Summary

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0. Structure of What we've learned.

	Electrostatics	Magnetostatics
Maxwell's Equation	$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ ① $\nabla\times\vec{E}=0$ ②	$\nabla\cdot\vec{B}=0$ ③ $\nabla\times\vec{B}=\mu_0\vec{J}$ ④
Superposition & Symmetry	$\vec{E}=\frac{1}{4\pi\varepsilon_0}\int\rho(r')\frac{\vec{r}-\vec{r}'}{\lvert\vec{r}-\vec{r}'\rvert^3}d\tau'$ (Gauss) $\oint\vec{E}\cdot d\vec{a}=\frac{Q_{enc}}{\varepsilon_0}$	(Biot-Savart) $\vec{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{\lvert\vec{r}-\vec{r}'\rvert^3}$ (Ampere) $\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{enc}$
Boundary Conditions	$E_{\perp out}-E_{\perp in}=\frac{\sigma}{\varepsilon_0}$ (from ①) $E_{\parallel out}=E_{\parallel in}$ (from ②)	$B_{\perp out}=B_{\perp in}$ (from ③) $B_{\parallel out}-B_{\parallel in}=\mu_0\vec{K}\times\hat{n}$ (from ④)
Potential & Relation	$\nabla\times\vec{E}=0\Rightarrow\vec{E}=-\nabla V$ $V=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')d\tau'}{\lvert\vec{r}-\vec{r}'\rvert}$ (Poisson) $\nabla^2V=-\frac{\rho}{\varepsilon_0}$ (from ①)	$\nabla\cdot\vec{B}=0\Rightarrow\vec{B}=\nabla\times\vec{A}$ $\vec{A}(r)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')d\tau'}{\lvert\vec{r}-\vec{r}'\rvert}$ (3 equations:) $\nabla^2\vec{A}=-\mu_0\vec{J}$ (from ④)
Multiple Expansion & Dipole	$V=\frac{1}{4\pi\varepsilon_0}\int\rho(r')d\tau'\sum_n\frac{(r')^n}{r^{n+1}}P_n(\cos\theta)$ $\vec{p}=\int\vec{r}'\rho(r')d\tau'$ $V_{dip}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$	$\vec{A}=\frac{\mu_0 I}{4\pi}\int d\vec{l}'\sum_n\frac{(r')^n}{r^{n+1}}P_n(\cos\theta)$ $\vec{m}=I\int d\vec{a}$ $\vec{A}_{dip}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$
Field in Matter	$\vec{P}\equiv$ dipole moment per volume $\vec{D}:=\varepsilon_0\vec{E}+\vec{P}$ s.t. $\nabla\cdot\vec{D}=\rho_f$ $\nabla\times\vec{D}=\nabla\times\vec{P}$ not necessarily 0 $D_{\perp out}-D_{\perp in}=\sigma_f$ $D_{\parallel out}-D_{\parallel in}=P_{\parallel out}-P_{\parallel in}$	$\vec{M}\equiv$ dipole moment per volume $\vec{H}:=\frac{\vec{B}}{\mu_0}-\vec{M}$ s.t. $\nabla\cdot\vec{H}=-\nabla\cdot\vec{M}$ not necessarily 0 $\nabla\times\vec{H}=\vec{J}_f$ $H_{\perp out}-H_{\perp in}=-(M_{\perp out}-M_{\perp in})$ $H_{\parallel out}-H_{\parallel in}=\vec{K}_f\times\hat{n}$
Bound Charge/Current	$\rho_b=-\nabla\cdot\vec{P}$ $\sigma_b=\vec{P}\cdot\hat{n}$ ($\hat{n}$ out of material)	$\vec{J}_b=\nabla\times\vec{M}$ $\vec{K}_b=\vec{M}\times\hat{n}$
Linear Material	$\vec{P}=\chi_e\vec{E}\Rightarrow\vec{D}=\varepsilon_0(1+\chi_e)\vec{E}=\varepsilon\vec{E}$ $\rho_b=-\nabla\cdot(\frac{\chi_e}{1+\chi_e}\vec{D})=-\frac{\chi_e}{1+\chi_e}\rho_f$ $\rho_f=0\Rightarrow\rho_b=0\Rightarrow\nabla^2V=0$ $\nabla\times\vec{P}=\chi_e\nabla\times\vec{E}=0$	$\vec{M}=\chi_m\vec{H}\Rightarrow\vec{B}=\mu_0(1+\chi_m)\vec{H}=\mu\vec{H}$ $\vec{J}_b=\nabla\times(\chi_m\vec{H})=\chi_m\vec{J}_f$ $\vec{J}_f=0\Rightarrow\nabla\times\vec{H}=0\Rightarrow\vec{H}=-\nabla V$ $\nabla\cdot\vec{M}=\frac{1}{\mu_0}\frac{\chi_m}{1+\chi_m}\nabla\cdot\vec{B}=0\Rightarrow\nabla^2V=\nabla\cdot\vec{M}=0$
Solving Problem	1) Image Charge 2) $\nabla^2V=0\Rightarrow$ Laplace Equation	$\nabla^2V=0\Rightarrow$ Laplace Equation $-\nabla V=\vec{H}$ (it also applies to $\vec{B}$)

1. Electric Field

1.1 Superposition

$E(\vec{r})=\frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{\lvert\vec{r}-\vec{r}'\rvert^3}$ for a point charge located at $\vec{r}'$

Discreate: $E(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{r - r_i}{|\vec{r} - \vec{r_i}|^3}$

Continuous: $E(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, d\tau'$

1.2 Gauss' Law

Conditions: Symmetry $\Rightarrow$ constant field over surface

Infinite line: $\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{r}$

Infinite plane: $\vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}$

2. Potential

2.1 Superposition

$V = -\int_0^{\vec{r}} \vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}$ for ∞ as reference point

$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'$

2.2 Energy

Discrete: $W = \frac{1}{2} \sum_i q_i V(\vec{r_i})$

Continuous: $W = \frac{1}{2} \int \rho(\vec{r}') V(\vec{r}') \, d\tau'$

Plug in $\rho = \varepsilon_0 \cdot (\nabla \cdot \vec{E})$, $W = \frac{1}{2} \varepsilon_0 \int (\nabla \cdot \vec{E}) V \, d\tau'$

Integrating for whole space: $W = \frac{1}{2} \varepsilon_0 \int E^2 \, d\tau'$

2.3 Conductors

① $\vec{E} = 0$ inside

② V is constant inside the conductor and on the surface

③ $\vec{E}_{out} = \frac{\sigma}{\varepsilon_0} \hat{n}$ is perpendicular to the surface

④ $f = \frac{F}{A} = \frac{\sigma}{2\varepsilon_0} \cdot \sigma = \frac{\sigma^2}{2\varepsilon_0}$

2.4 Capacitors

$C = \left| \frac{Q}{V} \right|$

$C = \frac{\varepsilon_0 A}{d}$ for parallel plates

3. Boundary Value Problem

3.1 Poisson's Equation

$\nabla \cdot \vec{E} = -\nabla^2 V = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$

Given $\rho(\vec{r})$ and V on the boundaries, solving V and thus $\vec{E}, \sigma$

3.2 Boundary Conditions

$$\left.\begin{array}{l} ①\ E^{\perp}_{up} - E^{\perp}_{down} = \frac{\sigma}{\varepsilon_0} \\ ②\ E^{\parallel}_{up} = E^{\parallel}_{down} \\ ③\ V \text{ is continuous} \end{array}\right\} \text{Always True, derived from Maxwells Equation}$$

3.3 Uniqueness Theorem

First: Potential V inside a volume has unique solution if:

① ρ inside is specified ② V on the boundary is specified

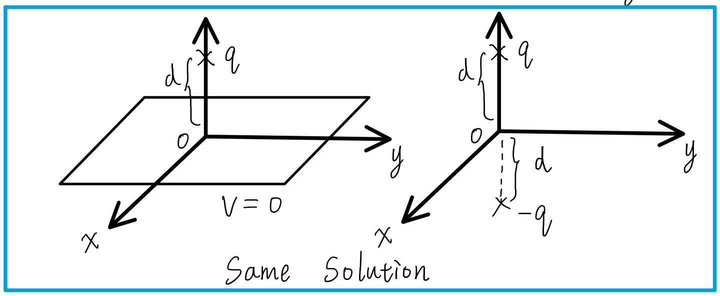
Second: In a region bounded by conductors, $\vec{E}$ has unique solution if:

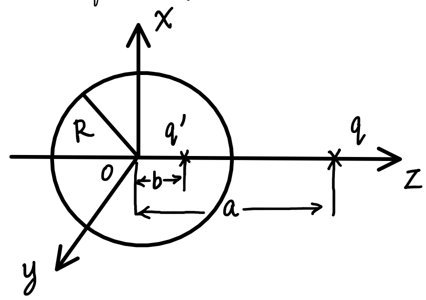
① ρ is known ② total Q of each conductor is specified

3.4 Image Charges

3.4 Image Charge

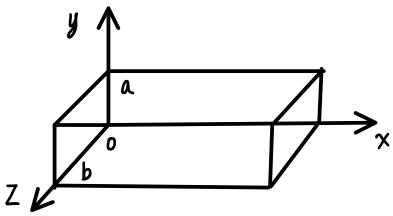
Put image charges outside of required region (ρ not changed), as long as boundary values are same, two distributions would given same (unique) solution.





3.5 Laplace Equation : $\nabla^2 V = 0$

3.5.1 Cartesian Coordinate



$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Two boundary surfaces $\Rightarrow$ Trig function

Direction of Propagation $\Rightarrow$ Exponential

Boundary Conditions $\rightarrow$ constants

Fourier transform $\rightarrow$ coefficients

Example : $V(0,y,z) = V_0 \Rightarrow V_0 = \sum_{m,n} V_{mn} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b}$

$$V_{mn} = \frac{4}{ab}\int_0^a\int_0^b V_0 \sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\, dz\, dy$$

3.5.2 Spherical Coordinate : Legendre Polynomials

Assume symmetry in φ, $V(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

Inside the shell, $B_l = 0$ since it diverges at center

Outside the shell, usually $A_l = 0$, unless there is $\vec{E}$ at $r = \infty$

$V_{in}(R) = V_{out}(R) \Rightarrow A_l R^{2l+1} = B_l$

$$\sigma = \varepsilon_0 \left(\frac{\partial V}{\partial r}\Big|_{in} - \frac{\partial V}{\partial r}\Big|_{out}\right) = \sum\left[l \cdot A_l R^{l-1} + (l+1)\frac{A_l R^{2l+1}}{R^{l+2}}\right] P_l(\cos\theta)\cdot\varepsilon_0$$

$$= \varepsilon_0 \sum (2l+1) A_l R^{l-1} P_l(\cos\theta) \quad \text{is the surface charge}$$

3.6 Multiple Expansion

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')\,d\tau'}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\varepsilon_0}\int\rho(r')\,d\tau' \sum_n \frac{(r')^n}{r^{n+1}} P_n(\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\underbrace{\int\rho(r')\,d\tau'}_{\substack{Q \\ \text{monopole}}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau'}_{\text{dipole}} + \cdots\right]$$

Dipole moment : $\vec{p} = \int \vec{r}'\rho(r')\,d\tau'$ $\qquad V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}$

4. Field in Dielectrics

4.1 Polarization

$\vec{P} \equiv$ dipole moment per unit volume is the polarization

We could decompose the dipole to surface charge and volume charge :

$\sigma_b = \vec{P}\cdot\hat{n}$ ($\hat{n}$ is pointing outward the dielectric) $\qquad \rho_b = -\nabla'\cdot\vec{P}$

edge of distribution $\qquad$ non-uniformity of polarization

4.2 Electric Displacement

Definition : $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

Maxwell Equations: $\nabla\cdot\vec{D} = \rho_f \,/\, \oint \vec{D}\cdot d\vec{a} = Q_f$, $\nabla\times\vec{D} = \nabla\times\vec{P}$ is not necessarily 0.

Boundary Conditions: $\begin{cases} D_{\perp out} - D_{\perp in} = \sigma_f \\ D_{/\!/ out} - D_{/\!/ in} = P_{/\!/ out} - P_{/\!/ in} \end{cases}$

4.3 Linear Dielectrics

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$ in linear dielectrics, $\vec{D} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E}$

$\therefore \rho_b = -\nabla\cdot\vec{P} = -\nabla\cdot\left(\frac{\chi_e}{1+\chi_e}\vec{D}\right) = -\frac{\chi_e}{1+\chi_e}\rho_f$

If $\rho_f = 0$, we could conclude $\rho_b = 0$ and solve Laplace Equation.

Boundary Conditions

$$\begin{cases} V_{in}(R) = V_{out}(R) \\ \varepsilon \frac{\partial V_{in}}{\partial r}(R) = \varepsilon_0 \frac{\partial V_{out}}{\partial r}(R) \quad \text{since } \rho_f = \sigma_f = 0 \\ V_{out} \to E \text{ as } r\to\infty \end{cases}$$

Tips: Look for $l=0$ and $l=1$, and other ls.

5. Magnetic Field

5.1 Current

$K = \sigma\vec{v}$ $\quad \vec{J} = \rho\vec{v}$

$I = \frac{dq}{dt}$ $\qquad I = \int \vec{K}\cdot d\vec{l} = \int \vec{J}\cdot d\vec{a}$ for surface and volume current

In magnetostatics, steady current ($\frac{d\vec{I}}{dt} = 0$) is studied.

5.2 Ampere's Law

$\int(\nabla\times\vec{B})\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l} = \int \mu_0\vec{J}\cdot d\vec{a} = \mu_0 I_{enc}$

Requires Symmetry: B points at same direction to l and is uniform.

5.3 Biot-Savart Law (Superposition)

$\vec{B}(r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

Tips: could use Biot-Savart Law to find the symmetry without evaluating the field; then apply Ampere's Law.

5.4 Vector Potential

$\vec{B} = \nabla\times\vec{A}$, choose $\vec{A}$ s.t. $\nabla\cdot\vec{A} = 0$

so $\nabla^2\vec{A} = \nabla^2 A_x\hat{x} + \nabla^2 A_y\hat{y} + \nabla^2 A_z\hat{z} = -\mu_0\vec{J}$ $\quad$ (3 Equations)

$\vec{A}(r) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')d\tau'}{|\vec{r}-\vec{r}'|}$

need to be calculated
in spherical / cylindrical Laplacian

5.5 Boundary Conditions

$$\begin{cases} B_{\perp out} - B_{\perp in} = 0 \\ B_{/\!/ out} - B_{/\!/ in} = \mu_0 \vec{K}\times\hat{n} \end{cases}$$

5.6 Multiple Expansion

$\vec{A}(r) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')d\tau'}{|\vec{r}-\vec{r}'|} \xrightarrow{\text{constant } I} \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$

Expand $\frac{1}{|\vec{r}-\vec{r}'|}$ to $\frac{1}{r}\sum_n\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$

$\vec{A}(r) = \frac{\mu_0 I}{4\pi}\Big[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\text{monopole, } 0} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta\, d\vec{l}'}_{\text{dipole}} + \cdots\Big]$

Dipole term: $\vec{A}(r) = \frac{\mu_0 I}{4\pi r^2}\oint(\hat{r}\cdot\vec{r}')d\vec{l}' = -\frac{\mu_0 I}{4\pi r^2}\hat{r}\times\int d\vec{a}'$ ← area vector

Define $\vec{m} = I\int d\vec{a} = I\vec{a}$ to be the magnetic dipole moment

$\vec{A}(r)$ … μ_0 … $\vec{m}\times\hat{r}$

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\ \ }{r^2}$

6. Magnetic Field in Matters

6.1 Magnetization

$\vec{M} \equiv$ magnetic dipole moment per unit volume

We could decompose the result field to 2 parts:

$\vec{J}_b = \nabla \times \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$

$$\vec{A}(r) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_b(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d\tau' + \frac{\mu_0}{4\pi}\int \frac{\vec{K}_b(\vec{r}')}{|\vec{r}-\vec{r}'|}\, da'$$

6.2 Auxiliary Field

$$\nabla \times \vec{B} = \mu_0(\vec{J}_b + \vec{J}_f) = \mu_0 \nabla \times \vec{M} + \mu_0 \vec{J}_f \Rightarrow \nabla\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f$$

Define $\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$, we have:

$$\begin{cases} \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \\ \nabla \times \vec{H} = \vec{J}_f \end{cases}$$

Boundary Conditions $\begin{cases} H_{\perp out} - H_{\perp in} = -(M_{\perp out} - M_{\perp in}) \\ H_{\parallel out} - H_{\perp in} = \vec{K}_f \times \hat{n} \end{cases}$

6.3 Linear Material

$$\vec{M} = \chi_m \vec{H} \Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu \vec{H}$$

Properties:

① $\nabla \cdot \vec{M} = \nabla \cdot \left(\frac{\vec{B}}{\mu_0} - \vec{H}\right) = \frac{\chi_m}{\mu_0(1+\chi_m)} \nabla \cdot \vec{B} = 0$

② $\vec{J}_b = \nabla \times \vec{M} = \chi_m \nabla \times \vec{H} = \chi_m \vec{J}_f$ $\quad \vec{J}_f = 0 \Rightarrow \vec{J}_b = 0$

Scalar Potential: When $\vec{J}_f = \vec{J}_b = 0$

$\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\nabla V$

$\nabla^2 V = -\nabla \cdot \vec{H} = \nabla \cdot \vec{M} = 0$ satisfies Laplace Equation

So we could apply general solution and boundary conditions above.

*Note: V is not necessarily continuous. It is when there is no surface current)