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PARTIAL DERIVATIVES OF THE EIGENFREQUENCIES OF A LATERALLY HETEROGENEOUS EARTH MODEL

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Abstract. We present a scheme for calculating the partial derivatives of the earth's free oscillation eigenfrequencies with respect to changes in an arbitrary laterally heterogeneous starting model. Such partial derivatives have previously been calculated only with respect to laterally homogeneous models. Numerical tests of the accuracy of the partial derivatives are presented. By means of the scheme presented in this paper it now is possible to apply linearized inverse theory to data such as singlet frequencies or waveforms to obtain a new laterally heterogeneous model, recalculate the forward solution, determine a new set of partial derivatives and residuals, and repeat this cycle until convergence is obtained. It also becomes possible to use a variety of laterally heterogeneous starting models to eliminate possible artifacts due to the choice of the initial model.

Introduction

To determine the lateral heterogeneity of the earth's elastic properties, density and anelasticity, an initial model must be altered, either by successive forward modeling or by use of formal inversion, until a reasonable fit to observed data is obtained. Forward modeling by (numerically) exact variational calculations yields considerably more accurate eigenfrequencies and eigenfunctions than presently used perturbation methods, but until now has not been formulated to give the partial derivatives necessary for the application of inverse theory. On the other hand, by using a variety of perturbation techniques, inverse formulations of the lateral heterogeneity problem have been presented. However, these perturbation approaches have the limitation that the partial derivatives are always computed with respect to a laterally homogeneous starting model.

The importance of being able to compute the partial derivatives with respect to any model, not just laterally homogeneous ones, is schematically illustrated in Figure 1. Iterated inversion to obtain a laterally heterogeneous earth model is analogous to the application of Newton's method to solving a non-linear equation. The eigenfrequencies of a laterally heterogeneous model are analogous to $f(x_i)$ in Figure 1, the model parameters are analogous to x_i, and altering the model to fit the observed eigenfrequencies is analogous to finding the root of the non-linear equation. The eigenfrequencies of a laterally heterogeneous model can be computed by the variational method [e.g., Morris and Geller, 1982; Tanimoto and Bolt, 1983; Morris et al, 1984]. The partial derivatives of eigenfrequency with respect to a change in the parameters of a laterally heterogeneous model are analogous to $f'(x_i)$ in Figure 1. In the present paper, we formulate these partial derivatives based on the variational method, and show by simple numerical examples that sufficient accuracy is obtained.

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Theory

The partial derivatives of eigenfrequency with respect to changes in the parameters of a spherically symmetric Earth model are concisely reviewed by Takeuchi and Saito (1972). We will follow their approach in deriving the partial derivatives of eigenfrequency with respect to changes in the parameters of a laterally heterogeneous model. We write the Lagrangian of the laterally hetero-

$$L = \omega^2 \langle k \mid \rho \mid k \rangle - \langle k \mid H \mid k \rangle \quad (1)$$

where $\mid k \rangle$ is the eigenfunction, ρ is the density, H is the integro-differential potential energy operator and ω is the eigenfrequency. We omit the boundary contribution for simplicity. When infinitesimal perturbations $\delta\rho$ and δH are introduced, the eigenfrequency is perturbed to $\omega+\delta\omega$ and the eigenfunction is unchanged. Taking the variation of (1), as $\delta L=0$, we have

$$\delta\omega = -\frac{\omega}{2}\frac{\langle k \mid \delta\rho \mid k \rangle}{\langle k \mid \rho \mid k \rangle} + \frac{\langle k \mid \delta H \mid k \rangle}{2\omega \langle k \mid \rho \mid k \rangle}$$

In explicit form

$$\delta\omega = \int \left\{ \frac{\partial\omega}{\partial\rho}\delta\rho + \frac{\partial\omega}{\partial\kappa}\delta\kappa + \frac{\partial\omega}{\partial\mu}\delta\mu \right\} \mathrm{dV} \quad (2)$$

where κ is the bulk modulus and μ the shear modulus. For the spherically symmetric case, we obtain partial derivatives directly from (2), e.g.,

$$\frac{\partial\omega}{\partial\rho} = -\frac{\omega r^2 W^2}{2\int \rho r^2 W^2 \mathrm{d}r}$$

for torsional free oscillations [e.g., Takeuchi and Saito,

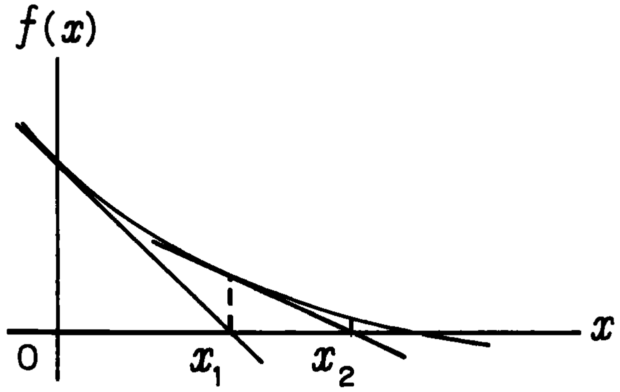

Fig. 1. A schematic illustration of why partial derivatives of eigenfrequency for a laterally heterogeneous model are required, using Newton's method of finding the root of an equation as an analog. If we use $x_0=0$ (a laterally homogeneous starting model) as an initial guess, and extrapolate linearly, using the derivative at $x=0$, it is necessary to refine the resulting root, x_1, by several additional cycles of successive iteration. We therefore must be able to compute the derivative of the curve at any arbitrary x. (I.e., we must be able to find the partial derivatives of eigenfrequency with respect to perturbations in the model parameters for an arbitrary laterally heterogeneous starting model.)

1972], where W is the eigenfunction. For the laterally heterogeneous case, we express the eigenfunction $| k >$ as the sum of eigenfunctions of a spherically symmetric model,

$$| k > = \sum_{i=1}^{N} c_i \; | n, l, m >_i \tag{3}$$

In principle (3) is a sum over all the basis functions, but for computational purposes (3) must be limited to a finite set of basis functions. We expand both the laterally heterogeneous starting model and the perturbations, δm , in terms of spherical harmonics as

$$m_0(r,\theta,\phi) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} m_s^t(r) Y_s^t(\theta,\phi)$$

and

$$\delta m(r,\theta,\phi) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \delta m_s^t(r) Y_s^t(\theta,\phi)$$

where $m_s^t=(\rho_s^t,\kappa_s^t,\mu_s^t)$, $\delta m_s^t=(\delta\rho_s^t,\delta\kappa_s^t,\delta\mu_s^t)$ and Y_s^t are the fully normalized complex spherical harmonics. When we calculate the eigenfrequencies and eigenfunctions of a laterally heterogeneous earth model by the variational method, we represent the Lagrangian in the same form as (1) and derive results analogous to (2) such as

$$\delta\omega = \int \sum_{s,t} \frac{\partial\omega}{\partial m_s^t} \delta m_s^t \mathrm{d}V$$

where δm_s^t is a perturbation to one of the parameters of the laterally heterogeneous model. The explicit form of the partial derivatives is

$$\left[\frac{\partial\omega}{\partial(m_s^t)}(r,\theta,\phi)\right] = \sum_{i=1}^{N} \sum_{j=1}^{N} c^*_i(k')\, c_j(k)\, M_s(i,j) \cdot (Y_l^{m'})^* Y_s^t Y_l^m \tag{4}$$

where $c_i(k')$ and $c_j(k)$ are the expansion coefficients of the k' th and k th singlets (as defined in (3)) and are obtained by solving the variational problem, and $M_s(i,j)$ is a reduced matrix element given elsewhere [e.g., Woodhouse, 1980].

Numerical results

The partial derivatives (4) are derived by using the fact that the eigenfunction remains fixed for an infinitesimal perturbation to the model, and then applying the variational principle to find the resulting change in the eigenfrequency (Rayleigh's Principle). Using these partial derivatives to find the change in frequency due to a finite perturbation is an approximation that is valid only to first order; we know that as we perturb the model, the eigenfunctions will also change. It therefore is necessary to verify numerically that extrapolation using (4) is sufficiently accurate for reasonable perturbations to an initial laterally heterogeneous model. The necessary numerical tests are quite straightforward.

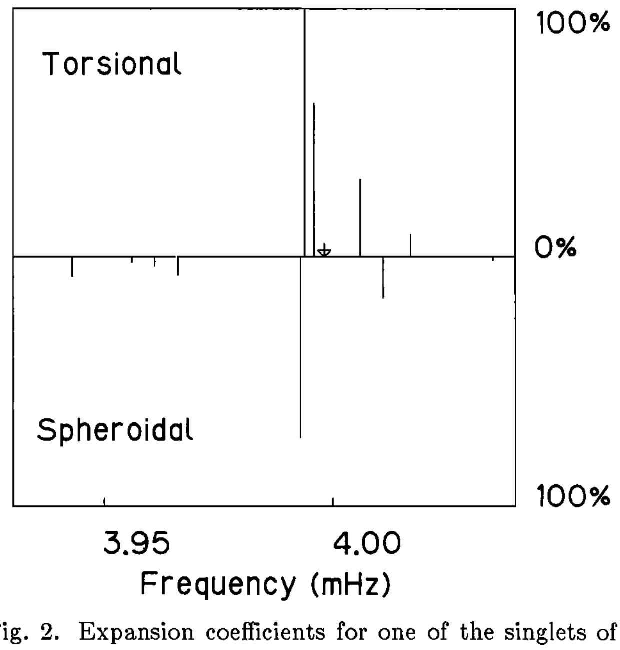

Fig. 2. Expansion coefficients for one of the singlets of the laterally heterogeneous starting model. Each line shows the total amplitude (square root of the sum of the squares) of the $2l+1$ degenerate modes of each of the original multiplets, and is plotted at the frequency of the degenerate multiplet of the spherically symmetric part of the model. The four lines above the axis are the contribution of the four toroidal multiplets; the lower ones are the spheroidal multiplets. Note the extensive degree of coupling between multiplets. The arrow shows the frequency of the split mode whose expansion coefficients are being plotted.

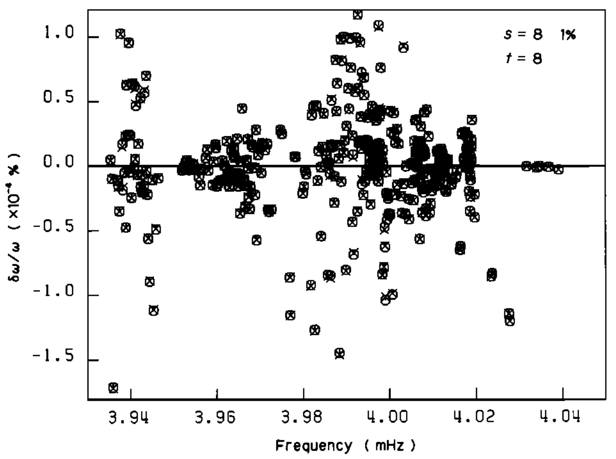

Fig. 3. Comparison of the eigenfrequency change predicted by extrapolation using the partial derivatives to the actual change (from a variational calculation). $\mu_8^{\pm 8}$ is perturbed by 1%. The vertical axis is the percentage of change of the eigenfrequency. The open circles represent extrapolation using the partials, and the crosses are the result of the variational calculation.

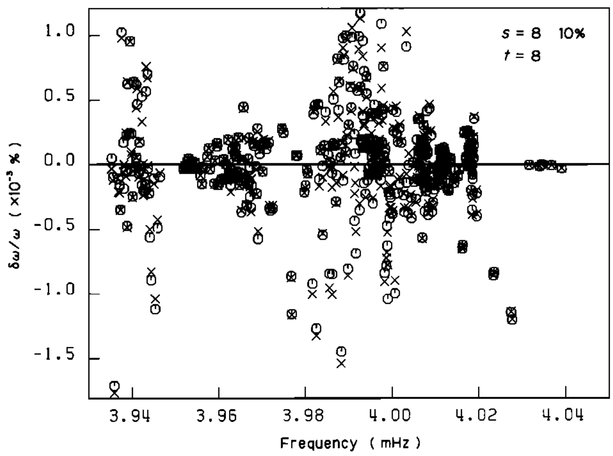

Fig. 5. Comparison of the eigenfrequency change. $\delta\mu_8^{\pm 8}$ is perturbed 10%. See caption of Fig. 3 for details.

We simply compute the changes in eigenfrequencies by linear extrapolation using the partial derivatives, and compare them to the exact $\delta\omega$, as determined by a variational calculation for the new model.

Dziewonski and Steim's (1982) laterally heterogeneous model, expanded in spherical harmonics up to angular order 40, is used as the starting model. The laterally heterogeneous parts of the model are in the depth range between 80 km and 216 km. Following Morris et al (1984), we use the variational method to calculate the normal modes of the laterally heterogeneous starting model at periods near 250 sec., using a basis set consisting of four toroidal multiplets (${}_0T_{31}$, ${}_1T_{19}$, ${}_2T_{14}$, ${}_3T_{10}$) and seven spheroidal multiplets (${}_0S_{32}$, ${}_1S_{20}$, ${}_3S_{16}$, ${}_4S_{11}$, ${}_6S_9$, ${}_7S_6$, ${}_{10}S_2$). The total number of singlets (N) is 351. For simplicity we did not include the effects of the earth's rotation, ellipticity or anelasticity in the numerical tests of the accuracy of the partial derivatives.

Figure 2 shows the expansion coefficients, c_i, of (3) for a typical singlet of the laterally heterogeneous starting model. Rather than plot the amplitude of each of the 351 expansion coefficients, we plot the total amplitude of all of the expansion coefficients for all of the basis functions from each of the 11 multiplets. For example, the line for ${}_0S_{32}$ is the square root of the sum of the squares of the amplitude of all 65 expansion coefficients. In addition to the Coriolis coupling between ${}_0S_{32}$ and ${}_0T_{31}$, there are substantial contributions to the eigenfunction from the spheroidal and toroidal overtone multiplets. Therefore neither first order degenerate perturbation theory nor quasi-degenerate perturbation theory, which do not include this coupling, are applicable, and the use of the variational method is required.

The strength of coupling between multiplets that is seen in Figure 2 leads directly to another important point. The partial derivatives of eigenfrequency with respect to m_s^t for a laterally homogeneous starting model are obtained from first order degenerate pertur-

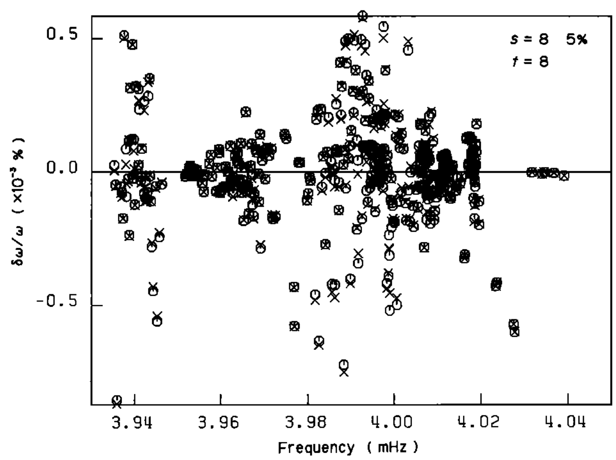

Fig. 4. Comparison of the eigenfrequency change. $\delta\mu_8^{\pm 8}$ is perturbed 5%. See caption of Fig. 3 for details.

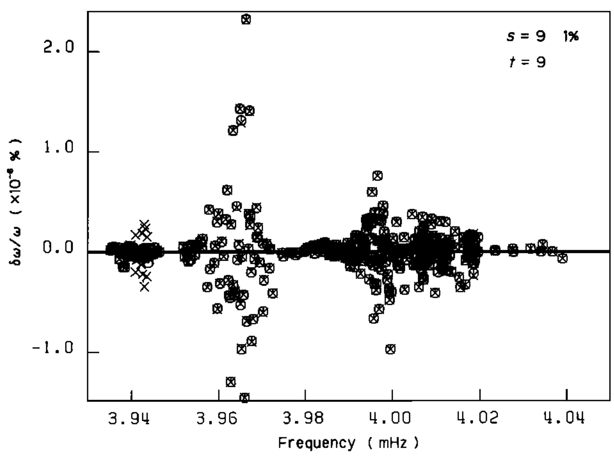

Fig. 6. Comparison of the eigenfrequency change. $\delta\mu_9^{\pm 9}$ is perturbed 1%. See caption of Fig. 3 for details.

bation theory [Jordan, 1978]. Because first order degenerate perturbation theory does not yield the coupling between multiplets shown in Figure 2, direct extrapolation from a laterally homogeneous starting model will not yield an accurate final model.

For our test we perturb only a single term in the spherical harmonic expansion of the laterally heterogeneous model, and then compare the eigenfrequencies obtained by extrapolation using the partial derivatives (4) to those from the variational calculation. We present results for two arbitrarily chosen test cases: perturbations to the shear modulus expansion coefficient, μ_s^t, for (a)$s = 8$, $|t| = 8$; and (b) $s = 9$, $|t| = 9$. The results are shown in Figures 3 through 5 for $\mu_8^{\pm 8}$ and Figure 6 for $\mu_9^{\pm 9}$. The perturbation for Figures 3 through 5 is of the form

$$\delta\mu_8^{\pm 8}(r) = f \ \mu_8^{\pm 8}(r)$$

where f is 1%, 5% and 10% respectively.

The sequence of results in Figures 3 through 5 shows that extrapolation using the partial derivatives predicts the change in the eigenfrequency accurately. For Figure 3, in which $\mu_8^{\pm 8}$ has been perturbed by 1%, the open circles (extrapolation by partial derivatives) and crosses (exact results) fall on top of each other. As the perturbations increase to 5% and 10% in Figures 4 and 5 respectively, the extrapolation using the partial derivatives gradually becomes less accurate. Nevertheless, even for the 10% case shown in Figure 5 almost all of the extrapolated $\delta\omega$'s are within $\pm 10\%$ of the correct $\delta\omega$. Also, for every case in which $\delta\omega$ is significant, the extrapolated $\delta\omega$ has the correct sign. Therefore, although a new model obtained by using the extrapolated $\delta\omega$'s would not be exact, it seems reasonable that successive iterations, following each of which the partial derivatives would be recomputed, would converge to a physically reasonable model.

If the starting model is laterally homogeneous the partial derivatives with respect to odd s components of m_s^t are exactly zero. To determine the sensitivity of the eigenfrequencies to perturbations in odd order components of a laterally heterogeneous starting model, we perturb an arbitrarily chosen ($s = 9$, $|t| = 9$) odd order component. Figure 6 shows the results of perturbing $\delta\mu_9^{\pm 9}(r)$ by 1%. The results indicate that the eigenfrequency change predicted by the partial derivatives is in excellent agreement with the actual calculations. By comparing the vertical scale of Figure 3 to that of Figure 6, we can see that the change in eigenfrequency caused by a perturbation to this odd spherical harmonic term, while non-zero, is two orders of magnitude smaller than that of the even case. Therefore to resolve the odd-order lateral heterogeneity of the earth it will probably be necessary to supplement the eigenfrequency data by data such as modal amplitudes.

Conclusion

The use of supercomputers as the HITAC S-810/20 makes it practical to apply the variational method to the study of the normal modes of the laterally heterogeneous Earth. The variational calculation clearly gives more accurate eigenfrequencies and eigenfunctions than first order degenerate perturbation theory. We now have shown that accurate partial derivatives for the change in eigenfrequency with respect to a perturbation to the parameters of a laterally heterogeneous starting model may be calculated by means of the variational method. Therefore by means of the variational method it now has become possible to apply inverse theory using an arbitrary laterally heterogeneous starting model.

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