Sublinear Expander Graphs and Cycle Decomposition Conjectures

Yu-Sheng Shih
GSS Lecturer: Matijia Bucic,
(Princeton University → University of Vienna)

National Taiwan University

October 2, 2025

Introduction

Expander graphs are graphs that stay very well-connected even when many edges are removed. They play an important role in many areas of mathematics and computer science.

Introduction

Expander graphs are graphs that stay very well-connected even when many edges are removed. They play an important role in many areas of mathematics and computer science.

In this talk, I'll introduce a special type called sublinear expanders. We'll see what makes them interesting and how they were recently used to solve the Erdős–Gallai cycle decomposition conjecture.

Outline

1. Linear and Sublinear Expanders
2. Pass to expander Lemma
3. Expander Decomposition
4. Application to Erdős-Gallai Cycle Decomposition Conjecture

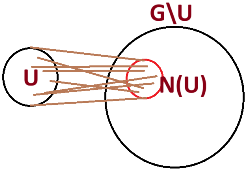
Linear Expander

Definition(ε-expander)

For $\varepsilon > 0$, a n–vertex graph $G = (V, E)$ is called a ε**-expander** if for all $U \subset V$ with $|U| \le n/2$, we have

$$|N(U)| > \varepsilon|U|$$

$N(U)$ is the set of neighbors of U that are outside U.
e.g. K_n is a 1-expander.



Diameter Bound of ε-expander

Diameter Lemma

If G is an n-vertex ε–expander graph, then

$$\mathrm{diam}(G) \le 2\lceil \log_{1+\varepsilon} \frac{n}{2} \rceil$$

Diameter Bound of ε-expander

Diameter Lemma

If G is an n-vertex ε–expander graph, then

$$\text{diam}(G) \leq 2\lceil \log_{1+\varepsilon} \frac{n}{2} \rceil$$

Proof

For a vertex $B^0(u) = \{u\}$, $B^{i+1}(u) = B^i(u) \cup N(B^i(u))$, then clearly

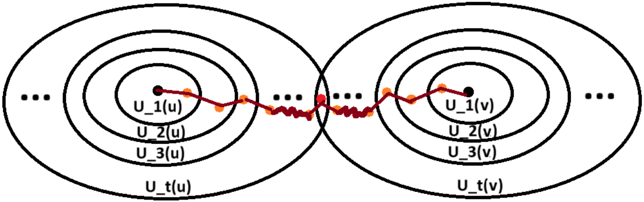
$$|B^{i+1}(u)| > (1+\varepsilon)|B^i(u)| \quad \text{if } |B_i(u)| \leq n/2$$

Then let $t = \lceil \log_{1+\varepsilon} \frac{n}{2} \rceil$, we have

$$|B^t(u)| > \frac{n}{2} \text{ for all } u$$

Then for all $u \neq v$, $|B^t(u) \cap B^t(v)| > 0$, dist$(u, v) \leq 2t$.

Diameter Bound of ε-expander



Sublinear Expander

Bigger graphs are stricter to be a strong expander.
We turn to a weaker class of expanders where $\varepsilon \to 0$ as n grows.

Definition(Sublinear Expander)

A class of graphs is called a sublinear expander if a n-vertex graph G is a $\varepsilon(n)$–expander, where $\varepsilon(n) = o(n)$.
In many cases, ε may be $\frac{1}{\text{polylog} n}$.

Sublinear Expander

Bigger graphs are stricter to be a strong expander.
We turn to a weaker class of expanders where $\varepsilon \to 0$ as n grows.

Definition(Sublinear Expander)

A class of graphs is called a sublinear expander if a n-vertex graph G is a $\varepsilon(n)$–expander, where $\varepsilon(n) = o(n)$.
In many cases, ε may be $\frac{1}{\text{polylog}n}$.

Example

Hypercube $Q(n)$ is a $\Omega(\frac{1}{\sqrt{\log N}})$–expander.

Sublinear Expander

Bigger graphs are stricter to be a strong expander.
We turn to a weaker class of expanders where $\varepsilon \to 0$ as n grows.

Definition(Sublinear Expander)

A class of graphs is called a sublinear expander if a n-vertex graph G is a $\varepsilon(n)$–expander, where $\varepsilon(n) = o(n)$.
In many cases, ε may be $\frac{1}{\text{polylog} n}$.

Example

Hypercube $Q(n)$ is a $\Omega(\frac{1}{\sqrt{\log N}})$–expander.

Why do we care about this kind of graphs?
Sublinear expanders' conditions are weak enough to allow them to appear everywhere, and still strong enough to possess great structures and properties.

Sublinear Expanders Properties

We have a similar diameter upper bound.

Diameter Bound

If G is an n-vertex $\frac{1}{\log n}$–expander graph, then

$$\mathrm{diam}(G) \le 2\lceil \log_{1+\frac{1}{\log n}} \frac{n}{2} \rceil = O(\log^2 n)$$

Sublinear Expanders Properties

We have a similar diameter upper bound.

Diameter Bound

If G is an n-vertex $\frac{1}{\log n}$–expander graph, then

$$\mathrm{diam}(G) \leq 2\lceil \log_{1+\frac{1}{\log n}} \frac{n}{2} \rceil = O(\log^2 n)$$

Sublinear expanders also guarantee the existence of long paths and cycles.

Lemma

An n-vertex $\frac{1}{\log n}$-expander graph G contains a path of length $\Omega(\frac{n}{\log n})$ and a cycle of size $\Omega(\frac{n}{\log n})$.

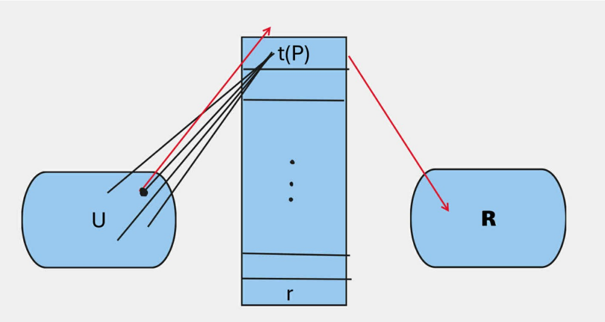
Lemma Proof : DFS Method

The existence of a long path can be proved by a so-called DFS method(I. Ben-Eliezer, M. Krivelevich, and B. Sudakov).

Lemma Proof : DFS Method

The existence of a long path can be proved by a so-called DFS method(I. Ben-Eliezer, M. Krivelevich, and B. Sudakov).

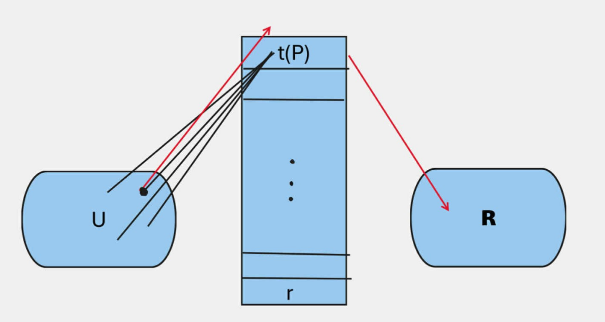
1. Initialize 3 sets $U = V(G) \backslash \{r\}, P = \{r\}$ and $R = \emptyset$. P is a stack with top element being $t(P) = r$.
2. At each step, choose any neighbor u of $t(P)$ in U if it exists, remove it from U, add it to P and it becomes the new $t(P)$.
3. If no such u exists, remove $t(P)$ from P and add it to R.
4. Repeat until $P = \emptyset$.



Lemma Proof: DFS Method

Observations: At any point,

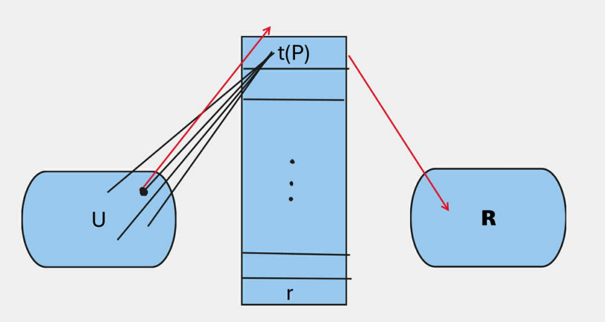
1. P always forms a path and $|U| + |P| + |R| = n$.
2. At each step, either $|U|- = 1$ or $|R|+ = 1$.
3. No edges between U and R
4. U is eventually $\emptyset$ and R is eventually $V(G)$.



Lemma Proof: DFS Method

Observations: At any point,

1. P always forms a path and $|U| + |P| + |R| = n$.
2. At each step, either $|U|- = 1$ or $|R|+ = 1$.
3. No edges between U and R
4. U is eventually $\emptyset$ and R is eventually $V(G)$.



There exists a moment when $|U| = |R| = k$.

If $k < \frac{n}{3}, |P| = n - 2k > \frac{n}{3}$, otherwise $|P| \geq |N(U)| > \frac{n}{3 \log n} = \Omega(\frac{n}{\log n})$.

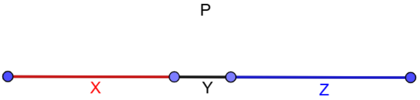
Large Cycle

We prove a simpler version: There exists a cycle of size $\Omega(\frac{n}{\log^2 n})$.

Large Cycle

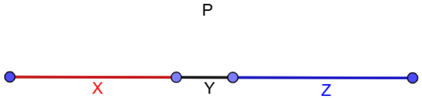
We prove a simpler version: There exists a cycle of size $\Omega(\frac{n}{\log^2 n})$.

Let P be a path with length $\ell \geq \frac{n}{3 \log n}$, partition P into 3 consecutive parts X, Y, and Z in this order, where $|Y| = \frac{n}{10 \log^2 n}$ and $|X| = |Z| \geq \frac{|P|}{3}$.



Large Cycle

We prove a simpler version: There exists a cycle of size $\Omega(\frac{n}{\log^2 n})$.
Let P be a path with length $\ell \geq \frac{n}{3\log n}$, partition P into 3 consecutive parts X, Y, and Z in this order, where $|Y| = \frac{n}{10\log^2 n}$ and $|X| = |Z| \geq \frac{|P|}{3}$.



Consider the graph $G \backslash Y$, if there is still a path from X to Z, then there's a cycle of size $\geq |Y| = \Omega(\frac{n}{\log^2 n})$ in G.
Otherwise, $G \backslash Y$ is disconnected, and let $X' \supseteq X, Z' \supseteq Z$ be 2 components of $G \backslash Y$, WLOG let $|X'| \leq \frac{n}{2}$

$$\frac{n}{10\log^2 n} = |Y| \geq |N(X')| \geq \frac{|X'|}{\log n} \geq \frac{|X|}{\log n} \geq \frac{|P|}{3\log n} \geq \frac{n}{9\log^2 n}$$

Outline

1. Linear and Sublinear Expanders
2. Pass to expander Lemma
3. Expander Decomposition
4. Application to Erdős-Gallai Cycle Decomposition Conjecture

Robust Sublinear Expander

Now we introduce a more robust version of a sublinear expander.

Robust Sublinear Expander

A graph G with n vertices is called a robust sublinear expander if for any $\varepsilon \geq 0$,

- choose any nonempty $U \subseteq V(G)$ with $|U| \leq n^{1-\varepsilon}$.
- choose any $F \subseteq E(G)$ with $|F| \leq \frac{\varepsilon}{4} \cdot d(G) \cdot |U|$ we have

$$N_{G-F}(U) \geq \frac{\varepsilon}{4} \cdot |U|$$

where $d(G)$ denotes the average degree of G.

Robust Sublinear Expander

Now we introduce a more robust version of a sublinear expander.

Robust Sublinear Expander

A graph G with n vertices is called a robust sublinear expander if for any $\varepsilon \geq 0$,

- choose any nonempty $U \subseteq V(G)$ with $|U| \leq n^{1-\varepsilon}$.
- choose any $F \subseteq E(G)$ with $|F| \leq \frac{\varepsilon}{4} \cdot d(G) \cdot |U|$ we have

$$N_{G-F}(U) \geq \frac{\varepsilon}{4} \cdot |U|$$

where $d(G)$ denotes the average degree of G.

Diameter Bound

An n-vertex robust sublinear expander has diameter at most $O(\log n \log \log n)$.

Pass to Expander

In fact, **every** non-empty graph G contains a large induced subgraph H as a robust sublinear expander.

Pass to expander Lemma

Every graph G has a subgraph H which is a robust sublinear expander with its average degree and minimum degree satisfying

$$2\delta(H) \geq d(H) \geq \frac{\log |H|}{\log |G|} d(G)$$

Pass to Expander

In fact, **every** non-empty graph G contains a large induced subgraph H as a robust sublinear expander.

Pass to expander Lemma

Every graph G has a subgraph H which is a robust sublinear expander with its average degree and minimum degree satisfying

$$2\delta(H) \geq d(H) \geq \frac{\log |H|}{\log |G|} d(G)$$

Proof Sketch

Choose a subgraph H maximizing the value of $\frac{d(H)}{\log |H|}$.
Suppose it's not a robust sublinear expander, choose a tuple of ε, U, F which satisfies the condition but fails the required property.
Construct a subgraph $H' \subseteq H$ with larger $\frac{d(H')}{\log |H'|}$ and get contradiction.

Application: Topological Clique

Application Idea:
Want to find a certain structure in a graph.
$\rightarrow$ Find a large enough robust sublinear expander subgraph.
$\rightarrow$ Find the desired structure inside this subgraph.

Application: Topological Clique

Application Idea:
Want to find a certain structure in a graph.
$\rightarrow$ Find a large enough robust sublinear expander subgraph.
$\rightarrow$ Find the desired structure inside this subgraph.

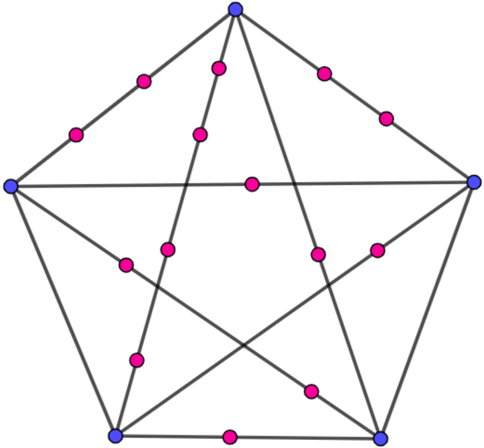
We'll show how it can be applied to find a large *topological clique* in a graph.

Topological Clique

A topological clique is a clique with each of its edges replaced by internally vertex-disjoint paths.
The number of the original vertices is the *order* of the topological clique.

Topological Clique Example

- Topological cliques of order 3 are exactly cycles.
- This is a topological clique of order 5



Large subgraph as a topological clique

In the ordinary Turán problem, we may ask how many edges we need to guarantee the existence of K_t as a subgraph, and for all $t \geq 3$, we need $\Omega(n^2)$ edges because of the construction of $K_{n/2,n/2}$.

Large subgraph as a topological clique

In the ordinary Turán problem, we may ask how many edges we need to guarantee the existence of K_t as a subgraph, and for all $t \geq 3$, we need $\Omega(n^2)$ edges because of the construction of $K_{n/2,n/2}$.

If we only want to guarantee a topological clique as a subgraph, the bound is usually much looser.
E.g. n edges guarantee the existence of a topological K_3.

Large subgraph as a topological clique

In the ordinary Turán problem, we may ask how many edges we need to guarantee the existence of K_t as a subgraph, and for all $t \geq 3$, we need $\Omega(n^2)$ edges because of the construction of $K_{n/2,n/2}$.

If we only want to guarantee a topological clique as a subgraph, the bound is usually much looser.
E.g. n edges guarantee the existence of a topological K_3.

B. Bollobás and A. Thomason 1998 & J. Komlós and E. Szemerédi 1996

Any graph G with average degree d contains a topological clique of order $\Omega(\sqrt{d})$

Note: This bound is tight when G is a disjoint unions of $K_{d,d}$

Proof Sketch of the simpler version

We sketch the proof of a simpler version of the theorem by showing that a topological clique of order $\Omega(\sqrt{d}/\log^2 n)$ exists.

Proof Sketch of the simpler version

We sketch the proof of a simpler version of the theorem by showing that a topological clique of order $\Omega(\sqrt{d}/\log^2 n)$ exists.

First, find a robust sublinear expander subgraph $H \subseteq G$ such that

$$\delta(H) \geq \frac{d(H)}{4} \geq \frac{d}{4\log n}$$

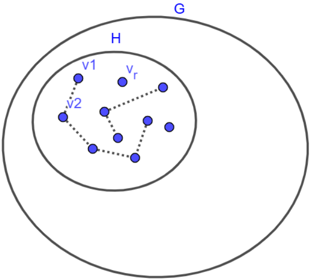
Proof Sketch of the simpler version

We sketch the proof of a simpler version of the theorem by showing that a topological clique of order $\Omega(\sqrt{d}/\log^2 n)$ exists.

First, find a robust sublinear expander subgraph $H \subseteq G$ such that

$$\delta(H) \geq \frac{d(H)}{4} \geq \frac{d}{4\log n}$$

Let $m = |H|, t = \Theta(\frac{\sqrt{d}}{\log^2 n})$ and arbitrarily choose $v_1, v_2, \cdots v_t \in V(H)$. We'll show there's a topological K_t where $v_1 \cdots v_t$ are anchor points.



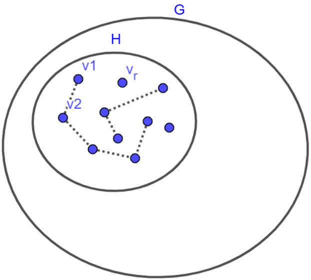
Proof Sketch of the simpler version

We only need to show that we can always build a path with length $\leq |O(\log^2 m)|$ between (v_i, v_j) with unused vertices.

The key steps are to observe that at any moment,

- The size of the used vertices $V_F \subseteq H \leq O(t^2 \log^2 m) = O(\frac{d}{\log^2 n})$
- $|B^1_{H\setminus F}(v_i)| = d_{H\setminus V_F}(v_i) \geq \delta(H) - |V_F| \geq \Omega(|V_F| \log m)$
- $|B^{\ell+1}_{H\setminus V_F}(v_i)| \geq |B^{\ell}_{H\setminus V_F}(v_i)| \cdot (1 + \frac{1}{O(\log m)})$ for all $\ell \geq 1$.

and apply a similar argument for proving the diameter bound.



Outline

1. Linear and Sublinear Expanders
2. Pass to expander Lemma
3. Expander Decomposition
4. Application to Erdős-Gallai Cycle Decomposition Conjecture

(ε, s)–expander

We can actually decompose a graph into very few sublinear expanders and very few leftover edges.

We can actually decompose a graph into very few sublinear expanders and very few leftover edges.
But we need a looser version of expander.

(ε, s)-expander

A n–vertex graph G is an (ε, s)–expander if for all $U \subseteq V(G), F \subseteq E(G)$ with $1 \leq |U| \leq \frac{2n}{3}, |F| \leq s|U|$, we have

$$|N_{G-F}(U)| \geq \varepsilon|U|.$$

Decomposition Lemma

Now we can state our decomposition lemma

Decomposition Lemma

Given any n-vertex graph G and a non-negative integer s, we can partition edge set $E(G)$ into graphs $G_1, G_2, \cdots, G_r, F$ such that

- $|F| \leq 4sn \log n$
- $\sum_{i=1}^{r} |V(G_i)| \leq 2n$
- each G_i is a $(\frac{1}{32 \log^2 |V(G_i)|}, s)$–expander.

Decomposition Lemma

Now we can state our decomposition lemma

Decomposition Lemma

Given any n-vertex graph G and a non-negative integer s, we can partition edge set $E(G)$ into graphs $G_1, G_2, \cdots, G_r, F$ such that

- $|F| \leq 4sn\log n$
- $\sum_{i=1}^{r} |V(G_i)| \leq 2n$
- each G_i is a $(\frac{1}{32\log^2|V(G_i)|}, s)$–expander.

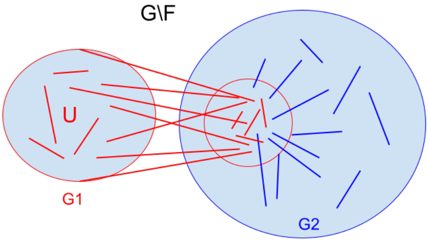
Corollary

Any n-vertex graph G has a subgraph H which is a $\frac{1}{32\log^2|H|}$-expander with $d(H) \geq d(G)/2$.

Proof Sketch

It can be proved by induction on n.

- If G is itself a $(\frac{1}{32\log^2 n}, s)$–expander, done.
- Choose U, F with $|F| \le s|U|$ such that $|N_{G-F}(U)| < \frac{1}{32\log^2 n}|U|$
- Let $G_1 = G[U \cup N_{G-F}(U) - F]$, $G_2 = G \backslash U - E(G_1) - F$.
- Decompose G_1, G_2, and union together.



Outline

1. Linear and Sublinear Expanders
2. Pass to expander Lemma
3. Expander Decomposition
4. Application to Erdős-Gallai Cycle Decomposition Conjecture

Erdős-Gallai Cycle Decomposition Conjecture

An application of the expander decomposition is to a famous conjecture posed by Erdős and Gallai in 1960s.

Erdős-Gallai cycle decomposition conjecture

Any n-vertex graph can be decomposed into $O(n)$ cycles and edges.

Erdős-Gallai Cycle Decomposition Conjecture

An application of the expander decomposition is to a famous conjecture posed by Erdős and Gallai in 1960s.

Erdős-Gallai cycle decomposition conjecture

Any n-vertex graph can be decomposed into $O(n)$ cycles and edges.

The state-of-the-art upper bound is

M. Bucić and R. Montgomery 2023

Any n-vertex graph can be decomposed into $O(nlog^* n)$ cycles and edges.

We later call it Theorem 0.

A reduction to another theorem

Before proving the $O(n \log^*(n))$ bound, we first look at another theorem.

Theorem 1

Any n-vertex graph with average degree d can be decomposed into $O(n)$ cycles and $n \log^{O(1)} d$ edges.

A reduction to another theorem

Before proving the $O(n\log^*(n))$ bound, we first look at another theorem.

Theorem 1

Any n-vertex graph with average degree d can be decomposed into $O(n)$ cycles and $n\log^{O(1)} d$ edges.

Claim: Theorem 1 $\implies$ Theorem 0

A reduction to another theorem

Before proving the $O(n \log^*(n))$ bound, we first look at another theorem.

Theorem 1

Any n-vertex graph with average degree d can be decomposed into $O(n)$ cycles and $n \log^{O(1)} d$ edges.

Claim: Theorem 1 $\implies$ Theorem 0

Proof of the claim

- Begin with graph G, decompose it into $O(n)$ cycles and $n \log^{O(1)} d$ edges.
- Remove the cycles in the previous step and let the new graph be G_1. Now $d(G_1) = \log^{O(1)} d(G)$
- Keep doing this $O(\log^* n)$ times then the average degree becomes constant.

Another reduction to another theorem

Now, we see another theorem

Theorem 2

Any n-vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.

Claim: Theorem 2 $\implies$ Theorem 1.

Another reduction to another theorem

Now, we see another theorem

Theorem 2

Any n-vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.

Claim: Theorem 2 $\Longrightarrow$ Theorem 1.

Proof of the claim(Sketch)

- Begin with a graph G with average degree d, remove all cycles in G having size $\geq d$. (At most $O(n)$ such cycles)
- Apply the expander decomposition lemma with $s = 0$, G split into $G_1 \cdots G_r$ with $\sum |G_i| \leq 2n$, each G_i is a $O(\frac{1}{32 \log^2 |G_i|})$–expander.
- $|G_i| \leq O(d \log^4 d)$, otherwise there exists a large cycle.
- Apply Theorem 2 to each G_i and union together.

Easier Version

So it remains to show that

Theorem 2

Any n-vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.

Easier Version

So it remains to show that

Theorem 2

Any n-vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.

But now we will prove an even simpler version again

Theorem 2'

Any n-vertex graph can be decomposed into $O(n)$ cycles and $O(n^{2-c})$ edges for constant $0 < c < 1$.

Easier Version

So it remains to show that

Theorem 2

Any n-vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.

But now we will prove an even simpler version again

Theorem 2'

Any n-vertex graph can be decomposed into $O(n)$ cycles and $O(n^{2-c})$ edges for constant $0 < c < 1$.

Note: This version can imply a $O(n \log \log n)$ bound for the Erdős-Gallai conjecture.

Proof Sketch: Reduce to Expander

First, we apply the expander decomposition with $s = n^{1-c}/\log n$ and split G into $G_1, \cdots G_r$ and $\leq 4n^{2-c}$ leftover edges, where each G_i is a $(\frac{1}{32\log^2|G_i|}, s)$–expander and $\sum |G_i| \leq 2n$.

Proof Sketch: Reduce to Expander

First, we apply the expander decomposition with $s = n^{1-c} / \log n$ and split G into $G_1, \cdots G_r$ and $\leq 4n^{2-c}$ leftover edges, where each G_i is a $(\frac{1}{32 \log^2 |G_i|}, s)$–expander and $\sum |G_i| \leq 2n$.

Suppose we can decompose each G_i into $O(|G_i|)$ cycles and $O(|G_i|^{2-c})$ edges; their sum is bounded by the linear bound, done.

Proof Sketch: Reduce to Expander

First, we apply the expander decomposition with $s = n^{1-c}/\log n$ and split G into $G_1, \cdots G_r$ and $\leq 4n^{2-c}$ leftover edges, where each G_i is a $(\frac{1}{32\log^2|G_i|}, s)$–expander and $\sum|G_i| \leq 2n$.

Suppose we can decompose each G_i into $O(|G_i|)$ cycles and $O(|G_i|^{2-c})$ edges; their sum is bounded by the linear bound, done.

It remains to show that Theorem 2' holds for a $(\frac{1}{32\log^2 n}, s)$ expander.

Theorem 2''

A n-vertex $(\frac{1}{32\log^2 n}, \frac{n^{1-c}}{\log n})$–expander can be decomposed into $O(n)$ cycles and $O(n^{2-c})$ edges for constant $0 < c < 1$.

Proof Sketch: Graph Partition and Lovász's Theorem

Before we continue, we first state a corollary from a theorem of Lovász from 1968

Corollary of L. Lovász 1968

Any n-vertex graph G can be decomposed into at most n paths, where each vertex is the endpoint of at most 2 of them.

Proof Sketch: Graph Partition and Lovász's Theorem

Before we continue, we first state a corollary from a theorem of Lovász from 1968

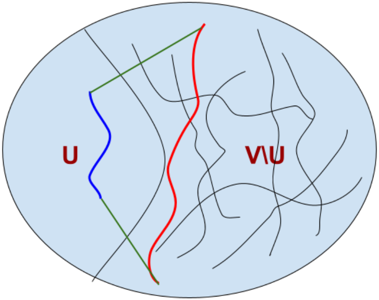
Corollary of L. Lovász 1968

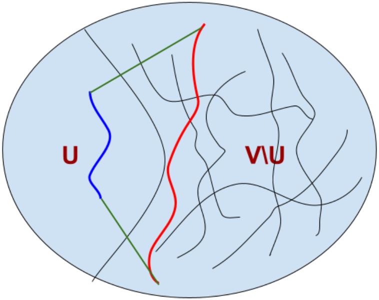
Any n-vertex graph G can be decomposed into at most n paths, where each vertex is the endpoint of at most 2 of them.

Strategy:

- Divide vertices into 2 sets U, $V \backslash U$, where $|U| \approx n^{1-c}$
- Use the corollary above to decompose $V \backslash U$ into paths
- For each path from v_1 to v_2 in $U \backslash V$, connect another path from v_1 to v_2 with all internal vertices within U, making it a cycle.

If this works, we have at most n cycles and at most n^{2-c} remaining edges.


U
V\U



Problems: U may not have a good expansion property(and is actually hard to have), we can't use expansion property of U to find path within U.

We need a more clever path-expanding argument and a good choice of U to find a good path.

Proof Sketch: Sampling and Robust neighborhood

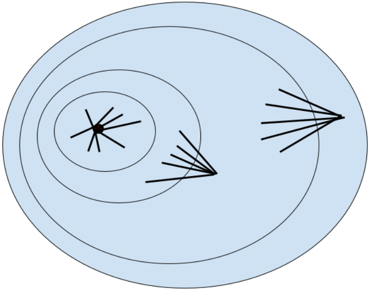
We randomly choose U with each vertex picked independently with probability $p = n^{-c}$.

Proof Sketch: Sampling and Robust neighborhood

We randomly choose U with each vertex picked independently with probability $p = n^{-c}$.

For each v, we define levels of robust neighborhoods as

- $R^0(v) = \{v\}, R^1(v) = N(v) \cup \{v\}$
- $R^\ell(v) = R^{\ell-1}(v) \cup \{u : \deg_{R^{\ell-1}(v)}(u) \geq \delta = n^{1-2c}/\log^2 n\}$ for $\ell \geq 2$.



Proof Sketch: Sampling and Robust neighborhood

We state some benefits of this approach without rigorous proofs.

- $|R^1(v)| \geq n^{1-c}/\log^2 n$ and $|R^{\ell+1}(v)| \geq (1+\frac{1}{32\log^2 n})|R^\ell(v)|$.
- w.h.p, $|U| \approx np$. and $U' = |R^r(u) \cap R^r(v) \cap U|$ is large.
- Due to the robustness, vertex in $R^{\ell+1}(v) \backslash R^\ell(v)$ has many neighbors in $R^\ell(v)$ even when $O(n \log^3 n)$ edges are forbidden.

By a good choice of $w \in U'$, there exists a path from w through the intersection of U and $R^i(u)$, $R^j(v)$ to u and v, resulting in a path of length $O(\log^3 n)$ avoiding all the used edges.