Summary

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1. Lagrangians

1.1 Hamilton Principle

The actual trajectory minimizes the action

$\Rightarrow \delta S = 0$ to first order

1.2 Euler-Lagrange Equation

$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \qquad i = 0, 1, 2, \cdots, S$

* is the corollary of Hamilton principle

1.3 Lagrangian of mechanical systems

$\mathcal{L} = T - U$, and is not unique

add a total derivative of some function $\frac{d}{dt}f(q,t)$,

$\mathcal{L}' = \mathcal{L} + \frac{df}{dt}$ satisfy the equation as well.

2. Application of Lagrangians

2.1 Calculus of variations

① Represent the infinitesimal change by generalized coordinates

(Ex. $dl = \sqrt{1+(\frac{dy}{dx})^2}\,dx$)

② For the integral to have equilibrium, apply E-L equations

(Ex. $f(y, \frac{dy}{dx}, x) = \sqrt{1+(\frac{dy}{dx})^2}$ and apply equation)

③ The solution will give a (indirect) parameterization of the curve

(Ex. $y' = \sqrt{\frac{c^2}{1-c^2}}$)

2.2 Lagronge Multiplrers with constraints - Finding constraint forces

* constraint is only for coordinates

Let $g(y,z) = 0$ be the constraint, y, z are generalized coordinates

$$\begin{cases} \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \\ \frac{\partial f}{\partial z} - \frac{d}{dt}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0 \\ g(y,z) = 0 \end{cases}$$ are the equations

And $\lambda\frac{\partial g}{\partial x_i}$ is the generolized constraint force for x_i

2.3 Conservation Laws

① Closed system → time is homogeneous → Energy is conserved $(\frac{\partial \mathcal{L}}{\partial t} = 0)$

② Closed system → space is homogeneous → Linear momentum is conserved

③ Isotropy of space → $\mathcal{L}$ is invariant under rotations → Angular momentum is conserved

3. Linear Stability Test

① Find the equilibrium solution r_0

② Add pertubation to the system $r = r_0 + r_1(t)$, with $\frac{r_1}{r_0} \ll 1$

③ Plug r in the equation of motion (usually 2nd order differential equation)
④ The 0th term would cancel out, 2nd and higher order could be neglected
⑤ Solve for the pertubation $r_1(t)$

4. Oscillators

4.1 Simple Harmonic Oscillator

$$\ddot{x} + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

4.2 Damped Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x(t) = A e^{i\omega_1 t} \qquad \omega_1 = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

① $\omega_0^2 > \beta^2$ Underdamped

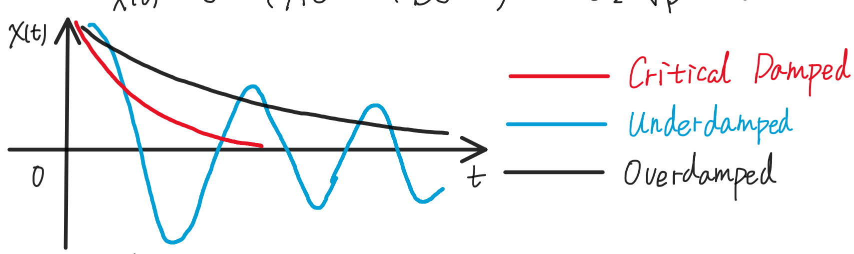
$$x(t) = e^{-\beta t}(A\cos\omega t + B\sin\omega t) \qquad \omega = \sqrt{\omega_0^2 - \beta^2}$$

② $\omega_0^2 = \beta^2$ Critical Damped

$$x(t) = e^{-\beta t}(A + Bt)$$

③ $\omega_0^2 < \beta^2$ Overdamped

$$x(t) = e^{-\beta t}(A e^{-\omega_2 t} + B e^{\omega_2 t}) \qquad \omega_2 = \sqrt{\beta^2 - \omega_0^2}$$



Critical damped oscillation returns to equilibrium faster, because at large t, $Be^{-(\beta-\omega_2)t}$ is the dominating term in overdamped oscillation. $\beta - \omega_2 < \beta$, so it takes more time for overdamped oscillation to return.

4.3 Driven Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

① Solution

$$x(t) = |A|\cos(\omega t + \varphi)$$

$$|A| = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2}}, \quad \tan\varphi = -\frac{2\beta\omega}{\omega_0^2 - \omega^2} < 0 \Rightarrow \text{phase Lagging respect to } F$$

② Resonance Frequency

$$\left.\frac{d|A|}{d\omega}\right|_{\omega_r} = 0 \Rightarrow \omega_r = \sqrt{\omega_0^2 - 2\beta^2}, \quad \text{lower than } \omega_0$$

③ Quality Factor

Near resonance, $\omega \approx \omega_0$, $(\omega_0^2 - \omega^2) = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0(\omega_0 - \omega)$

$$\therefore \ |A| = \frac{1}{2\omega_0}\frac{F_0/m}{[(\omega_0-\omega)^2 + \beta^2]^{1/2}}$$

$$|A|_{max} = |A|(\omega = \omega_0), \quad |A|(\omega - \omega_0 = \pm\beta) = \frac{1}{\sqrt{2}}|A|_{max}$$

$$\therefore \ Q := \frac{\omega_r}{2\beta} \quad \left(\text{For small damping, } Q \approx \frac{\omega_0}{2\beta}\right)$$

$$\because \ \omega = \frac{2\pi}{T}, \quad 2\beta = \frac{1}{\tau}$$

$$\therefore \ Q \propto \frac{\tau}{T}$$

④ Fourier Series Solution

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

$\hat{L} = \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) \Rightarrow \hat{L}x(t) = f(t)$

$f(t) = \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t)$

$x_n(t)$ are solutions of $\hat{L}x(t) = a_n \cos n\omega t$

$y_n(t)$ are solutions of $\hat{L}x(t) = b_n \sin n\omega t$

$\therefore x(t) = \sum_{n=0}^{\infty} x_n(t) + y_n(t)$

⑤ Green's Function: Driven Force $\Rightarrow$ Instantaneous Pulses

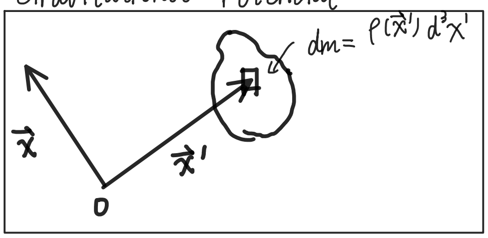
Find solutions of $\hat{L}x(t) = \delta(t)$

Find solutions of $\hat{L}x(t,t') = \delta(t-t')$ $\quad x(t,t') := G(t,t')$

$x(t) = \int_{-\infty}^{\infty} f(t')\, G(t,t')\, dt'$

5. Gravity

5.1 Gravitational Potential



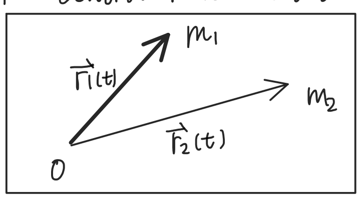
$\phi(\vec{x}) = -G\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$

$g(\vec{x}) = -\nabla\phi(\vec{x})$

In practice, we choose appropiate coordinates and use symmetry to reduce the integral to 1 dimensional

5.2 Two Body Problem

5.2.1 Central Potential and reduced mass



Set the origin at center of mass, let $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\therefore \vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}, \quad \vec{r}_2 = -\frac{m_1}{m_1+m_2}\vec{r}$

the reduced mass, $\mu = \frac{m_1 m_2}{m_1+m_2}$

$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(|\vec{r}|)$

Two body problem $\rightarrow$ central potential problem

5.2.2 Kepler's Problem

① Equation

$U(r) = -\frac{k}{r}, \quad k = GM_\odot$

$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$

② Effective Potential

$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$

$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) = \frac{\partial\mathcal{L}}{\partial\theta} = 0 \Rightarrow mr^2\dot{\theta} = l$ is a constant

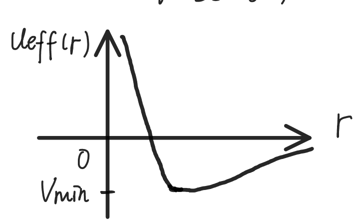
Substitute $\dot{\theta} = \frac{l}{mr^2}$ into E

$E = \frac{1}{2}m\dot{r}^2 + \underline{\frac{1}{2}\frac{l^2}{mr^2} + U(r)}$ depends only on position

$U_{eff}(r) = \frac{1}{2}\frac{l^2}{mr^2} + U(r)$

③ Trajectory

$r = \frac{a}{1+\varepsilon\cos\theta}, \quad a = \frac{l^2}{mk}, \quad \varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$



$E > 0$: motion is unbounded	$\varepsilon > 1$	Hyperbola
$E = 0$: unbounded	$\varepsilon = 1$	Parabola
$V_{min} < E < 0$: bounded	$\varepsilon < 1$	Ellipse
$E = V_{min}$: bounded	$\varepsilon = 0$	Circle
$E < V_{min}$: fall on center		

④ Period : Kepler's Third Law

$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m}$ is a constant (Kepler's Second Law)

$\therefore A = \pi a b = \frac{L}{2m} T$

$\therefore T = \frac{2m}{L} \pi a b = \pi \sqrt{\frac{4m}{K}} a^{3/2}$

$T^2 \propto a^3$

6. Hamiltonian

6.1 Legendre Transform

$H = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} = \sum_i p_i \dot{q}_i - \mathcal{L}$

6.2 Conservation of Energy

If $\frac{\partial \mathcal{L}}{\partial t} = 0$, $H = T + U = E$ is conserved

6.3 Equations of Motion

$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases}$ 2 first order ODEs instead of a second order