the following expression for $A^{1,2,3}_{uv}$:

$$A^1_{uv} = \langle u,0|\delta H\,\mathbf{G}_v\,\delta H|0,v\rangle,$$

$$\begin{aligned}A^2_{uv} = {} & \langle u,0|\delta H\,\mathbf{G}_v\,\delta H\,\mathbf{G}_v\,\delta H\,|0,v\rangle \\ & + \langle u,1|\mathscr{P}\delta H\,\mathbf{G}_v\,\delta H|0,v\rangle + \langle u,0|\delta H\,\mathbf{G}_v\,\delta H\mathscr{P}|1,v\rangle + \langle u,1|\mathscr{P}\delta H\mathscr{P}|1,v\rangle \\ & + \delta_1\omega_v\,\delta_1\omega_u\,\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle\end{aligned},$$

$$\begin{aligned}A^3_{uv} = {} & \langle u,0|\delta H\,\mathbf{G}_v\,\delta H\,\mathbf{G}_v\,\delta H\,\mathbf{G}_v\,\delta H|0,v\rangle + \langle u,1|\mathscr{P}\delta H\,\mathbf{G}_v\,\delta H\,\mathbf{G}_v\,\delta H|0,v\rangle \\ & + \langle u,0|\delta H\,\mathbf{G}_v\,\delta H\,\mathbf{G}_v\,\delta H\mathscr{P}|1,v\rangle + \langle u,2|\mathscr{P}\delta H\,\mathbf{G}_v\,\delta H|0,v\rangle + \langle u,0|\delta H\,\mathbf{G}_v\,\delta H\mathscr{P}|2,v\rangle \\ & + \langle u,1|\mathscr{P}\delta H\,\mathbf{G}_v\,\delta H\mathscr{P}|1,v\rangle + \langle u,2|\mathscr{P}\delta H\mathscr{P}|1,v\rangle + \langle u,1|\mathscr{P}\delta H\mathscr{P}|2,v\rangle \\ & - \delta_1\omega^2_v\,\delta_1\omega^2_u[\langle u,1| + \langle u,0|[\delta K - \frac{\mathbf{B}}{2\omega_v^{(0)}}]]\mathbf{G}_v[[\delta K - \frac{\mathbf{B}}{2\omega_v^{(0)}}]|0,v\rangle + |1,v\rangle] \\ & + (\delta_1\omega_v\delta_2\omega_u+\delta_1\omega_u\,\delta_2\omega_v-\delta_1\omega_u\delta_1\omega_v\frac{\delta_1\omega_u+\delta_1\omega_v}{2\omega_v^{(0)}})\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle \\ & + \delta_1\omega_v\delta_1\omega_u(\langle u,1|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle+\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|1,v\rangle)\end{aligned} \tag{2.4}$$

In the same way, we can express (2.2) in terms of the power series of ϵ, which gives

$$\begin{aligned}\underline{\langle u|v\rangle} = {} & \langle u,0|0,v\rangle \\ & + \langle u,1|\mathscr{P}|0,v\rangle + \langle u,0|\mathscr{P}|1,v\rangle + B^1_{uv} \\ & + \ldots \\ & + \langle u,0|\mathscr{P}|n,v\rangle + \langle u,n|\mathscr{P}|0,v\rangle + B^n_{uv} \\ & + \ldots,\end{aligned} \tag{2.5}$$

where

$$B^1_{uv} = \langle u,0|\delta K|0,v\rangle - \langle u,0|\frac{B}{2\omega_v^{(0)}}|0,v\rangle,$$

$$B^2_{uv} = \langle u,1|1,v\rangle + \langle u,0|\delta K|1,v\rangle + \langle u,1|\delta K|0,v\rangle - \langle u,1|\frac{B}{2\omega_v^{(0)}}|0,v\rangle - \langle u,0|\frac{B}{2\omega_v^{(0)}}|1,v\rangle + \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega_v^{(0)}}\langle u,0|\frac{B}{2\omega_v^{(0)}}|0,v\rangle,$$

$$B^3_{uv} = \langle u,2|1,v\rangle + \langle u,1|2,v\rangle + \langle u,0|\delta K|2,v\rangle + \langle u,2|\delta K|0,v\rangle + \langle u,1|\delta K|1,v\rangle + \frac{\delta_1\omega_u+\delta_1\omega_v}{2\omega_v^{(0)}}(\langle u,1|\frac{B}{2\omega_v^{(0)}}|0,v\rangle+\langle u,0|\frac{B}{2\omega_v^{(0)}}|1,v\rangle) + (\frac{(\delta_1\omega_u+\delta_1\omega_v)^2}{4\omega_v^{(0)2}} - \frac{\delta_2\omega_u+\delta_2\omega_v}{2\omega_v^{(0)}})\langle u,0|\frac{B}{2\omega_v^{(0)}}|0,v\rangle. - \langle u,2|\frac{B}{2\omega_v^{(0)}}|0,v\rangle - \langle u,0|\frac{B}{2\omega_v^{(0)}}|2,v\rangle - \langle u,1|\frac{B}{2\omega_v^{(0)}}|1,v\rangle \quad (2.6)$$

Equating each term of order ϵ^n for (2.5) and each term of order ϵ^{n+1} for (2.3), we then obtain the two equations

$\langle u,n|\mathcal{P}|0,v\rangle + \langle u,0|\mathcal{P}|n,v\rangle + B^n_{uv} = 0,$

$\langle u,n|\mathcal{P}H|0,v\rangle + \langle u,0|H\mathcal{P}|n,v\rangle + A^n_{uv} = 0,$

which give the expression of $\langle u,0|n,v\rangle$, using

$\mathcal{P}H|0,v\rangle = \delta_1\omega^2_v\,|0,v\rangle.$

We then have

$$\langle u,0|n,v\rangle = \frac{1}{\delta_1\omega^2_v - \delta_1\omega^2_u} A^n_{uv} - \frac{\delta_1\omega^2_v}{\delta_1\omega^2_v - \delta_1\omega^2_u} B^n_{uv}. \quad (2.7)$$

In the case where v is the same as u, we note that the $n+1^{th}$ order eigenfrequency associated to the n^{th} order eigenmode approximation verifies

$$-\omega_v^{(n+1)2} K|v^{(n)}\rangle + \omega_v^{(n+1)} B|v^{(n)}\rangle + A|v^{(n)}\rangle = 0 + o(\epsilon^{n+1}). \quad (2.8)$$

so that

$\langle v^{(n)}| - \omega_v^{(n+1)2} K + \omega_v^{(n+1)} B + A|v^{(n)}\rangle = 0 + o(\epsilon^{n+1}),$

whenever

$\langle v|K - 1/(2\omega_v)\,B|v\rangle = 1 + o(\epsilon^{n+1}).$

Finally, only the relation (2.5) are needed, and assuming that the phase of the eigenmode (which still remains unconstrained) is such that $\langle v|0,v\rangle$ is real. We have thus, equating each term of order ϵ^n to zero:

$$\langle v,0|n,v\rangle = -1/2\, B^n_{vv}. \quad (2.9)$$

Let us now look for the secular terms. Using the secular operators $\mathcal{K}$ and $\mathcal{N}$:

$\mathcal{K} = \mathcal{P}[\delta H + \delta H\, G_v\, \delta H + \delta H\, G_v\, \delta H\, G_v\, \delta H + \ldots]\mathcal{P},$

$$\mathcal{N} = \mathcal{P}^*[1 + G_v\delta H + G_v\delta H\, G_v\delta H + ..]^* \left[1+\delta K - \frac{B}{2\omega_v^{(0)}}\right][1 + G_v\delta H + G_v\delta H\, G_v\delta H + ..]\mathcal{P},$$

one sees clearly that (2.3) can also be written in the form:

$$\begin{aligned}\langle u| &- \omega_v{}^2 K + \omega_v B + A|v\rangle = 0\\ &+ \langle u,0|\mathcal{K}|0,v\rangle\\ &+ \langle u,1|\mathcal{P}\mathcal{K}|0,v\rangle + \langle u,0|\mathcal{K}\mathcal{P}|1,v\rangle + \mathcal{N}^1{}_{uv}\\ &+ \ldots\\ &+ \langle u,n|\mathcal{P}\mathcal{K}|0,v\rangle + \langle u,0|\mathcal{K}\mathcal{P}|n,v\rangle + \mathcal{N}^n_{uv}\\ &+ \ldots,\end{aligned} \quad (2.10)$$

and in the same manner, that (2.5) can be written in the form:

$$\begin{aligned}\underline{\langle u|v\rangle} &= \langle u,0|0,v\rangle\\ &+ \langle u,0|\mathcal{P}\mathcal{N}|1,v\rangle + \langle u,1|\mathcal{P}\mathcal{N}|0,v\rangle + \mathcal{B}^1{}_{uv}\\ &+ \ldots\\ &+ \langle u,0|\mathcal{P}\mathcal{N}|n,v\rangle + \langle u,n|\mathcal{N}\mathcal{P}|0,v\rangle + \mathcal{B}^n_{uv}\\ &+ \ldots,\end{aligned} \quad (2.11)$$

where $\mathcal{A}^{1,2}_{uv}$ and $\mathcal{B}^{1,2}_{uv}$ are now defined as

$$\mathcal{A}^1{}_{uv} = 0,\ \mathcal{B}^1{}_{uv} = 0,$$

$$\mathcal{A}^2{}_{uv} = \langle u,1|\mathcal{P}\delta\mathbf{H}\mathcal{P}|1,v\rangle + \delta_1\omega_v\ \delta_1\omega_u\ \langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle,$$

$$\mathcal{B}^2{}_{uv} = \langle u,1|\mathcal{P}|1,v\rangle + \langle u,0|\delta\mathbf{K}\mathcal{P}|1,v\rangle + \langle u,1|\mathcal{P}\delta\mathbf{K}|0,v\rangle$$
$$- \langle u,1|\frac{\mathcal{P}\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle - \langle u,0|\frac{\mathbf{B}\mathcal{P}}{2\omega_v^{(0)}}|1,v\rangle + \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega_v^{(0)}}\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle.$$

In the same manner as before, we see that, if $|0,v\rangle$ is solution of the secular equation

$\mathcal{K}|0,v\rangle = \delta_1\omega^2_v\ \mathcal{N}|0,v\rangle$,

we retrieve the same relation as (2.7) and (2.9), equating each term order ϵ^n in relations (2.10) and (2.11) to zero. The projection of the first order perturbation is now zero, which leads to the following expressions for $\mathcal{A}^n_{uv}$ and $\mathcal{B}^n_{uv}$:

$$\mathcal{A}^1{}_{uv} = 0,\ \mathcal{B}^1{}_{uv} = 0,$$

$$\mathcal{A}^2{}_{uv} = \delta_1\omega_v\ \delta_1\omega_u\ \langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle,\ \mathcal{B}^2{}_{uv} = \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega_v^{(0)}}\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle,$$

$$\mathcal{A}^3{}_{uv} = -\delta_1\omega^2_v\ \delta_1\omega^2_u\ [\langle u,1| + \langle u,0|[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}]]\mathbf{G}_v[[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}]|0,v\rangle + |1,v\rangle]$$
$$+ (\delta_1\omega_v\delta_2\omega_u + \delta_1\omega_u\ \delta_2\omega_v - \delta_1\omega_u\delta_1\omega_v\frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega_v^{(0)}})\langle u,0|\frac{\mathbf{B}}{2\omega_v{}^{(0)}}|0,v\rangle$$
$$+ \delta_1\omega_v\delta_1\omega_u(\langle u,1|\frac{\mathbf{B}}{2\omega_v{}^{(0)}}|0,v\rangle + \langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|1,v\rangle)$$

$$\mathcal{B}^3{}_{uv} = -(\delta_1\omega^2_v + \delta_1\omega^2_u)\ [\langle u,1| + \langle u,0|[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}]]\mathbf{G}_v[[\delta\mathbf{K} - \frac{\mathbf{B}}{2\omega_v^{(0)}}]|0,v\rangle + |1,v\rangle]$$
$$+ \frac{\delta_1\omega_u + \delta_1\omega_v}{2\omega_v^{(0)}}(\langle u,1|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle + \langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|1,v\rangle)$$
$$+ (\frac{(\delta_1\omega_u + \delta_1\omega_v)^2}{4\omega_v^{(0)2}} - \frac{\delta_2\omega_u + \delta_2\omega_v}{2\omega_v^{(0)}})\langle u,0|\frac{\mathbf{B}}{2\omega_v^{(0)}}|0,v\rangle$$

Appendix 3: Density renormalisation

Let us suppose that the aspherical density is non zero in V_0, reference domain of the SNREI model. We define the renormalised ket as

$$|u_r\rangle = \sqrt{\frac{\rho}{\rho_0}}\,|u\rangle.$$

It is quite obvious that the density-renormalisation permits to retrieve the inertial terms of the reference model, such as the inertial Coriolis and kinetic terms. For the expression of the operator, instead of expression (3), let us start from its eulerian form (Valette, 1986), which may be directly used if all discontinuities of ρ are included in those of the reference density field ρ_0 :

$$\langle v|\mathbf{A}|u\rangle = \int_V \left\{\delta_e\mathbf{T}(\mathbf{u})_{ij}\ (\mathbf{d}^{-1})^{ijkl}\ \delta_e\mathbf{T}(\mathbf{v}^*)_{kl} + \frac{\mathbf{u}.\mathbf{g}\ \mathbf{v}^*.\mathbf{g}}{|\mathbf{g}|^2}\ N^2\right\}dV -$$
$$\int_E \frac{\mathbf{Grad}(\psi(\mathbf{u}))\cdot\mathbf{Grad}(\psi(\mathbf{v}^*))}{4\pi G}\,dV + \int_{\Sigma_\rho}[\rho](\mathbf{g}.\mathbf{n})(\mathbf{u}.\mathbf{n})(\mathbf{v}^*.\mathbf{n})\,d\Sigma. \quad (3.1)$$

where $\delta_e\mathbf{T}(\mathbf{u})$, N^2 are respectively the eulerian stress-tensor perturbation and the Brunt-Vaissala parameter, equal to

$$\delta_e\mathbf{T}(\mathbf{u})^{ij} = \mathbf{d}^{ijkl}\mathbf{D}_k u_l + \rho\ \mathbf{g}.\mathbf{u}\ \mathbf{g}^{ij},$$

$$N^2 = (\ \mathbf{grad}\rho - \rho^2\ (\mathbf{d}^{-1})^{ik}_{ik}\ \mathbf{g}\ ).\mathbf{g}.$$

The first part of this expression represents the perturbation from the elastic state, the second the entropy perturbation and the two last ones the gravitational perturbation. Note that every case of

non-positiveness of the operator **A** is due either to elastic unstability, i.e. a non-positive definite tensor **d** or a negative Brunt Vaisala frequency, a condition for convective unstabilities, or a $[\rho]$ **g.n**, which is a condition for gravitational instabilities of interfaces. Let us express relation (3.1) with the renormalised function $\mathbf{u}_r$ and $\mathbf{v}_r$. We obtain for the different parts:

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$$\delta_e T(\mathbf{u})^{ij} = d^{ijkl} D_k \left(\sqrt{\rho_0/\rho}(u_r)_l \right) + \sqrt{\rho\rho_0}\,\mathbf{g}.\mathbf{u}\; g^{ij} = \sqrt{\rho_0/\rho}\{d^{ijkl} D_k (u_r)_l - 1/2 (u_r)_l D_k \mathrm{Ln}(\rho/\rho_0)\} + \rho_0 \mathbf{g}.\mathbf{u}_r g^{ij}$$

which finally gives for the elastic perturbation

$\delta_e T(\mathbf{u})_{ij}\ (d^{-1})^{ijkl}\ \delta_e T(\mathbf{v}^*)_{kl} = \delta_e T_r(\mathbf{u}_r)_{ij}\ (d_r^{-1})^{ijkl}\ \delta_e T_r(\mathbf{v}_r^*)_{kl}$,

where the renormalised stress and stiffness tensor are

$$d_r^{ijkl} = \frac{\rho_0}{\rho} d^{ijkl},$$

$$\delta_e T_r(\mathbf{u})^{ij} = d_r^{ijkl} \left\{ D_k u_l - \frac{1}{2} u_l D_k \mathrm{Ln}(\rho/\rho_0) \right\} + \rho_0 \mathbf{g}.\mathbf{u}\; g^{ij}.$$

Renormalisation of entropy perturbation

After substitution, we get the term

$\rho_0/\rho\ (\ \mathbf{grad}\rho - \rho^2 (d^{-1})^{ik}_{ik}\ \mathbf{g}\ ) = (\ \mathbf{grad}\rho_0 - \rho_0^2 (d_r^{-1})^{ik}_{ik}\ \mathbf{g}\ ) + \rho_0 \mathbf{grad}(\mathrm{Ln}(\rho/\rho_0))$

which gives for the entropy perturbation

$$\frac{\mathbf{u}_r.\mathbf{g}\ \mathbf{v}_r^*.\mathbf{g}}{|\mathbf{g}|^2} N^2_r,$$

where N^2_r is the renormalised Brunt Vaissala parameter given by

$N^2_r = (\ \mathbf{grad}\rho_0 - \rho_0^2 (d_r^{-1})^{ik}_{ik}\ \mathbf{g}\ ).\mathbf{g} + \rho_0 \mathbf{grad}(\mathrm{Ln}(\rho/\rho_0)).\mathbf{g}$.

Renormalisation of the boundary perturbation

The renormalised boundary term here simply becomes.

$[\rho_0]\ \mathbf{g}.\mathbf{n}\ \mathbf{u}_r.\mathbf{n}\ \mathbf{v}_r^*.\mathbf{n}$

Renormalisation of the pre-stress

The pre-stress, which appears in the tensor **d**, is simply renormalised by a multiplication by ρ_0/ρ. Let us define the renormalised pre-stress as

$\sigma_r^{ij} = \rho_0/\rho\ \sigma^{ij}$.

Starting from the equilibrium equations of the reference and aspherical states,

$D_i \sigma_0^{ij} + \rho_0\ g_0^j = 0$

$D_i \sigma^{ij} + \rho\ g^j = 0$

we obtain , multipling by ρ_0/ρ,

$D_i(\sigma_r^{ij} - \sigma_0^{ij}) - \sigma_r^{ij}\ D_i(\mathrm{Ln}(\rho/\rho_0)) = 0$,

which gives to first order for a hydrostatic reference model

$D_i(\sigma_r^{ij} - \sigma_0^{ij}) + p_0\ D^j(\mathrm{Ln}(\rho/\rho_0)) = 0$,

Renormalisation of the redistribution potential

There is no simple way to renormalise the redistribution potential. Nevertheless, the effect of this renormalisation is a second order effect and may be neglected, except for modes with very low angular order. In this case, since:

$$\int_E \frac{\mathbf{Grad}(\psi(\mathbf{u}_r)).\mathbf{Grad}(\psi(\mathbf{v}^*_r))}{4\pi G} dV = \int_V \mathbf{Grad}(\psi(\mathbf{u}_r)).\rho \mathbf{v}^*_r dV .$$

one has to take into account the perturbation to the mass-redistribution force, which can be written as :

$$\mathbf{grad}(\psi(\mathbf{v}_r)) = \sqrt{\rho_0/\rho}\ \mathbf{grad}\ (\psi_0(\mathbf{v})) + \delta\Psi,$$

where ψ_0 and $\psi(\mathbf{v}_r)$ are respectively solutions of the spherical and aspherical redistribution potential equation:

$$\nabla^2\psi_0(\mathbf{v}) = 4\pi G\ \mathrm{div}(\rho_0\mathbf{v}),$$

$$\nabla^2\psi(\mathbf{v}_r) = 4\pi G\ \mathrm{div}(\rho\mathbf{v}_r).$$

Using all these expressions, neglecting the perturbation of the mass-renormalisation forces, the final expression of the renormalised operator A_r is thus given by

$$\langle \mathbf{v}|A_r|\mathbf{u}\rangle = \int_V \left\{ \delta_e T_r(\mathbf{u})_{ij}\ (d_r^{-1})^{ijkl}\ \delta_e T_r(\mathbf{v}^*)_{kl} + \frac{\mathbf{u}.\mathbf{g}\ \mathbf{v}^*.\mathbf{g}}{|\mathbf{g}|^2}\ N_r^2 \right\} dV$$

$$- \int_E \frac{\mathbf{Grad}(\psi_0(\mathbf{u})).\mathbf{Grad}(\psi_0(\mathbf{v}^*))}{4\pi G}\ dV + \int_{\Sigma_\rho} [\rho_0](\mathbf{g}.\mathbf{n})(\mathbf{u}.\mathbf{n})(\mathbf{v}^*.\mathbf{n})\ d\Sigma \quad (3.2).$$

Apppendix 4: Error Analysis and perturbation in the product space

Let us first use the Krylow-Weinstein theorem (Dautray & Lions, 1985), valid for a self-adjoint operator **A**, as in the case of the elastic case.

Krylow-Weinstein Theorem

For every $|\mathbf{v}\rangle$ included in the definition domain of the operator **A**, let us define the real α and β such as

$\alpha = \langle \mathbf{v}|\mathbf{A}|\mathbf{v}\rangle/\langle \mathbf{v}|\mathbf{v}\rangle$,

$\beta = \sqrt{\langle \mathbf{v}|\mathbf{A}^2|\mathbf{v}\rangle/\langle \mathbf{v}|\mathbf{v}\rangle}$.

Then the operator **A** has a real eigenvalue λ, such that

$\alpha - \sqrt{\beta^2-\alpha^2} < \lambda < \alpha + \sqrt{\beta^2-\alpha^2}$.

We can apply this theorem for a non-rotating Earth, and consider an n^{th} order approximation for the solution of the renormalised equation (34). Using (19), we obtain

$$-\omega_v^{2(n)}\ |\mathbf{v}^{(n)}\rangle + \mathbf{A}_r|\mathbf{v}^{(n)}\rangle = |\mathbf{n},\mathbf{r},\mathbf{v}\rangle + o(\epsilon^{n+1})\ . \quad (4.1)$$

and using the Krylow-Weinstein theorem, we have the approximation α for the quadratic frequency of singlet $|\mathbf{v}^{(n)}\rangle$, by definition the Rayleigh quotient:

$$\alpha = \omega_v^{2(n)} + \langle \mathbf{v}^{(n)}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle/\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle = \omega_v^{2(n+1)} + o(\epsilon^{n+1}), \quad (4.2)$$

i.e. the $n+1^{th}$ order approximation of eigenfrequency. As $\mathbf{A}_r$ is a self-adjoint operator, we have for the second term β^2, using its definition given in part 4.3 and equation (4.1) :

$$\beta^2 = (\omega_v^{2(n)})^2 + 2\ \omega_v^{2(n)}\ \langle \mathbf{v}^{(n)}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle/\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle + \langle \mathbf{v},\mathbf{r},\mathbf{n}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle/\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle,$$

which finally gives, for the error bound on quadratic eigenfrequency α:

$$\sqrt{\beta^2-\alpha^2} = \sqrt{\langle \mathbf{v},\mathbf{r},\mathbf{n}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle/\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle - \langle \mathbf{v}^{(n)}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle^2/\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle^2}$$

which yields, using $\langle \mathbf{v}^{(n)}|\mathbf{v}^{(n)}\rangle \simeq 1$,

$$\sqrt{\beta^2-\alpha^2} = \sqrt{\langle \mathbf{v},\mathbf{r},\mathbf{n}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle - \langle \mathbf{v}^{(n)}|\mathbf{n},\mathbf{r},\mathbf{v}\rangle^2}. \quad (4.3)$$

We note that the error is the norm of the projection of the remaining ket $|\mathbf{r},\mathbf{n},\mathbf{v}\rangle$ on the orthogonal of $|\mathbf{v}^{(n)}\rangle$.

In the non renormalised case, starting from

$$-\omega_v^{2(n)}\ \mathbf{K}|\mathbf{v}^{(n)}\rangle + \mathbf{A}|\mathbf{v}^{(n)}\rangle = |\mathbf{n},\mathbf{r},\mathbf{v}\rangle + o(\epsilon^{n+1})\ , \quad (4.4)$$

we have, after multiplication by $K^{(1/2)}$:

$$- \omega_v^{2(n)} \mathbf{K}^{1/2}|v^{(n)}> + \mathbf{A}_r \mathbf{K}^{1/2}|v^{(n)}> = \mathbf{K}^{1/2}|n, r, v> + o(\epsilon^{n+1}),$$

which proves that the expressions for α and β are the same to order n+1.

Let us now generalise this result in the case of a rotating, elastic Earth. Instead of using the second order equation

$$-\omega^2 |v> + \omega \mathbf{B}|v> + \mathbf{A}|v> = 0, \tag{4.5}$$

we shall use the equivalent first order equation written in the product space (Valette, 1989) of generalized kets $|\underline{v}>$

$$|\underline{v}> = \begin{bmatrix} 1/\omega \mathbf{A}^{1/2} \mathbf{v} \\ \mathbf{v} \end{bmatrix}.$$

This equation can thus be written as:

$$\omega |\underline{v}> = \mathbf{H} |\underline{v}>, \tag{4.6}$$

where $\mathbf{H}$ the hamiltonian associated to the operators $\mathbf{A}$ and $\mathbf{B}$, defined as

$$\mathbf{H} = \begin{bmatrix} 0 & \mathbf{A}^{1/2} \\ \mathbf{A}^{1/2} & \mathbf{B} \end{bmatrix}. \tag{4.7}$$

Here $\mathbf{A}^{1/2}$ is the square root of the operator $\mathbf{A}$, defined if no unstable modes are included in the spectrum of $\mathbf{A}$. It can be proven (Valette, 1989) that the hamiltonian is self-adjoint, and thus all properties and results of the perturbation theory and Krylow-Weinstein theorem applications can be done in the product space. For exemple, for any n^{th} order approximation of the renormalized quadratic equation (4.5):

$$- \omega_v^{(n)2} |v^{(n)}> + \omega_v^{(n)} \mathbf{B}|v^{(n)}> + \mathbf{A}|v^{(n)}> = |n, r, v> + o(\epsilon^{n+1}),$$

we can define an n^{th} order approximation in the product space $|\underline{v}^{(n)}>$ and its associated residual $|n, r, \underline{v}>$ as

$$|\underline{v}^{(n)}> = \begin{bmatrix} 1/\omega_v^{(n)} \mathbf{A}^{1/2} v^{(n)} \\ v^{(n)} \end{bmatrix},$$

$$|r,n,\underline{v}^{(n)}> = \begin{bmatrix} 0 \\ -1/\omega_v^{(n)} r^{(n)} \end{bmatrix},$$

where $r_v^{(n)}$ is $|n,r,v^{(n)}>$.

This n^{th} order approximation will verify equation (4.6) to within the residual:

$$\omega_v^{(n)} |\underline{v}^{(n)}> = \mathbf{H} |\underline{v}^{(n)}> + |n, r, \underline{v}> + o(\epsilon^{n+1}).$$

We see that the n^{th} order perturbation of frequency is now obtained by the projection of the residual part on the zero order $|\underline{v}^{(0)}>$:

$$<\underline{v}^{(0)}|n,r,\underline{v}^{(n)}> = - <v^{(0)}|n,r,v>/\omega_v^{(n)}.$$

The Krylow-Weinstein theorem can now be used, with relations equivalent to (4.2), (4.3) and (4.4), but with the product space inner product. We have thus for α:

$$\alpha = <\underline{v}^{(n)}|\mathbf{H}|\underline{v}^{(n)}>/<\underline{v}^{(n)}|\underline{v}^{(n)}>,$$

which gives, using (4.7)

$$\alpha = \omega_v^{(n)} - <\underline{v}^{(n)}|r,n,\underline{v}>/<\underline{v}^{(n)}|\underline{v}^{(n)}>,$$

and for β

$$\beta = <\underline{v}^{(n)}|\mathbf{H}^2|\underline{v}^{(n)}>/<\underline{v}^{(n)}|\underline{v}^{(n)}>.$$

which gives

$$\beta^2 - \alpha^2 = <\underline{v},r,n|n,r,\underline{v}>/<\underline{v}^{(n)}|\underline{v}^{(n)}> - <\underline{v}^{(n)}|n, r, \underline{v}>^2/<\underline{v}^{(n)}|\underline{v}^{(n)}>^2.$$

These results can now be expressed in the usual space, using

$\langle \underline{v}^{(n)} | \underline{v}^{(n)} \rangle = \langle v^{(n)} | v^{(n)} \rangle + \langle v^{(n)} | A | v^{(n)} \rangle / \omega_v^{(n)2}$,

which gives, using equation (4.4):

$\langle \underline{v}^{(n)} | \underline{v}^{(n)} \rangle = 2 \langle v^{(n)} | v^{(n)} \rangle - \langle v^{(n)} | B | v^{(n)} \rangle / \omega_v^{(n)} + \langle v^{(n)} | r,n,v \rangle / \omega_v^{(n)2}$,

and thus, if $\langle v^{(n)} | v^{(n)} \rangle \simeq 1$,

$\alpha = \omega_v^{(n)} + \delta^2 \omega_v^{(n)} / 2\omega_v^{(n)} + o(\epsilon^{n+1})$

which is nothing else but the squared root of the n+1 order quadratic frequency. In the same way, the Krylow-Weinstein error may be expressed as :

$\sqrt{\beta^2 - \alpha^2} = \sqrt{2\langle v,r,n|n,r,v\rangle - \langle v^{(n)}|n,r,v\rangle^2}$.

In this case, this error is thus bigger than those defined in the quadratic case. In the non-renormalised case, all relations may be applied, to the same order.

Appendix 5: The transformation method

The Legendre forward and backward transformation of a tensorial field U of order N are given by

Backward transformation

$$U^n(\mu_i, \phi_j) = \sum_{m=-M}^{m=+M} \exp(-im\phi_j) \sum_{\ell=|m|}^{\ell=\ell max} U_\ell^{nm} P_\ell^{nm}(\mu_i). \tag{5.1}$$

Forward transformation

$$U_\ell^{nm} = \sum_{i=1}^{i=nmax} w_i P_\ell^{nm}(\mu_i) \frac{1}{M} \sum_j \exp(im\phi_j) U^n(\mu_i, \phi_j) \tag{5.2}$$

In this appendix, we shall give an optimised discrete, generalized Legendre Transformation based on a direct decomposition of the field U into the four symmetry subsets S_{1234}. Let us take a field U in one of these subsets, with associated eigenvalues η_t, η_p. After P reflection:

$$U^{-n}(-\mu_i, \phi_j + \pi) = \sum_{m=-M}^{m=+M} \exp(im\phi_j) \sum_\ell (-1)^\ell U_\ell^{-nm} P_\ell^{nm}(\mu_i)$$
$$= \eta_p U^n(\mu_i, \phi_j)$$

and thus

$$U_\ell^{nm} = \eta_p (-1)^\ell U_\ell^{-nm}. \tag{5.3}$$

After T reflection:

$$(-1)^n U^{-n}(\mu_i, \phi_j)^* = \sum_{m=-M}^{m=+M} \exp(im\phi_j) \sum_{\ell=|m|} (-1)^m U_\ell^{-n-m^*} P_\ell^{nm}(\mu_i)$$
$$= \eta_t U^n(\mu_i, \phi_j)$$

and thus

$$U_\ell^{Nm} = \eta_t (-1)^m U_\ell^{-n-m^*}. \tag{5.4}$$

The intrinsic symmetries are very important to provide an optimisation of the discrete transformation. We start from equation 5.1, using the symmetry of Gauss points, the symmetry relation of the generalised Legendre functions:

$$P_\ell^{nm}(-\mu_i) = (-1)^{\ell+m} P_\ell^{-nm}(\mu_i), \tag{5.5}$$

and the property of the Fourier Transformation:

$$\frac{1}{M}\sum_{j}(-1)^m \exp(im\phi_j)\, U^n(\mu_i,\phi_j) = \frac{1}{M}\sum_{j}\exp(im\phi_j)\, U^n(\mu_i,\phi_j-\pi),$$

Expression 5.1 then becomes:

$$U_\ell^{nm} = \sum_{i=nmax/2}^{i=nmax} w_i\,(-1)^\ell\, P_\ell^{-nm}(\mu_i)\,\frac{1}{M}\sum_{j}\exp(im\phi_j)\, U^n(-\mu_i,\phi_j-\pi)$$
$$+\sum_{i=nmax/2}^{i=nmax} w_i\, P_\ell^{nm}(\mu_i)\,\frac{1}{M}\sum_{j}\exp(im\phi_j)\, U^n(\mu_i,\phi_j)$$

but, using the P and T reflections:

$$U^n(-\mu_i,\phi_j-\pi) = \eta_p\, U^{-n}(\mu_i,\phi_j) = (-1)^n\,\eta_p\,\eta_t\, U^{n*}(\mu_i,\phi_j)$$

using the functions

$$A_\ell^{nm}(\mu_i) = \{\, P_\ell^{nm}(\mu_i) + (-1)^{n+\ell}\,\eta_p\,\eta_t\, P_\ell^{nm}(\mu_i)\,\}$$
$$B_\ell^{nm}(\mu_i) = \{\, P_\ell^{nm}(\mu_i) - (-1)^{n+\ell}\,\eta_p\,\eta_t\, P_\ell^{nm}(\mu_i)\,\}$$

we finally have the expression:

$$U_\ell^{nm} = \sum_{i=nmax/2}^{i=nmax} w_i\, A_\ell^{nm}(\mu_i)\,\frac{1}{M}\sum_{j}\exp(im\phi_j)\,\mathrm{Real}\{\, U^n(\mu_i,\phi_j)\} +$$
$$\sum_{i=nmax/2}^{i=nmax} w_i\, B_\ell^{nm}(\mu_i)\,\frac{1}{M}\sum_{j}\exp(im\phi_j)\,\mathrm{Imag}\{\, U^n(\mu_i,\phi_j)\}. \qquad (5.6)$$

Thus, we note that only components for positive n and m index are necessary. For scalar fields, the number of computations is divided by two, only A_ℓ or B_ℓ being non zero. This is the symmetric/antisymmetric decomposition used in the scalar Legendre Transformation (Butel, 1984).

The backward transformation optimisation uses the same symmetry. Starting from equation 5.2, we have for the real part of U^n:

$$\mathrm{Real}\{\, U^n(\mu_i,\phi_j)) = 1/2\,\{\, U^n(\mu_i,\phi_j) + U^{n*}(\mu_i,\phi_j)\}$$
$$= 1/2\sum_{m=-M}^{m=+M}\exp(-im\phi_j)\sum_{l=|m|}^{\ell=\ell max} U_\ell^{nm}\, P_\ell^{nm}(\mu_i) + U_\ell^{n-m*}\, P_\ell^{n-m}(\mu_i), \qquad (5.7)$$

but

$$U_\ell^{n-m*} = \eta_p\eta_t(-1)^{\ell+m}\, U_\ell^{nm},$$

and thus

$$\mathrm{Real}\{\, U^n(\mu_i,\phi_j)) = \sum_{m=-M}^{m=+M}\exp(-im\phi_j)\sum_{\ell=|m|}^{\ell=\ell max} U_\ell^{nm}\; 1/2\; A_\ell^{nm}(\mu_i). \qquad (5.8)$$

For the imaginary part, we easily find the same expression, with B instead of A

$$\mathrm{Imag}\{\, U^n(\mu_i,\phi_j)) = \sum_{m=-M}^{m=+M}\exp(-im\phi_j)\sum_{\ell=|m|}^{\ell=\ell max} U_\ell^{nm}\; 1/2\; B_\ell^{nm}(\mu_i). \qquad (5.9)$$

Here also, the computation is necessary for positive index terms U^{nm} only in the half spectral space, these terms being respectively hermitian or antihermitian. Using the symmetries of the functions A and B,

$$A_\ell^{n-m} = +(-1)^{\ell+m}\,\eta_p\,\eta_t\, A_\ell^{Nm},$$
$$A_\ell^{-nm} = +(-1)^{\ell+n}\,\eta_p\,\eta_t\, A_\ell^{Nm},$$
$$B_\ell^{n-m} = -(-1)^{\ell+m}\,\eta_p\,\eta_t\, B_\ell^{Nm},$$
$$B_\ell^{-nm} = -(-1)^{\ell+n}\,\eta_p\,\eta_t\, B_\ell^{Nm}, \qquad (5.10)$$

odd and even parts of ℓ are separated. For a vector, the number of computations is 3 times larger of course and generally, is equal to $(2N+1)(\ell max+1)nmax(M+1)$ for the forward transformation and $(2N+1)(2\ell max+1-M)nmax(M+1)$ for the backward transformation. We recall that ℓmax is the maximum order of the Legendre Transformation, nmax is the number of Gauss points.