Geophys. J. R. astr. Soc. (1986) **86**, 167–176

The free oscillations of an anisotropic and heterogeneous earth

E. Mochizuki *Geophysical Institute, Faculty of Science, University of Tokyo, Tokyo, Japan*

Accepted 1986 January 6. Received 1986 January 6; in original form 1985 May 16

Summary. Generalized spherical harmonics are used to simplify the calculation of the perturbation matrix elements (coupling coefficients) for the free oscillations of an anisotropic and laterally heterogeneous earth. In the asymptotic limit of large angular order, the local frequency pertubration which depends on the azimuth and on the location of Earth's surface is defined, and the correspondence to surface waves is established.

Key words: anisotropy, free oscillation, surface waves

1 Introduction

The effect of aspherical perturbations on the Earth's free oscillations has been studied by many authors. The theoretical results are summarized by Woodhouse & Dahlen (1978) and Woodhouse (1980). The perturbations from spherical symmetry include diurnal rotation, ellipticity and lateral heterogeneity. However, it has been usual to neglect elastic anisotropy except for transverse isotropy (Woodhouse & Dziewonski 1984). In Section 2, we extend the theory to the case of general anisotropy. The procedure is the same as for the isotropic case, and the only difference lies in the derivation of the perturbation matrix elements (expressing coupling between modes due to the aspherical pertubations). This is accomplished by using the generalized spherical harmonics of Phinney & Burridge (1973).

The correspondence between free oscillations and surface waves for an isotropic and aspherical earth has been established by Jordan (1978), Woodhouse & Girnius (1982), Woodhouse & Dziewonski (1984) and Mochizuki (1986). In Section 3, these results are extended to the anisotropic case following the procedure of Mochizuki (1986), and the immediate application of our theory is discussed.

2 Free oscillations

First we consider the free oscillations of a laterally homogeneous and transversely isotropic sphere as a reference model (Dziewonski & Anderson 1981). Following Phinney & Burridge (1973) and Woodhouse (1980), we define complex, spherical, contravariant components of displacement s for each singlet

$$s^{\pm} = \frac{1}{\sqrt{2}}(\mp s_\theta + i s_\phi)$$

$$= \gamma\, \Omega_0^l\, ({}_nV_l^q \pm i\, {}_nW_l^q)\, Y_l^{\pm 1 m} \tag{1}$$

$$s^0 = s_r$$

$$= \gamma_n\, U_l^q\, Y_l^{0m} \tag{2}$$

where $\gamma_l = [(2l+1)/4\pi]^{1/2}$ and $\Omega_N^l = [(l+N)(l-N+1)/2]^{1/2}$. In equations (1) and (2), (s_r, s_θ, s_ϕ) are displacement components for spherical polar coordinates, $Y_l^{Nm}(\theta,\phi)$ are the generalized spherical harmonics and $U(r)$, $V(r)$, $W(r)$ (hereafter the indices are omitted) are radial eigenfunctions. The parameters n, l, m are radial order, angular order and azimuthal order, respectively, and q denotes mode type; T(toroidal) or S(spheroidal). Because of spherical symmetry, singlets which have the same (q, n, l) are degenerate and make a multiplet. We use the symbol k to denote the multiplet with degenerate frequency ω_k. The contravariant components of strain are

$$\epsilon^{\pm\pm} = \gamma_l\, \Omega_0^l \Omega_2^l\, r^{-1}(V \pm iW)\, Y_l^{\pm 2m}$$

$$\epsilon^{00} = \gamma_l\, \dot{U}\, Y_l^{0m}$$

$$\epsilon^{+-} = -(1/2)\,\gamma_l f\, Y_l^{0m}$$

$$\epsilon^{0\pm} = (1/2)\,\gamma_l\, \Omega_0^l\, (X \pm iZ)\, Y_l^{\pm 1m} \tag{3}$$

where

$$f = r^{-1}[2U - l(l+1)V]$$

$$X = \dot{V} + r^{-1}(U - V)$$

$$Z = \dot{W} - r^{-1}W \tag{4}$$

and '$\cdot$' denotes differentiation with respect to r.

We proceed to the laterally heterogeneous and anisotropic case. We must evaluate the perturbation matrix element (Woodhouse & Dahlen 1978)

$$\langle k'm' | \mathbf{C} | km \rangle$$

$$= \int (\epsilon')^* : \mathbf{C} : \epsilon\, dV \tag{5}$$

$$= \int (\epsilon'^{\alpha\beta})^*\, C^{\alpha\beta\gamma\delta}\, \epsilon^{\gamma'\delta'}\, e_{\gamma\gamma'}\, e_{\delta\delta'}\, dV \tag{6}$$

where $e_{00} = 1$, $e_{+-} = e_{-+} = -1$ and $e_{\gamma\gamma'} = 0$ if $\gamma + \gamma' \neq 0$. In equations (5) and (6) ϵ is the strain, as shown in (3) with (k, m) and ϵ' with (k', m'), '*' denotes complex conjugation, and $\mathbf{C}$ is the perturbation of the elastic tensor from the spherically symmetric and transversely isotropic starting model.

We expand the contravariant components of C, which are shown explicitly in Appendix A, in terms of generalized spherical harmonics

$$C^{\alpha\beta\gamma\delta}(r,\theta,\phi) = \sum_{s=|N|}^{\infty} \sum_{t=-s}^{s} C_{st}^{\alpha\beta\gamma\delta}(r)\, Y_s^{Nt}(\theta,\phi) \tag{7}$$

where $N = \alpha + \beta + \gamma + \delta$. For convenience, we use the notation $\mathbf{C}^{(N)}$ whose contravariant components have $\alpha + \beta + \gamma + \delta = N$. When the perturbation is transversely isotropic ($N = 0$), (7) coincides with the usual expansion by spherical harmonics Y_s^t except for the normalization factor ($Y_s^t = \gamma_s Y_s^{0\,t}$). To evaluate (6) we use the formula (Edmonds 1960)

$$\int (Y_{l_1}^{N_1\, m_1})^* \, Y_{l_2}^{(N_1 - N_3)\, m_2} \, Y_{l_3}^{N_3\, m_3} \, d\Omega$$

$$= 4\pi (-1)^{N_1 + m_1} \begin{pmatrix} l_1 & l_2 & l_3 \\ -N_1 & N_1 - N_3 & N_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \tag{8}$$

where integration is over the surface of the unit sphere and

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}$$

is a Wigner $3-j$ symbol which is zero unless $l_i \geqslant |m_i|$, $|l_1 - l_2| \leqslant l_3 \leqslant l_1 + l_2$ and $m_1 = m_2 + m_3$. Straightforward calculation shows

$$\langle k'm' | \mathbf{C} | km \rangle = \sum_{N=-4}^{4} \langle k'm' | \mathbf{C}^{(N)} | km \rangle \tag{9}$$

where $\langle k'm' | \mathbf{C}^{(N)} | km \rangle$

$$= \sum_s 4\pi\gamma_{l'}\gamma_l (-1)^{m'} \begin{pmatrix} l' & s & l \\ -m' & m' - m & m \end{pmatrix} \sum_i \int K_{Ni} r^2 dr. \tag{10}$$

The kernels K_{Ni} which are related to $\mathbf{C}^{(N)}$ are shown in Appendix B.

Next we discuss the matrix element required for first-order perturbation theory ($k' = k$). Using properties of the $3-j$ symbols (Edmonds 1960), we obtain the following results.

(1) All of the K_{Ni} are zero unless s is even and $s \leqslant 2l$.

(2) K_{Ni} and K_{-Ni} are proportional.

(3) At the asymptotic limit ($l \gg s$), K_{Ni} with odd N tend to zero (Appendix C). This result is related to the azimuthal dependence of surface wave phase velocity as shown in the next section.

3 Asymptotic approximation

Following Jordan (1978) and Woodhouse & Girnius (1982) the multiplet location parameter is defined for each multiplet k

$$\lambda_k = \frac{\sum_{mm'} R_k^{m'} \langle km' | \mathbf{C} | km \rangle S_k^m}{2\omega_k \sum_m R_k^m S_k^m} \tag{11}$$

where the receiver vector R_k^m and the source vector S_k^m are given explicitly in Woodhouse & Girnius (1982). Because our purpose is the extension of previous results to the anisotropic case, the effects of other perturbations (density and boundary) are neglected.

For the asymptotic limit ($l \gg s$) the matrix elements may be approximated by (using the formulae of Appendix C)

$$\langle km' \,|\mathbf{C}^{(2N)}|\, km\rangle$$

$$\sim \frac{2l+1}{4\pi} \int 2\omega_k\, \delta\omega_k^{(2N)}\, Y_l^{Nm'*}\, Y_l^{-Nm}\, d\Omega \tag{12}$$

where

$$2\omega_k\, \delta\omega_k^{(0)}(\theta,\phi) = \int [\dot{U}^2 C^{0000} + (1/2)\, l(l+1)(l+2)(l-1) r^{-2}(V^2 + W^2) C^{++--} + f^2 C^{+-+-} - 2f\dot{U}C^{+-00} - l(l+1)(X^2 + Z^2) C^{+0-0}]\, r^2\, dr \tag{13}$$

$$2\omega_k\, \delta\omega_k^{(2)}(\theta,\phi) = l(l+1) \int [r^{-1} V\dot{U}C^{++00} - (1/2)(X^2 - Z^2) C^{+0+0} - r^{-1} VfC^{+++-}]\, r^2\, dr$$

$$2\omega_k\, \delta\omega_k^{(-2)}(\theta,\phi) = l(l+1) \int [r^{-1} V\dot{U}C^{--00} - (1/2)(X^2 - Z^2) C^{-0-0} - r^{-1} VfC^{---+}]\, r^2\, dr$$

$$2\omega_k\, \delta\omega_k^{(4)}(\theta,\phi) = (1/4)\, l(l+1)(l+2)(l-1) \int (V^2 - W^2) C^{++++}\, dr$$

$$2\omega_k\, \delta\omega_k^{(-4)}(\theta,\phi) = (1/4)\, l(l+1)(l+2)(l-1) \int (V^2 - W^2) C^{----}\, dr.$$

For odd N, the matrix elements tend to zero as stated in Section 2. Substituting (12) into (11) and following the procedure of Mochizuki (1986), the multiplet location parameter may be approximated by, in the asymptotic limit,

$$\lambda_k \sim \sum_{N=-2}^{2} (1/2\pi) \int \delta\omega_{\mathrm{local}}^{(2N)}\, ds \tag{14}$$

where

$$\delta\omega_{\mathrm{local}}^{(2N)}(\theta,\phi,\psi) = (-1)^N \exp(2iN\psi)\, \delta\omega_k^{(2N)}(\theta,\phi) \tag{15}$$

and the azimuth ψ is measured from south to east. The integration in (14) is over the great circle containing the source and receiver, and the correspondence to surface waves (great circle measurement) is fairly clear. The local frequency perturbations $\delta\omega_{\mathrm{local}}^{(2N)}$, which depend on the azimuth for the anisotropic case ($N \neq 0$), are related to the phase velocity of surface waves.

The application of $\delta\omega_{\mathrm{local}}^{(0)}$ to minor arc measurement has been discussed by Woodhouse & Dziewonski (1984) and Mochizuki (1986). In addition to the apparent frequency shift (multiplet location parameter) the apparent distance shift is introduced, and the amplitude perturbation is incorporated into the phase perturbation. These results are also valid for the anisotropic case.

$\delta\omega_{\mathrm{local}}^{(2N)}$ depend on $\mathbf{C}^{(2N)}$ as shown in (13) and (15). This is consistent with the plane-stratified model of Smith & Dahlen (1973) where the canonical harmonics components of $\mathbf{C}$ are used (Appendix A).

4 Discussion

Recently Tanimoto & Anderson (1985) obtained the distribution of $\delta\omega_{\mathrm{local}}^{(2N)}(\theta, \phi, \psi)$ using a surface wave approach. However, they used the spherical harmonics expansion

$$\delta\omega_{\text{local}}^{(2N)}(\theta, \phi, \psi) = \sum_{st} A_s^{2N\ t} Y_s^t(\theta, \phi) \exp(2iN\psi). \tag{16}$$

Because $\delta\omega_{\text{local}}^{(2N)}$ depends on $\mathbf{C}^{(2N)}$ which is expanded by the generalized spherical harmonics (7), expression (16) is inconvenient when we wish to obtain the distribution of $\mathbf{C}$. The generalized spherical harmonics expansion is suggested,

$$\delta\omega_{\text{local}}^{(2N)}(\theta, \phi, \psi) = \sum_{st} A_s^{2N\ t} Y_s^{2N\ t}(\theta, \phi) \exp(2iN\psi). \tag{17}$$

The waveform inversion of Woodhouse & Dziewonski (1984) can be extended to the anisotropic case using (13) and (15). However, as long as we rely on the asymptotic approximation, $\mathbf{C}^{(N)}$ with odd N cannot be determined. To determine these components the application of the Born approximation (Woodhouse 1983) with the matrix elements given in (10) may be useful.

Acknowledgments

I would like to thank M. Kumazawa for arousing my interest in anisotropy and for reading the manuscript. I also thank T. Tanimoto for pointing out errors in earlier version of the manuscript.

References

Backus, G. E., 1970. A geometrical picture of anisotropic elastic tensors, *Rev. Geophys. Space Phys.*, **8**, 633–671.

Edmonds, A. R., 1960. *Angular Momentum in Quantum Mechanics*, Princeton University Press, New Jersey.

Dziewonski, A. M. & Anderson, D. L., 1981. Preliminary Reference Earth Model (PREM), *Phys. Earth planet. Int.*, **25**, 297–356.

Jordan, T. H., 1978. A procedure for estimating lateral variations from low-frequency eigenspectra data, *Geophys. J. R. astr. Soc.*, **52**, 441–455.

Mochizuki, E., 1986. Free oscillations and surface waves of an aspherical earth, *Geophys. Res. Lett.*, in press.

Phinney, R. A. & Burridge, R., 1973. Representation of the elastic-gravitational excitation of a spherical earth model by generalized spherical harmonics, *Geophys. J. R. astr. Soc.*, **34**, 451–487.

Smith, M. L. & Dahlen, F. A., 1973. The azimuthal dependence of Love and Rayleigh wave propagation in a slightly anisotropic medium, *J. geophys. Res.*, **78**, 3321–3333.

Takeuchi, H. & Saito, M., 1972. Seismic surface waves, in *Methods in Computational Physics*, **11**, 217–295, Academic Press, New York.

Tanimoto, T. & Anderson, D. L., 1985. Lateral heterogeneity and azimuthal anisotropy of the upper mantle: Love and Rayleigh wave 100–250 s, *J. geophys. Res.*, **90**, 1842–1858.

Woodhouse, J. H., 1980. The coupling and attenuation of nearly resonant multiplets in the earth's free oscillation spectrum, *Geophys. J. R. astr. Soc.*, **61**, 261–283.

Woodhouse, J. H., 1983. The joint inversion of seismic wave forms for lateral variations in earth structure and earthquake source parameters, in *Proc. Enrico Fermi International School of Physics*, **LXXXV**, eds Kanamori, H. & Boschi, E., North-Holland, Amsterdam.

Woodhouse, J. H. & Dahlen, F. A., 1978. The effect of a general aspherical perturbation on the free oscillations of the earth, *Geophys. J. R. astr. Soc.*, **53**, 335–354.

Woodhouse, J. H. & Dziewonski, A. M., 1984. Mapping the upper mantle: three-dimensional modeling of earth structure by inversion of seismic waveforms, *J. geophys. Res.*, **89**, 5953–5986.

Woodhouse, J. H. & Girnius, T. P., 1982. Surface waves and free oscillations in a regionalized earth model, *Geophys. J. R. astr. Soc.*, **68**, 653–673.

Appendix A

The contravariant components of the elastic tensor are shown. As they have symmetries

$$C^{\alpha\beta\gamma\delta} = C^{\beta\alpha\gamma\delta} = C^{\gamma\delta\alpha\beta},$$

21 components are independent. Also the relation to the canonical harmonics components C_σ^{lmp} (Backus 1970; Smith & Dahlen 1973) is shown. Note the correspondence between $N = \alpha + \beta + \gamma + \delta$ and m.

$$C^{0000} = C_{rrrr}$$
$$= 3\,C_s^{00c} - 12\,C_s^{20c} + 8\,C_s^{40c}$$

$$C^{++--} = \tfrac{1}{4}\,C_{\theta\theta\theta\theta} + \tfrac{1}{4}\,C_{\phi\phi\phi\phi} - \tfrac{1}{2}\,C_{\theta\theta\phi\phi} + C_{\theta\phi\theta\phi}$$
$$= 2\,C_s^{00c} - C_A^{00c} + 4\,C_s^{20c} - 2\,C_A^{20c} + 2\,C_s^{40c}$$

$$C^{+-+-} = \tfrac{1}{4}\,C_{\theta\theta\theta\theta} + \tfrac{1}{4}\,C_{\phi\phi\phi\phi} + \tfrac{1}{2}\,C_{\theta\theta\phi\phi}$$
$$= 2\,C_s^{00c} + \tfrac{1}{2}\,C_A^{00c} + 4\,C_s^{20c} + C_A^{20c} + 2C_s^{40c}$$

$$C^{+-00} = -\tfrac{1}{2}\,(C_{\theta\theta rr} + C_{\phi\phi rr})$$
$$= -C_s^{00c} - C_A^{00c} + C_s^{20c} + C_A^{20c} + 4\,C_s^{40c}$$

$$C^{+0-0} = -\tfrac{1}{2}\,(C_{\theta r\theta r} + C_{\phi r\phi r})$$
$$= -C_s^{00c} + \tfrac{1}{2}\,C_A^{00c} + C_s^{20c} - \tfrac{1}{2}\,C_A^{20c} + 4\,C_s^{40c}$$

$$C^{\pm 000} = \mp\frac{1}{\sqrt{2}}\,C_{\theta rrr} + \frac{i}{\sqrt{2}}\,C_{\phi rrr}$$
$$= \mp\frac{1}{\sqrt{2}}\,(3\,C_s^{21c} - 4\,C_s^{41c}) + \frac{i}{\sqrt{2}}\,(3\,C_s^{21s} - 4\,C_s^{41s})$$

$$C^{\pm\pm\mp 0} = \pm\frac{1}{2\sqrt{2}}(C_{\theta\theta\theta r} + 2\,C_{\theta\phi\phi r} - C_{\phi\phi\theta r}) + \frac{i}{2\sqrt{2}}\,(C_{\theta\theta\phi r} - 2\,C_{\theta\phi\theta r} - C_{\phi\phi\phi r})$$
$$= \pm\frac{1}{\sqrt{2}}\,(2\,C_s^{21c} - C_A^{21c} + 2\,C_s^{41c}) - \frac{i}{\sqrt{2}}\,(2\,C_s^{21s} - C_A^{21s} + 2C_s^{41s})$$

$$C^{+-\pm 0} = \pm\frac{1}{2\sqrt{2}}\,(C_{\theta\theta\theta r} + C_{\phi\phi\theta r}) - \frac{i}{2\sqrt{2}}\,(C_{\theta\theta\phi r} + C_{\phi\phi\phi r})$$
$$= \pm\frac{1}{2\sqrt{2}}\,(4\,C_s^{21c} + C_A^{21c} + 4\,C_s^{41c}) - \frac{i}{2\sqrt{2}}\,(4\,C_s^{21s} + C_A^{21s} + 4\,C_s^{41s})$$

$$C^{\pm\pm 00} = \tfrac{1}{2}\,(C_{\theta\theta rr} - C_{\phi\phi rr}) \mp i\,C_{\theta\phi rr}$$
$$= C_s^{22c} + C_A^{22c} - C_s^{42c} \mp i\,(C_s^{22s} + C_A^{22s} - 2\,C_s^{42s})$$

$$C^{\pm 0\pm 0} = \tfrac{1}{2}\,(C_{\theta r\theta r} - C_{\phi r\phi r}) \mp i\,C_{\theta r\phi r}$$
$$= C_s^{22c} - \tfrac{1}{2}\,C_A^{22c} - C_s^{42c} \mp i\,(C_s^{22s} - \tfrac{1}{2}\,C_A^{22s} - 2\,C_s^{42s})$$

$$C^{\pm\pm+-} = -\tfrac{1}{4}\,(C_{\theta\theta\theta\theta} - C_{\phi\phi\phi\phi}) \pm \frac{i}{2}\,(C_{\theta\theta\theta\phi} + C_{\theta\phi\phi\phi})$$

$$= -(3\,C_s^{22c} + \tfrac{1}{2}\,C_s^{42c}) \pm i\,(3\,C_s^{22s} + C_s^{42s})$$

$$C^{\pm\pm\pm 0} = \mp \frac{1}{2\sqrt{2}}\,(C_{\theta\theta\theta r} - 2\,C_{\theta\phi\phi r} - C_{\phi\phi\theta r}) + \frac{i}{2\sqrt{2}}\,(C_{\theta\theta\phi r} + 2C_{\theta\phi\theta r} - C_{\phi\phi\phi r})$$

$$= \mp\sqrt{2}\,C_s^{43c} - i\sqrt{2}\,C_s^{43s}$$

$$C^{\pm\pm\pm\pm} = \tfrac{1}{4}\,C_{\theta\theta\theta\theta} + \tfrac{1}{4}\,C_{\phi\phi\phi\phi} - \tfrac{1}{2}\,C_{\theta\theta\phi\phi} - C_{\theta\phi\theta\phi} \mp i\,(C_{\theta\theta\theta\phi} - C_{\theta\phi\phi\phi})$$

$$= 2\,C_s^{44c} \mp i\,2\,C_s^{44s}.$$

For transverse isotropy only five components ($N = 0$) are non-zero.

$$C^{0000} = C$$

$$C^{++--} = 2N$$

$$C^{+-+-} = A - N$$

$$C^{+-00} = -F$$

$$C^{+0-0} = -L,$$

where A, C, F, L and N are the usual transversely isotropic elastic moduli (Takeuchi & Saito 1972).

Appendix B

The kernels K_{Ni} which appear in (9) are shown. For transverse isotropy ($N = 0$),

$$K_{01} = \dot{U}'\,\dot{U}B^{(0)+}C_{st}^{0000}$$

$$K_{02} = (1/2)r^{-2}[(V'V + W'W)B^{(2)+} + (V'W - W'V)iB^{(2)-}]\,C_{st}^{++--}$$

$$K_{03} = f'fB^{(0)+}C_{st}^{+-+-}$$

$$K_{04} = -(f'\dot{U} + \dot{U}'f)B^{(0)+}C_{st}^{+-00}$$

$$K_{05} = -\,[(X'X + Z'Z)B^{(1)+} + (X'Z - Z'X)iB^{(1)-}]C_{st}^{+0-0}$$

where U', V', W' are radial eigenfunctions for k', and

$$B^{(N)\pm} = (1/2)\,[1 \pm (-1)^{l'+s+l}]\,[(l'+N)\,!\,(l+N)\,!\,/(l'-N)\,!\,(l-N)\,!\,]^{(1/2)}$$
$$\times\,(-1)^N \begin{pmatrix} l & s & l \\ -N & 0 & N \end{pmatrix}$$

(Woodhouse 1980). For other case ($N \neq 0$),

$$K_{11} = -\left[\Omega_0^{l'} \begin{pmatrix} l' & s & l \\ -1 & 1 & 0 \end{pmatrix} (X' - iZ')\dot{U} + \Omega_0^{l} \begin{pmatrix} l' & s & l \\ 0 & 1 & -1 \end{pmatrix} \dot{U}'(X - iZ)\right] C_{st}^{+000}$$

$$K_{-11} = -\left[\Omega_0^{l'}\begin{pmatrix} l' & s & l \\ 1 & -1 & 0 \end{pmatrix}(X'+iZ')\dot{U} + \Omega_0^{l}\begin{pmatrix} l' & s & l \\ 0 & -1 & 1 \end{pmatrix}\dot{U}'(X+iZ)\right]C_{st}^{-000}$$

$$K_{12} = -\Omega_0^{l'}\Omega_0^{l}\left[\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ -2 & 1 & 1 \end{pmatrix} r^{-1}(V'-iW')(X+iZ)\right.$$

$$\left. + \Omega_2^{l}\begin{pmatrix} l' & s & l \\ 1 & 1 & -2 \end{pmatrix} r^{-1}(X'+iZ')(V-iW)\right] C_{st}^{++-0}$$

$$K_{-12} = -\Omega_0^{l'}\Omega_0^{l}\left[\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ 2 & -1 & -1 \end{pmatrix} r^{-1}(V'+iW')(X-iZ)\right.$$

$$\left. + \Omega_2^{l}\begin{pmatrix} l' & s & l \\ -1 & -1 & 2 \end{pmatrix} r^{-1}(X'-iZ')(V+iW)\right] C_{st}^{--+0}$$

$$K_{13} = \left[\Omega_0^{l}\begin{pmatrix} l' & s & l \\ 0 & 1 & -1 \end{pmatrix} f'(X-iZ) + \Omega_0^{l'}\begin{pmatrix} l' & s & l \\ -1 & 1 & 0 \end{pmatrix}(X'-iZ')f\right] C_{st}^{+-+0}$$

$$K_{-13} = \left[\Omega_0^{l}\begin{pmatrix} l' & s & l \\ 0 & -1 & 1 \end{pmatrix} f'(X+iZ) + \Omega_0^{l'}\begin{pmatrix} l' & s & l \\ 1 & -1 & 0 \end{pmatrix}(X'+iZ')f\right] C_{st}^{-+-0}$$

$$K_{21} = \left[\Omega_0^{l'}\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ -2 & 2 & 0 \end{pmatrix} r^{-1}(V'-iW')\dot{U}\right.$$

$$\left. + \Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ 0 & 2 & -2 \end{pmatrix} r^{-1}\dot{U}'(V-iW)\right] C_{st}^{++00}$$

$$K_{-21} = \left[\Omega_0^{l'}\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ 2 & -2 & 0 \end{pmatrix} r^{-1}(V'+iW')\dot{U}\right.$$

$$\left. + \Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ 0 & -2 & 2 \end{pmatrix} r^{-1}\dot{U}'(V+iW)\right] C_{st}^{--00}$$

$$K_{22} = \Omega_0^{l'}\Omega_0^{l}\begin{pmatrix} l' & s & l \\ -1 & 2 & -1 \end{pmatrix}[(X'X-Z'Z)-i(X'Z+Z'X)]\, C_{st}^{+0+0}$$

$$K_{-22} = \Omega_0^{l'}\Omega_0^{l}\begin{pmatrix} l' & s & l \\ 1 & -2 & 1 \end{pmatrix}[(X'X-Z'Z)+i(X'Z+Z'X)]\, C_{st}^{-0-0}$$

$$K_{23} = -\left[\Omega_0^{l'}\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ -2 & 2 & 0 \end{pmatrix} r^{-1}(V'-iW')f\right.$$

$$\left. + \Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ 0 & 2 & -2 \end{pmatrix} r^{-1}f'(V-iW)\right] C_{st}^{+++-}$$

$$K_{-23} = -\left[\Omega_0^{l'}\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ 2 & -2 & 0 \end{pmatrix} r^{-1}(V' + iW')f\right.$$

$$\left. + \Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ 0 & -2 & 2 \end{pmatrix} r^{-1}f'(V + iW)\right] C_{st}^{---+}$$

$$K_{31} = -\Omega_0^{l'}\Omega_0^{l}\left[\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ -2 & 3 & -1 \end{pmatrix} r^{-1}(V' - iW')(X - iZ)\right.$$

$$\left. + \Omega_2^{l}\begin{pmatrix} l' & s & l \\ -1 & 3 & -2 \end{pmatrix} r^{-1}(X' - iZ')(V - iW)\right] C_{st}^{+++0}$$

$$K_{-31} = -\Omega_0^{l'}\Omega_0^{l}\left[\Omega_2^{l'}\begin{pmatrix} l' & s & l \\ 2 & -3 & 1 \end{pmatrix} r^{-1}(V' + iW')(X + iZ)\right.$$

$$\left. + \Omega_2^{l}\begin{pmatrix} l' & s & l \\ 1 & -3 & 2 \end{pmatrix} r^{-1}(X' + iZ')(V + iW)\right] C_{st}^{---0}$$

$$K_{41} = \Omega_0^{l'}\Omega_2^{l'}\Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ -2 & 4 & -2 \end{pmatrix} r^{-2}[(V'V - W'W) - i(V'W + W'V)]\, C_{st}^{++++}$$

$$K_{-41} = \Omega_0^{l'}\Omega_2^{l'}\Omega_0^{l}\Omega_2^{l}\begin{pmatrix} l' & s & l \\ 2 & -4 & 2 \end{pmatrix} r^{-2}[(V'V - W'W) + i(V'W + W'V)]\, C_{st}^{----}.$$

Appendix C

For first-order perturbation theory ($l' = l$), the following formulae for $3-j$ symbols are useful. These are valid for even s and unaffected by multiplying the lower three numbers of the $3-j$ symbol by (-1).

$$\begin{pmatrix} l & s & l \\ -1 & 0 & 1 \end{pmatrix} = -\left[1 - \frac{s(s+1)}{2l(l+1)}\right]\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -2 & 0 & 2 \end{pmatrix} = \frac{1}{2(l+2)(l+1)l(l-1)}\,[2l^2(l+1)^2 - 4\,l(l+1) - 4l(l+1)s(s+1)$$

$$+ s^2(s+1)^2 + 2\,s(s+1)]\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -1 & 1 & 0 \end{pmatrix} = -\frac{1}{2}\sqrt{\frac{s(s+1)}{l(l+1)}}\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -2 & 1 & 1 \end{pmatrix} = \sqrt{\frac{s(s+1)}{(l+2)(l-1)}}\ \frac{3\,l(l+1) - s(s+1)}{2\,l(l+1)} \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -1 & 2 & -1 \end{pmatrix} = \sqrt{\frac{s(s+1)}{(s+2)(s-1)}} \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} l & s & l \\ -2 & 2 & 0 \end{pmatrix} = \frac{1}{2} \sqrt{\frac{s(s+1)}{(s+2)(s-1)(l+2)(l+1)\,l(l-1)}}$$

$$\times\ [(s+2)(s-1) - 2\,l(l+1)] \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -2 & 3 & -1 \end{pmatrix} = -\frac{1}{2} \sqrt{\frac{(s+3)(s-2)\,s(s+1)}{(l+2)(l-1)(s+2)(s-1)}} \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l & s & l \\ -2 & 4 & -2 \end{pmatrix} = \sqrt{\frac{(s+3)(s-2)\,s(s+1)}{(s+4)(s-3)(s+2)(s-1)}} \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}.$$