Summary

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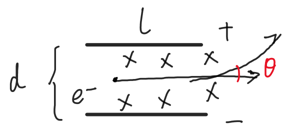
1. The Atomic Nature

1.1 Electron and Charge

① Electrolysis (Faraday)

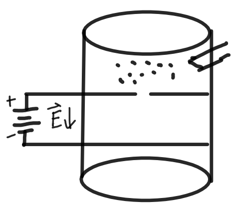
$m = \frac{qM}{FZ}$ there is charge inside atoms

② Discovery of electron (J.J. Thomson)



$\frac{e}{m} = \frac{V\theta}{B^2 L d}$ is a constant

③ Measurement of elementory charge (Millikan)



$q_i = \frac{mg}{E} \frac{V_{ri} + V_f}{V_f}$ $r = 3\sqrt{\frac{\eta V_f}{2\rho g}}$

ratio of q_0, q_j are rational numbers

$V_f\,(w/o\ \vec{E}) \to r \to m \xrightarrow{V_r(w/\vec{E})} q_i$

1.2 Atomic Models

① Rutherford's Model

High concentration of mass and charge at center (nucleaus)

Small volume of nucleaus compares to the whole atom

Orbiting electrons

② Bohr's Model

$\frac{dE}{dL} = \omega \Rightarrow dL = \hbar$ Angular momentum is quantized

$r_n = \frac{n^2 \hbar^2}{m_e k e^2}$ $r_1 = \frac{\hbar^2}{m_e k e^2} = a_0$ Bohr radius

$E_n = -\frac{ke^2}{2a_0} \frac{Z^2}{n^2}$

2. Particle Nature of Waves

2.1 Planck's Law

$u(\lambda, T) \cdot d\lambda = \frac{\#\text{ of oscillators} \cdot \langle E \rangle}{\text{volume}}$ ← average energy of on oscillator

Energy is quantized: $E_n = nhf$ $P(E_n) = e^{-nhf/k_B T}$

$\langle E \rangle = \sum E_n P(E_n) / \sum P(E_n)$

# of oscillators: $\frac{4\pi k^2 dk}{(\pi/L)^3} \cdot \frac{1}{8} \cdot 2$

↑ $k > 0$ ↖ 2 polarization

$\therefore u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/k_B T\lambda} - 1}$

$u(f, T) = \frac{8\pi h}{c^3} \cdot \frac{f^3}{e^{hf/k_B T} - 1}$

} Planck's Law

Small $\lambda \to$ Wien's Law

Large $\lambda \to$ Rayleigh-Jeans Law

2.2 Photoelectric Effect

$$K_{max} = hf - \phi$$

hf ~ Energy of photon as a particle

ϕ ~ Work function (need to free the electron)

2.3 Compton Scattering

Inelastic scattering ⇒ Particle Nature

$$\lambda' - \lambda = \Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$$

$\frac{h}{mc}$ ~ Compton Wavelength Related to rest energy

2.4 Wave-Particle Duality : De Broglie Wave

$\lambda = \frac{h}{p} = \frac{h}{\gamma m v}$ ~ De Broglie Wavelength Related to momentum

3. Wave Functions

3.1 Schrodinger Equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x,t)\psi$$

Meaning : $|\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$ is the probability density function of finding the particle at (x,t)

3.2 Operators

Operators are applied on wave functions to get specific physical quantities

$\hat{Q}\psi(x,t) = q\psi(x,t)$, q is the eigenvalue and $\psi(x,t)$ is the eigenstate

- Some operators

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad \hat{x} = -\frac{\hbar}{i}\frac{\partial}{\partial p}, \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

- Expectation value of operators

$$\langle\hat{Q}\rangle = \int_{-\infty}^{\infty}\psi^*(x)\,\hat{Q}\,\psi(x)\,dx$$

$i\hbar\frac{d}{dt}\langle\hat{Q}\rangle = \langle[\hat{Q},\hat{H}]\rangle$ ⇒ if $\hat{Q}$ and $\hat{H}$ commutes [i.e. $[\hat{Q},\hat{H}]=0$], its expectation value would be a constant

3.3 Time-independent Schrodinger Equation

Assumption : V is only a function of x ⇒ We could separate variables

Result : $\psi(x,t) = \psi(x)\,e^{-iEt/\hbar}$

Equation : $\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi = E\psi$ E is the energy of particle

3.3.1 Stationary States

Quantum states with all observables independent of t.

Example. Probability density

$$\int_{-\infty}^{\infty}\psi^*(x,t)\,\psi(x,t)\,dx$$

$$= \int_{-\infty}^{\infty}\psi^*(x)\,e^{-iEt/\hbar}\,\psi(x)\,e^{iEt/\hbar}\,dx$$

$= \int_{-\infty}^{\infty}|\psi(x)|^2\,dx$ is independent of time

3.3.2 Wave Packets

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ violates the uncertainty principle, thus $\psi(x)$ is a superposition of eigenstates $\psi_n(x)$ with different "weight" C_n.

$\psi(x) = \sum_{n=1}^{\infty} C_n\psi_n(x)$ where $C_n = \int_{-\infty}^{\infty}\psi_n^*(x)\psi(x,0)\,dx$

$|C_n|^2$ is the probability of returning E_n in a measurement, $\sum_{n=1}^{\infty}|C_n|^2 = 1$

Add the time-dependence :

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty} C_n\psi_n(x,t)$$

Phase Velocity : Velocity of the carrier wave $v_p = \frac{\omega}{k}\Big|_{k=k_0}$

Group Velocity : Velocity of the wave packet $v_g = \frac{d\omega}{dk}\Big|_{k=k_0}$

3.3.3 General Procedure of solving a problem

① $\frac{d^2\psi}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\psi$

② Solve this 2nd order ODE, the solution is $\begin{cases} \text{oscillatory} & V_0<E \\ \text{exponential} & V_0>E \end{cases}$

③ Determine the constants using boundary conditions ($\psi(x)$ is continuous, and $\frac{d\psi}{dx}$ is continuous as long as no delta potential)

④ Determine the last constant by normalization of $\psi(x)$

3.3.4 Examples of potentials

① Square Well: $V(x)=0$ inside the well

Infinite: $\psi(x) = \begin{cases} A\cos kx + B\sin kx & k=\frac{\sqrt{2mE}}{\hbar} \\ 0 & \text{outside of the well} \end{cases}$

$\psi(0)=\psi(L)=0 \Rightarrow kL = n\pi,\ k=\frac{n\pi}{L}$

$\therefore E_n = \frac{\hbar^2k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}$

Finite: Follows the same procedure, but $\psi(x)$ is not 0 outside the well

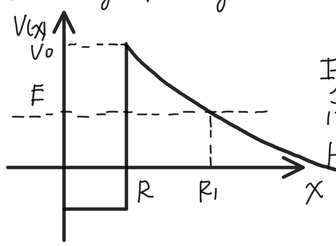
Penetration Depth: $\delta = \frac{1}{k} = \frac{\hbar}{\sqrt{2m(V_0-E)}}$ when ψ_1 decreased by $\frac{1}{e}$

② Harmonic Oscillator: $V(x)=\frac{1}{2}m\omega^2x^2$

$\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/2} e^{-\frac{m\omega x^2}{2\hbar}}$ is a Gaussian Distribution

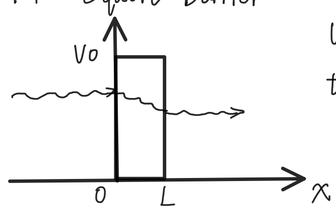
$E_0 = \frac{1}{2}\hbar\omega,\ E_n = (n+\frac{1}{2})\hbar\omega$

3.4 Tunneling: Ability to "Cross" the barrier



If the energy of particle, E, is less than V_0, classically it's impossible to cross the barrier.
However, in Quantum Mechanics, it happens.

3.4.1 Square Barrier



When the wave hits the barrier, both reflection and transmission happens. $T+R=1$

$T = \left[1+\frac{V_0^2\sinh^2(k_2L)}{4E(V_0-E)}\right]^{-1}$ for bound states $E<V_0$

if $E>V_0$, $k_2L = n\pi \Rightarrow T=1$, full transmission since two reflection waves have a phase difference of π and cancels out.

3.4.2 Generalized Barrier

$T = T_1\cdot T_2\cdot T_3 \cdots \approx \exp\left(\frac{2}{\hbar}\sqrt{2m}\int\sqrt{V(x)-E}\,dx\right)$

Example. Alpha Decay

$E = V(R_1) = \frac{2kZe^2}{R_1}$ (Z is the atomic number of atom after decay)

$T \approx \exp\left(-4\pi Z\sqrt{\frac{E_0}{E}} + 8\sqrt{\frac{RZ}{r_0}}\right)$, where $r_0 = \frac{\hbar^2}{m_\alpha ke^2}$, $E_0 = \frac{ke^2}{2r_0}$

3.4.3 Delta Potential: Discontinuity of $\frac{d\psi}{dx}$

$V(x) = -S\delta(x)$

$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\psi\,dx - S\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi(x)\,dx = \int_{-\varepsilon}^{\varepsilon}E\psi(x)\,dx$

$\Rightarrow \left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-} = -\frac{2mS}{\hbar^2}\psi(0)$ reveals discontinuity of $\frac{d\psi}{dx}$ at $x=0$

4. Statistical Mechanics

4.1 Distribution of Particles

4.1.1 Discrete Distribution

There are different arrangments of N particles with total Energy E. Each arrangment corresponds to several microstates. Number of states corresponding to an energy level is called degeneracy, g_i.

4.1.2 Continuous Distribution

$$n(E)\,dE = g(E) f(E)\,dE$$ General Formula for all distributions

$n(E)$: number of particles per volume between E and $E+dE$

$g(E)$: density of states

$f(E)$: probability that a particle occupies energy level E

4.2 Classical Statistics : Distinguishable Particles

4.2.1 Maxwell-Boltzmann Distribution

$$f_{MB}(E) = A e^{-E/k_BT}$$

$\because E = \frac{1}{2}mv^2$ for ideal gas

$$n(v) = \frac{4\pi N}{V}\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_BT}} = g(v) f_{MB}(v)$$

4.2.2 Velocities of ideal gas

① Most probable velocity

$$\frac{d}{dv} n(v)\Big|_{v=v_{mp}} = 0$$

$$\therefore v_{mp} = \sqrt{\frac{2k_BT}{m}}$$

② Averge velocity

$$\overline{v} = \frac{\int_0^\infty v\,n(v)\,dv}{\int_0^\infty n(v)\,dv} = \sqrt{\frac{8k_BT}{\pi m}}$$

③ Root-mean-square velocity

$$v_{rms} = \sqrt{\langle v^2\rangle}$$

$$= \left(\frac{\int_0^\infty v^2 n(v)\,dv}{\int_0^\infty n(v)\,dv}\right)^{1/2} = \sqrt{\frac{3k_BT}{m}}$$

$v_{rms} > \overline{v} > v_{mp}$

4.2.3 Equipartition Theory

1 degrees of freedom $\longrightarrow \frac{1}{2}k_BT$ Energy

- Some examples

1D oscillators : $2\cdot\frac{1}{2}k_BT = k_BT$

3D atoms : $3\cdot\frac{1}{2}k_BT = \frac{3}{2}k_BT$

3D oscillators : $6\cdot\frac{1}{2}k_BT = 3k_BT$

4.2.4 Validity of Classical Statistics

Maxwell-Boltzmann Distribution is valid when wave functions do not interact.

In other words, distance between adjacent particles, d, is greater than Δx.

$\Delta p_x = \sqrt{\overline{p_x^2} - \overline{p_x}^2}$ $\overline{p_x} = 0$ because velocity has 2 directions

$\therefore \Delta p_x = \sqrt{\overline{p_x^2}} = \sqrt{mk_BT}$ $\Delta x \Delta p_x \geq \frac{\hbar}{2}$

$\therefore d = \left(\frac{V}{N}\right)^{1/3} \gg \frac{\hbar}{2\Delta p_x}$

$\therefore \frac{N}{V}\frac{\hbar^3}{8(mk_BT)^{3/2}} \ll 1$

So, ① High concentration of particles, ② Low temperature and ③ small mass particle would violate classical distribution.

4.3 Quantum Statistics : Indistinguishable Particles

4.3.1 Bosons and Fermions

Bosons : Spins $(0, 1, 2, \cdots)$, could have many in a state

Fermions : Spins $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\cdots)$, obeys Pauli's Exclusion Principle, only 1 in a quantum state.

4.3.2 Bose-Einstein Distribution

$$f_{BE} = \frac{1}{e^{(E-\mu)/k_BT} - 1}$$ μ is the chemical potential

4.3.3 Fermi-Dirac Distribution

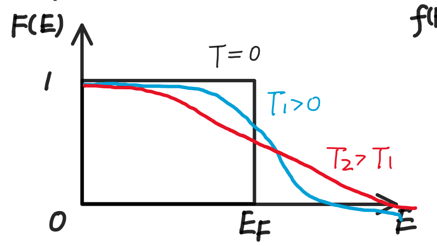
$$f_{FD} = \frac{1}{e^{(E-\mu)/k_BT} + 1}$$

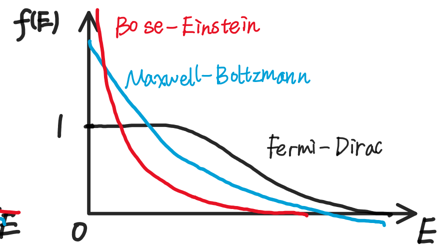
4.3.4 Fermi Energy

$\mu(T=0K) = E_F$ is Fermi Energy

It's defined as the energy difference between highest and lowest occupied single-particle states of non-interacting Fermions at $T=0K$.

$\therefore f_{FD}(E_F) = \frac{1}{2}$





4.4 Applications of Quantum Statistics

Bose-Einstein Distribution:

4.4.1 Blackbody Radiation

$$u(E)\,dE = E\,n(E)\,dE = E\,g(E) f_{BE}(E)\,dE$$
$$= \frac{8\pi E^3}{(hc)^3 (e^{E/k_BT}-1)}\,dE$$

Substitute $E = hf$, $dE = h\,df$

$$u(f,T) = \frac{8\pi h f^3}{c^3 (e^{hf/k_BT}-1)}$$ Planck's Law

4.4.2 Specific Heat of Solid

Atoms as Quantum Harmonic Oscillators

$$U = 3N_A \bar{E} = \frac{3N_A \hbar\omega}{e^{\hbar\omega/k_BT}-1}$$

$$C_V = \frac{\partial U}{\partial T} = 3R\left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T}-1)^2}, \quad \theta_E = \frac{\hbar\omega}{k_B}$$ is Einstein temperature

① High temperature: $e^{\theta_E/T} \approx 1 + \frac{\theta_E}{T}$ $\quad C_V \approx 3R$ like classical theories

② Low temperature: $C_V \approx 3R\left(\frac{\theta_E}{T}\right)^2 e^{-\theta_E/T}$

Fermi-Dirac Distribution:

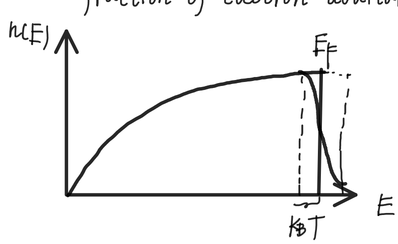
4.4.3 Free Electron Gas Model of Metals

Problem: C_V of metal $< \frac{9}{2}R$ predicted, due to the electron's part $(\frac{3}{2}R)$

$$n(E) = g(E) f_{FD}(E) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E} \cdot \frac{1}{e^{(E-E_F)/k_BT}+1}$$

should use $n(E)$, but at low temperature approximately same

fraction of electron available: Fermi Sea ✓



$$\frac{k_BT\,g(E_F)}{\int_0^\infty g(E)\,dE} = \frac{k_BT\sqrt{E_F}}{\int_0^{E_F}\sqrt{E}\,dE} = \frac{3}{2}\frac{k_BT}{E_F} = \frac{3}{2}\frac{T}{T_F}$$

where $T_F = \frac{E_F}{k_B}$ is Fermi Temperature

$\frac{T}{T_F} \approx \frac{1}{200}$, so specific heat of electron is much less than predicted. $(\frac{3}{2}R)$