Graphs of zero Turán density in a hypercube

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- Vertex set can be divided into **layers**, the i-th layer $\mathbf{V_i}$ is the set of vertices $\binom{[n]}{i}$
- The i-th **edge layer** $\mathbf{L_i}$ is the graph induced by $V_i \cup V_{i-1}$

Examples

Example

$$
Q_3 : \quad
\begin{array}{cccccc}
V_3 : & & & \{1,2,3\} & & \\
 & & / & | & \backslash & \\
V_2 : & \{1,2\} & & \{1,3\} & & \{2,3\} \\
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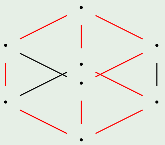
It is 9 and its maximum is attained in this construction



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Consider the $2n$ subhypercubes $\{S\colon i \in S\}$ and $\{S\colon i \notin S\}$ for $i = 1 \dots n$ of Q_n, each of dimension $n-1$.

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It leads to the next property,

- $\lim_{n\to\infty} \mathrm{ex}(Q_n, H)/\|Q_n\|$ exists for all H.

And the limit is called the **Turán density** of H in hypercube.

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In this talk, we will introduce a graph condition that guarantees the graph to have zero Turán density.

Partite hypergraph

Before we move on to the condition, let's first talk about a hypergraph property.

Definition

A $k-$uniform hypergraph H is called $k-$**partite graph** if its vertex set can be partitioned into k parts and its edge each has exactly one vertex in each parts.

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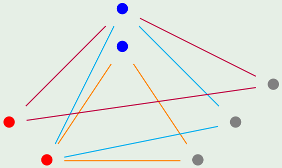
This is a 3-partite 3-uniform graph.



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And these are the properties that we'll applied to embed a graph.

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Definition

A graph H has a **partite representation** if $\exists n, k$ such that H is isomorphic to a subgraph H' of the layer L_k in Q_n, and $V(H') \cap V_k$ forms an edge set of a $k-$partite $k-$uniform graph $\mathcal{H}$.

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An 8-cycle C_8 has a 2-partite representation as follows.

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On the other hands, C_4 and C_6 have no partite representation.

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Theorem (Axenovich 2023)

Let H be a graph whose blocks each have a partite representation, then H has zero Turan density in hypercube.

Recall that a *block* of a graph is a maximal two-connected subgraph or a bridge, and the blocks form a tree-like structure.

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Equivalently, we can show that for any sequences of graphs $\{G_n\}$ with $G_n \subseteq Q_n \; \forall n$, if $\|G_n\|/\|Q_n\|$ is lower-bounded by a constant γ, then there exists a subgraph of $G_n \cong H$ for all large enough n.

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Main strategy : Find a dense layer L_i of Q_n such that $\|G_n \cap L_i\|/\|L_i\|$ is lower bounded by $c\gamma$ (c is constant) , then find a subgraph of $G_n \cap L_i$ isomorphic to H.

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Observation : Most edges of Q_n is located near the middle layers.

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By some asymptotic argument, the following facts hold.

Property

- $\sum_{|n/2-i|\leq n^{2/3}} |V_i| = (1-o(1))|Q_n|$
- $\sum_{|n/2-i|\leq n^{2/3}} \|L_i\| = (1-o(1))\|Q_n\|$

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If we require $\|G_n\|/\|Q_n\| \geq 2\gamma$, removing the high layers and the low layers hardly effects the edge density when n is large.

And we can find some $i \in [n/2 - n^{2/3}, n/2 + n^{2/3}]$ satisfying $\|G_n \cap L_i\|/\|L_i\| \geq \gamma$.

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Formally speaking.

Lemma (1)

Let H be a connected bipartite graph **whose blocks each have partite representation**, *and $\gamma \in (0, 1)$ be a constant.*
Then for any $n > n_0(\gamma, H)$, if $G \subseteq L_j$ in Q_n satisfying $j \in [n/2 - n^{2/3}, n/2 + n^{2/3}]$ and $\|G\|/\|L_j\| \geq \gamma$, then G contains a subgraph isomorphic to H.

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To prove this lemma, we will do induction on the number of blocks.

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Base Case Proof Idea

Suppose H has a k-partite representation, it has two bipartite parts X, Y and X can correspond to a $k-$uniform $k-$partite hypergraph $\mathcal{H}$.
And $j > k$ satisfies $j \in (\frac{n}{2} - n^{2/3}, \frac{n}{2} + n^{2/3})$ and $\|G \cap L_j\| \geq \gamma \|L_j\|$.

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Idea : Applying Erdos' result we previously stated.

Theorem (Erdos(1964))

For any k-partite graph $\mathcal{H}$, $ex_k(n, \mathcal{H}) = o(n^k)$.

We want to find at least $\Omega(n^k)$ $k-$uniform edges in $V_j \cap G$ so that $\mathcal{H}$ is a subgraph, then we can embed X into V_j.
And try to embed Y later on.

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2. From the set chosen in step 1, choose only the vertices in V_j that have full degree (i.e. k) in $L_j \cap G$.
   If every point's degree is full, the desired assignment exists.

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- $Up(x)$ = Subgraph induced by $\{y \subseteq [n] : x \subseteq y\}$, $x \in V_{j-k}$.
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Due to our previously stated purpose, we hope that there is a $x \in V_{j-k}$ such that u_x is large, or more precisely, $u_x \geq \Omega(n^k)$.

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Applying Jensen's inequality

$$\sum_{y \in V_j} \binom{d_{G \cap L_j}(y)}{k} \geq |V_j| \binom{\|G \cap L_j\| / |V_j|}{k} \geq |V_j| \binom{\gamma \|L_j\| / |V_j|}{k}$$

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Then we know that there is a $x \in V_{j-k}$ such that

$$\begin{aligned} u_x &\geq \frac{|V_j|}{|V_{j-k}|} \binom{\gamma\|L_j\|/|V_j|}{k} \\ &= \Omega(n^k) \end{aligned}$$

(Derived from $|j - n/2| \leq n^{2/3}$)

Base Case Proof

Now we have a x with $u_x = \Omega(n^k)$, and this implies $Up(x) \cap V_j \cap G = T_x$ contains $\Omega(n^k)$ vertices and each have k neighbors in $Up(x) \cap V_{j-1} \cap G$.

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Remark: We can also assign X in V_{j-1} and Y in V_j instead with similar approach, but we do not go into details now.

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Suppose H has $\ell > 1$ blocks and each block has a partite representation. Let $H = E \cup F$, where F is a leaf block and $E \cap F = \{\hat{v}\}$.

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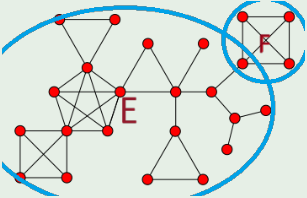

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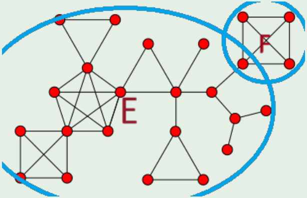

Note: it's just an example of leaf block and has nothing to do with partite representation.

We want to find a E^*, F^* in $L_j \cap G$ such that $E \cong E^*$ and $F \cong F^*$ and $E^* \cap F^*$ is a vertex in $V_j \cap G$.

Induction Proof Idea

For convenience, we write $E = A \cup B \cup \{\hat{v}\}$, where $A \cup \{\hat{v}\}$ is one of the bipartite part, and B is another.

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1. The $\hat{v}$ of E^* and that of F^* should coincide at the same vertex.

   That is, there should be a vertex v that can play the role of $\hat{v}$ for both E^* and F^*.

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And if all these three conditions hold, $E^* \cup F^* \subseteq G$ and $E^* \cup F^*$ is isomorphic to H.

And we will fulfill these conditions by steps.

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For some future purposes, we hope all the vertices in the set has large degree to guarantee the edge set dense, so we further choose

$$V = \{v \in \widetilde{V} \colon d_{G \cap L_j}(v) \geq \frac{\gamma}{2} \frac{\|L_j\|}{|V_j|}\}$$

Now we have

$$V = \{v \in V_j \cap G \colon d_{G \cap L_j}(v) \geq \frac{\gamma}{2}\frac{\|L_j\|}{|V_j|} \text{ and } v \text{ can serve as } \hat{v} \text{ of } E^*\}$$

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Notice that by induction hypothesis, we can assume n is so big that

$$\|G[V_j \backslash \widetilde{V}, V_{j-1}]\| \leq \frac{\gamma}{4} \|L_j\|$$

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Combining two results, we get

$$\|G[V, V_{j-1}]\| \geq \frac{\gamma}{4} \|L_j, \| \quad |V| \geq \frac{\gamma}{4} |V_j|$$

Induction Proof

We have a subset $V \subseteq V_k$ with $\|G[V, V_{j-1}]\| / \|L_j\|$at least constant ratio, so we can construct F^* in $G[V, V_{j-1}]$, and use its $\hat{v}$ to construct the corresponding $E^* \subseteq G \cap L_j$.

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Before we start random coloring, remember that each $v \in V$ correspond to a copy E_v of E in L_j, we also denote its corresponding A and B as A_v and B_v. So $A_v \subseteq V_j$ and $B_v \subseteq V_{j-1}$.

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Notice that in this coloring, all vertices in V' are red by definition of "good" vertex.
And for any $v \in V'$, A_v is colored all blue, so "$A_v \cap V' = \emptyset$!!!"

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So we hope to construct F^* in $G[V', V_{k-1}]$, and suppose $v \in V'$ play the role of $\hat{v}$, then find its corresponding E_v, and in this case, $A_v \cap F^* = \emptyset$.

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Do we finish now?

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To allow F^m to be embedded, we need the following property.

Lemma

F^m has partite representation for all m.

Actually we can show this by proving

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If graphs A, B have partite representation, and $|A \cap B| = 1$, which lies in both their top(or bottom) layer, then $A \cup B$ has a partite representation.

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$$
\begin{array}{ccccc}
\{1,2\} & & \{2,3\} \\
| & \diagdown & | \\
\{1\} & & \{2\}
\end{array}
\quad + \quad
\begin{array}{ccccccc}
\{4,5\} & & \{5,6\} & & \\
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$\square$

Induction Proof

So for any given m, F^m must have partite representation, and if we $m = |E|$, so we can embed $F^m = F_1 \cup F_2 \cup \ldots F_{|E|}$ into $G[V', V_{j-1}]$, and we say $u \in V'$ is the intersection of all F_i.

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Clearly $|E| > |B_u|$, hence there exists $q \in [m]$ such that $F_q \cap B_u = \emptyset$, and in this case, $F_q \cap E_u = \{u\}$.

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So now $F_q \cup E_u \cong H$, and $F_q \cup E_u \subseteq G \cap L_j$, proof done.

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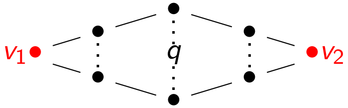

That is, q disjoint paths of length 4 from v_1 to v_2.

We also define a class of graph $Y(q)$, such that $Y(q)$ is composed of two copies of $X(q)$, which share only one vertex, which is v_1 of a copy and a neighbor of v_1 in another copy. i.e.

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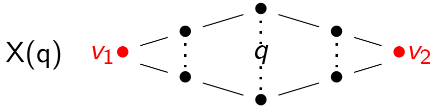

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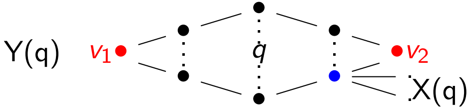

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To prove this, we need to show 3 properties.

Property 1

$X(q)$ has partite representation.

Proof.

We can assign the vertices of its path this way:

$$\{1\} — \{1,t\} — \{t\} — \{2,t\} — \{2\}$$

Since that $\{1, 2\} \times \{3, 4, \ldots, q + 2\}$ is bipartite, it's a 2-partite representation of $X(q)$. $\square$

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Proof.

By property 1 we know that each of $Y(q)$'s block, which are isomorphic to $X(q)$, has partite representation.
Observing the previous construction, it's also not difficult to see that $Y(q)$ is cubical.

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Observing the previous construction, it's also not difficult to see that $Y(q)$ is cubical.
Applying the previous theorem, it has zero Turan density. $\square$

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Proof.

Suppose $Y(q)$ has k-partite representation and can be embedded into L_k such that $Y(q) \cap V_k$ forms a $k-$uniform k-partite graph. Notice that for the two copies of $X(q)$, their v_1 are adjacent, so one of them lies in V_k, we can suppose $v_1, v_2 \in V_k$.

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Suppose a_1, a_2 are the binary vectors corresponded by v_1, v_2, because $d(v_1, v_2) = 4, d_{Ham}(a_1, a_2) \leq 4$, and v_1, v_2 are on the same layer implies that $2|d_{Ham}(a_1, a_2)$, so $d_{Ham}(a_1, a_2)$ =2 or 4.

But in the case $d_{Ham}(a_1, a_2) = 4$, then there are at most $2 < q$ disjoint path between v_1, v_2, so it must have $d_{Ham}(a_1, a_2) = 2$. $\square$

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Let a path from v_1 to $v2$ be $v_1 - p - q - r - v_2$.

Since that $|v_1| = |p| + 1$ and they're adjacent, we can suppose $s = (1|s')$ and five vertices must correspond to vectors as follows:

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$$v_1 : 101s'$$

$$p : 100s'$$

$$q : 110s'$$

$$r : 010s'$$

$$v_2 : 011s'$$

□

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Notice that in our assumption v_1, q, v_2 all correspond to $k-$-uniform edges of a k-partite graph, and they're $(101s'), (110s')$ and $(011s')$.

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Combining the 3 properties, the claim is proved.

Discussion

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Open question :
Is there a larger class of graphs that guarantees zero Turán density?
Or is this property already characterize all cubical graphs with zero Turán density.

References

Axenovich A class of graphs of zero Turán density in a hypercube
https://www.researchgate.net/publication/373298158_A_class_of_graphs_of_zero_Tur'an_density_in_a_hypercube

Conlon An extremal theorem in the hypercube
https://arxiv.org/abs/1005.0582

Erdos On extremal problems of graphs and generalized graphs
https://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=B816E6ADCC4F8D14C825B38C9B12B3D9?doi=10.1.1.210.6421&rep=rep1&type=pdf