2

Sequences and Difference Equations

The introductory chapter explained the central role that difference equations will play throughout this course. In this chapter you will learn the basic properties of difference equations, and the mathematical terminology and notation that are used in the study of difference equations.

Sequences

We will begin with an example based on a chart printed in *USA Today*[1]. The chart depicted world oil consumption for several years in the past, as well as a predicted level of consumption for the future. The numbers contained in the chart are shown in Table 2.1. These figures are reported in units of millions of barrels of oil per day. So, for 1991, the table entry of 66.6 indicates that on the average the world consumed 66.6 million barrels of oil each day.

This example has an important feature that shows up over and over again in applications: the data are a series of numerical values that go with a series of times, and the times are equally spaced. In this case, we have one value for each year. In other cases we might have data for each day, or for each hour. The common feature is that the data are separated by equal periods of time. We usually refer to data of this type as *discrete*

Year	1991	1992	1993	1994	1995
Oil Used	66.6	66.9	66.9	67.6	68.4

TABLE 2.1
World Oil Consumption in Millions of Barrels per Day

[1] From *USA Today,* June 16, 1994

data. The collection of data values is referred to as a *sequence,* and the individual data values are referred to as *terms* of the sequence.

Many many applications are analyzed using discrete data. In its most general meaning, the word *discrete* simply indicates a separation between values. Often, this separation is introduced in the process of recording data: we have a first reading, a definite next reading, and so on. These are discrete data values because they are separated, and can be listed in a definite order. In contrast, it is easy to imagine a situation where data are not separated, at least conceptually. Consider the way air temperature changes over the course of the day. Perhaps at noon the temperature is 70 degrees. You might also have the temperature at 1 minute after noon, or 1 second, or a tenth of a second. There is no one *next* time to consider. Instead, the time can be thought of as a continuous range of possible values, like the points on a line. As we will see later, that is connected with the idea of a *continuous* variable. But for now we will concentrate on sequences of discrete data. Usually these will correspond to measurements made at regularly spaced times, such as daily temperatures or annual rain fall amounts. But discrete data don't have to be associated with regularly spaced times, or with times at all. In one example, we will consider various prices a company can charge for one of its products, and how the price affects sales. In that problem, the data will reflect sales levels at a sequence of prices, not at a sequence of times. The point to keep in mind is this: when you think of discrete data, think of separation. A good image to keep in mind is a list of separate data values.

It is customary in this context to use a shorthand notation consisting of a label (usually a single letter) and a number. For example, we might choose to use the letter C (standing for consumption), and then write C_{1991} for the average daily oil consumption in 1991. Of course for any other year we use a similar notation: C_{1995} stands for the daily consumption in 1995. Reading the data from the table, we see that $C_{1992} = 66.9$ million barrels.

Notice that the number is written in small size type and below the line. It is called a *subscript*. It is important to pay close attention to the position and size of symbols when subscripts are used. Compare C_{1991+4} with $C_{1991}+4$. For the first one, all of the symbols in $1991+4$ are below the line of type, and in the small type size, so these symbols are all part of the subscript attached to the C. Because $1991 + 4 = 1995$, you should recognize C_{1991+4} to mean the same thing as C_{1995}, which is 68.4. In contrast, in $C_{1991} + 4$, only 1991 is part of the subscript attached to C. This time, you are supposed to add 4 to the numerical value of C_{1991}. Since $C_{1991} = 66.6$, that means $C_{1991} + 4 = 66.6 + 4 = 70.6$. Be sure you understand the difference between C_{1991+4} and $C_{1991} + 4$. To practice, write down in the space below the numerical value of C_{1994-2} and $C_{1994} - 2$. The answers are given in a footnote, but don't peek until after you write down your own answers.[2]

You will be using subscripts throughout this course. The examples above illustrate the importance of reading subscript notation carefully. It is also very important to write carefully. It can be very hard to tell in handwritten work what is part of a subscript, and what is not. There can be a big difference in the meaning of two very similar looking expressions. This is illustrated by the example above. For C_{1991+4} refers to the oil

[2] Answers: $C_{1994-2} = C_{1992} = 66.9$; and $C_{1994} - 2 = 67.6 - 2 = 65.6$

consumption 4 years after 1991; $C_{1991} + 4$ concerns adding 4 to the oil consumption in 1991. In one case, the 4 is added to the year; in the other case it is added to the amount of oil consumed. These are clearly completely different ideas.

For the examples above, we have used years such as 1991 and 1994 as subscripts, but there are other alternatives. We might have chosen simply to number the data values. In that approach, C_1 is the first data value, or 66.6, C_2 is the second, and so on. We will see in the next section that it is often convenient to number the starting data value with a 0 instead of a 1. In that case, C_0 is the consumption for 1991, C_1 the value for 1992, and so on. This choice permits a simple verbal description of the numbering scheme: *the subscript on each term indicates a number of years after 1991.* That is, C_1 denotes the daily consumption one year after 1991, C_2 stands for the daily consumption two years after 1991, and so on. It even makes some kind of sense to say that C_0 is the consumption zero years after 1991—that is, in 1991.

There is nothing magical about the letter C in the preceding paragraph. Any letter can be used as part of the shorthand for a sequence of numbers. Usually, we try to pick a letter that helps us remember something about the data. In a problem involving population data, we might use the letter p. For example, if the data gives the US population in 1950, 1960, 1970, and so on, we could abbreviate the data as p_0, p_1, etc.

So far, the subscripts have all been given as numerical values. Often it is useful to use a variable as a subscript. This idea can be illustrated using the oil data again. Consider the following three statements:

C_2 is the daily oil consumption 2 years after 1991
C_3 is the daily oil consumption 3 years after 1991
C_4 is the daily oil consumption 4 years after 1991.

These statements form an easily recognized pattern, and communicate quite well what is meant by the C notation. Indeed, after reading the three statements, everyone can agree what C_5 ought to mean. This same pattern can be condensed into a single statement:

C_n is the daily oil consumption n years after 1991.

Here, the point of using n instead of a specific number is to convey the idea that the statement holds for every number. Replace n with 2, or with 3, or with 234. In each case the resulting statement is true. This kind of notation will be used repeatedly. Rather than list several examples and hope that a pattern is apparent, we will make just one statement involving a variable. By replacing that variable with various numbers, you can create your own list of examples.

This idea can be illustrated using an example from Chapter 1. Suppose the following statement is made:

a_n is the amount of carbon dioxide in the atmosphere (in parts per million) n years after 1965.

To help understand what that means, rewrite the statement changing each n to a number, say 7.

a_7 is the amount of carbon dioxide in the atmosphere (in parts per million) 7 years after 1965.

So a_7 gives the parts per million of carbon dioxide in 1972. Repeat this process with several different numbers until you see what the pattern is. With a little practice, you will be able to understand what the pattern is just by looking at the single statement with the variable.

This completes the introduction of the concept of discrete data and sequences, as well as the shorthand notation using subscripts. In the next section these ideas are used to introduce difference equations.

Difference Equations

The oil consumption table was printed in *USA Today* in 1994, before the figures for 1994 or 1995 were available. The oil consumption shown for those years are projections, or estimates. How can estimates like those be made? Generally speaking, the answer involves patterns in the known data. If the data follow some recognizable pattern, we imagine that the pattern will continue into the future and make a projection on that basis. For the *USA Today* article, patterns were probably found in many kinds of data that contribute to oil consumption, such as population growth, industrialization, and economic development patterns. To keep this discussion simple, we will only look at the data displayed in the table, and only consider a very simple pattern. In discussing the simple pattern, we will pretend that the USA Today data are true values, ignoring the fact that the last two values are actually projections.

For a moment, look only at the first two data values. Notice that the consumption rose from 66.6 in 1991 to 66.9 in 1992, an increase of 0.3. One very simple assumption you could make is that the same increase will occur every year. This assumption would be stated in the following form:

> The average daily oil consumption each year is .3 more than in the preceding year.

It is easy to see from the data that this pattern is not correct. At least, it does not exactly fit the data. If it did, the oil consumption for 1993 would be $66.9 + .3 = 67.2$, which is not the figure shown in the table. However, we do not demand that the pattern fit the data perfectly. After all, a pattern is a kind of mathematical model, and mathematical models always involve simplification and approximation. In this course we will see many different kinds of models, some of which fit real data quite well. Our aim now, though, is not to find a highly accurate model for the oil data. Rather, the oil example is intended only to illustrate some notation and terminology that will be used throughout the course. The basic conceptual idea is this: it is often possible to find patterns that approximate a data set. We will study the patterns, eventually comparing them to the original data.

So let us consider the pattern described above. The first data value we have is 66.6. According to the pattern, the next value should be .3 more or 66.9. Then the next value should be 67.2, then 67.5, and so on. We can continue in this way to generate one number after another as long as we choose. At this point, recognizing that these numbers are not exactly the same as the data, we should introduce a new label. We will use the letter c.

Then the previous remarks can be summarized as follows:

$$c_0 = 66.6$$
$$c_1 = 66.6 + .3 = 66.9$$
$$c_2 = 66.9 + .3 = 67.2$$
$$c_3 = 67.2 + .3 = 67.5$$

There is a pattern here. Changing the way the equations are written will make the pattern clearer. The first equation says that $c_0 = 66.6$. But 66.6 appears again in the second equation. Let us write c_0 in that equation, in place of 66.6. This emphasizes that we are adding $.3$ to the initial data value (c_0), rather than emphasizing exactly what that data value is (66.6). In the same way, the 66.9 that appears in the third equation can be replaced with c_1, and the 67.2 in the last equation can be replaced with c_2. Then the equations become

$$c_0 = 66.6$$
$$c_1 = c_0 + .3$$
$$c_2 = c_1 + .3$$
$$c_3 = c_2 + .3$$

As in previous examples, there is a clear pattern here. It is much more compactly expressed using variable subscripts:

$$c_{n+1} = c_n + .3 \tag{1}$$

This one equation contains the same information as the last three equations above. For example, make $n = 1$, and the equation says $c_2 = c_1 + .3$. But the equation with n says more, because it can be used with any number in place of n. If n is changed to 100, we get $c_{101} = c_{100} + .3$, meaning *the one hundred and first data value equals the one hundredth value plus* $.3$.

There is one pattern at work here, and we have seen three different ways to communicate what the pattern is. The pattern can be described verbally:

Each data value is $.3$ *greater than the preceding data value.*

The pattern can be illustrated with several equations:

$$c_1 = c_0 + .3$$
$$c_2 = c_1 + .3$$
$$c_3 = c_2 + .3.$$

Or the pattern can be expressed in the single equation

$$c_{n+1} = c_n + .3.$$

This final approach is the most compact. It is a kind of shorthand for describing a pattern. Like any shorthand, it is only useful after it becomes so familiar that it is understood immediately. That requires practice. The shorthand will be used repeatedly throughout the text. At first, it will be good practice to take the time each time you see such an equation to substitute several different values for the variable in the subscript and to write out a verbal description.

For example, if you see this equation:

$$p_{k+1} = 4p_k - 3$$

you should replace the variable in the subscript (in this case, k) with several numbers. Replacing k with 1 gives $p_{1+1} = 4p_1 - 3$ or $p_2 = 4p_1 - 3$. Replacing k with 2 gives $p_{2+1} = 4p_2 - 3$ or $p_3 = 4p_2 - 3$. Writing several equations in this way should help you get a feel for what the pattern is. Then you should write a description of the pattern in words:

Each data value is found by multiplying the preceding data value by 4 *and subtracting* 3.

Finally, pick a starting value for the data, and use it to compute several more data values. If the starting value is 2, the verbal description says to multiply that by 4 and subtract 3. This gives 5 as the next data value. Repeat the process: multiply the 5 by 4 and subtract 3; 17 is the next data value. Using the equations rather than the verbal description gives the same results. If the starting value is $p_1 = 2$. Using $k = 1$ in the equation leads to $p_2 = 4p_1 - 3 = 4 \times 2 - 3 = 5$.

As you can see, there is a lot of information packed into an equation like Eq. (1), and sometimes it takes a bit of effort to unpack all that information. When equations of this type are used, it is very important to state verbally what the variables stand for. For Eq. (1), we should state:

The value c_n is the model value for daily oil consumption n years after 1991.

Please note that there is a difference between c in this statement and the C that was used at the start of the chapter: C_n represents a true data value, while c_n stands for a value that comes from the pattern. The same letter is used because both represent oil consumption, but we use the distinction between upper case (that is, capital) letters and lower case letters to keep separate the actual data and the model. Of course, the whole point is to choose a model that gives a good approximation to the real data, so that c_n and C_n are nearly equal. In fact, comparing c_n and C_n is a good way to see how accurate the model is, and that is one of the reasons to use the two symbols c and C.

In much of this text, we will be concerned mainly with models, and we won't have to use one kind of symbol for the actual data and another for the model values. But when it is necessary to keep these two distinct, you will need to be sensitive to subtle differences in notation. In other courses you may see other devices used to get at the same idea. Sometimes, when a variable like x is used to represent actual data, the symbol $\hat{x}$ will stand for the model value.

The simple pattern expressed in Eq. (1) is characteristic of a very large class of patterns that will form the core of this course. It is called a *difference equation*. In the case of Eq. (1), the pattern shows how any term in the sequence can be computed from the preceding term. This is a general feature of difference equations: a difference equation always shows how to compute one term of a sequence from one or more preceding terms. The equation itself can be stated in several different forms. For example, one alternative to Eq. (1) is

$$c_n = c_{n-1} + .3 \tag{2}$$

Compare the two equations. Each says that the term on the left side of the equal sign is equal to the preceding term plus .3. But they use variables in a slightly different way. In Eq. (2), n is the subscript of the new term that is being computed. In Eq. (1) n is the subscript of the old term that is used for the calculation. There is no general reason to prefer one form over the other. You should be comfortable with both, and understand how they each describe the same pattern.

Here is another alternative to Eq. (1):

$$c_{n+1} - c_n = .3 \tag{3}$$

In this equation, the left-hand side represents the difference between two successive terms of the sequence. In fact, this version emphasizes the fact that the difference between two successive terms is *constant*. Pick a pair of terms next to each other in the sequence, and no matter which pair you pick, the difference between the terms always has the same numerical value. This idea is important for two reasons. First, it is this idea that gives us the name *difference equation.* Second, the appearance of constants in a model is of fundamental importance in science and mathematics. Mathematicians refer to such constants as *invariants*; physicists talk about *conservation* principles. But the idea is the same: something that can be depended on not to change. It is the foundation of many kinds of analysis. We will see several kinds of application where the models we use express our belief that something remains constant.

There is one more comment that should be made about the various difference equations that have been displayed here. Return for a minute to Eq. (2). Do you see that it doesn't work for $n = 0$? In words, we can't compute c_0 from the preceding term, because there is no preceding term! In fact, Eq. (2) only makes sense for $n = 1, 2, 3$, etc. In the same way, Eq. (1) only makes sense for $n = 0, 1, 2, 3$, etc. When difference equations are used, it is generally necessary to keep in mind which values are permitted for the subscript variable (in this case n). Notice how many things we need to keep straight in this modeling problem: the real data, the model, the notation, the subscript variable, and the difference equation. To emphasize this even more, all of this information is restated in a compact way below:

C_n represents average daily world oil consumption in millions of barrels n years after 1991. A model for the data is given by the numbers c_n, where $c_0 = C_0$ and for $n > 0$ $c_n = c_{n-1} + .3$.

This kind of information is typical of what is needed to understand the meaning of a difference equation. It provides a context within which the equation is related to a real problem. As you work with difference equation problems, you should try to make a habit of writing out a brief description like this when possible. Usually, you can't write it all down at once, just as we didn't get it all down at once in this chapter. Instead, as you work with a problem, little bits and pieces will show up at different times. When you choose a letter for the original data, you get one piece. When you formulate a pattern as a difference equation, you get another. But at the end, when you are ready to write a description of how you solved the problem, you should begin with a summary statement like that above. This is an organizational tool. Even though it is not the way you figured the problem out, it is a good way to record the problem and its solution for future reference. Then, if you reread the problem solution at a future date, or if someone else reads it, the context for the solution is made clear at the beginning.

Now we have introduced the basic tool for the course: difference equations. To complete this chapter, we will briefly discuss the three types of analysis mentioned in the Introduction: numerical, graphical, and theoretical. To focus the discussion we consider two questions that are typical in using models:

> According to the model, what will the daily consumption be in 1999?

and

> According to the model, when will the daily consumption reach 70 million barrels?

The Numerical Approach

The difference equation Eq. (1) can be used to generate model data for as many years as we like. As shown earlier,

$$c_0 = 66.6,$$
$$c_1 = 66.6 + .3 = 66.9,$$
$$c_2 = 66.9 + .3 = 67.2.$$

We can continue this process to find answers for the two questions. First, to determine the daily consumption in 1999, recall that c_n refers to n years after 1991. Since 1999 is 8 years after 1991, we need the figure for c_8. Just continue the calculations for 6 more steps:

$$c_3 = 67.2 + .3 = 67.5,$$
$$c_4 = 67.5 + .3 = 67.8,$$
$$c_5 = 67.8 + .3 = 68.1,$$
$$c_6 = 68.1 + .3 = 68.4,$$
$$c_7 = 68.4 + .3 = 68.7,$$
$$c_8 = 68.7 + .3 = 69.0.$$

The model predicts a daily consumption of 69.0 million barrels of oil in 1999.

For the second question we can continue the data even further until we see a daily oil consumption of 70 or more:

$$c_9 = 69.0 + .3 = 69.3$$
$$c_{10} = 69.3 + .3 = 69.6$$

$$c_{11} = 69.6 + .3 = 69.9$$

$$c_{12} = 69.9 + .3 = 70.2$$

The model predicts that daily world oil consumption will go above 70 million barrels per year 12 years after 1991, or in 2003.

These calculations are exemplary of the numerical approach to using the model. Notice the pattern of calculations. We keep performing the same operation over and over, each time using the result of the last computation as the starting point for the next. This type of activity is referred to as *recursion*, and sometimes a difference equation is called a recurrence for that reason. The process of computation is sometimes described as computing values of oil consumption recursively. Whatever name you give it, it is a fundamental activity for exploring how a difference equation model behaves. It is also easy to do on a graphing calculator.

The Graphical Approach

In many cases, using diagrams or figures can help us gain additional insights about relationships. In the study of difference equations, one standard kind of diagram is a bar chart. Fig. 2.1 is a bar chart for the real data at the start of the chapter.

This gives some visual impressions of the data, but there are too few numbers to gain much in the way of insight. Next, we will look at a bar chart representation for the model, but this time include many more values. See Fig. 2.2. Here we immediately gain a new insight: in the model, the tops of the bars form a straight line. In later chapters the significance of this observation will be explored further. But one immediate conclusion can be drawn: if we have experience that shows that oil consumption graphs do not usually follow straight lines, the model we are using here will probably not be very accurate. More will be said about this a little later.

A commonly used alternative to the bar chart is called a line graph. Here, we use one point (corresponding to the center of the top of the bar) in place of the entire bar, and

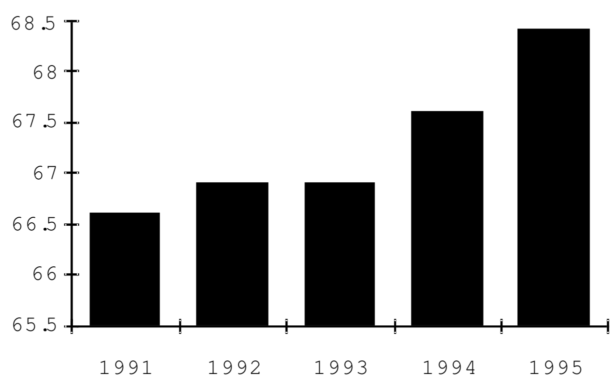


FIGURE 2.1
Bar Graph for Oil Consumption Data

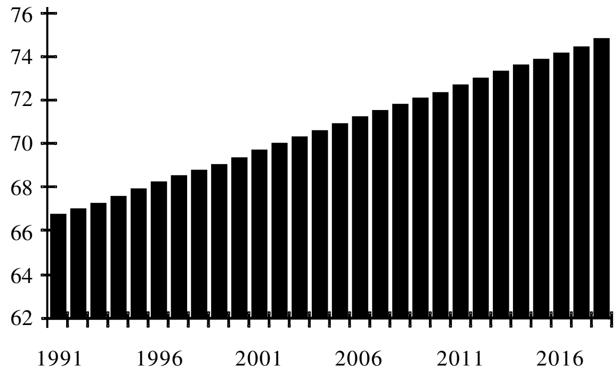


FIGURE **2.2**
Bar Graph for Oil Consumption Model

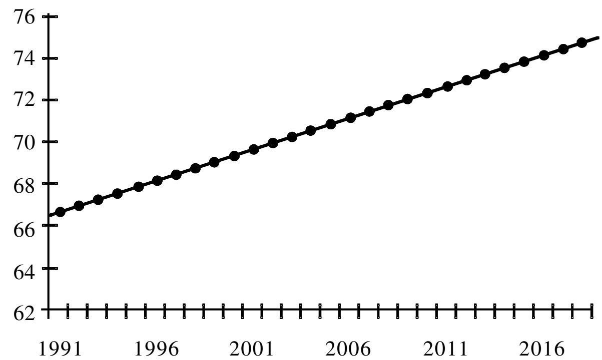


FIGURE **2.3**
Line Graph for Oil Consumption Model

then connect the points with lines. A graph of this type is shown in Fig. 2.3. In the line graph it is even more apparent that the model data fall on a straight line. Although the points are emphasized in this figure, with so many points it is more customary to make them smaller, or to leave them out altogether. In that case the figure would simply appear as one straight line.

Using either the bar or line graph, it is possible to answer again the same questions we considered before. That is, we can read off either chart the daily oil consumption level for 1999, and find out the year when the consumption reaches 70. However, the numerical approach is much more accurate for answering questions of this type. The graphical approach is better suited to identifying trends and visual patterns in the data or model. Because the graphical approach conveys impressions about an entire data set, it can also be very useful for getting a first estimate for a problem. Then, a numerical approach can be used to refine the estimate.

The Theoretical Approach

In the previous sections we have used Eq. (1). This is a recursive equation for the model: in order to find c_n for a particular value of n, we first have to find all of the preceding values. Here is a better way to find values of c_n:

$$c_n = n \cdot .3 + 66.6 \tag{4}$$

For example, to find c_8 we just set $n = 8$

$$c_8 = 8 \cdot .3 + 66.6 = 2.4 + 66.6 = 69.0$$

It is just as easy to compute c_{100}

$$c_{100} = 100 \cdot .3 + 66.6 = 30 + 66.6 = 96.6$$

Eq. (4) is an example of a *functional equation*. It allows us to compute c_n directly from n without first computing all the preceding terms of the sequence. In this situation we say that Eq. (4) gives c_n *as a function of* n, meaning just that c_n can be directly computed as soon as we know n. It is also said in this case that the value of c_n is *determined* by the value of n.

At this point you are probably wondering where Eq. (4) came from. The answer will have to wait until the next chapter, where you will learn how to find a functional equation for difference equations like Eq. (1). In later chapters you will learn how to find functional equations for other kinds of difference equations, as well.

Finding a functional equation is just one aspect of the theoretical method. As another sample, here is how the theoretical approach is used to find when the daily oil consumption reaches a level of 70. In Eq. (4), set c_n to 70 and leave the other side of the equation unchanged:

$$70 = n \cdot 0.3 + 66.6$$

This equation can be solved for n using the methods of algebra:

$$\begin{aligned} 70 - 66.6 &= n \cdot 0.3 \\ 3.4 &= n \cdot 0.3 \\ 3.4/0.3 &= n \\ 11.333\ldots &= n \end{aligned}$$

The final value for n can be found by dividing 3.4 by 0.3 on a calculator (the numerical approach) or by multiplying the top and bottom of the fraction by 10 to get $34/3$ and converting to a mixed fraction $11\frac{1}{3}$ (the theoretical approach). As is often the case, the numerical approach gives an approximate answer, and the theoretical approach gives an exact answer. Either way, in this problem we see that daily oil consumption reaches 70 more than 11 years and less than 12 years after 1991. So the first year that it exceeds 70 will be 12 years after 1991, or 2003.

These examples of the theoretical approach reveal something of its power, and illustrate how one theoretical result can lead to another. There is another advantage to the theoretical approach—it has greater generality than either the numerical or the graphical approach. As an example, instead of asking when the consumption will reach 70, what if we ask when it will reach 80? Or 90? The theoretical approach allows us to answer all of these questions at once. Using methods of algebra very much like those above, it is possible to find a new equation. Without going into all the steps, the new equation turns out to be

$$n = (c_n - 66.6)/0.3 \tag{5}$$

Now to find when a value of 80 is reached, just substitute 80 for c_n

$$n = (80 - 66.6)/0.3 = 13.4/0.3 = 44.666\cdots$$

showing that it will be 45 years after 1991. And any other number than 80 could be handled as easily. When we say that the theoretical method is general, it means that it can be adapted to solve many closely related problems at once.

It should also be pointed out that what has been done here is to express n as a function of c_n. Notice that in Eq. (5) the n is all by itself on one side of the equation. That allows the value of n to be directly calculated using the value of c_n. In words, we compute the number of years past 1991, given the level of oil consumption. This reverses the situation of Eq. (4) in which c_n is all by itself on one side of the equation. Using that equation, the oil consumption level is computed directly from the number of years past 1991. This illustrates the idea of *inverting* a function.[3] It is an idea we shall meet often. We will first have an equation with one variable standing alone, and another variable mixed up with numbers and operations. Then we will find a new equation in which it is the second variable that stands alone. In the next chapter you will see how to do this for equations like Eq. (4). That will show the steps that led to Eq. (5).

Back to Reality

The last several sections have focused mainly on the model defined in Eq. (1). But how good a model is it? Does it match the real data we started with? As usual, we can study this question using each of the three methods. Numerically, we can compare the true data C_n with the model values c_n by computing the differences $C_n - c_n$. This leads to the data in Table 2.2. The first two values agree exactly because we designed the model that way. After that, the errors grow steadily. Still, it is not possible to say in the abstract whether this model is good or bad. It depends on the purpose for which it is to be used. This may be a good enough model for some purposes. The point of the table is simply to illustrate a numerical approach to studying how well the model agrees with the data.

A graphical approach to the same question might be to graph both the original data and the model results on one graph. That is shown in Fig. 2.4. Again, it is visually

[3] It may be helpful to think of *inverting* as another way to say *reversing*. In Eq. (4) we start with a year and use it to figure out an oil consumption level. Eq. (5) allows us to reverse this process—we can start with a consumption level and figure out a year. It is this reversal that you should think of in connection with inverting a function.

Year	1991	1992	1993	1994	1995
n	0	1	2	3	4
C_n	66.6	66.9	66.9	67.6	68.4
c_n	66.6	66.9	67.2	67.5	67.8
Difference	0.0	0.0	−0.3	0.1	0.6

TABLE 2.2
Data and Model Comparison

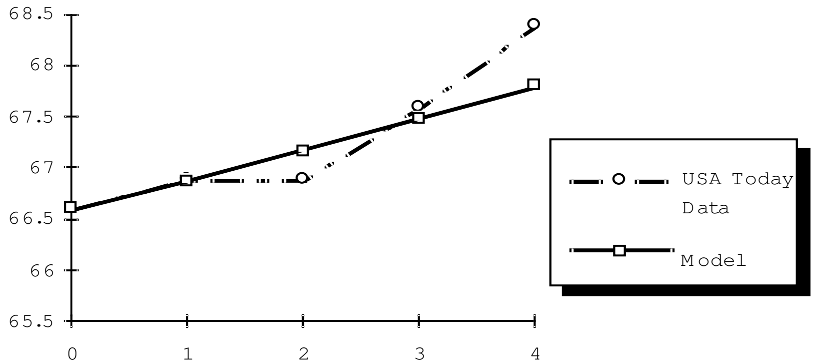


FIGURE 2.4
Line Graphs for Oil Consumption Data and Model

clear that the two graphs seem to be separating as they are followed to the right, but the significance of the differences depends on the problem context.

The theoretical approach might focus on a slightly different question: how could the model be modified to fit the data better? For example, what if we use a different amount in place of 0.3? If the model says that $c_{n+1} = c_n + .25$, is that better? This type of exploration leads naturally to the question of choosing the *best* value to replace 0.3 with. There is a theoretical method for finding the answer to that question, but we won't go into that here. However, most of this course is designed around questions of that type. The models we use are grouped together based on *forms* of difference equations. For example, the following equations all have the same form:

$$c_{n+1} = c_n + .3$$

$$c_{n+1} = c_n + .4$$

$$c_{n+1} = c_n + .5$$

We can condense them into a single line by using a variable for the number at the end of each equation:

$$c_{n+1} = c_n + d$$

This is another kind of shorthand notation. When you read

Let us look at equations that have the form

$$c_{n+1} = c_n + d,$$

you should be thinking:

> *For example, the* d *might be* 4*; that would give the equation* $c_{n+1} = c_n + 4$*. Or,* d *might be* 1.5*, and then the equation would be* $c_{n+1} = c_n + 1.5$*. What if* d *is a negative number? Say* $d = -2$*, then* $c_{n+1} = c_n - 2$*.*

In this way you can get an idea of what is meant by the compact statement $c_{n+1} = c_n + d$.

The equation

$$c_{n+1} = c_n + d$$

defines a whole family of closely related difference equations. They have similar properties (for example, they all have graphs that form straight lines), and similar methods of analysis work for all of them. In this situation, the variable d is referred to as a *parameter*. In any actual model, it will have a specific constant numerical value. When we want to talk about properties shared by all of the models in the family, we represent the parameter with a variable instead of any specific number.

Here is a similar example. In the equation

$$c_{n+1} = r \cdot c_n$$

r is a parameter. The equation gives the form of a whole family of difference equations. Some of the equations in this family are

$$c_{n+1} = 2c_n$$

$$c_{n+1} = 1.78c_n$$

$$c_{n+1} = .1c_n$$

We will study several different families of models. Each family is identified by a particular type of difference equation with one or more parameters. When we try to apply a difference equation of this type to a real problem, one common problem is choosing the best possible values for the parameters. Another important question is how to recognize which kind of model will work well for which kind of problem. Over the course you will be introduced to several kinds of models that work well in many problem areas. You will learn what properties of a model type make it especially suitable for problems of a certain type. In the process, you will see how applied mathematicians develop models for real problems.

As a preview of some of these ideas, the next section introduces a few more difference equations.

Sample Difference Equations

The difference equation for the oil consumption model has the form

$$c_{n+1} = c_n + .3$$

It is part of a family of closely related difference equations, all of the form

$$c_{n+1} = c_n + d$$

where d is a parameter. Models featuring this kind of difference equation exhibit what is called arithmetic or linear growth. These equations will be studied in depth in the next chapter.

As a second example, consider a bank account that collects interest. The interest is paid once per month and amounts to one half of a percent of the balance just before the interest payment is made. If the amount after n interest payments is a_n, what will the amount be after one more payment? As an example, suppose the balance is \$27. Then the interest paid is one half percent of \$27. This is computed as $.005 \times 27$. After the interest is paid, the new balance will be $27 + .005 \times 27$. More generally, if the balance is represented by the variable b, the interest payment will be $.005 \times b$ and the new balance will be $b + .005 \times b$. This leads to the difference equation

$$a_{n+1} = a_n + .005 \times a_n$$

As we shall see later, this is an example of geometric growth.

A famous example of a difference equation is

$$f_{n+1} = f_n + f_{n-1}$$

In words, each new f is found by adding the two preceding f's. If the first two numbers in the sequence are 1 and 1, the next will be 2, then 3, then 5, and so on. This pattern was described by the mathematician Leonardo Fibonacci in 1202, and the sequence is named in his honor the Fibonacci numbers. The original context discussed by Fibonacci was a puzzle involving rabbits that reproduce according to the following pattern. Each month a pair of rabbits produces one pair of children, a male and a female. Each new pair produces children of its own, one pair per month, starting 2 months after being born. The puzzle is this: if you start with one pair of new children, how many rabbits will there be after 10 months? After 15 months? After any number of months? Fibonacci's solution was to show that, if f_n is the number of rabbits after n months, then $f_0 = f_1 = 1$ and for $n > 0$

$$f_{n+1} = f_n + f_{n-1}$$

Although Fibonacci became interested in the pattern in connection with a puzzle, mathematicians have found that the Fibonacci numbers reveal a large number of interesting patterns. What is more, these numbers arise naturally in biology. As one example, we consider the problem of counting ancestors for a common bee[4]. Now, male bees are produced asexually by the queen, so each male has a mother but no father. Each queen bee, by contrast, has both a father and a mother. Starting with a queen, we can chart the ancestors in a diagram as in Fig. 2.5. In the diagram, each black circle is a male, and each white circle is a female. The circle at the top of the diagram is the queen bee

[4] This example is taken from *Concrete Mathematics,* by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. Addison Wesley, Reading, Mass., 1989. See page 277.

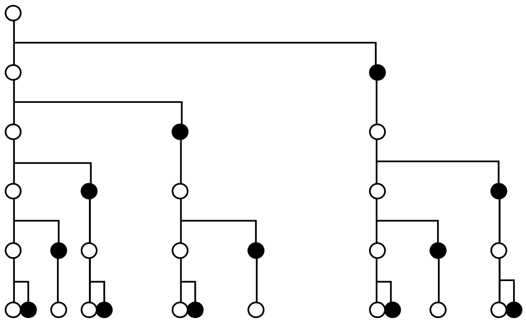


***FIGURE* 2.5**
Ancestry for a Queen Bee

we are interested in. The white circle directly below is the mother of the queen, while the black circle off to the right is the father of the queen. In fact, for any white circle, there is a white circle directly below, indicating the mother, and a black off to the side for the father. In contrast, for black circles, there is only one parent, shown by a white circle directly below. We can continue this diagram for as many generations of bees as we wish. Observe that the queen has 2 parents, 3 grandparents, 5 great-grandparents, 8 great-great-grandparents, and so on. It is this pattern of numbers, 2, 3, 5, 8, etc., that we are interested in. To be specific, let a_n be the number of ancestors that the queen has in the nth preceding generation. Then a_1, the number of ancestors in the first preceding generation (that is the parents), is 2, $a_2 = 3$, $a_3 = 5$, and so on. The question is, how can we compute a_n for any n? The answer is that the a's are just the Fibonacci numbers.

An explanation for this result can be found in the diagram. In Fig. 2.6 parts of the ancestry chart are highlighted to show the ancestors of the queen's mother and paternal grandmother. There are two points to consider. First, since the mother and grandmother are also queen bees, their ancestry charts look just like the one we started with. Second,

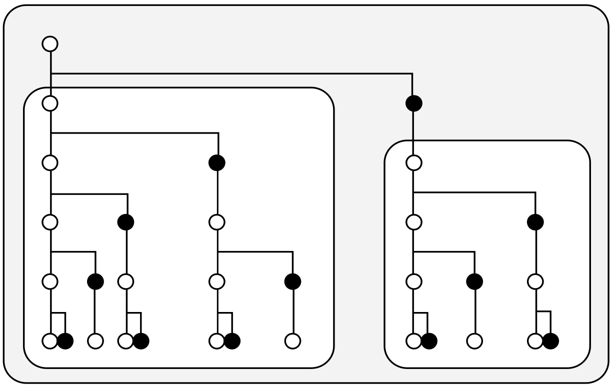


***FIGURE* 2.6**
Ancestry of Mother and Grandmother Bees

we can find all of the ancestors for our queen by looking at the ancestors of the mother and grandmother. For example, the number of ancestors for our queen 4 generations back can be found by adding the mother's ancestors 3 generations back to the grandmother's ancestors 2 generations back. How many ancestors does the mother have 3 generations back? The answer is a_3, because all queens have the same number of ancestors at each generation. How many ancestors does the grandmother have 2 generations back? a_2. So, we can see that $a_4 = a_3 + a_2$. The same argument works for any number of generations. This shows that $a_{n+1} = a_n + a_{n-1}$, which is the same difference equation as the one found by Fibonacci. This example illustrates how difference equations can be found by examining patterns.

Sometimes difference equations are just made up without any connection to an application. Here is one with a funny pattern:

$$a_{n+1} = a_n / a_{n-1}$$

Pick a couple of numbers to start the pattern, say $a_1 = 4$ and $a_2 = 7$, and work out the next 8 or 9 numbers in the pattern, following the difference equation, and you will see what the pattern is.

As a final example, we will develop a difference equation that can be used to model how drugs are metabolized from the body, or how impurities are washed out of a lake. To simplify the explanation, we will look here at a water tank that holds 10 gallons. Suppose someone accidentally spills a pollutant into the tank. For the sake of discussion, we will imagine that 4 pounds of salt fall into the tank, and completely dissolve. In an attempt to purify the water, we drain the tank and refill it with pure water. Unfortunately, not all of the water drains out. The drain is located in such a way that one gallon is left at the bottom and doesn't go out of the drain. So when we put in fresh water, we only put in 9 gallons, and that mixes with the one salty gallon that was left in the tank. If we repeat the process several times, the salt is reduced further and further. The difference equation will describe how much salt is left in the tank after each draining and refilling.

Initially, there are 4 pounds of salt in the water. When we remove 9 gallons from the tank, that is $9/10$ of the total. The remaining one gallon is one-tenth of the total, so it contains one-tenth of the salt. That is $.4$ pounds. Now when we refill the tank and the water mixes up, there are $.4$ pounds of salt dissolved in the 10 gallons. We remove 9 gallons, leaving behind one-tenth of the water, and with it, one-tenth of the $.4$ pounds of salt. That is, we leave $.04$ pounds of salt. The same pattern always applies. If there are s pounds of salt dissolved in the ten gallons in the tank, when we drain the tank, we leave behind one-tenth of the water, and so one-tenth of the salt. That means that there will be $.1s$ pounds of salt. After refilling the tank with pure water, the same $.1s$ pounds of salt remain. Starting with s pounds of salt, if we drain and refill the tank once, there will then be $.1s$ pounds of salt. This gives the difference equation

$$s_{n+1} = .1s_n$$

where s_n is the amount of salt after draining and refilling the tank n times.

This type of difference equation arises in many many applications. If there is a lake with some pollution in it, we can think of the amount of water flowing into the lake and out of the lake each day as being like draining part of the tank and refilling it. The difference equation for the pollution left in the lake will be of the same form as the one for the tank. For medical applications, the water tank concept provides a good first approximation to the way the body removes a drug from the blood stream. In each hour, a certain fraction of the blood is purified, leaving some of the drug behind. If the body purifies about nine-tenths of the blood, then about one-tenth of the drug remains in the blood at the end of the hour. The difference equation for this situation is

$$d_{n+1} = .1d_n$$

where d_n is the amount of drug in the body after n hours.

A modification of this equation applies when more medicine is added on a regular basis. If the patient takes a pill every hour, and adds .5 milligrams of the medicine, then the difference equation becomes

$$d_{n+1} = .1d_n + .5$$

This shows that each hour all but $1/10$ of the drug already in the body is removed, and that .5 new milligrams of the drug are added. This model can be used to understand how medicines can build up in the body. It also has uses in setting drug doses.

Summary

The main point of this chapter has been to introduce the idea of difference equations, and the terminology and notation that are used with them. In addition, the basic ideas of developing and applying mathematical models involving difference equations were discussed. A particular example involving oil consumption data was referred to throughout the chapter to illustrate the ideas that were presented. Numerical, graphical, and theoretical techniques were illustrated as approaches to two typical questions that arise in modeling. One of the fundamental questions in this regard concerns how well a model fits real data. Often a model can be improved by making a slight modification to one or more parameters. This leads to the idea of studying families of models with very similar difference equations, an approach that will be followed throughout the course. Several different kinds of difference equations were given as examples at the end of the Chapter. These show some of the kinds of patterns that difference equations can lead to.

Exercises

Reading Comprehension

1. Explain what is meant by each of the following terms, introduced in the reading: sequence of numbers, discrete data, subscript, terms, difference equation, invariant, recursion.

2. What is the advantage of using a variable for a subscript? Why not just use numerical subscripts?

3. Explain why the following difference equations all express the same pattern: $a_n = a_{n-1}/3$, $a_{n+1} = a_n/3$, $a_n - a_{n-1}/3 = 0$.
4. For each part of this problem, write a mathematical equation that expresses the written statement:
   a. In a set of data, the 4th number is equal to the 3rd number plus .65.
   b. In a set of data, the 5th number is equal to .85 times the 4th number.
   c. In a set of data, each number is 2.38 more than the preceding number.
   d. In a set of data, each number is found by multiplying the two preceding numbers.
   e. In a set of data, each number is 20% less than the preceding number.
5. The following is given: $a_{n+1} = (a_n - 2)^2$ and $a_1 = 5$. A student is asked to figure out a_4, and makes the following computations:

$$\begin{array}{rcccccr} a_2 &=& (a_1-2)^2 &=& (5-2)^2 &=& 9 \\ a_3 &=& (a_2-2)^2 &=& (9-2)^2 &=& 49 \\ a_4 &=& (a_3-2)^2 &=& (49-2)^2 &=& 2,209 \end{array}$$

   Is this student using a difference equation or a functional equation? Is the method recursive? Explain.
6. The following is given: $a_n = 3n^2 - 2n + 5$. A student is asked to figure out a_4 and a_7, and makes the following computations:

$$\begin{array}{rcccccr} a_4 &=& 3\cdot 4^2 - 2\cdot 4 + 5 &=& 48-8+5 &=& 45 \\ a_7 &=& 3\cdot 7^2 - 2\cdot 7 + 5 &=& 147-14+5 &=& 138 \end{array}$$

   Is this student using a difference equation or a functional equation? Is the method recursive? Explain.
7. Explain what a parameter is, and give an example showing how a parameter is used.

Mathematical Skills

1. For each part of this problem, a difference equation and one or more initial values are given. Use the difference equation to compute additional terms as specified.
   a. If $a_0 = 1,000$ and $a_{n+1} = a_n + 0.006a_n$, compute a_1, a_2, and a_3.
   b. If $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$, compute f_3, f_4, and f_5.
   c. If $a_1 = 1$ and $a_{n+1} = a_n + 2$, compute a_1, through a_{10}.
2. For each part of this problem a sequence of numbers is shown. See whether you can find a pattern for each sequence. Describe the pattern in words, and, if possible, with either a difference equation or a functional equation.
   a. $2, 4, 6, 8, 10, \ldots$
   b. $1, 3, 5, 7, 9, \ldots$
   c. $1, 4, 9, 16, 25, \ldots$
   d. $1, 3, 6, 10, 15, \ldots$
   e. $1, 2, 4, 8, 16, 32, \ldots$
   f. $1, 4, 5, 9, 14, 23, \ldots$
   g. $1, 2, 6, 24, 120, 720, \ldots$

3. For the difference equation $b_{n+1} = b_n/b_{n-1}$ there is a simple pattern starting with any two initial values. Make up your own values of b_1 and b_2 and use the difference equation to figure out the next several b's until you can see the pattern. It might take 6 or more steps for you to recognize the pattern. Then pick different starting values of b_1 and b_2 and repeat the process. Do this for several more starting choices of b_1 and b_2. Finally, write a verbal description that fits all of these patterns.

4. Several familiar patterns of numbers are described by difference equations. For each equation below, work out enough terms of the sequence to see what pattern is generated.

   a. $a_{n+1} = a_n + 2$, $a_1 = 2$
   b. $a_{n+1} = a_n + 5$, $a_1 = 5$
   c. $a_{n+1} = a_n + 10$, $a_1 = 10$
   d. $a_{n+1} = -a_n$, $a_0 = 1$
   e. $a_{n+1} = 2a_n$, $a_1 = 2$

5. Functional equations are given below for each of the patterns in the preceding problem. Match each functional equation with the correct difference equation.

   a. $a_n = (-1)^n$
   b. $a_n = 5n$
   c. $a_n = (2)^n$
   d. $a_n = 2n$
   e. $a_n = 10n$

6. For each of the difference equations in the two preceding problems, figure out a_{10} using (a) the difference equation and (b) the functional equation. Which is easier to use? Which would you want to use to find a_{1000}?

7. In the discussion of the world oil consumption example, the notation c_n was used to represent the oil consumption n years after 1991. This problem is concerned with finding the correct n for a given year. For example, given the year 1993, which c_n gives the oil consumption? The answer is c_2 because with $n = 2$, this gives the oil consumption 2 years after 1991. Which c_n gives the oil consumption for each of the following years?

   a. 1992   b. 1995   c. 2000
   d. 2050   e. 1991   f. 1984

8. Referring to the preceding problem, this problem is concerned with finding the year that corresponds to a given n. For example, c_8 represents oil consumption for 1999, because 1999 is $n = 8$ years after 1991. Tell which year each of the following refers to:

   a. c_5 b. c_{25} c. c_0 d. c_{-2} e. c_{-10}

Problems in Context. This set of problems refers to the carbon dioxide data discussed in Chapter 1. For that discussion, an equation was expressed in the form

$$\text{Carbon Dioxide Level} = 1.366 \cdot \text{Year} - 2365 \tag{6}$$

Using this equation, you will generate data and then follow many of the steps of analysis used in this chapter's discussion of oil consumption.

1. The first data point in the carbon dioxide example was for the year 1965. Using c_n to stand for the carbon dioxide level n years after 1965, find c_n for $n = 1$ through 10. [For each n, figure out the correct year, and use Eq. (6).]
2. Find the change from each year to the next. That is, calculate $c_2 - c_1$, $c_3 - c_2$, $c_4 - c_3$, etc. What pattern do you observe?
3. Express the pattern from the preceding problem as a difference equation. That is, write it in the form $c_{n+1} = ?$, but in place of the question mark write down something that involves c_n.
4. Write out a paragraph like the one in the box on page 17 describing c_n and C_n, but this time you should be writing about the carbon dioxide data. For C_0 use 319.19.
5. Make a bar graph and a line graph for the data you worked out in question 1.
6. Use a graphical and numerical approach to estimate the year when the carbon dioxide level will reach 350.
7. Compare the data in Table 1.1 with the values you have been working with in this problem. Make a table similar to Table 2.2.

   For the next three problems some algebra is needed. These problems are recommended for students who have studied algebra previously and feel comfortable with the subject. The correct methods will be explained in the next chapter.
8. Can you determine the functional equation for c_n? It should be of the form $c_n = ?$, but in place of the question mark there should be something that involves n.
9. Use the equation from the preceding problem to express n as a function of c_n. This time the form should be $n = ?$, and the question mark should be something involving c_n.
10. Use the equation in the preceding problem to find the year when the carbon dioxide level will reach 350. Compare this to your answer from problem 6.

Solutions to Selected Exercises

Reading Comprehension

4. a. $a_4 = a_3 + .65$
   b. $a_5 = .85a_4$
   c. $a_{n+1} = 2.38 + a_n$
   d. $a_{n+1} = a_n \cdot a_{n-1}$
   e. $a_{n+1} = a_n - .20 \cdot a_n$
5. This is a difference equation. The student uses a_1 to get a_2, then uses a_2 to get a_3, then a_3 to get a_4. That is a recursive method.
6. This is a functional equation. To find a_4, the student changes the n to a 4 and computes a_4 directly, without using any of the other a_n values. Similarly with a_7. This is not a recursive process because the result of the first step is not used in the second step.

Mathematical Skills

1. a. $a_1 = a_0 + .006a_0 = 1,000 + 6 = 1,006$. $a_2 = a_1 + .006a_1 = 1,006 + 6.036 = 1,012.036$

b. Use $n = 1$ so that $f_{n+2} = f_{1+2} = f_3$. This gives $f_3 = f_{1+2} = f_1 + f_{1+1} = f_1 + f_2 = 1 + 1 = 2$. Similarly, $f_4 = f_3 + f_2 = 2 + 1 = 3$.

2. a. Each number is 2 more than the preceding number. A difference equation is $a_{n+1} = a_n + 2$. A functional equation is $a_n = 2n$. How do you find that? You have to look for a pattern of the right type. For a difference equation, the pattern tells how to go from a_1 to a_2, then from a_2 to a_3, and so on. In this problem, the question is how to get from 2 to 4, then from 4 to 6, then from 6 to 8 and so on. The answer is *add* 2 *each time*, and that gives the difference equation. For the functional equation the idea is different. This time we have to get from 1 to a_1, from 2 to a_2, from 3 to a_3, and so on. So for this particular problem I need to explain how to get from 1 to 2, from 2 to 4, from 3 to 6, and so on. Once you understand the question, you just have to try to think of the right pattern.

b. Again, the pattern is that each number is 2 more than the one before it. The difference equation is the same as before: $a_{n+1} = a_n + 2$. Of course, there are different starting values in this problem and the preceding one, so, although the difference equation is the same for both, the number patterns are different. That means the functional equations will be different. Remember that we need to go from 1 to a_1 from 2 to a_2, and so on. The pattern we are after might be symbolized this way: $1 \to 1$, $2 \to 3$, $3 \to 5$, $4 \to 7$. The pattern is: double the first number and subtract 1. The equation is $a_n = 2n - 1$.

c. This time the functional equation might be more obvious than the difference equation. Did you notice that $1 = 1 \times 1$, $4 = 2 \times 2$, $9 = 3 \times 3$, and so on? The functional equation is $a_n = n \times n$. The difference equation is more obscure. You might have noticed how the odd numbers show up in the pattern. Start at 1, add **3** to get 4, add **5** to get 9, add **7** to get 16. Each time you go to the next number, you add the next odd number. The difference equation would be something like this: $a_{n+1} = a_n +$ the next odd number. But what is the next odd number? The answer is closely related to the functional equation from the preceding problem. The difference equation can be written $a_{n+1} = a_n + (2n + 1)$ This is admittedly pretty obscure. It shows that some simple patterns are not easy to express using difference equations.

d. These numbers follow a different kind of pattern: 1, $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, and so on. A difference equation for this pattern can be written in the form $a_n = a_{n-1} + n$ with $a_1 = 1$. There is a functional equation but it is not obvious. In Chapter 5 you will learn how to find functional equations for this type of pattern.

e. Here, each number is twice the preceding number. The difference equation can be written either as $a_{n+1} = 2a_n$ or $a_{n+1} = a_n + a_n$. The functional equation can be written $a_n = 2^n$ with $a_0 = 1$. We will study this kind of pattern in Chapter 9.

f. This is a pattern like the Fibonacci numbers—each number is the sum of the two preceding numbers. The difference equation can be written $a_{n+1} = a_n + a_{n-1}$. The first two numbers, 1 and 4, were not picked according to any pattern. But all the numbers after the 4 follow the pattern of the difference equation. There is a functional equation for this pattern but it is too complicated to present here.

g. The pattern here is $1, 1 \cdot 2, 1 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6, \ldots$ This is most easily represented by the difference equation $a_n = a_{n-1} \cdot n$ with $a_1 = 1$.

3. If you start with $b_1 = 3$ and $b_2 = 5$, the next number will be $b_3 = b_2/b_1 = 5/3$; then $b_4 = b_3/b_2 = (5/3)/5 = 1/3$; then $b_5 = (1/3)/(5/3) = 1/5$; $b_6 = (1/5)/(1/3) = 3/5$; $b_7 = (3/5)/(1/5) = 3$; and $b_8 = 3/(3/5) = 5$. From that point on, the pattern repeats: 3, 5, 5/3, 1/3, 1/5, 3/5. If the first two numbers are a and b, then the pattern that will occur is a, b, b/a, $1/a$, $1/b$, a/b repeated over and over.

4. a. The even numbers
   b. Counting by 5s
   c. Counting by 10s
   d. Switching back and forth between 1 and -1
   e. Powers of 2: 2^1, 2^2, 2^3, ...

5. a. matches d
   b. matches b
   c. matches e
   d. matches a
   e. matches c

7. $1992 \to c_1$, $1995 \to c_4$, $2000 \to c_9$, $1991 \to c_0$, and $1984 \to c_{-7}$.

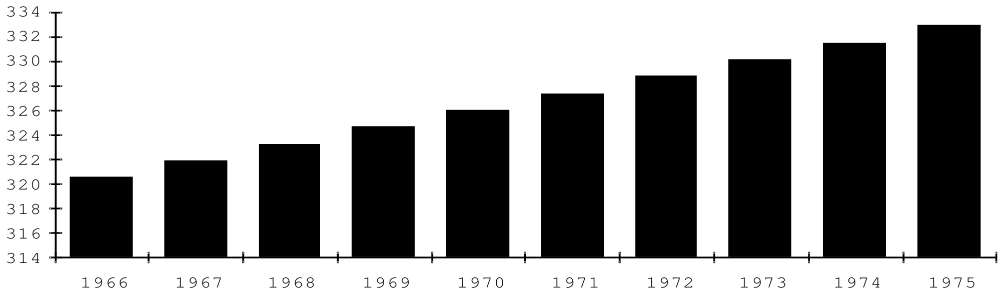
8. $c_5 \to 1996$, $c_{25} \to 2016$, $c_0 \to 1991$, $c_{-2} \to 1989$

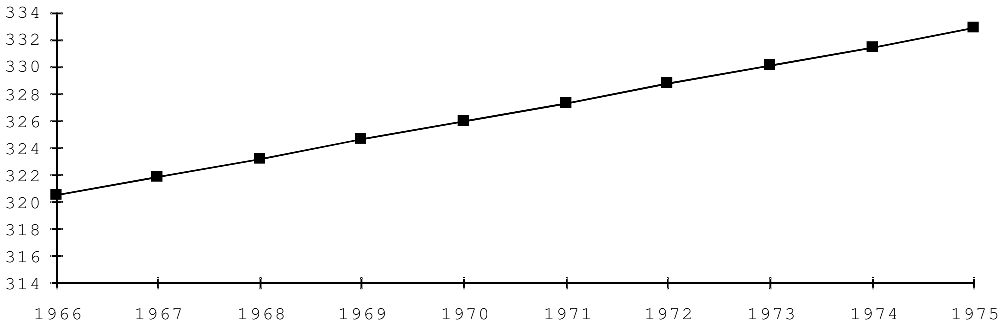
Problems in Context

1. For c_1 the year is 1966. The carbon dioxide level from the equation is $1.366 \cdot 1966 - 2365 = 320.556$. The values for the first 10 c's are given below:

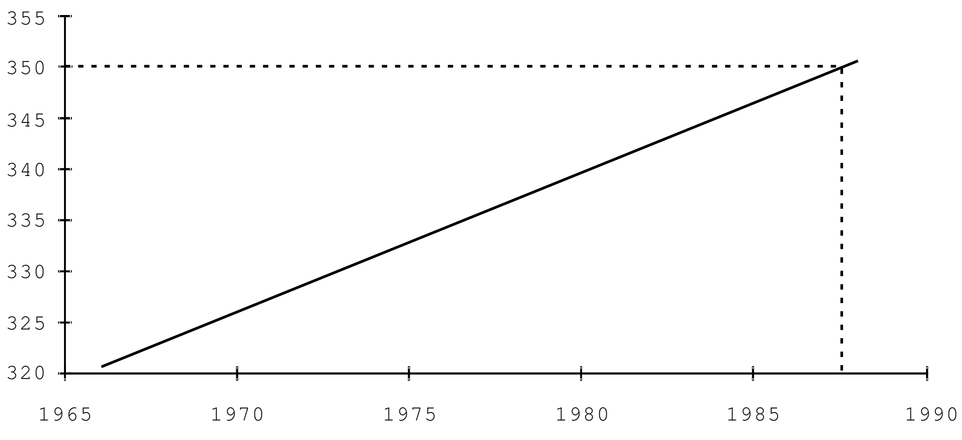
n	1	2	3	4	5
c_n	320.556	321.922	323.288	324.654	326.020
n	6	7	8	9	10
c_n	327.386	328.752	330.118	331.484	332.850

2. Each year the difference is 1.366.
3. $c_{n+1} = c_n + 1.366$.
4. C_n represents the amount of carbon dioxide in the atmosphere, in parts per million, n years after 1965. A model for the data is given by the numbers c_n, where $c_0 = 319.19$ and for $n > 0$ $c_n = c_{n-1} + .366$.
5. The graphs are shown below:





6. In the graph below, the line graph has been extended to show where a carbon dioxide level of 350 will be. It appears that level will be reached in about 1988. Using the equation and the year 1988 the carbon dioxide level is found to be $1988 \cdot 1.366 - 2365 = 350.6$. That is pretty close, but a little too high. Putting in 1987.5 gives $1987.5 \cdot 1.366 - 2365 = 349.9$, which is closer to the exact figure.



7. The data from Table 1.1 only include four years. For each of those years the model value from the equation and the original data value are included in the table below. The errors are all less than 1.2, which is not very significant for data values on the order of 330.

Year	1965	1970	1980	1988
n	0	5	15	23
C_n	319.9	325.3	338.5	351.3
c_n	319.19	326.02	339.68	350.61
Difference	0.71	−0.72	−1.18	0.69

8. $c_n = 1.366n + 319.19$
9. $n = (c_n - 319.19)/1.366$
10. $n = (350 - 319.19)/1.366 = 22.555$ is the number of years after 1965. This gives the year as 1987.555, which is pretty close to the approximate answer 1987.5 found earlier.