Proofs

What is a proof?

- Proof: a series of logical steps that together make a convincing argument
- We start with a “claim” – the thing we want to prove
- We can depend on “definitions” - given facts we can assume
  - Let i be an integer
  - Assume k is even
  - $\forall i, j \ni i \neq j$-- for all i and j, such that i is not equal to j
  - S = {i∈$\mathbb{Z}$|i<0}-- S is the set of negative integers
  - Let pi=3.14159
  - Let s = “this is a string”

Proof by Cases

- Prove: if n is an integer, then $3n^2 + n + 14$ is even
- n is either even or odd – first assume n is even
- re-write n as 2k
- rewrite $3n^2 + n + 14$ as $3(2k)^2 + 2k + 14$
- simplify to $3(4k^2) + 2k + 14$
- simplify to $12k^2 + 2k + 14 = (2*6)k^2 + (2*1)k + (2*7)$
- factor out the common constant: $2(6k^2 + k + 7)$ which must be even

Proof by Cases

- Prove: if n is an integer, then $3n^2 + n + 14$ is even
- n is either even or odd – now assume n is odd
- rewrite n as $2k+1$
- rewrite $3n^2 + n + 14$ as $3(2k+1)^2 + (2k+1) + 14$
- simplify to $3(4k^2 + 4k + 1) + 2k + 15 = 12k^2 + 12k + 3 + 2k + 15$
- simplify to $12k^2 + 14k + 18 = (2*6)k^2 + (2*7)k + (2*9)$
- factor out the common constant: $2(6k^2+7k+9)$ which must be even

Proof by Contradiction - Example

- There are no integers y and z, such that $24y + 12z = 1$
- Assume the opposite, that integers $y=a$ and $z=b$ exists, such that
- $24a+12b=1$
- Factor $24a+12b=1$ into $12(2a+b)=1$
- Divide both sides by 12 to get $2a+b = 1/12$
- But $1/12$ is a fraction
- It is impossible for 2 integers to add up to a fraction
- Hence, the assumption was false, so the original statement is true

Proof by Induction - Example

- To show some predicate P(n) is true for all $n \geq b$:
- 1. Show that P(b) is true for the base case $n=b$
- 2. Assume that P(k) is true, then show P(k+1) is also true

Proof by Induction - Example

Prove that the sum of the first n integers equal $\frac{n(n+1)}{2}$, for all n>0

Base Case: n=1

Substitute n=1 into $\frac{n(n+1)}{2}$ to get $\frac{1(1+1)}{2}$ = $\frac{1(2)}{2}$ = 1

Indeed the sum of the first integer (1) is 1

Hence the base case is proved ✅

Proof by Induction - Example

Prove that the sum of the first n integers equal $\frac{n(n+1)}{2}$, for all n>0

Induction Step: Assume sum(1..k) = $\frac{k(k+1)}{2}$

Add k+1 to both sides: sum(1..k) + (k+1) = $\frac{k(k+1)}{2}$ + (k+1)

Simplify left side to: sum(1..k+1) = $\frac{k(k+1)}{2}$ + (k+1)

Rewrite "(k+1)" as "$\frac{2(k+1)}{2}$": sum(1..k+1) = $\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

Combine two fractions: sum(1..k+1) = $\frac{(k+1)(k+2)}{2} = \frac{(k+1)([k+1]+1)}{2}$ ✅