Summary

2019年10月27日 星期日 下午8:25

1. Vectors

1.1 Vector Operator: $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

① $\nabla \cdot \vec{V} = \mathrm{div}\,\vec{V}$; $\nabla \times \vec{V} = \mathrm{curl}\,\vec{V}$

② $\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ is the laplacian (scalar) of a function

③ $\nabla \cdot (\phi \vec{V}) = (\nabla \phi) \cdot \vec{V} + \phi (\nabla \cdot \vec{V})$

④ $\nabla \times (\nabla \times \vec{V}) = \nabla (\nabla \cdot \vec{V}) - (\nabla \cdot \nabla)\vec{V} = \nabla(\mathrm{div}\,\vec{V}) - \nabla^2 \vec{V}$

1.2 Curl of cross product

$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - (\vec{A} \cdot \nabla)\vec{B} - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A}$

2. Multivariable Calculus

2.1 Stokes Theorem

$$\int_M d\omega = \int_{\partial M} \omega$$

① Green's Theorem

$$\iint_A \mathrm{curl}_z \vec{F}\, dA = \oint_{\partial A} \vec{F} \cdot d\vec{r}$$

$$\iint_A \mathrm{div}\,\vec{F}\, dA = \oint_{\partial A} \vec{F} \cdot \hat{n}\, ds$$

② Stokes Theorem

$$\iint_S \mathrm{curl}\,\vec{F} \cdot \hat{n}\, dA = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

③ Divergence Theorem

$$\iiint_W \mathrm{div}\,\vec{F}\, dV = \iint_{\partial W} \vec{F} \cdot \hat{n}\, dA$$

2.2 Surface Integral

$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$ if $\phi(x,y,z)$ is the surface

Surface area: $ds = \frac{1}{\cos\theta} dA$

If $z = f(x,y)$ $\frac{1}{\cos\theta} = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}$

3. Complex Number

3.1 Conjugate

$Z = \frac{a+bi}{c+di}$ $\bar{Z} = \frac{a-bi}{c-di}$

3.2 Disk of convergence: ratio test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ means that the series is convergent

3.3 Complex functions

① Exponential and logrithms

$e^{i\theta} = \cos\theta + i\sin\theta$

$\ln(z) = \ln(re^{i\theta}) = \ln r + i\theta$

② Powers and roots

$Z^n = \cos n\theta + i\sin n\theta$

$Z^{\frac{1}{n}} = \sqrt[n]{r}\,(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$ Normally have n roots

③ Hyperbolic functions

$\sinh(y) = \frac{e^y - e^{-y}}{2}$ $\cosh(y) = \frac{e^y + e^{-y}}{2}$

$i\sinh(y) = \sin(iy)$ $\cosh(y) = \cos(iy)$

$$\begin{cases} \cosh^2(z) - \sinh^2(z) = 1 \\ \frac{d}{dz}\cosh(z) = \sinh(z) \end{cases}$$

4. Fourier Series and Fourier Transform

4.1 Fourier Series

① real form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

② Complex form

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$C_n=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-inx}\,dx$$

Relation: $C_n=\frac{a_n+ib_n}{2}$, $C_{-n}=\frac{a_n-ib_n}{2}=C_n^*$

Generalization to period of $2L$

$\sin\frac{n\pi}{L}x$, $\cos\frac{n\pi}{L}x$ forms a basis

4.2 Fourier Transform

$$f(x)=\int_{-\infty}^{\infty} g(k)e^{ikx}\,dk \qquad g(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

① sine transform

$$f_s(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} g_s(k)\sin kx\,dk \qquad g_s(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(x)\sin kx\,dx$$

② cosine transform

$$f_c(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} g_c(k)\cos kx\,dk \qquad g_c(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} g_c(k)\cos kx\,dx$$

5. Laplace Transform

5.1 Definition

A transform from t (real space) to p (complex)

$$\mathcal{L}[f(t)]=\int_0^{\infty} e^{-pt}f(t)\,dt=F(p) \qquad f(t)=0 \text{ for } t<0$$

5.2 Linearity

$$\mathcal{L}[af_1(t)+bf_2(t)]=a\mathcal{L}[f_1(t)]+b\mathcal{L}[f_2(t)]$$

5.3 Laplace Transform of Specific Functions

5.3.1 Derivatives

$$\mathcal{L}[y^{(n)}]=p^nY-p^{n-1}y(0)-\cdots-py^{(n-2)}(0)-y^{(n-1)}(0)$$

5.3.2 Dirac Delta Function

① Definition

$$\int_a^b \delta(t-t_0)f(t)\,dt=\begin{cases} f(t_0) & \text{if } t_0\in(a,b) \\ 0 & \text{otherwise}\end{cases}$$ open interval

② Laplace Transform

$$\mathcal{L}[\delta(t-t_0)]=\int_0^{\infty} e^{-pt}\delta(t-t_0)\,dt=e^{-pt_0}$$

$$\mathcal{L}[g(t-t_0)]=\int_0^{\infty} e^{-pt}g(t-t_0)\,dt=e^{-pt_0}\int_0^{\infty} e^{-p(t-t_0)}g(t-t_0)\,d(t-t_0)$$

$$=e^{-pt_0}\mathcal{L}[g(t)]$$

③ Derivatives

$\delta'(x-x_0)$ is meaningless on its own, but could act on test functions like an operator

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-x_0)\,dx=(-1)^n f^{(n)}(x_0)$$

④ Properties

$$\delta(x)=\delta(-x),\quad \delta'(x)=-\delta'(-x),\quad \delta(ax)=\frac{1}{|a|}\delta(x)\quad (a\neq 0)$$

5.4 Applications of Laplace Transform

5.4.1 Convolution of functions: Inverse Transform of products

$$f(t),\ g(t)\xrightarrow[\text{Transform}]{\text{Laplace}} F(p),\ G(p)$$

$$\mathcal{L}^{-1}[F(p)G(p)]=f*g$$

$f*g=\int_0^t f(t-\tau)g(\tau)\,d\tau$ is a function of only t

5.4.2 Solving ODEs with Laplace Transform

$$Ay''+By'+Cy=f(t) \qquad y(0)=y'(0)=0$$

$Y(p)=T(p)\cdot\mathcal{L}[f(t)]$ where $T(p)=\frac{1}{Ap^2+Bp+C}$

$y(t)=\mathcal{L}^{-1}[Y(p)]=f(t)*\mathcal{L}^{-1}[T(p)]\Rightarrow$ find the inverse transform, and calculate the convolution

6. Series Solutions of Differential Equations

6.1 General Procedure

6.1.1 Simple Solution

Let $y(x)=\sum_{n=0}^{\infty} a_n x^n$, then $y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}$, $y''(x)=\sum_{n=2}^{\infty} n(n-1)x^{n-2}$

① Substitute the polynomial expressions into orignal equation

② Shift the sum to make the powers to be same

③ Adjust the starting point to combine the sum (write some terms explicitly)

④ $\begin{cases} \text{Coefficient inside the sum is } 0 \Rightarrow \text{Recursion Relation} \\ \text{Coefficients of explicit terms are } 0 \Rightarrow \text{Constraints (ex. } a_1 + a_0 = 0) \end{cases}$

⑤ Use the recursion relation to find a_n in terms of a_0 (and $a_1, \cdots$ depends on the order)
Nth order $\rightarrow a_{N-1}$

⑥ Substitute a_n and get the solution

6.1.2 Generalized Solution

Let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$, addition of s allows negative and fractional powers

Since the first term is $a_0 x^s$, we assume $a_0 \neq 0$

Now, the coefficients of explicit terms (with a_0) give N possible values of s for an Nth order equation

Plug in different values of s, solve for solutions

The linear combination of all solutions gives the general solution

6.2 Orthogonality and Completeness

6.2.1 Orthogonality

$\int_a^b A^*(x) B(x)\, dx = 0 \Rightarrow A(x), B(x)$ are orthogonal on (a, b)

$\int_a^b A_m^*(x) A_n(x)\, dx = C\,\delta_{mn} \Rightarrow \{A_n(x)\}$ is an orthogonal set on (a, b)

6.2.2 Completeness

A set of orthogonal functions is complete on (a,b) if there is no other function orthogonal to all of them on interval (a,b)

On interval $(0, \pi)$ $\sin nx$ and $\cos nx$ are both complete sets because we could construct function on $(-\pi, 0)$ to make it odd/even.

6.3 Legendre Polynomials

6.3.1 Legendre Equation

$$(1-x^2)y'' - 2xy' + l(1+l)y = 0$$

Appears in PDEs in Spherical coordinates, like wave function and heat

6.3.2 Legendre Polynomials

The series solution is $y = a_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}x^4 - \cdots\right] + a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \cdots\right]$

even series　　　　odd series

Ratio test suggests that it converges on $x \in (-1, 1)$

To make the series converge at $x = \pm 1$, we choose l specifically to:

① truncate even series, discard odd series (by setting $a_1 = 0$)

or ② truncate odd series, discard even series (by setting $a_0 = 0$)

The remaining polynomial after truncation is Legendre Polynomial, $P_l(x)$, satisfying $P_l(1) = 1$

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2 - 1)$

6.3.3 Orthogonality

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2l+1}\delta_{lm}$$

$\therefore \{P_l(x)\}$ is an orthogonal set on $(-1, 1)$

6.3.4 Completeness

$\{P_l(x)\}$ is a complete set on $(-1, 1)$

6.3.5 Normalization

$\int_{-1}^{1} P_l^2(x)\, dx = \frac{2}{2l+1}$, $\sqrt{\frac{2}{2l+1}}$ is the norm of $P_l(x)$

$\therefore \sqrt{\frac{2l+1}{2}} P_l(x)$ is the normalized Legendre Polynomials

6.3.6 Rodrigues' Formula: Construct Legendre Polynomials

$$P_l(x) = \frac{1}{2^l l!} \frac{d^{(l)}}{dx^{(l)}} (x^2 - 1)^l$$

6.3.7 Legendre Series: Using $P_l(x)$ as a complete, orthonormal basis

$$f(x) = \sum_{n=0}^{\infty} C_l P_l(x)$$

$$C_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x)\, dx$$

6.3.8 Generating Functions

$$\Phi(x, h) = (1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} h^l P_l(x)$$

$$P_l(x) = \frac{1}{l!} \frac{d^{(l)}}{dh^{(l)}} \Phi(x, h)\Big|_{h=0}$$

Another way to define $P_l(x)$, and is directly connected to multipole expansion in electrostatics