ON FREE OSCILLATIONS OF THE EARTH

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Abstract. This survey concerns mostly the theory of free oscillations, with a section on experimental work included at the end. Developments over the last 15 years are examined. The general theory of free oscillations is reviewed, and the effect on free oscillations of such factors as heterogeneity, the Earth's rotation and non-sphericity, and the source of the oscillations are discussed. Earth models, which have been obtained from oscillation data, are reviewed, and their use in forming theoretical seismograms is described.

1. Introduction

Free oscillations have become important recently as a geophysical tool. Using their data, it becomes possible to obtain information about the detailed structure of the Earth. In particular it becomes possible to obtain values of density within the Earth, which cannot be obtained from other seismic vibrations or waves. This review examines developments over the last 15 years, dealing mainly with theoretical aspects but also looking at the experimental work.

The free oscillations of the Earth can be compared to the vibrations of a drum. When a drum is hit, it vibrates. In a similar way the earthquake 'hits' the Earth and oscillations result. The mathematical formulation of both examples is essentially the same. Each has an equation of motion, with appropriate boundary conditions. Each gives a solution consisting of eigenfrequencies with appropriate eigensolutions.

The equation of motion for the Earth is that for an elastic body:

$$\varrho \partial^2 u_i / \partial t^2 = \partial (\lambda \Theta \delta_{ij} + 2\mu e_{ij}) / \partial x_j + F_i ,$$

where

u_i is the displacement vector,

e_{ij} is the strain tensor,

λ and μ are Lame's constants,

ϱ is density,

F_i is the body force vector,

Θ is the dilatation,

and the summation convection is assumed.

If F_i is neglected the equation is a modified wave equation. The solution is best expressed in terms of spherical co-ordinates. Taking u, v, and w as the components of displacement with respect to r, θ and ϕ we obtain two sets of solutions.

The first set has the form:

$$\begin{aligned} u &= U(\omega, r)\, X_l^m(\theta, \phi)\, e^{i\omega t} \\ v &= V(\omega, r)\, (\partial X_l^m / \partial \theta)\, e^{i\omega t} \\ w &= (\sin\theta)^{-1}\, V(\omega, r)\, (\partial X_l^m / \partial \phi)\, e^{i\omega t}, \end{aligned}$$

* See note at p. 426.

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where

U and V are functions of r and ω,

$X_l^m(\theta, \phi)$ is a spherical harmonic,

$\omega/2\pi$ is the frequency, and

l and m are integers.

The second set has the form:

$$\begin{aligned} u &= 0 \\ v &= (\sin\theta)^{-1}\, W(\omega, r)\,(\partial X_l^m/\partial\phi)\, e^{i\omega t} \\ \omega &= -\, W(\omega, r)\,(\partial X_l^m/\partial\theta)\, e^{i\omega t}, \end{aligned}$$

where W is a function of r and ω.

The boundaries which condition the solution are: (i) the centre of the Earth and (ii) its surface. At the centre of the Earth an absence of singularities is required, and at the surface an absence of stress. These two conditions can only be satisfied by certain values of ω, which in turn determine the functions U, V, and W. In addition there are also colatitudinal and longitudinal boundary conditions, but these are implied by the formulation of the spherical harmonic X_l^m.

The first set of solutions represents spheroidal oscillations and the second set toroidal oscillations. Toroidal oscillations involve particle motions over concentric spherical surfaces with no displacement perpendicular to them (along the radial axis); the motion is shear with no dilatation. Spheroidal oscillations involve radial particle motion as well; and there is dilatation.

The effect of spherical harmonics is seen from the general expression for X_l^m,

$$X_l^m = P_l^m(\cos\theta)\, e^{im\phi},$$

where $P_l^m(\cos\theta)$ is an associated Legendre polynomial and m takes values from -1 to $+1$. $2m$ gives the number of nodes of the oscillation along every parallel of latitude, while $l-|m|$ gives the number of nodes along every meridian. As a result, l and m determine the pattern of motion in every spherical shell within the sphere. Examples of spherical harmonics are shown in Figure 1.

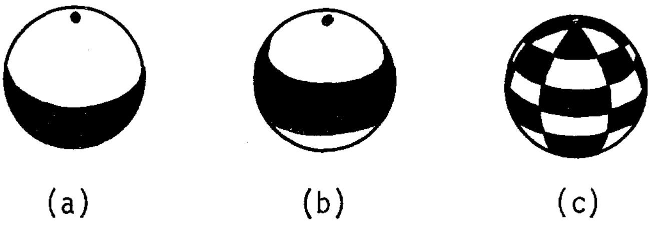

Fig. 1. Examples of surface spherical harmonics: (a) $P_1{}^0(\cos\theta)$; (b) $P_2{}^0(\cos\theta)$; (c) $P_8{}^3(\cos\theta\cos 3\phi)$, (Garland, 1971; with permission, W. B. Saunders Co.).

For each spherical harmonic X_l^m a number of eigenfrequencies ω_n can be obtained with corresponding eigenfunctions U_n, V_n, and W_n. ω_0 is called the fundamental, and

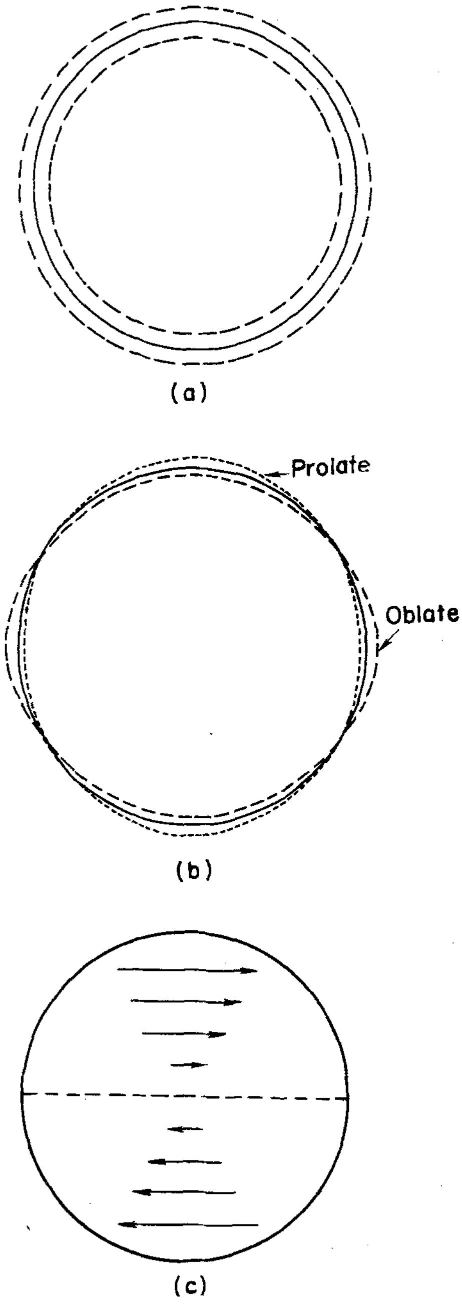

Fig. 2. Modes of free oscillations: (a) ${}_0S_0{}^0$; (b) ${}_0S_2{}^0$; (c) Instantaneous motion in ${}_0T_2{}^0$, (Stacey, 1969; with permission, John Wiley & Sons, Inc.).

higher frequencies are called overtones. The number of nodes of oscillation along the radial axis varies for each function and increases with n.

Examples of simple oscillations are shown in Figure 2. Spheroidal and toroidal oscillations are symbolised by ${}_nS_l^m$ and ${}_nT_l^m$ respectively. Figure 2a shows ${}_0S_0^0$. Since $l=0$, the motion is spherically symmetric and is only radial. Higher values of n give similar motion, but the number of nodes along the radial axes increases. Figure 2b shows ${}_0S_2^0$. There are two nodal parallels of latitude. As the sphere oscillates, it is distorted alternately into an oblate and prolate spheroid. Figure 2c shows ${}_0T_2^0$. There is a nodal line for displacement around the equator, and the two hemispheres oscillate in anti-phase. (Bullen, 1963; Garland, 1971).

In the above presentation the effect of gravity has been neglected. It enters into the initital equations through the body force term F_i and through changes in the stress term

$$\lambda\Theta\delta_{ij} + 2\mu\, e_{ij}.$$

The effect appears only when there are changes in density, which in turn are related to changes in dilatation. Since toroidal oscillations do not cause dilatation changes, they will not be affected by gravity. Spheroidal oscillations will, however, be affected. It is found that gravity tends to lower the frequencies of oscillation and to have a de-stabilising influence (Pekeris and Jarosch, 1958).

The free oscillations of a uniform sphere were first considered by Lamb (1882) and the additional effect of gravity was included by Love (1911). In 1952 Benioff (1954) observed what he thought to be a free oscillation on his seismograph. This stimulated research on the subject, which is described in Section 2, 'Heterogeneous Models'. In 1960 several observations of free oscillations were obtained from the Chile earthquake, confirming Benioff's hypothesis. It was also found that some frequency peaks were split, indicating that the value of m was an influencing factor.

Two possible causes for this phenomena were thought to be the Earth's rotation and its non-sphericity. They are discussed in the third and fourth sections.

Later sections deal with the effect of the source and with Earth models. The latter includes the effect of lateral inhomogeneities and anelasticity on free oscillations. There is a section on theoretical seismograms, which have been derived from various Earth models, and one on experimental work.

2. Heterogeneous Models

In 1952, Benioff (1954) obtained from the Kamchatka earthquake, a ground motion on his seismograph of period 57 min. He suggested that this motion may have been from a free oscillation. His suggestion stimulated studies of free oscillation theory.

Up till then the main theory had been on a homogeneous Earth (Lamb, 1882; Love 1911). In order to compare theoretical with experimental results the theory had to be extended to heterogeneous models. The earlier theory had been, on the whole, analytical; but it was found that for heterogeneous models non-analytical methods had to be used to obtain results. Three methods were used: (i) numerical integration; (ii) the variational method; and (iii) Thomson-Haskell matrices. We shall outline them later in this section. In the models examined the heterogeneity varied with radius only. There was no lateral heterogeneity.

Results from numerical integration gave a period of about 53.5 min for the spheroidal oscillation ${}_0S_2$ (Alterman *et al.*, 1959). ${}_0S_2$ was presumed to have been the oscillation observed by Benioff, and the observed theoretical value can be compared with the observed value of 57 min. Results were obtained using Bullen's model B, Bullard's model I and II, a homogeneous mantle plus homogeneous core (model α), and a homogeneous Earth (model β). All the models gave approximately the same results except that of a homogeneous Earth, which gave a period of 44.3 min. This showed the influence of the core on free oscillations. Calculations using the variational method (Jobert, 1957) gave similar results. Periods of toroidal oscillations were also calculated using numerical integration methods (Alterman *et al.*, 1959), variational methods (Jobert, 1956; Takeuchi, 1959), and Thomson-Haskell matrices (Gilbert and MacDonald, 1960). The three methods gave, on the whole, similar results.

In 1960 free oscillations were observed from the Chile earthquake (Alsop *et al.*, 1961; Benioff *et al.*, 1961; Bogert, 1961; and Ness *et al.*, 1961). These confirmed Benioff's earlier observation, and also showed up more oscillations. In fact, all spheroidal and toroidal oscillations up to S_7 and T_7 were observed and a few higher modes as well. A splitting of peaks was also observed; its meaning will be discussed in later sections.

Calculations were carried out on several models to see if they fitted the observed toroidal oscillations. It was found that Gutenberg's models fitted better, and this gave evidence to substantiate Gutenberg's hypothesis of a low-velocity layer in the upper mantle (Pekeris *et al.*, 1961a; MacDonald and Ness, 1961).

The original numerical integration calculations also gave a 'core oscillation' of 101 min period, where most of the displacement was confined to the core (Alterman *et al.*, 1959). The model used assumed a liquid core with no rigidity. When rigidity, however, was introduced, the period decreased and the amplitude spread into the mantle.

Using this method, an attempt was made to explain an 86 min oscillation from the 1960 earthquake observed on a gravimetrical record (Ness *et al.*, 1961). The period could be obtained for a suitable rigidity but the theoretical amplitude was much less than the observed value (Pekeris *et al.*, 1963). The effect of mantle-core coupling on oscillations was also investigated (Takeuchi, 1961; MacDonald and Ness, 1961).

Among the theoretical periods calculated were those for larger values of l. At these values it becomes convenient to look at a free oscillation as a combination of two surface waves travelling in opposite directions. A toroidal oscillation will be connected with Love waves and a spheroidal oscillation with Rayleigh waves. The connection was investigated by comparing numerical results from toroidal oscillations with experimental results on Love waves. A certain amount of agreement was observed (Takeuchi, 1959; Gilbert and MacDonald, 1960). In addition, a phenomenon on the mantle core boundary was computed which appeared to be connected with Stoneley waves (Alsop, 1963a).

The final part of this section deals mostly with the methods employed in the theory. As mentioned before, there are three main methods: numerical integration, the variational method, and Thomson-Haskell matrices.

In numerical integration, the differential equations of motion in a heterogeneous Earth are simulated on a computer. They are then integrated step-by-step from the centre of the Earth to its edge. In more detail, the differential equations are broken up into a set of first order differential equations. Values of the solution at the Earth's centre are chosen such that they fit the boundary conditions there. These values are then substituted into the first order differential equations, which will give the derivatives of these values at the centre of the Earth. Using these derivatives, the values of solutions at a small radius δr can be computed. The process is repeated to give values at $2\delta r$, $3\delta r$, etc. Ultimately, values at the surface of the Earth are obtained. At this point the method runs into a problem. The solution at the surface must fit the boundary conditions there. So, in order to get the right solution, one must estimate the right value when starting the numerical integration. The problem can be solved by starting with two values for the expected period of oscillation. Two sets of value will be obtained at the surface of the Earth. These can be compared with the requirements of the boundary conditions, and a better value for the period can be obtained through linear interpolation. The process is repeated until a suitably accurate solution is obtained.

The first-order differential equations for a spheroidal oscillation ${}_nS_l$ which are used in these computations are:

$$
\begin{aligned}
\dot{y}_1 &= -\frac{2\lambda y_1}{(\lambda+2\mu)\,r} + \frac{y_2}{(\lambda+2\mu)} + \frac{\lambda n(n+1)\,y_3}{(\lambda+2\mu)\,r}, \\
\dot{y}_2 &= \left[-\sigma^2\varrho_0 r^2 - 4\varrho_0 g_0 r + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)}\right]\frac{y_1}{r^2} - \frac{4\mu y_2}{(\lambda+2\mu)\,r} + \\
&\quad + \left[n(n+1)\,\varrho_0 g_0 r - \frac{2\mu(3\lambda+2\mu)\,n(n+1)}{(\lambda+2\mu)}\right] y_3/r_2 + \\
&\quad + n(n+1)\,y_4/r - \varrho_0 y_6, \\
\dot{y}_3 &= -y_1/r + y_3/r + y_4/\mu, \\
\dot{y}_4 &= \left[g_0\varrho_0 r - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)}\right]\frac{y_1}{r^2} - \frac{\lambda y_2}{(\lambda+2\mu)\,r} + \\
&\quad + \left\{-\varrho_0\sigma^2 r^2 + \frac{2\mu}{(\lambda+2\mu)}\left[\lambda(2n^2+2n-1) + 2\mu(n^2+n-1)\right]\right\} \times \\
&\quad \times y_3/r^2 - 3y_4/r - \varrho_0 y_5/r, \\
\dot{y}_5 &= 4\pi G\varrho_0 y_1 + y_6, \\
\dot{y}_6 &= -4\pi G\varrho_0 n(n+1)\,y_3/r + n(n+1)\,y_5/r^2 - 2y_6/r,
\end{aligned}
$$

where

$$
\begin{aligned}
y_1 &= U, \\
y_2 &= \lambda X + 2\mu\dot{U},
\end{aligned}
$$

$y_3 = V,$
$y_4 = \mu(\dot{V} - V/r + U/r),$
$y_5 = P,$
$y_6 = \dot{P} - 4\pi G \varrho_0 U,$
U = the part of the radial displacement dependent on radius,
V = the part of the non-radial displacement dependent on radius,
X = the part of the dilatation dependent on radius,
P = the part of the gravitational potential dependent on radius,
λ and μ = Lame constants dependent only on radius,
ϱ_0 = unperturbed density dependent only on radius,
g_0 = unperturbed gravity dependent only on radius,
G = gravitational constant, and
$\sigma/2\pi$ = frequency of the oscillation.

The dot indicates differentiation with respect to radius. $y_1 - y_6$ are chosen so as to avoid differentiation of the empirically determined quantities ϱ_0, μ, and λ (Alterman *et al.*, 1959).

In place of integrating outwards, it is also possible to integrate inwards. This is done when calculating higher modes of oscillation (Sato *et al.*, 1960). Methods have been given for reducing the equations of motion for toroidal oscillations to one first-order differential equation. This simplifies the numerical integration (Carr, 1970; Randall, 1970).

The variational method is based on the calculus of variations. We begin with an integral

$$I = \int f(\mathbf{u}(\mathbf{r}), \mathbf{r}) \, \mathrm{d}v ,$$

where $\mathbf{r}$ is position in the Earth; $\mathbf{u}$ is the displacement at that point; f is a certain function containing frequency; and the integral is taken through the whole volume of the Earth.

The function f is chosen such that the minimality of I gives the differential equations of motion within the Earth. The displacement $\mathbf{u}$ is assumed to be a certain function of $\mathbf{r}$ and is chosen to satisfy the boundary conditions. It can be shown that I will be a minimum, when $\mathbf{u}(\mathbf{r})$ is an eigenfunction and the frequency within f is the corresponding eigenfrequency $\mathbf{u}$ is assumed to be a certain function depending upon n parameters $a_1, a_2, \ldots, a_n$. To obtain the best approximation to the correct solution, the parameters a_1 to a_n must be varied until I is a minimum. This is equivalently expressed by n simultaneous equations. $\partial I/\partial a_1 = 0$, $\partial I/\partial a_2 = 0, \ldots, \partial I/\partial a_n = 0$. The solution of these n equations gives n frequencies, which normally are the first n harmonics of the oscillation. To obtain more accurate solutions, other functions of $\mathbf{u}$ can be chosen with more parameters. By Rayleigh's principle it can be shown that the frequencies obtained by this method approach the exact value from above (Jarosch, 1962). Thus the lower the frequency value the greater its accuracy.

In the method which uses Thomson-Haskell matrices the Earth is divided into a series of homogeneous spherical shells. Since each shell is homogeneous, a general solution can be found for it of the form

$$\psi_n(\mathbf{r}) = A_n f_n(\mathbf{r}) + B_n g_n(\mathbf{r}),$$

where n is the no. of the particular shell; $\mathbf{r}$ is a point in the shell; f_n and g_n are known functions; and A_n and B_n are unknown constants.

The unknown constants can be calculated by matching functions at boundaries between shells, and matching the functions in the extreme shells to boundary conditions at the Earth's surface and centre. A series of simultaneous equations are obtained, which give non-trivial solutions for certain discrete frequencies only. These are the frequencies of the free oscillations (Gilbert and MacDonald, 1960).

In addition to these methods it is also possible to obtain analytic solution of specific heterogeneous models (Ben-Menahem, 1959; Bhattacharya, 1972). A theory of spheroidal oscillations has also been given for relatively simple models of the Earth (Zharkov, 1967).

3. Rotation

The free oscillations from the 1960 earthquake showed, among other things, splittings from various peaks. It was suggested that a possible cause for the splittings was the rotation of the Earth. Relatively simple calculations were put forward to support this theory (Pekeris *et al.*, 1961a). The Earth was compared to a rotating cylindrical bowl of water. Using theoretical calculations (Lamb, 1932) it was shown that the free oscillations in the bowl were split. From this model it was shown that the theoretical split in frequencies was the same order of magnitude as the observed split. The splitting is a mechanical counterpart to the Zeeman effect – the splitting of atomic spectral lines in a magnetic field (Pekeris *et al.*, 1961b).

The splitting is considered to be due to the Coriolis force. It occurs only on oscillations whose amplitude varies with longitude. These oscillations are, like all the other oscillations, standing waves, and can be represented by two waves travelling in opposite directions, one with the rotation of the Earth (eastwards) and one against it (westwards). Because of the Coriolis force, the frequency of the eastward-travelling wave decreases and that of the westward-travelling wave increases. The effect can be compared to that of a Foucault pendulum. It can also be shown that the Coriolis force changes the direction of vibration of the oscillations. This causes a coupling between spheroidal and toroidal oscillations. Because of this, toroidal oscillations, which normally give horizontal displacement alone, may also give vertical displacement. As a result, a vertical seismometer may register toroidal oscillations in this case, while normally it would not (MacDonald and Ness, 1961).

A full theory of rotational splitting has been developed. We shall outline it at the end of this section. Results obtained from the theory have been compared with observed results. The oscillations, which were examined, were ${}_0S_2^m$ and ${}_0S_3^m$. In S_2, m can equal -2, -1, 0, 1, 2, and in S_3, $m = -3$, -2, -1, 0, 1, 2, 3, i.e., S_2 can split into 5 peaks and S_3 into 7 peaks. Amplitude calculations however show that in

S_2 only $m = -1, 1$ will be dominant, and in S_3 only $m = -2, 2$, with $m = 0$ coming afterwards. These calculated results agree with observations. The amplitudes are found by calculating the oscillations from a source, which simulates an earthquake. Among the sources used are an explosive compressional point-source, a vertical impulsive force, a horizontal impulsive force pointing south, and an impulsive couple obtained by differentiating the south force in the east direction. All of them gave good results except the vertical impulsive force (Backus and Gilbert, 1961; Pekeris *et al.*, 1961c).

An odd feature in the observed splittings was that they appeared only in spheroidal oscillations, not in toroidal. This may be due to attenuation of the toroidal oscillations, which would broaden the frequency band of the split peaks until they cannot be distinguished (Backus and Gilbert, 1961).

Peak splitting may also lead to wrong identification of peaks. A side peak may be taken to be a central peak because: (i) noise may cover over some of the other split peaks and/or (ii) there is spectral broadening and smothering of each peak due to dissipation within the Earth. This misidentification may lead to errors in calculations on the Earth's properties (Gilbert and Backus, 1965).

The full theory of rotational splitting is based on perturbation theory. The effect of rotational splitting is assumed to be small. All the functions used in free oscillation calculations for a non-rotating Earth, are altered. In place of a typical function

$$f = f_0(\mathbf{r}, t)$$

we put

$$f = f_0(\mathbf{r}, t) + (\Omega/\omega) f_1(\mathbf{r}, t) + (\Omega/\omega)^2 f_2(\mathbf{r}, t) \ldots,$$

where Ω is the angular frequency of the Earth's rotation, and ω is the angular frequency of the free oscillation.

$f_1, f_2 \ldots$ are unknown functions, independent of Ω and ω. Similarly, ω is altered to

$$\omega' = \omega + (\Omega/\omega)\,\omega_1 + (\Omega/\omega)^2\,\omega_2 \ldots$$

The altered functions are substituted into the relevant differential equations, and in place of a differential expression $D_0 = 0$ we get

$$D_0 + \omega(\Omega/\omega)\,D_1 + (\Omega/\omega)^2\,D_2 \ldots = 0\,.$$

We take the coefficient of each $(\Omega/\omega)^n$ for each n and put it $= 0$. $n = 0$ gives the equations for a non-rotating Earth with the solutions f_0 and frequency ω. $n = 1$ gives

$$D_1 = 0.$$

D_1 contains f_0, f_1, and ω, ω_1. Since f_0 and ω are already known, f_1 and w_1 can be obtained. $(\Omega/\omega)\,\omega_1$ is the 1st order perturbation in frequency due to the rotation of the Earth. The process can be continued for 2nd order perturbations.

A criterion of the smallness of the rotational effect is the ratio of Ω to ω. In radians per hour $\Omega = 2\pi/24$, while the lowest ω is about 2π. Thus, the highest Ω/ω is about 1/24, which is small. The effect of Ω/ω is shown in above equations.

The smaller (Ω/ω) is the smaller is the error of a perturbation from the actual result. Also, the smaller (Ω/ω) the smaller is the actual perturbation (Backus and Gilbert, 1961; Pekeris *et al.*, 1961c).

4. Non-Spherical Earth

As we mentioned before, the free oscillations from the 1960 Chile earthquake showed splitting of various peaks. A possible cause was thought to be the rotation of the Earth, which has already been discussed. A second possible cause was thought to be non-sphericity in the Earth.

In calculations on non-sphericity the Earth is considered to be an ellipsoid. The ellipticity has a value of about 1/298, which is small. As a result, the changes due to ellipticity are small, and a perturbation theory may be used in the calculations.

Results for a homogeneous Earth show that a toroidal oscillation T_l is split up into $l+1$ peaks as opposed to $2l+1$ from rotational effects. The percentage differences in frequencies of the split peaks due to ellipticity can be compared to those due to rotation (Usami and Sato, 1962). Toroidal oscillations from a thick homogeneous shell with a spherical inside surface and an ellipsoidal outside surface have also been examined. It is pointed out that the splittings are only a few parts in a thousand, which would be noticeable only in lower modes (Caputo, 1963).

The studies described in this and the previous section looked at the effects of rotation and non-sphericity separately. It is, however, possible to look at the combined effects, and this has been done in a general way using Rayleigh's principle (Dahlen, 1968). The method will be outlined towards the end of this section.

Calculations using Rayleigh's principle show that rotation splits an oscillation S_l or T_l into $2l+1$ separate peaks, while ellipticity gives only $l+1$ peaks, as mentioned above.

The new eigenfrequency ω_l^m can be expressed in the form

$$\omega_l^m = \omega_l(1 + \alpha_l + m\beta_l + m^2\gamma_l),$$

where

$\alpha_l = A(\Omega/\omega_l)^2 + B\varepsilon$,
$\beta_l = C(\Omega/\omega_l)$,
$\gamma_l = D(\Omega/\omega_l)^2 + E\varepsilon$,
ω_l = eigenfrequency of S_l or T_l,
m = number of split peak, and is an integer between $-l$ and $+l$ inclusive,
Ω = frequency of rotation of the Earth,
ε = effect of Earth's ellipticity, and
A, B, C, D, E = constants.

This equation gives ω_l^m to the 1st order in ellipticity and the 2nd order in rotation.

Besides showing the different effect of rotation and ellipticity, the equation also shows that the centre frequency of a group of split peaks differs by a factor α_l from the frequency of the unsplit peak. Also, because of the last m^2 term, the splitting will be asymmetric about the peak $n=0$.

For low-order modes the effect of rotation is much greater than that of ellipticity. In fact, the first-order effect due to ellipticity is about the same as the 2nd order effect due to rotation. At higher modes, where ω_l is larger, the rotation factor Ω/ω_l decreases with respect to the ellipticity factor. As a result, ellipticity effects dominate. This will give an increase in the m-term, which in turn will increase asymmetry in the split peaks.

These theoretical results were compared with the split peaks of ${}_0S_2$ observed in the Chile earthquake. It was found possible to measure β_l, but not the asymmetry factor γ_l, because of noise (Dahlen, 1968).

Besides changes in frequency, it is also possible to calculate the new perturbed field. It is found that a typical ${}_nS_l^m$ mode will have a first order perturbed field

$$= \tau_{l+1}^m + \tau_{l-1}^m + \sigma_{l+2}^m + \sigma_{l-2}^m ,$$

and ${}_nT_l^m$ will give

$$\sigma_{l-1}^m + \sigma_{l+1}^m + \tau_{l-2}^m + \tau_{l+2}^m ,$$

where τ_l^m and σ_l^m are components from T_l^m and S_l^m respectively. Ellipticity effects contribute to all the terms, but rotational effects contribute only to $l-1$ and $l+1$ terms.

The equations show that coupling can occur between spheroidal oscillations. If the two coupled modes have almost equal eigenfrequencies the coupling between them will be very strong, and calculations using Rayleigh's principle will become invalid. The phenomenon is called pseudo-degeneracy. A possible way to treat the problem is to consider both modes as one big mode. If the two modes split into $2l_1+1$ and $2l_2+1$ peaks, respectively, then they can be considered as one mode splitting into $(2l_1+1)+(2l_2+1)$ peaks. The calculation is then done on this enlarged mode (Dahlen, 1968).

It can be shown that coupling effects occur between modes only if one mode appears in the first-order perturbation field of the other. Thus, coupling will occur only between: (i) S_l^m and $T_{l\pm1}^m$, (ii) S_l^m and S_{l+2}^m, and (iii) T_l^m and T_{l+2}^m. Incidentally, all coupled modes will have the same m, because rotational and ellipticity effects are independent of longitude.

Strong coupling occurs between ${}_0S_l$ and ${}_0T_{l+1}$ when $l=10$ to 25. In this range, each pair of S_l and T_{l+1} have approximately the same frequency to within 2%. The strength of coupling is found to increase as the percentage difference in frequency between the two modes decreases. Coupling between S_l and S_{l+2} is significant only for ${}_2S_1$ and ${}_1S_3$. However, it is rather weak, because the two eigenfrequencies are not that close together (Dahlen, 1969).

Although coupling effects exist for $l=10$ to 25 they may be only observable for $l\leqslant 22$. Above $l=22$, the effect of lateral heterogeneities (e.g., crustal structure) may begin to make itself felt. The effect may be observed at the North and South Poles. There only the peak with $m=0$ from the group of split peaks will appear. However, if some effect dependant on longitude begins to affect the oscillations other peaks will be observed. Since crustal structure is not independent of longitude, observation at the Poles should show whether it affects free oscillations and, if so, on what modes. Because many peaks are absent at the Poles, observations there should also be the clearest for showing coupling between modes (Dahlen, 1969).

We shall now outline the method of calculation. An Earth model is chosen with certain properties. We take an angular frequency of vibration ω and a function of vibration $\mathbf{S}(\mathbf{r})$, which fits the boundary conditions on the Earth's surface. From this function expressions for the total kinetic and potential energies (T and V respectively) are formed. It can then be shown that

$$L = T - V$$

is 0. Since L contains both ω and $\mathbf{S}(\mathbf{r})$, we can say that $L=0$ gives implicitly ω as a function of $\mathbf{S}(\mathbf{r})$. Rayleigh's principle then states that if $\mathbf{S}(\mathbf{r})$ varies slightly, ω will remain stationary to the 1st order if and only if: (i) ω is the eigenfrequency of a free oscillation and (ii) $\mathbf{S}(\mathbf{r})$ is the corresponding eigenfunction. The principle can be proved by varying $\mathbf{S}(\mathbf{r})$ in $L=0$, while ω is kept stationary. This will give the equations of motion within the Earth.

This property can be used for perturbations in the following manner. We start off with free oscillation solutions for an Earth model in which rotational and ellipticity effects have not been included. These effects are introduced into L, and the new L is put $=0$. This would define the new eigenfrequency $\omega+\delta\omega$, except that we do not know the new eigenfunction $\mathbf{S}(\mathbf{r})+\delta\mathbf{S}(\mathbf{r})$. However, Rayleigh's principle states that if $\mathbf{S}(\mathbf{r})+\delta\mathbf{S}(\mathbf{r})$ varies $\omega+\delta\omega$ remains stationary. If so, let us change $\mathbf{S}(\mathbf{r})+\delta\mathbf{S}(\mathbf{r})$ to $\mathbf{S}(\mathbf{r})$, which we know. Then $L=0$ defines $\delta\omega$, and the change in frequency of oscillation has been found.

The advantage of the method is that it gives a relatively simple way of calculating $\delta\omega$. Only the changes in the Earth model and eigenfunctions of the unperturbed Earth model need to be known. Also, the changes in the Earth model can be arbitrary, as long as they are small. Thus, the method is not only useful for rotation and ellipticity effects but also for calculations involving lateral heterogeneity.

An eigenfrequency ω_l can be degenerate, and can give rise to the $2l+1$ split peaks which we have been discussing. This means that each ω_l has $2l+1$ independent eigenfunctions $\mathbf{S}^{-l}(\mathbf{r}), \ldots, \mathbf{S}^{0}(\mathbf{r}), \ldots, \mathbf{S}^{l}(\mathbf{r})$. When the peaks are split we do not know which eigenfunction or combination of eigenfunctions correspond to a particular split peak. The problem can be solved by supposing a correct solution

$$\mathbf{S}(\mathbf{r}) = \sum_{k=-l}^{+l} a_k \mathbf{S}^k(\mathbf{r})$$

and solving for the a_k's using a matrix eigenvalue equation. A similar method can be used for pseudo-degenerate cases (Dahlen, 1968).

It can be shown to the 1st order that the average frequency of the split peaks in one group is equal to the frequency of the unperturbed peak. Using this fact – the diagonal sum rule – it is possible to find from seismograms with split peaks the frequencies of the unperturbed peaks (Gilbert, 1971). The splitting of T_l has been specially examined (Zharkov *et al.*, 1969). Theory has also been developed for free oscillations of a thermoviscoelastic rotating Earth (Saastamoinen, 1970).

5. Sources

The properties of free oscillations can be classified into two sections: (i) their number and frequency, and (ii) their amplitudes. (i) is determined by the Earth's structure, and is what we have been considering mostly up to now. Section (ii), however, depends upon the properties of the source, which is usually an earthquake. This means that examination of amplitudes can lead to knowledge about sources. The method of calculating free oscillation amplitudes from a given source is outlined at the end of this section.

Results from a simple source – a compressional point source or explosion – show that the amplitudes of S_l for low l should have about the same value (Alterman *et al.*, 1959). For an earthquake, a more accurate source is a couple or a torque. Comparison between these two sources shows that the torque does not generate spheroidal oscillations, but the couple does. The amplitudes of toroidal oscillations from both sources are found to be less in the overtones than in the fundamentals. They also decrease as the source depth increases. The amplitudes from the couple decrease as l increases, but those from the torque increase. The difference is presumed due to the former source generating spheroidal oscillations. The decrease of amplitude with source-depth has a comparison in surface waves. There it is found that deeper earthquakes give rise to smaller Love waves (Gilbert and MacDonald, 1960). The double-couple source, which is also considered to be a simulation of an earthquake, has also been examined (Ben-Menahem, 1964).

In an effort to achieve a closer simulation of earthquakes, sources of finite extent have been used in calculations. The effects of a dip-slip and a strike-slip movements have been examined on the oscillation ${}_0T_2$. It was found that a vertical strike-slip movement gave an effective amplitude, but a vertical dip-slip or a horizontal strike-slip movement did not (Sato, 1969a, b). The effect of an internal dislocation on free oscillations has also been examined (Singh and Ben-Menahem, 1969a, b, c).

Formulas have been derived which give oscillation amplitudes for a general Earth model and general source (Saito, 1967). Also, tables have been derived, which show the amplitudes of an oscillation for a given structural model and dislocation source of arbitrary orientation and depth (Ben-Menahem *et al.*, 1971). The tables give results, which agree with data obtained from the 1960 Chile and 1964 Alaska earthquake (Ben-Menahem, 1971; Ben-Menahem *et al.*, 1972).

The method of calculating free oscillation amplitudes is fairly standard. We start off with a given Earth model for which the eigenfrequencies and eigenfunctions of the free oscillations have been found. A source is chosen. It gives out a pulse, and we consider the Earth at the moment the pulse is emitted. The displacement ($\mathbf{u}$) will be zero at all points except at the source. We can express $\mathbf{u}$ in terms of the eigenfunctions by the relation

$$\mathbf{u}(\mathbf{r}, t_0) = \sum a_n \mathbf{e}_n (\mathbf{r}, t_0),$$

where $\mathbf{e}_n$ is an eigensolution; $\mathbf{r}$ is a point in the Earth; t_0 is the time of pulse emission; and a_n is a constant.

Since $\mathbf{u}$ is known at t_0, all the a_n's can be found. Then, at later t, $\mathbf{u}$ will be of the form

$$\mathbf{u}(\mathbf{r}, t) = \sum a_n \mathbf{e}_n (\mathbf{r}, t).$$

6. Earth Models

In this section we look at Earth models which have been fitted to the observed properties of free oscillations. The topic is wider than that covered in our paper, since the Earth models used have also to be fitted to geophysical data other than those of free oscillations. The whole problem of fitting Earth models to geophysical data – the inverse problem in geophysics – is a subject to itself, and will only be dealt with summarily in this paper. We shall give this summary at the beginning of the section, and later on look at the information obtainable from Earth models.

The usual basis of the inversion procedure is a trial and error method. Models are chosen and compared with the data. The model is altered and the process repeated until suitable agreement is achieved (Keilis-Borok and Yanovskaya, 1967). Various methods can be used to alter the model to fit the observations, but most, if not all of them, suffer from the fact that they can give a non-unique solution, which is in itself a problem (Dziewonski, 1970). There are two main methods for altering the models. One is based on minimizing the least-squares difference between theoretical and observed calculations (Pekeris, 1966; Backus and Gilbert, 1967; Gilbert and Backus, 1968; Wiggins, 1972; Derr, 1969b). The second is based on the Monte Carlo method (Press, 1968 and 1970a; Anderssen and Seneta, 1971; Anderssen *et al.*, 1972; Worthington *et al.*, 1972).

We now look at some of the information which has been obtained about different parts of the Earth. We start at the centre of the Earth and work outwards. Earlier work on the core was concentrated on estimating its rigidity as a whole (Pekeris *et al.*, 1962; Takeuchi *et al.*, 1963). It was followed by work on the inner core (Alsop, 1963b), in particular whether the inner core was solid. This appears to be the case (Derr, 1969b; Dziewonski and Gilbert, 1971) and it is incorporated in some models (Bullen and Haddon, 1973). The relation between inner core oscillations and the geomagnetic field has also been investigated (Won and Kuo, 1973). At the mantle-core boundary the core radius is apparently larger than that from previous models by 10–20 km (Dorman *et al.*, 1965). There may, however, be other changes in its place which would fit the observed data (Dziewonski, 1970; Muirhead and Cleary 1969;

Haddon and Bullen 1969). The density of the core near its boundary is considered to be near 10.0 gm/cm^3 (10^4 kg/m^3) (Pekeris, 1966; Press, 1970a).

In the lower mantle the density is suggested to be constant between 1600 to 2800 km depth (Landisman *et al.*, 1965), while in a thin layer near the core, it has been suggested that the shear velocity decreases (Dorman *et al.*, 1965). In the upper mantle, several models give a low-velocity layer (Pekeris, 1966; Gilbert and Backus, 1968; Press, 1970a, b; Derr, 1969b; Mizutani and Abe, 1972) usually at about a 100 km depth. High density in this region is also suggested (Press, 1970a; Mizutani and Abe, 1972), and there has been discussion on its exact form (Worthington *et al.*, 1972).

The effect of the crust in models is also pointed out, especially the different effects of oceanic and continental crust (Derr, 1967).

The different effects of oceanic and continental crust are part of the general effect of lateral heterogeneities in the Earth. Nearly all the models presented up till here assumed that properties of the Earth varied with its radius alone. However, to see the full effect of the Earth on free oscillations, heterogeneities, which are not radial, have to be considered also. The calculations (Zharkov and Lyubimov, 1970a,b; Saito, 1971; Madariaga, 1972; and Arkani-Hamed, 1972) are based on perturbation theory and usually on the work of Dahlen (1968). Lateral heterogeneities give rise to splitting or widening of spectral lines. Numerical calculations show that the spectral lines of large-period toroidal oscillations, are split or widen up to 1% (Usami, 1971) or maybe even more (Zharkov and Lyubimov, 1971).

Besides splitting, lateral homogeneities also give rise to coupling of different modes of oscillations (Arkani-Hamed, 1972). The coupling can be examined by taking a spherical harmonic expansion of the lateral heterogeneity and looking at the effect of each harmonic. In this way it is possible to derive selection rules of coupling (Madaraiga, 1972). If there is no quasi-degeneracy i.e., two modes with nearly equal frequencies (Dahlen, 1968), the odd spherical harmonics do not cause any coupling; for example, if the Earth had one hemisphere as a continent and one as an ocean (Madariaga, 1972).

This theory can be applied to the interpretation of long-period surface waves, where the ray-path theory can be apparently faulty. The application is possible because the eigenfunction of each degenerate eigenfrequency (ω_l^m say) leads asymptotically (for large l) to a surface wave propagating along paths determinated by l and m. The coupling effect causes effective scattering of the wave, and a complicated interference pattern is produced on the surface of the Earth. However, interpretation of surface waves using this theory is limited because of the odd harmonics – it is impossible to derive their amplitudes since they do not give rise to any coupling. However, if coupling between quasi-degenerate modes is examined, then the effect of odd harmonics can be observed. This theory is more complicated (Madariaga and Aki, 1972), but it has been developed (Luh, 1973).

The models presented here have assumed perfect elasticity throughout the whole Earth. Theory however has been put forward for taking into account anelestic properties (Anderson and Archambeau, 1964; Zharkov and Lyubimov, 1967). From computations of amplitude decay versus period of toroidal oscillations and Love waves, it has been possible to obtain an idea of the distribution of the quality factor Q within the Earth. The models of Q which satisfied the data have a broad, highly-attenuating zone in the upper mantle and have high Q (low attenuation) in the lower mantle (Anderson and Archambeau, 1964; Zharkov *et al.*, 1967).

7. Theoretical Seismograms

Assuming specific Earth models, it has been possible to compute theoretical seismograms for hypothetical source events. The seismograms show free oscillations among the other seismic waves. The subject is wider than that covered in our paper, and has been reviewed (Landisman *et al.*, 1970). The basic method employed is to solve the equations of motion of the Earth model for its modes of free oscillations. The amplitudes and phases of the free oscillations are adjusted according to the source function. The oscillations are then combined and seismograms can be generated. Although much work has been carried out on simplified Earth models e.g., homogeneous mantle and core, more realistic layered models have also been used (Alterman and Aboudi, 1969). The seismograms show body waves, surface waves, diffracted waves, and Stoneley waves along the mantle-core boundary (Alterman and Aboudi, 1969; Landisman *et al.*, 1970). The seismograms show that the fundamental radial modes of spheroidal and toroidal oscillations give rise to Rayleigh and Love waves, respectively, and that higher radial modes (overtones) give rise to body and diffracted waves, together with higher-mode surface waves. The seismograms also show that the seismic radiation from a source in a low-velocity layer is poorly recorded at the surface (Landisman *et al.*, 1970). The expression of body waves in terms of free oscillations leads tothe general problem of correspondence of betweennormal modesand rays (Odaka and Usami, 1972).

8. Experimental Work

The main instruments used for observations of free oscillations are the strain seismograph (Benioff, 1935) and gravimeter-accelerometers (Ness *et al.*, 1961). Most of the observations of free oscillations have already been reviewed (Derr, 1969a).

The main problem at present in experimental observations of free oscillations is identification of modes – especially for high orders and overtones. Most of the fundamental modes can be identified by assuming a plausible Earth model and calculating their periods from it. The method tends to break down for overtones, especially when an overtone interferes with another mode. Such interference occurs when the two modes have roughly equal frequencies. In cases like these several methods can by used to distinguish the two modes (Dziewonski and Gilbert, 1972).

All the methods assume that l, m and n of the two modes are already known, but that they cannot be separately distinguished in the frequency spectrum and their exact periods cannot be obtained. Approximate periods of the modes are obtained from a priori Earth models.

In the first method, horizontal and vertical displacements are compared. The theory sometimes predicts that one mode will have a large component of displacement in one direction, while the other one will not. Observation of different components should then resolve the modes. A similar method can be used to separate spheroidal and toroidal modes (Nowroozi, 1972).

The second method involves differences in attenuation. If the two modes are observed at different times one may have attenuated more than the other. Since *a priori* predictions can be obtained of attenuation in each mode, it then becomes possible to identify them. Some of the overtones have quality factors (Q) much greater than those of fundamentals (Dratler *et al.*, 1971).

The third method involves observations of group velocities of surface waves. As already mentioned, each mode of free oscillation can be considered as a combination of two surface waves travelling in opposite directions. If two modes have approximately the same period they will have different orders, and therefore different group velocities. As a result, the free oscillations can be separated and identified as surface waves.

The fourth method can be used when the periods of the two modes can be distinguished and measured, but it is not known which period corresponds to which mode. If data on nearby modes are known it is possible to build an Earth model which satisfies the data. The Earth model will then predict the period of one of the unidentified modes. It is found that one of the observed periods corresponds more closely to that period than the other. As a result, the two periods are identified (Dziewonski and Gilbert, 1972).

Recently a new method – the excitation interior – has been suggested. It involves comparison of observed spectra with theoretical spectra, predicted from a known source and an Earth model. If the source and the model are known it is possible to calculate for a particular mode its sign and amplitude at different points on the surface of the Earth. If observed spectra are added together with the appropriate sign the particular mode of interest will be reinforced and the effect of other modes will be reduced. The method can be extended to include data from several earthquakes, if the effect of the different source mechanisms can be eliminated. (Mendiguren, 1973).

Identification of more modes can be attained by better observations, in particular improvement of signal-to-noise ratio. To this end, an improved accelerometer has been developed (Block and Moore, 1970), and noise reduction by data processing has been applied (Wiggins and Miller, 1972). The improvement in instrumentation has resulted in free oscillations being observed from a relatively small magnitude ($M=6.5$) earthquake (Block *et al.*, 1970).

9. Summary

In this article we have reviewed developments in free oscillations over the last 15 years. We have reviewed the general theory of free oscillations and have discussed the effect on free oscillations of such factors as heterogeneity, the Earth's rotation and non-sphericity, and the source of the oscillations. Earth models, which have been obtained from oscillation data, have been discussed and their use in forming theoretical seismograms is described. Experimental work has also been reviewed.

The subject of free oscillations has advanced from a question of their existence to a vast range of information. At present, data from free oscillations are being used, together with all other geophysical data, to obtain models of the Earth through inversion procedures. As such, the theory of free oscillations has now become a general subject covering other aspects of geophysics. On the experimental side, most of the effort is now being concentrated on identifying and estimating the periods of as many modes as possible, since it is this information, which is needed in inversion procedures. Recently, much progress has been made in this line, especially in the identification of overtones. In addition, effort is now being concentrated on estimating the effects of lateral heterogeneities on free oscillations.

NOTE

Ziporah Alterman (1925–1974)

Professor Ziporah Alterman passed away suddenly on 26 April 1974 in Sydney, Australia, while this article was in press. Her co-workers express their grief at this loss.

Professor Alterman was a valued member of the Editorial Board of *Geophysical Surveys*.

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