3

Arithmetic Growth

In the last chapter the following difference equation was considered as a model for daily world oil consumption year by year:

$$c_{n+1} = c_n + .3 \tag{1}$$

In this chapter you will see that this is just one example of *arithmetic growth*. Additional examples of this kind of model will be presented, and a general procedure for finding a functional equation will be described. The functional equation is more convenient than the difference equation for answering questions about the behavior of the model.

The Arithmetic Growth Difference Equation

The general idea of arithmetic growth can be stated verbally as follows:

> *Growth of a variable concerns the way the variable changes over time. Under the assumption of Arithmetic Growth, equal periods of time result in equal increases of the variable.*

For example, let the variable represent daily world oil consumption for each year. An arithmetic growth assumption means that in any year the daily oil consumption is expected to increase by the same amount as any other year. If daily oil consumption increases by .3 million barrels from 1991 to 1992, then it must increase by an equal amount in any other year. For the oil consumption model, the assumed increase we first considered was .3. But in general in an arithmetic growth model the increase can be any amount. For this reason, we represent the increase by a parameter, d. This leads to a general difference equation

$$a_{n+1} = a_n + d \tag{2}$$

Here, the letter a is used to emphasize that this equation applies in a wider array of problems than just the oil consumption case where we used the variable c.

Here are some examples. In a study of a flu epidemic, the equation

$$p_{n+1} = p_n + 500$$

might be used to indicate that the number of people who have been infected is going up by 500 per month; the parameter d is 500. The equation

$$f_{n+1} = f_n + 10$$

could be used to represent the fine for an overdue book at the library. The parameter d is 10 and indicates that the amount of the fine increases by 10 cents a day. And we have already seen the equation

$$c_{n+1} = c_n + .3$$

for the oil consumption model. In that case, the parameter d is .3. In all of these examples, the equation has the same form as Eq. (2); however, a particular numerical value appears in place of d, and each equation uses a different variable, p, a, and c.

The assumption of arithmetic growth, as described verbally in the box, leads to a difference equation of the form of Eq. (2). For the equation to make sense, the meaning of a_n should be made clear. For example, in the oil consumption problem, we defined c_n to be the average daily oil consumption n years after 1991. Then, assuming arithmetic growth, c_n must increase by the same amount each year. The amount of that increase is d. Actually, the arithmetic growth assumption implies more than the difference equation. It implies that the oil consumption should increase by the same amount each half year, or each month. This broader idea of arithmetic growth will be taken up in greater detail later. The point of emphasis here is that, if you agree to use an arithmetic growth model, as described verbally, then you are led to the difference equation Eq. (2).

This illustrates an important idea—using a common sense verbal description to devise a difference equation. Here the verbal idea of equal growth in equal periods of time leads to a difference equation of the form shown in Eq. (2). In other chapters we will consider different kinds of verbal descriptions, and the difference equations they inspire.

Although the expression *arithmetic growth* suggests that something is getting bigger, it is sometimes used to describe a variable that is getting smaller over time. In this case, equal periods of time would result in equal decreases in the variable, and the parameter d in Eq. (2) would be a negative number. For example, suppose you are modeling the reserves of a nonrenewable resource, such as the fuel on board a satellite. If the satellite uses up the same amount of fuel each month, the supply of fuel will decrease. This would be a good application for an arithmetic growth model, even though the variable is shrinking rather than growing. Perhaps it would be reasonable to consider this to be an example of negative growth. In any case, we will include arithmetic decay (meaning shrinkage) among our examples of arithmetic growth.

The arithmetic growth model is one of the simplest that can be used. For this reason, it is a good starting point for many kinds of problems. In the next section we will consider three different examples of modeling problems that can be started in this way. Later, we will return to the models with mathematical tools that are more advanced, and which provide better accuracy. But for now, we will be content to study arithmetic growth.

Examples

The first example is population growth. An instance of this is the tracking of school-age populations by school boards. If the population is increasing, additional schools may have to be built. If the population decreases, it may become too expensive to keep all existing schools open, particularly since state and federal funding is often based on the number of students attending school. Over the short term, say for a five-year period, an arithmetic growth model may be accurate enough to make some reasonable predictions about future population figures. It is also a fairly natural model to consider. If you find that the number of elementary school students has increased by about 5,000 for each of the last two or three years, it is easy to imagine that the population might grow in the same way in future years. This would be expressed in a difference equation as follows. First, we need to define a variable for the population size. Let p_n be the number of school-aged children in the district in n years. The difference equation is

$$p_{n+1} = p_n + 5{,}000$$

By itself, this equation does not give us predicted values for future years. We also have to know the starting value, p_0. If the current population size is $p_0 = 100{,}000$ then after one year the model predicts a population of size $p_1 = 105{,}000$.

The second example involves *interest*, as paid on loans and savings accounts. In most real situations, some sort of compound interest is used, and the result is *not* arithmetic growth. However, in a very special simplified kind of interest calculation, arithmetic growth does occur. This is called *simple interest*.

Here is an example of simple interest. Suppose that you borrow \$1,000 from a relative to help pay for your education. You agree to pay 5% interest per year. The interest for one year is 5% of \$1,000, or \$50, so that at the end of a year you owe \$1,050. Unfortunately, you are unable to pay off the loan that year, and another year goes by. So you add another \$50 of interest, bringing the total owed to \$1,100. Each year that goes by increases the amount owed by another \$50. If a_n is the amount owed at the end of n years, then $a_0 = 1{,}000$ and the difference equation is $a_n = a_{n-1} + 50$. This is of the same form as Eq. (2), and shows that simple interest corresponds to arithmetic growth of the amount owed.

The final example concerns a model that is used in business applications. In many situations, the price that is charged for an item has an effect on how many items will be sold, particularly for items that are not necessities. A student may choose to purchase a soda between classes, but she can certainly get along without the soda. If the price is very low, say 10 cents, it is likely that many students will choose to buy a soda. If the price is very high, say five dollars, it is unlikely that many sodas will be sold. Between these extremes, it is a reasonable expectation that the number of sodas that can be sold will decrease as the price increases. In this kind of situation, the number of items that can be sold is called the *demand* for the item. Using this terminology, the preceding remarks can be restated as follows: the demand goes down as the price goes up.

To investigate the relationship between price and demand, sales surveys are often performed. The basic idea of a sales survey is to determine how many items can be sold at various prices. Students in a finite math class conducted a survey of this type for canned soda one semester. They found they could sell 141 cans at a price of 40 cents per can, and for each nickel increase in the price of the soda, the number they could sell was reduced by about 12. Of course, the survey data did not fit this pattern exactly. The pattern is a simplified description of the actual price-and-demand data; it gives a *model* for the effect of price on demand. This model fit the data fairly well for prices from 40 cents up to about a dollar.

We can formulate this pattern as an arithmetic growth model. It is different from previous examples in one important respect. Here, we are not interested in how the demand changes over time. Rather, we consider the way demand changes in response to hypothetical changes in price. In this context, arithmetic growth means that equal changes in the price of a soda result in equal changes in the number that can be sold. If raising the price from 40 to 45 cents results in a loss of 12 sales, then raising the price from 60 to 65 cents will also result in a loss of 12 sales. That is what arithmetic growth means. To put this into the framework of difference equations, we can begin with a base price of 40 cents per can and refer to the expected number of sales at the base price as $s_0 = 141$. Now imagine raising the price a nickel at a time, and estimating from the survey the expected number of sales at each price. Let s_1 be the sales at 45 cents a can, s_2 the number of sales at 50 cents a can, and so on. Then the pattern observed in the data says that each s is 12 lower than the preceding s. In other words, s_1 is 12 less than s_0; s_2 is 12 less than s_1; and so on. Using the kind of careful description used earlier, we can describe the model as follows.

> At a base price of 40 cents, $s_0 = 141$ cans of soda can be sold; s_n is the number of cans of soda that can be sold after raising the price by a nickel n times. A model for this situation is given by the difference equation $s_{n+1} = s_n - 12$.

This is an example of arithmetic growth where the parameter d is negative. The difference equation has the same form as before, and can be analyzed using the same methods that apply to the other examples. This illustrates that arithmetic growth can be used in models that predict how changes in one variable (price) can affect another variable (number of sales), as well as in models where we wish to predict how changes occur over time.

The examples in this section all illustrate the idea of arithmetic growth. Each one is described by a difference equation that is of the form shown in Eq. (2). The next section will discuss numerical, graphical, and theoretical properties of arithmetic growth models.

Numerical, Graphical, and Theoretical Properties

Numerics. Numerically, an arithmetic growth model can be explored easily, once the starting value is specified. In the school population example, the difference equation is $p_n = p_{n-1} + 5{,}000$. If we start with $p_0 = 100{,}000$, then we can compute $p_1 = 100{,}000 + 5{,}000 = 105{,}000$, $p_2 = 105{,}000 + 5{,}000 = 110{,}000$, and so on. However, if the population starts at 65,000, then the successive values will be 70,000, 75,000, etc. Look back at the examples of the preceding section. Can you see the importance of the starting values in each of those examples? It should be clear that a starting value is essential for the use of any difference equation. Once a starting value is given, the rest of the values are easily computed as far ahead as we wish. Without the starting value, the difference equation can't be used to tell us what any of the following values will be. For this reason, choosing the starting value is an important part of developing a model which involves a difference equation. The starting value is also given a special name—it is called the *initial value*.

Graphics. The graphical representation of an arithmetic growth model as either a bar graph or a line graph produces a straight line. For the bar graph, it is the tops of the bars that line up, while for a line graph the individual points for the sequence values all fall on a line. The value of the parameter d is reflected in the steepness of the line. If d is positive, the line slopes up to the right, whereas for negative d the line slopes down to the right. The bigger d is, the more steeply the line slopes. In the next chapter you will learn more specifics about the sloping of the lines. The fact that the graphs always involve straight lines explains why arithmetic growth models are sometimes referred to as *linear* models.

Theory. Do you recall the distinction between functional equations and difference equations discussed in Chapter 2? The difference equation is used *recursively*; we have to use the initial value to find the next value, then use that to find the next, and so on. To find the 100th value in the sequence, we have to compute all of the 99 values that come before it. A functional equation is direct. We can use it to find the 100th data value in a single computation. That is what makes functional equations so useful.

A theoretical analysis allows us to find a functional equation that can be applied in any case of arithmetic growth. To illustrate with an example, the following table shows the first several values of the school population, as described in the example on p. 39. We begin with $p_0 = 50{,}000$.

$$
\begin{aligned}
p_0 &= 50{,}000 \\
p_1 &= p_0 + 5{,}000 \\
p_2 &= p_1 + 5{,}000 = p_0 + 2 \times 5{,}000 \\
p_3 &= p_2 + 5{,}000 = p_0 + 3 \times 5{,}000 \\
p_4 &= p_3 + 5{,}000 = p_0 + 4 \times 5{,}000
\end{aligned}
$$

There is a clear pattern here, which can be condensed into the following equation:

$$p_n = p_0 + 5{,}000n \tag{3}$$

Although the equation can be understood simply as a pattern of data, it also follows in a logical way from an understanding of the difference equation. Remember that p_n is the population after n years. Each year the population goes up by $5{,}000$. Clearly, in n years it will go up by $5{,}000 \times n$. If the starting value is p_0, after n years it will be at $p_0 + 5{,}000 \times n$.

The p_0 that shows up in these equations would actually be a number in any specific case. To apply the equation in the population example, we need to know that $p_0 = 50{,}000$. Then Eq. (3) becomes $p_n = 50{,}000 + 5{,}000n$. But we leave p_0 in the general form of the equation, as a reminder of the role played by the initial value of the sequence. Notice that the p_0 is being treated here in much the same way that d was treated earlier. It is a parameter. It is thought of as being a specific number in any particular application of the equation, but we realize that it will be different numbers in different applications.

To return to the main point of this discussion, observe that Eq. (3) is a functional equation for p_n. It gives p_n as a function of n, because, as soon as you replace n with a particular number, the equation tells exactly how to compute p_n. You can now compute p_{100} directly using $n = 100$ without first computing the preceding 99 values.

This whole line of reasoning applies for any arithmetic growth model. If the difference equation is $a_{n+1} = a_n + d$, then term number n will be reached after adding d repeatedly n times. Starting from a_0 this results in a value of $a_n = a_0 + dn$. This is the general form for functional equations for all arithmetic growth models. We formulate it as a general principle for future reference.

> If an arithmetic growth model has an initial value of a_0 and obeys the difference equation $a_{n+1} = a_n + d$, then for any $n \geq 0$, $a_n = a_0 + dn$.

This can be applied immediately to get a functional equation for any arithmetic growth model. Just fill in the values of the parameters a_0 (the starting value) and d (the constant amount added at each step of the difference equation). For the simple interest problem, the initial value was $\$1{,}000$ and the amount added each year was $\$50$. The functional equation is then $a_n = 1{,}000 + 50n$. In a similar way, in the demand model the number of sodas that can be sold at an initial price of 40 cents is 141. That number decreases by 12 as you go to each higher price. Therefore, $s_n = 141 - 12n$.

This last example is a little confusing because n is such a strange variable—the number of five cent increases. A much more natural variable to use would be the price itself. In fact, this is a good place to introduce an idea that will be important later: in many applications it is useful to consider difference equations for and relationships between several different variables at the same time. Here, we will illustrate this idea by looking at two variables, the price per can of soda and the number of cans that can be sold. We have already worked out the functional equation for s_n, the number of cans sold after n increases in price. Now let us look at the price itself. Let p_n be the price after n nickel increases. Since the starting price is 40 cents, we have $p_0 = 40$. Then, with each nickel increase, the price goes up by 5. So, the prices follow an arithmetic growth model, and we can conclude that $p_n = 40 + 5n$. Now the model makes a little more sense. For any n, we can compute both the price p_n and the demand s_n. As an example, with $n = 5$ we have a price of $p_5 = 65$, and we can sell $s_5 = 81$ sodas at that price.

The functional equation that goes with an arithmetic growth model is a powerful tool for studying the model. With it, you can easily determine what will happen in the future, or determine when in the future something of interest will happen. For this reason, it is a good idea to figure out the functional equation whenever an arithmetic growth model is used. In fact, once you have determined that an arithmetic growth model is appropriate, there is a routine outline that you should follow. This outline is illustrated in the following example. The statements in **boldface** are the steps of the outline.

Example Problem. A cellular phone company has equipment that can service 100 thousand customers. In 1990 they had 70 thousand customers. Over the last few years the number of customers has been increasing by about 4,500 per year. Assuming that the growth continues at the same pace, what will the situation be in the year 2000? 2010? How long will it be before additional equipment will be needed?

Step 1: Formulate an arithmetic growth model. In this case, the idea that the number of customers goes up each year by the same amount, 4,500, identifies this as an arithmetic growth model. What is it we wish to study? The number of customers. So make up a name for the variable: *Let c_n be the number of customers, in thousands, n years after 1990.* Notice that this statement does three things: it gives a letter for the data values, it indicates that the data will be in units of thousands of customers, and it establishes the initial year or starting year for the model. You should reread the problem and clarify in your own mind why these choices were made. Why is the starting year 1990? Why are units of thousands of customers used?

Step 2: Formulate the arithmetic growth difference equation. Since the number of customers goes up by 4,500 per year, the difference equation should say that each c is 4,500 more than the preceding c. Be careful. Note that 4,500 customers is four and one-half *thousands* or 4.5 thousands of customers. The difference equation is

$$c_{n+1} = c_n + 4.5$$

Step 3: Formulate the initial value and any restrictions on n. The problem says there were 70 thousand customers in 1990. That gives $c_0 = 70$ (Why?) The difference equation makes sense for any $n \geq 0$. (Why?)

Step 4: Draw a graph. The visual portrayal of the model gives you another way to look at things. Sometimes you can recognize mistakes or errors because something about the difference equation or numerical results does not agree with the graph.

Step 5: Formulate the functional equation. The functional equation tells us directly what c_n will be. If the number of customers goes up by 4.5 thousand each year, then after n years it will have increased by $4.5n$. Since it started at 70, that gives $c_n = 70 + 4.5n$. If you prefer working with formulas, use $c_n = c_0 + nd$ where $c_0 = 70$ is the initial number of customers and $d = 4.5$ is the amount of increase each year. The end result is the same: $c_n = 70 + 4.5n$.

Step 6: Answer any specific questions. Restate the questions in terms of the model. What will the situation be in the year 2000? That will be 10 years after 1990, so $n = 10$. Then, $c_{10} = 70 + 4.5 \cdot 10 = 115$ means there will be 115 thousand customers in the year 2000. The year 2010 corresponds to $n = 20$. In 2010 there will be $c_{20} = 70 + 4.5 \cdot 20 = 160$ thousand customers, according to the model. When will there be 100 thousand customers? First we ask, for what n does $c_n = 100$? This problem can be approached graphically and numerically, or we can use algebra. For the latter, use the functional equation for c_n and solve for n:

$$\begin{aligned}
c_n &= 70 + 4.5n \\
100 &= 70 + 4.5n \\
100 - 70 &= 4.5n \\
30 &= 4.5n \\
\frac{30}{4.5} &= n \\
6.6666\cdots &= n
\end{aligned}$$

That means that after 6 and $2/3$ years we expect there to be 100 thousand customers. That will occur during the year 1996.

The exact order of these steps is not critical. What is important is to understand the overall approach of formulating the model in regular English, then as a difference equation, and then using a functional equation. In the exercises, you will again have a chance to work with this kind of problem.

Proportional Reasoning and Continuous Models

The idea of arithmetic growth leads to two additional ideas that are important in modeling. The first is the idea of proportional reasoning. Here is an example:

> *At a price of* 40 *cents a can,* 120 *cans of soda can be sold. If the price is raised to* 55 *cents, then only* 90 *cans will be sold. If the price is set at* 60 *cents per can, how many do you think can be sold?*

Take a minute now to answer this question just using common sense.

Did you predict 80 cans would be sold at 60 cents per can? Most people do, arguing as follows:

> *Raising the price by* 15 *cents caused a reduction of* 30 *in sales. That means that each time you raise the price a nickel you lose* 10 *sales. So, by raising the price from* 55 *to* 60 *we expect to lose another* 10 *sales, giving a result of* 80 *cans. That is, if the price is set at* 60 *cents per can, we expect to sell* 80 *cans.*

This kind of reasoning is very common. It is based on the assumption that two quantities change in such a way that their ratio always remains constant or invariant. In this example, we are assuming that the ratio *(price increase)/(sales loss)* gives the same value when the price increase is 20 cents as it does when the price increase is 15 cents. Proportional reasoning can always be expressed in terms of ratios. The connection of this kind of proportional reasoning to arithmetic growth models is quite direct: an arithmetic growth assumption will always lead to the same results as using proportional reasoning.

So far, we have seen that assuming arithmetic growth leads to a particular kind of difference equation. As mentioned before, the original arithmetic growth assumption says a little bit more than the the difference equation that it leads to. Whereas the difference equation might say that the same growth occurs every year (for example), the principle of arithmetic growth indicates that the same growth must also occur every half-year, every month, every week, and so on. In fact, if we know how much growth occurs in a year, we can actually figure out how much growth occurs in any other period of time.

To get an idea of why this is so, consider again the simple interest model. As developed on p. 39, the amount a student owes for a loan increases by $50 every year. That is what gave us the difference equation. But the arithmetic growth assumption also tells us that there must be a $25 increase for each *half* of a year. After all, the increase in the first half of the year must be the same as in the second half of the year, and the total increase over the year is $50. Using similar arguments, we can figure out the increase for any fraction of a year. This is another instance of proportional reasoning. The end result is this: the functional equation that we derived for arithmetic growth makes sense even when the values of n are not whole numbers. In the interest example, the amount owed after n years was found to be $1{,}000 + 50n$. This result was obtained based on the assumption that n is a whole number. But even for fractional values of n this makes sense. For example, if $n = 3.5$, we find the amount owed is $\$1{,}000 + 50 \times 3.5 = \$1{,}175$, and this is the amount that would be owed after three and one-half years. As so often before, we again encounter an important general principle in an example. We will see frequently the idea that a functional equation derived thinking of n as a whole number still makes sense when n is replaced with fractional values. When a variable (such as n) is thought of as being restricted to whole number values, it is described as a discrete variable. Even if the values are not whole numbers, the variable can be discrete if it is restricted to a particular set of separate values. For example, if we make measurements every tenth of a second, the time values are .1, .2, .3, .4, etc. These are elements of a discrete set. In contrast, when a variable is allowed to take on all possible fractional values (possibly restricted to fall between some given maximum and minimum values), the variable is said to be continuous. One good way to test whether a variable is discrete or continuous is to ask whether each value is followed by a specific next value. If n stands for a whole number, then $n = 8$ is followed by a definite next value $n = 9$. However, if n stands for any fraction between 0 and 1, the value $n = .5$ is not followed by a definite next value. It could be .6, but .51 is also a possibility, as are .501, .5001, .50001, and so on. There is no closest next value. That is the idea of a continuous variable.

This issue came up from considering the functional equation $a_n = 1{,}000 + 50n$, which came out of an analysis in which n was supposed to be a whole number. The functional equation still makes sense if we make n a continuous variable, allowing n to be any fraction. Then, both the amount of time (n) and the amount of money a are continuous variables, linked by the functional equation. In the next several chapters you will study many properties of continuous variables related by simple functional equations.

Beyond Arithmetic Growth: Oil Reserves

The development of a mathematical model is a cumulative process. A model for one phenomenon is used as a basis for modeling a related phenomenon. This process continues, leading to ever more involved models. To illustrate this process, we will refer once again to the oil consumption model.

Given a model for average daily oil consumption, it is a simple matter to proceed to a model for oil reserves. Using 1991 as the baseline year, as before, let us consider the total oil reserves that exist in each succeeding year. According to the International Petroleum Encyclopedia for 1994, the world oil reserves in 1991 amounted to 999.1 billion barrels = 999,100 million barrels. How much oil was left in 1992? In 1993? If you subtract what was consumed in a year from the oil reserves at the start of the year, the result gives the oil reserves at the start of the next year. This leads directly to a new difference equation.

Let us agree to use r_n to represent the oil reserves n years after the start of 1991. This gives r_0 as the oil reserves for 1991, so $r_0 = 999{,}100$. What is r_1, the oil reserves for 1992? Recalling that c_0 is the daily consumption during 1991, observe that over an entire year the world will consume 365 days worth of oil, or $365c_0$. Accordingly, $r_0 - 365c_0$ gives the reserves for the next year, that is, r_1. In equation form, $r_1 = r_0 - 365c_0$ Reasoning similarly, $r_2 = r_1 - 365c_1$, $r_3 = r_2 - 365c_2$, and so on. This pattern gives a difference equation for the reserves:

$$r_{n+1} = r_n - 365c_n$$

Here we see again a situation where it is useful to have two different variables in the model. Combining the reserves model with the consumption model gives us the following, richer, description.

> c_n represents average daily world oil consumption in millions of barrels n years after 1991. r_n represents the total supply of oil available at the start of the year, n years after 1991. Our model includes the following initial values and difference equations:
>
> $$c_0 = 66.6$$
>
> $$r_0 = 999,100$$
>
> $$c_{n+1} = c_n + .3$$
>
> $$r_{n+1} = r_n - 365c_n$$

We can use this model to compute both the annual consumption and the reserves for any year after 1991. If we are mainly interested in the reserves, we can streamline the process a little by using the functional equation $c_n = 66.6 + .3n$. Just replace c_n with $66.6 + .3n$ in the difference equation $r_{n+1} = r_n - 365c_n$, and we obtain a difference equation for r alone

$$r_{n+1} = r_n - 365(66.6 + .3n)$$

So, building on the model for oil *consumption*, we have formulated a difference equation for oil *reserves*.

An obvious question to ask, now, is: *When will the oil run out?* That is, in how many years will the reserves reach 0? In terms of the model, the problem is to find the n for which $r_n = 0$. This is very similar to questions we have considered before. Numerical and graphical methods can be applied, and you will do that in the exercises. A preferred approach would be to come up with a functional equation for the reserves. The methods for deriving this equation are not yet available to you, but the final result turns out to be

$$r_n = 999,100 - 24,254.25n - 54.75n^2$$

You will be asked to verify this for a few small values of n in the exercises, too. In future chapters you will learn how to obtain the functional equation for r_n and how to use it to figure out when the reserves will reach 0.

The point of this example is to illustrate how models can be built on top of other models. From a simplistic model for oil consumption we are led to a more complicated model for oil reserves. A completely reasonable idea for modeling the reserves using the consumption model leads quickly to much more complicated mathematical questions. In a real application, any conclusion about when the oil will run out would carry the following qualification: *This conclusion assumes a linear model for consumption.* By using alternate consumption models we can obtain quite different conclusions. We will explore this example again in future chapters.

Summary

In this chapter we have discussed arithmetic growth. Arithmetic growth models all share many common features, including the form of their difference and functional equations. They all have straight-line graphs, and they all embody proportional thinking. Arithmetic growth led us into the idea of continuous variables. We also saw how one model can be built on another in the example of oil reserves.

Exercises

Reading Comprehension

1. Write a short essay, about one page in length, on the topic of arithmetic growth. Tell what features are shared by all arithmetic growth models, and give details about graphical properties, difference equations, and functional equations of these models. What are the limitations of using arithmetic growth models to approximate real phenomena? How can you tell whether a particular application might be an appropriate place to use an arithmetic growth model?

2. Write brief paragraphs to explain each of the following concepts:*simple interest, initial value, demand, proportional reasoning, continuous variable.* Include examples in your paragraphs as appropriate.

3. Why are arithmetic growth models often referred to as linear models?
4. In the reading the development of mathematical models is described as being cumulative in nature. What is meant by that? Give an example.

Mathematical Skills

1. For each difference equation find the corresponding functional equation:
   a. $a_{n+1} = a_n + 2;\ \ a_0 = 1$
   b. $a_{n+1} = a_n + 2;\ \ a_0 = 5$
   c. $a_{n+1} = a_n + 2;\ \ a_0 = -312$
   d. $b_{n+1} = b_n - 1.3;\ \ b_0 = 100$
   e. $p_n = p_{n-1} + .8;\ \ p_0 = 11.3$
   f. $a_{n+1} = a_n + 2;\ \ a_3 = 12$
2. Find the graph for each part of the preceding problem. It is recommended that you use a graphing calculator or a computer graphing tool for this problem.
3. For each part, a functional equation is given. Find the corresponding difference equation, and the initial value a_0.
   a. $a_n = 15 - 3n$ b. $a_n = 15 + 3n$
   c. $a_n = 20 - 3n$ d. $a_n = 20 + 3n$
   e. $a_n = 20 - 5n$ f. $a_n = 20 + 5n$
   g. $a_n = 20n - 3$ h. $a_n = 20n + 3$
4. For each part of problem 3 find a_8.
5. In part (a) of problem 3 find the n for which $a_n = 3$.
6. In part (b) of problem 3 find the n for which $a_n = 3$.
7. For each part of problem 3 find an equation for n as a function of a_n. [For example, for part a, since $a_n = 15 - 3n$, we have $3n = 15 - a_n$ so $n = 5 - a_n/3$. This equation expresses n as a function of a_n.]

Problems in Context

1. A weather balloon carries a battery-powered radio transmitter which sends weather data back to the ground. When the balloon is sent up, the battery carries a charge of 30 units. It uses up 2.4 units of charge per hour. Let q_n represent the charge on the battery n hours after the balloon is sent up.
   a. Using a numerical method, find q_1, q_2, and q_3.
   b. What is the difference equation for q_n?
   c. What is the functional equation for q_n?
   d. What will the charge be 4 hours after launch?
   e. Find an equation expressing n as a function of q_n.
   f. The radio transmitter cannot continue to work once the charge on the battery falls below 4 units. How many hours will that take?
2. A student borrows $\$5{,}000$ from his aunt. He promises to pay the money back as soon as possible, with simple interest. The interest will be calculated at one half of a

percent per month. Let p_n be the amount of money the student will have to pay the aunt if he makes the payment after n months. For example, if he makes the payment after a year, that is 12 months. At half a percent per month, the interest will be 6 percent in 12 months. Now 6 percent of $\$5,000$ is $0.06 \times 5{,}000 = 300$. So if the loan is paid back after 12 months, the amount that has to be paid is $\$5,300$, the original loan of $\$5,000$ plus the \$300 interest. That is p_{12}. In contrast, p_6 is what the student has to pay if he makes the payment after 6 months.

a. What is p_0? Does that make sense?

b. What is the difference equation for p_n?

c. What is the functional equation for p_n?

d. Use the functional equation to figure out how much must be paid if the payment is made after 18 months.

e. Use the functional equation to find an equation for n as a function of p_n. This is the inverse equation.

f. Use the inverse equation to find how long it will be before the student owes twice the amount she originally borrowed. That is, find n so that $p_n = \$10{,}000$.

3. A scientist studying the spread of a new disease in a small town decides to use an arithmetic growth model. She estimates that 3,700 people have the disease at the start of her study, and that there are 45 new cases each day. Follow steps similar to those in the last two problems to analyze the arithmetic growth model for this epidemic.

4. This is a continuation of the preceding problem. The scientist has also found that about 3 percent of the people who get the disease require treatment with a special medicine. The local hospital had 500 doses of the medicine on hand at the start of the study. According to the model, how long will it be before this medicine is used up?

5. (Continuation of the preceding problems.) Suppose that the small town in the study is isolated—very few people arrive or leave. Given what you know about the way diseases spread, do you think an arithmetic growth model is reasonable? Consider both predictions made over a short period of time, and those over much longer periods of time.

6. Oil Reserves. Review the discussion of the oil reserves model on p. 46. There, the following functional equation was given:

$$r_n = 999{,}100 - 24{,}254.25n - 54.75n^2$$

Use this equation to compute r_n for several different values of n. To check that the answers are correct, also compute r_n using the difference equation $r_{n+1} = r_n - 365(66.6 + .3n)$ that is based on the oil consumption model. The starting value is $r_0 = 999{,}100$. The point of this exercise is to provide you with some evidence that the functional equation above is correct. Later you will see how the equation was obtained.

7. Using the functional equation for r_n in the previous problem, use numerical techniques to study two questions: What will the oil reserves be in the year 2000? And

when will half the oil that was available in 1991 be used up?

8. Consider again the exercises about the epidemic of a disease in a small town. Modify the previous discussion as follows. Suppose that the spread of the disease is increasing, and that the researcher observes 45 new cases on the first day of the study, 50 new cases on the next day of the study, and 55 new cases the day after that. She models the number of new cases per day using an arithmetic growth model. Here, let c_n be the number of new cases of the disease in day n of the study. So $c_1 = 45$, $c_2 = 50$, and so on. Study the way the number of new cases each day increases. Then use numerical methods to figure out how many days the hospital's medicine supply will last.

Group Activity

1. Look through newspapers or magazines for a graph that appears to be nearly a straight line for all or part of the data. Create an arithmetic growth model for this graph, and write a short report on the model. Your report should define what is being modeled, what assumptions are being made, and over what range of data the model is expected to be valid. Also discuss the kinds of predictions that can be made using your model, and why these predictions might be useful.
2. Using the data from your news article, or from the carbon dioxide example (see Fig. 1.1 on page 2), find the best line you can for the data. Do this by eye. Simply put a ruler on the graph and draw a line as close as possible to the data points. What is the difference equation for the points on this line? What is the functional equation?
3. Again using the data from the preceding problem, compute the difference between each data value and the preceding data value. Are these differences fairly constant? If so, compute the average difference and use it to define an arithmetic growth difference equation. Compare the model based on this difference equation with what you found in the preceding questions.

Solutions to Selected Exercises

Mathematical Skills

1. a. $a_n = 1 + 2n$

   c. $a_n = -312 + 2n$

   e. $p_n = 11.3 + .8n$

   f. Starting with $a_3 = 12$, can you find a_2? a_1? a_0? The difference equation tells us that the sequence of a's goes *up* by 2 each time. So if $a_3 = 12$, a_2 must have been 10, $a_1 = 8$, and $a_0 = 6$. Therefore, $a_n = 6 + 2n$. As a check, this gives $a_3 = 6 + 2 \times 3 = 12$, which is what we expected.

3. There are three different ways to approach these problems: (1) use the general form of the functional equation to identify the parameters a_0 and d; (2) use the functional equation to work out a few terms of the sequence so that you can see the pattern of

the difference equation; and (3) use algebra to derive the difference equation. Each method will be illustrated below.

a. Using method 1: compare these equations:

$$a_n = a_0 + nd$$
$$a_n = 15 - 3n$$

The two equations will be the same if $a_0 = 15$ and $d = -3$. So the initial value for this problem is $a_0 = 15$, and the difference equation is $a_{n+1} = a_n - 3$.

c. By method 2: Using the equation $a_n = 20 - 3n$ we can compute several numbers in the sequence. Starting with $n = 0$ the equation gives $a_0 = 20 - 3 \cdot 0 = 20$. Similarly, $a_1 = 20 - 3 \cdot 1 = 17$; $a_2 = 20 - 3 \cdot 2 = 14$; and $a_3 = 20 - 3 \cdot 3 = 11$. So, the pattern is 20, 17, 14, 11, ... It should be clear that the difference equation for this pattern is $a_{n+1} = a_n - 3$, and the value of $a_0 = 20$ was already found.

e. Using method 3: The difference equation begins with a_{n+1} so that is what we use in the functional equation. The result is $a_{n+1} = 20 - 5(n+1) = 20 - 5n - 5$. But $20 - 5n = a_n$, so $a_{n+1} = a_n - 5$. That gives the difference equation. The initial value is given by $a_0 = 20 - 5 \cdot 0 = 20$.

g. Answer: $a_{n+1} = a_n + 20$; $a_0 = -3$.

5. The functional equation is $a_n = 15 - 3n$, and we want $a_n = 3$. So set the left side of the functional equation equal to 3: $3 = 15 - 3n$. You can solve this by trial and error, just guessing different values of n and seeing if they work. Or, use algebra:

$$\begin{aligned} 3 &= 15 - 3n \\ 1 &= 5 - n \\ n + 1 &= 5 \\ n &= 4 \end{aligned}$$

Check your answers, if possible. Here, we can go back to the functional equation and check for $n = 4$: $a_4 = 15 - 3 \cdot 4 = 3$. That is what the problem required.

7. For part c:

$$\begin{aligned} a_n &= 20 - 3n \\ a_n + 3n &= 20 \\ 3n &= 20 - a_n \\ n &= \frac{20 - a_n}{3} \end{aligned}$$

The answer for part g is $n = (a_n + 3)/20$

Problems in Context

1. a. The starting charge is 30. Each hour the charge goes down by 2.4. So after one hour the charge is 27.6, this is q_1; after another hour the charge is 25.2, which is q_2; and after another hour the charge is 22.8, which is q_3.

b. $q_{n+1} = q_n - 2.4$.

c. We start at 30. q_n is found by reducing the original amount by 2.4 per hour for n hours. The total reduction is $2.4n$, so $q_n = 30 - 2.4n$.

d. The answer is $q_4 = 30 - 2.4 \cdot 4 = 20.4$.

e. Using algebra on the functional equation

$$\begin{aligned} q_n &= 30 - 2.4n \\ q_n + 2.4n &= 30 \\ 2.4n &= 30 - q_n \\ n &= \frac{30 - q_n}{2.4}. \end{aligned}$$

f. We want to know when $q_n = 4$ or less. Set $q_n = 4$, and use the equation from the previous problem: $n = (30 - 4)/2.4 = 10.833333$. So after 10.8333 hours (or 10 hours and 50 minutes), the charge will be less than 4, and the transmitter will stop working.

3. Following the outline starting on page 43: First, define the variables. Let s_n be the number of sick people n days after the start of the study. Next, assuming 45 new cases per day, the difference equation is $s_{n+1} = s_n + 45$. Next, observe that the starting value is $a_0 = 3{,}700$. Now we can make a graph. It is recommended that you use a graphing calculator or a computer, but it is possible to do it by hand. Observe that the first several data values will be 3,700, 3,745, 3,790, 3,835, 3,880, etc., always increasing by 45 cases per day. The functional equation for this problem is $s_n = 3{,}700 + 45n$.

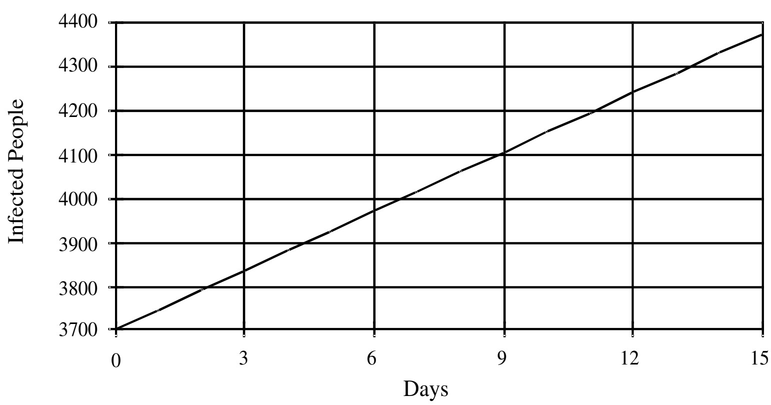

FIGURE 3.1
Graph for problem 3.

4. In this problem, there are a lot of different parts to keep straight. At first, it is normal to feel a little lost or overwhelmed. When that happens, a good strategy is to look at specific cases and get a feel for the numbers. In this particular problem,

you can do this by systematically working out how much medicine is used up after 1 day, then after 2 days, and so on. At the very start, with 3,700 people infected, how much medicine is required? 3,700 doses? No, only 3 percent of the people need the medicine. What is 3 percent of 3,700? It is computed as $.03 \cdot 3{,}700 = 111$ doses. After one day, another 45 cases come in. How many of them will need the medicine? Right, 3 percent. Now 3 percent of 45 is $.03 \cdot 45 = 1.35$. Of course, you will not use up 1.35 doses of medicine that day. But the idea of 3 percent of patients needing the medicine is an average. So some days you might give one dose, others you will give two doses, but in the long run, it will average out to 3 percent. For this reason, we will act as if 1.35 doses were given the first day. Then the total medicine used up is $111 + 1.35 = 112.35$. Do you see that each day another 1.35 doses will be used? And that therefore there is an arithmetic growth model for the amount of medicine used up? Let d_n be the number of doses used up by end of day n. We know that $d_0 = 111$. Then $d_1 = 111 + 1.35$, and d_2 will be the d_1 amount plus 1.35, and so on. The difference equation is $d_{n+1} = d_n + 1.35$. That means the functional equation is $d_n = 111 + 1.35n$. We want to know when this will reach the total 500 doses available. That is, we want to solve $111 + 1.35n = 500$. As usual, you can use graphical and numerical methods to get an idea of what the answer will be. For an exact answer, we use algebra: $1.35n = 389$; $n = 389/1.35 = 288.15$. This shows that the medicine will last 288 days, and will be used up by the next day, assuming the epidemic continues to grow according to the arithmetic growth model.

6. Here are several sample calculations

$$r_0 = 999{,}100 - 24{,}254.25 \cdot 0 - 54.75 \cdot 0^2 = 999{,}100$$

$$r_1 = 999{,}100 - 24{,}254.25 \cdot 1 - 54.75 \cdot 1^2 = 974{,}791$$

$$r_2 = 999{,}100 - 24{,}254.25 \cdot 2 - 54.75 \cdot 2^2 = 950{,}373$$

$$r_3 = 999{,}100 - 24{,}254.25 \cdot 3 - 54.75 \cdot 3^2 = 925{,}845$$

Using the difference equation, we start with $r_0 = 999{,}100$, $r_1 = 999{,}100 - 365(66.6 + .3 \cdot 1)$. This gives the same answer as the functional equation. In fact, all of the answers check out.

7. The first part is a matter of direct use of the functional equation. The year 2000 is 9 years after 1991, so the oil reserves at that time will be

$$r_9 = 999{,}100 - 24{,}254.25 \cdot 9 - 54.75 \cdot 9^2 = 776{,}377$$

For the other part of the question, use trial and error guessing different n's. We are looking for an oil reserve that is half the original amount of 999,100, that is, for a reserve of 499,550. We already found that $r_9 = 776{,}377$ which is larger than 499,550. So we need to try an n that is larger than 9. How about $n = 15$? That gives

$$r_{15} = 999{,}100 - 24{,}254.25 \cdot 15 - 54.75 \cdot 15^2 = 622{,}968$$

That is still too big. Try $n = 20$.

$$r_{20} = 999{,}100 - 24{,}254.25 \cdot 20 - 54.75 \cdot 20^2 = 492{,}115$$

This is a little less than the answer we are looking for. Continuing in this way, you can test different values of n. This trial-and-error approach is often a feature of numerical methods. In this problem, we find $r_{19} = 518{,}505$, which is more than half of the 1991 oil supply. So it will be between 19 and 20 years, according to this model, before the oil is half gone.