4

Linear Graphs, Functions, and Equations

In the last chapter, we found that the graphs of arithmetic growth models always appear as straight lines. We also found that the functional equations for arithmetic growth models always appear in the same form. This form defines what is called a *linear equation*, so called because the graph is a straight line. In this chapter we will study linear equations a little more carefully, covering ideas connected with functions, graphs, and solving equations. Additional important topics are *continuous* models and *proportional reasoning*.

Linear Functions and Equations

The general form for the functional equation of an arithmetic growth model is

$$a_n = a_0 + d \cdot n$$

In this equation, a_0 and d are parameters. That is, they stand for numerical values that can change from one problem to another, but are always thought of as remaining fixed or constant. For one of the examples we used, $a_0 = 66.6$ and $d = .3$ giving us

$$a_n = 66.6 + .3n \tag{1}$$

Recall that this equation expresses a_n as a function of n because it permits us to compute a value of a_n as soon as we know n. To be more specific, we refer to this as a *linear* function because the functional equation is a linear equation.

A word is in order here about terminology, and in particular about the terms *equation*, *expression*, and *function*.. It is a common error to use the word *equation* for just about anything that can be written down with algebraic symbols. Many students refer to all of the following as equations:

$$y = 3x + 4 \qquad 3x + 2 \leq 5y \qquad 7x^3 + 5xy - 2$$

To be technically correct, only the first of these is an *equation*, because only the first has an equal sign. The middle example should properly be called an inequality, while the third is simply an expression. An expression has no equal sign or inequality. It can be thought of as a recipe for computing something. Indeed, if you replace every variable in an expression with a number, you end up with something that can be computed. In this chapter, the focus is on examples that are *linear*. To use the terms properly, $y = 3x + 4$ should be called a linear *equation,* while $3x + 4$ should be called a linear *expression*.

Both equations and expressions are closely connected to the concept of *function*. The accepted mathematical definition of a function is closest conceptually to a set of (x, y) data points. So when we say that a_n is a function of n, it is most correct to think of the graph, where each point is of the form (n, a_n). Normally, we use an equation to find these data points. For the equation $a_n = 66.6 + 3n$, we can start with any n we like, say $n = 3$, and compute $a_3 = 66.6 + 9 = 75.9$. That gives $(3, 75.9)$ as one of the points that make up the function. In cases like this, it is often convenient to refer to either the equation itself, or to the expression that appears on the right side of the equation, as the function. For example, you will sometimes see statements like

For the function $y = 3x + 4$ ...

or

Note that $3x + 4$ is a function of x ...

In each case, it is understood from the context that the function being considered is made up of points (x, y) and that for each point, if you are given the x, you can compute the y as $3x+4$. Although technically it is not completely correct to refer to expressions or equations as functions in this way, it usually does not lead to any confusion. Throughout this book, the convention will be to use *expression* and *function* more or less interchangeably, and to use *equation* to emphasize the idea that there is an equal sign present. To illustrate this distinction using the earlier examples, $y = 3x + 4$ will be called a linear *equation,* while $3x + 4$ will be called a linear *function*.

So far, linear equations and functions have been discussed only in connection with specific examples. Let us proceed to a definition of the concept of linearity. It is based on the mathematical operations used to perform a calculation. When we use Eq. (1) to compute a_n, what operations are performed? The variable n is multiplied by one constant, and added to another. These simple operations define the concepts of linear function and equation:

Linear functions and linear equations involve just two types of operations: (1) addition/subtraction, and (2) multiplication/division by constants.

For example, consider the operations that have to be performed in

$$z = 3(x - 7y) + w/5$$

In each multiplication at least one of the two quantities multiplied is a number, and the only division that appears is division by 5, a number. There are also several additions and subtractions. This is a linear equation. It defines z as a linear function of x, y, and w. In contrast, the following examples are not linear equations

$$xy + 2 = z$$
$$x^2 + y^2 = z^2$$
$$y = -2 + \sqrt{4 - 3x}$$
$$y = \frac{3x + 4}{4x - 3}$$

In each of these there are operations other than addition, subtraction, and multiplication or division by constants. For example, in the second equation there appears an x^2. This is an abbreviation for $x \cdot x$ and so requires multiplication of two non-constant quantities. You should try to decide why each of the other equations is not a linear equation.

The definitions of linear equation and linear function above are formulated in terms of numerical operations. There is an alternative description that emphasizes visual appearance:

> A linear function is a combination of variables and constants in which (1) variables have no exponents, (2) variables are not multiplied by other variables, (3) variables do not appear in denominators, (4) variables are not acted on by other operations, such as square roots and logarithms.

This is not really a complete definition since it doesn't specify exactly what the other operations in item (4) are, but it provides a useful way to think about linear functions.

A linear equation may involve as many variables as we choose, but for most of the course we will be especially interested in linear equations in two variables, like Eq. (1). There, one of the variables is n and the other is a_n. As you are aware, it is possible to rewrite one equation in many different forms by applying the rules of algebra. The equation $3x + 4y = 2$, for example, can also be written as $4x + 3y = 2$, $3x + 4y - 2 = 0$, $3x = 2 - 4y$, and in many other ways. Among all the possible ways to express a linear equation in two variables, there are a few standard formats that are usually used. For example, in Eq. (1) the constant 66.6 would normally be moved to the end of the equation, giving the form

$$a_n = .3n + 66.6 \tag{2}$$

The significance of this and of other standard forms will be discussed later in this chapter.

Continuous Models

An important aspect of Eq. (2) is the fact that n is thought of as a whole number. Do you remember the discussion in Chapter 2 of discrete data? There we emphasized the idea of data sequences with a first value, a second value, and so on. The variable n was introduced as a label or counter for the sequence values, and that is why n made sense for whole number values: $n = 1$ for the first data value, $n = 2$ for the second, and so on. But in many applications it is reasonable to think about fractional values for n, especially when the data values are for a series of times. Suppose, for example, a model for the AIDS epidemic uses a_n as the number of infected people after n months. Then $a_{3.5}$ would mean the number of infected people after three and one-half months. In fact, we can reasonably interpret an a with any decimal value for the subscript: $a_{4.687}$ would mean the number of infected people after 4.687 months. However, this notation is rarely used. Instead of expressing the decimal as a subscript, it is written in parentheses. That is, $a(4.687)$ would be used instead of $a_{4.687}$.

The preceding example illustrates the idea of a continuous variable. If we only allow n to be a whole number, the model that is based on n is a discrete model. If we allow n to be any real number (meaning any decimal, or any point on a number line), then n is a continuous variable, and may be part of a continuous model. Continuous and discrete models can also be distinguished by the kinds of graphs that are used. In the discrete case, the graphs are defined by a series of individual separate points, as in Fig. 2.1 and Fig. 2.3. In the continuous case, we imagine that there are actually data points for each n on the horizontal axis. If we could individually graph all of these points, they would crowd together completely covering a line without gaps or separations. This is the image that inspires the term *continuous*.

Which is better, discrete or continuous? For our purposes, the discrete approach will be easier to understand when it comes to describing how changes occur. It is the context in which we formulate difference equations. However, when we combine the theoretical approach with a difference equation, the end result is frequently a functional equation which makes sense with continuous variables. This is what happened in the case of the oil consumption model. The difference equation was formulated by thinking about how the consumption next year might depend on the consumption this year. In that setting, and in particular in the equation

$$c_{n+1} = c_n + .3$$

it was natural to think of n as a whole number. That is, we think of n as a discrete variable. Eventually we obtained the functional equation

$$c_n = .3n + 66.6$$

Now it is natural to allow n to be any decimal, so that we can talk about the oil consumption after 3.4 years, for example. When we use the functional equation, it is reasonable to think of n as a continuous variable. As we proceed, we will follow this same progression many times: starting with a discrete model and a difference equation; using it to find a functional equation; and then using the functional equation as part of a continuous model.

This discussion of continuous variables appears here to prepare for the ensuing consideration of linear equations. In the remaining sections of the chapter you will learn about algebraic and graphical aspects of linear equations, as well as applications and the use of proportional reasoning. In all of these topics, continuous (rather than discrete) variables will be used.

Algebra and Solving Linear Equations

Consider again an arithmetic growth model for oil consumption:

$$c_n = .3n + 66.6$$

What is the daily oil consumption when $n = 10$? The answer to this question is simply a matter of computation: Replace n by 10 and carry out the operations on the right side of the equation. In contrast, the following question is not so easily answered: When will the daily oil consumption equal 75? In this case, we are specifying c_n and asking for the value of n. Replacing c_n with 75 leads to

$$75 = .3n + 66.6$$

The task is to find the n for which this becomes a true statement. Finding that n is referred to as solving the equation.

There is a general method for solving linear equations, and usually, the result is a single answer. We will see later that other kinds of equations can be more difficult to solve, and may produce multiple answers.

The Importance of Solving Equations. The general idea of solving an equation is one of the most fundamental concepts in algebra. Why is it so important? The answer brings up the topic of functions again. Notice that in the first question, the one that is easy to answer, we are given n and seeking c_n. Since the equation gives c_n as a function of n, that is an easy task. In the second question the roles are reversed. We are asked to find the n that produces a certain value of c_n. This is sometimes referred to as *inverting* the function, since we want to begin with the result of the function computation and figure out what the starting point was. It occurs repeatedly with functions. For example, suppose we have a model that predicts the ozone depletion as a function of time. The model will predict for any time what the level of ozone depletion will be. Then someone will ask when a particular critical ozone level will be reached. That question calls for inverting the function in the model: it specifies the ozone depletion and asks for the time. For another example, we may have a model that predicts how many voters will choose our candidate as a function of the amount of money that will be spent on advertising. If you spend 2 million dollars, the model predicts how many votes you will get. But what the candidate wants to know is, how much must be spent to win? You know how many votes are needed, and want to determine how much money it will take. That calls for inverting the function in the model. In all of these examples, if the function in the model is given by an equation, then the process of inverting the function amounts to solving an equation. That is why solving equations is such an important part of algebra.

The Process of Solving Equations. When algebra is used to solve an equation, there is one basic process that is used over and over again: replace the equation you have with a simpler equation without changing the solution. Here is an example. If the original equation is $3x + 4 = 10$, then we replace it with the simpler equation $3x = 6$. We can argue that the new equation has the same solution as the original as follows. If $3x+4$ and 10 are equal, then subtracting 4 from them both will produce equal results. That is $3x$

and 6 must be equal. That is another way of saying that the same value of x solves both equations. For linear equations, there are only a few necessary operations that simplify equations. You can

- algebraically rearrange either side
- add the same amount to both sides
- subtract the same amount from both sides
- multiply both sides by the same amount
- divide both sides by the same amount (but not by zero)

Using these operations, any linear equation can be reduced to the simple form *variable = number*. Here is a somewhat involved example:

$$\begin{aligned}
2(3x-5)+6 &= 2-4x+2(x-4)\\
6x-10+6 &= 2-4x+2x-8\\
6x-4 &= -2x-6\\
6x-4+2x &= 2x-2x-6\\
8x-4 &= -6\\
8x-4+4 &= -6+4\\
8x &= -2\\
8x/8 &= -2/8\\
x &= -1/4
\end{aligned}$$

In this example, each step has been done separately to emphasize what operations are being performed. Normally, when you solve an equation, you will be able to do several steps at once and shorten the process significantly.

No Solution; Every Number a Solution. It is possible to have linear equations without any solution, or for which every number is a solution. Although these equations rarely occur in practice, it is not always obvious when they do occur. For example, the following equation looks like a typical linear equation.

$$3x+2=5+3(x-4)$$

A few steps of algebra lead to the equation

$$2=-7$$

This is an impossible situation, and is never true no matter what x is. So the original equation has no solution. On the other hand, this equation

$$2-4x=4(3-x)-10$$

is true for every value of x. Try a few. If you make $x=0$ the equation becomes $2=12-10$, which is certainly true. What do you get if $x=1$? $x=-1$? In all these cases the resulting equation is true. Now no number of examples can *prove* that this equation is true for every x. However, this conclusion *can* be reached by using algebra. Applying the same kind of steps as in the preceding examples, we reach the equation $2 = 2$. Since this is true for all values of x, so is the original equation. As already stated, equations of this type rarely actually occur in practice, although they do appear once in a while, especially as a result of an error. It is important to understand what they mean when they do occur. Then, if an error has been made, it will be easier to spot.

More than One Variable. So far the examples have all involved just one variable. The same kinds of operations can be applied in equations with more than one variable. In that case, it is often useful to simplify the equation to a form with one of the variables isolated on one side of the equation, and all other terms on the other side of the equation. As an example, we will again use the oil consumption model

$$c_n = .3n + 66.6$$

This equation gives c_n as a function of n. Using algebra, we can replace it with one that has n isolated on one side of the equation:

$$\begin{aligned} c_n &= .3n + 66.6 \\ c_n - 66.6 &= .3n \\ (c_n - 66.6)/.3 &= n \\ n &= (c_n - 66.6)/.3 \end{aligned}$$

This process is described as solving for n in terms of c_n. Notice that the final equation gives n as a function of c_n, because as soon as the value of c_n is substituted, the value of n can be immediately computed. As described before, solving the equation for n in this way amounts to *inverting* the original equation, the one giving c_n as a function of n. The new equation for n as a function of c_n is very useful. Using it, we can easily determine for any level of oil consumption, when that level will be reached. Following the convention mentioned earlier for replacing subscripts with quantities in parentheses, we can express the original equation in the form

$$c(n) = .3n + 66$$

where we think of n as any whole number, fraction, or decimal. In the same way, the inverse equation can be written

$$n(c) = (c - 66.6)/.3$$

This emphasizes that n is expressed as a function of the consumption c.

Graphs of Linear Equations

We have seen in earlier work how to create a line graph for the discrete model

$$c_n = .3n + 66.6$$

In that discussion, we used whole number values for n and plotted a point for each n. If we now think of n as a continuous variable, it would be impossible to plot a point for every possible value of n. However, methods of higher mathematics can be used to show that the result of plotting all possible points would be a smooth continuous curve, and in the case of a linear equation, a straight line. Graphing on a computer is performed by plotting many points so closely spaced that the appearance of a smooth curve results. The case of a straight line is special. We need only plot two points, and then draw a line through them using a straight edge.

What is more interesting is to develop a qualitative feel for the connections between parts of the equation and aspects of the graph. This insight can be useful in two ways. First, given a graph showing a line, we might want to determine the equation of the line. Second, if we have an equation, we might want to form an idea of the appearance of the line. Examples of each case will be given in more detail. Before moving on to those examples, we introduce a slight change in notation. To emphasize the idea that the variables are continuous, we will represent them as single letters, without using a subscript. In the context of a specific problem, the letters will be chosen to help remind us what the variables stand for. For the case of oil consumption, we will use c for consumption and n for number of years. For general discussions that are not connected with a specific problem, x will be used for the variable on the horizontal axis and y for the variable on the vertical axis.

Special Forms for Linear Equations. There are three different forms that are often used for linear equations. The first is the one we mentioned earlier. For the oil consumption example, it was

$$c = .3n + 66.6$$

The more general form is

$$y = mx + b \tag{3}$$

In this form the constants m and b have special significance. The number b indicates the point at which the line crosses the y axis. It is called the *y-intercept*. For example, the graph of the equation $y = 3x + 5$ crosses the y axis at 5 (Fig. 4.1). Similarly, the graph of $y = 3x - 5$ crosses the y axis at -5 (Fig. 4.2). Notice that in the equation $y = mx + b$, b is the value of y that results when x is set to 0. An x value of 0 always gives a point on the y axis.

Slope. The number m also has a special meaning. It is called the slope and tells how steeply the line slopes up or down as you trace it from left to right. The numerical value of the slope tells how much the line moves up for every unit you move to the right. For the equation $y = 3x + 5$, the slope is 3. This indicates that the line moves up 3 units for every unit you move to the right. This is illustrated in Fig. 4.3. For the line $y = -3.2x + 5$, the slope of -3.2 indicates that the line moves *down* 3.2 units for each unit to the right (Fig. 4.4). When the slope is a fraction, it has an alternate interpretation. For $y = \frac{3}{4}x + 2$, the line moves up 3 units for every 4 units to the right (Fig. 4.5). In this context, think of $3/4$ as 3 *over* 4, and think of the slope as *up* 3 *over* 4.

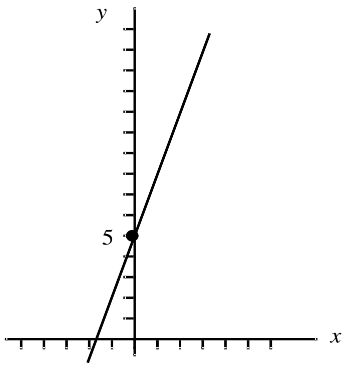

FIGURE 4.1
Positive *y* intercept

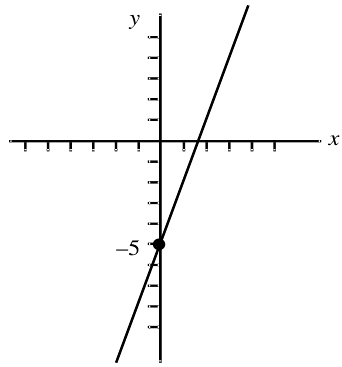

FIGURE 4.2
Negative *y* intercept

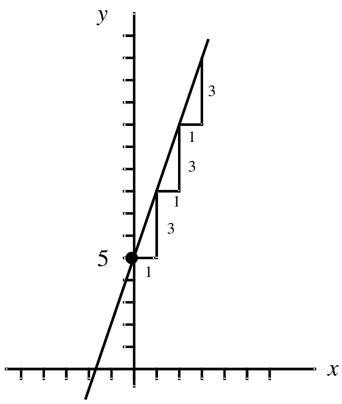

FIGURE 4.3
Positive slope

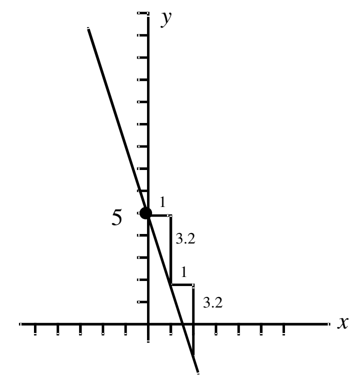

FIGURE 4.4
Negative slope

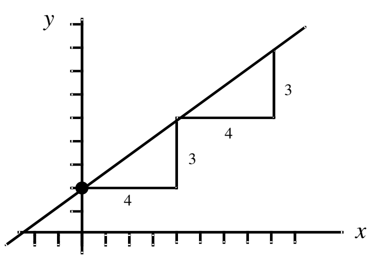

FIGURE 4.5
Slope is a fraction

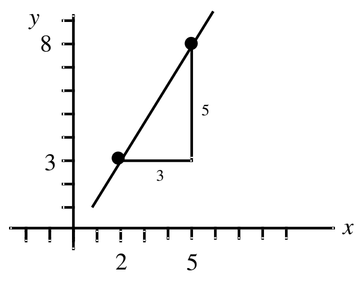

FIGURE 4.6
Slope between two points

The slope of the line can be computed using any two points of the line. Draw a triangle as in Fig. 4.6. Then moving from left to right along the line, the distance you go up is called the rise (it is negative if you go down). The distance you go to the right is called the run. The slope is the rise over the run. From the point $(2,3)$ to the point $(5,8)$ the rise is 5 (from 3 to 8) and the run is 3 (from 2 to 5), so the slope is $5/3$. This is another way of saying that for every 3 units you go to the right, you also go 5 units up.

A slope is actually a rate. In a purely geometric discussion, a slope of 3.2 indicates that the line rises 3.2 units per unit traveled to the right. In the context of a specific application, the slope acquires whatever units of measurement are used on the two axes. For the oil consumption problem, the equation $c = .3n + 66.6$ relates the daily consumption c in millions of barrels of oil to the number n of years after 1991. The slope .3 indicates that daily oil consumption goes up by .3 (million barrels of oil) for every 1 (year) increase in n. The slope therefore has units of millions of barrels of oil per year. The statement *daily oil consumption is increasing at the rate of* $.3$ *million barrels of oil per year* is a direct translation of the mathematical statement that the slope is .3.

The Slope–Intercept Form. As explained in the preceding discussion, when an equation is expressed in the form $y = mx + b$, we can immediately see the slope and the y–intercept of the line. For example, the graph of $y = 2.4x - 7$ has a slope of 2.4 and a y–intercept of -7. With this in mind, $y = mx + b$ is called the *slope-intercept* form of a linear equation. It gives an immediate visual understanding of the graph. Start at the point b on the y axis, and proceed on a diagonal going up m units for every unit you go right. You can actually generate a sequence of points following the recipe *up* m *over* 1 over and over. For a numerical example, if the equation is $y = \frac{1}{2}x + 3$, start out at 3 on the y axis, go up 1 and over 2 to a new point, then up 1 and over 2 to another new point, etc. This is illustrated in Fig. 4.7. If you think about this process, you will see that it is intimately connected to the idea of arithmetic growth. The vertical positions of the points grow arithmetically as you generate them from left to right.

The slope-intercept form of a line allows you to visualize the graph given the equation. It is also useful for the reverse problem: determine the equation given the graph. For example, in Fig. 4.8 a line has been drawn. What is the equation? A visual inspection

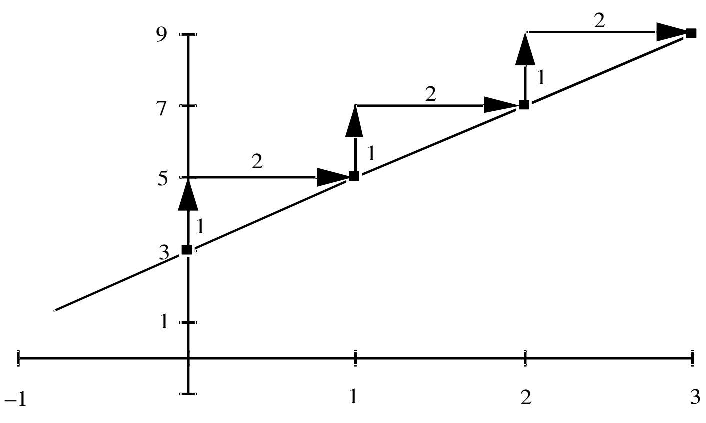

FIGURE 4.7
Up 1 and Over 2

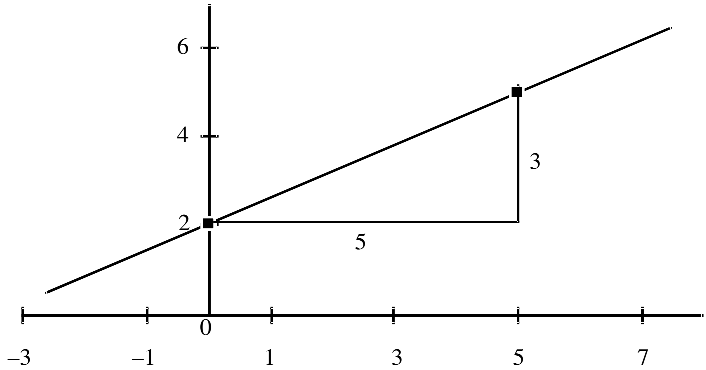

FIGURE **4.8**
What is the equation?

of the graph reveals that the y intercept is 2. A little careful measurement of the triangle that is shown in the figure will convince you that the slope is $3/5$. So the equation can be immediately written down as $y = \frac{3}{5}x + 2$.

The Point–Slope Form. Sometimes you have a graph in which it is inconvenient to find the y intercept. In Fig. 4.9 the axes have not been drawn in the usual locations. That is, they do not meet at the 0 point of each axis. Because our notion of y intercept assumes that x is 0 on the vertical axis, it is an error to think of the y intercept as the point where the line crosses the vertical axis for this figure. Instead, the true y intercept, and the y axis itself, would actually be far to the left of the figure. It is still possible to determine the equation in a simple way. First, using the two points shown, $(73, 45)$ and $(83, 51)$, the slope is easily computed. The graph goes over 10 (from 73 to 83) and up 6 (from 45 to 51) so the slope is $6/10$ or .6. Now imagine another point on the line at an

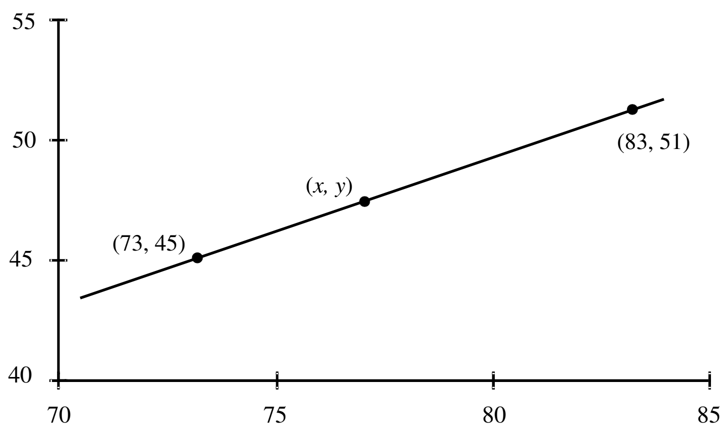

FIGURE **4.9**
No y intercept

unknown location (x, y). Let us recompute the slope using this unknown point and the first point, $(73, 45)$. The rise is $y - 45$ (from 45 to y), and the run is $x - 73$ (from 73 to x). This gives the slope as $(y - 45)/(x - 73)$. Since we already know the slope is .6, we conclude that for any point on the line,

$$\frac{y - 45}{x - 73} = .6$$

or

$$y - 45 = .6(x - 73).$$

This is referred to as the *point-slope* form of a linear equation. It allows you to write down the equation for a line as soon as you know the slope and one point of the line. In this case, we know the slope is .6 and one point of the line is $(73, 45)$. For another example, Fig. 4.10 shows a graph for soda prices (p) and sales (s). If the price is set at 80 cents, 200 sodas can be sold. If the price is raised to 90 cents, only 160 sodas will be sold. The slope between these two points is $-40/10 = -4$ in units of sodas per penny. That is, the sales drop by 4 sodas per penny increase in price. Knowing the slope (-4) and one point $(80, 200)$, we can immediately write the equation: $s - 200 = -4(p - 80)$.

The general form of the point-slope equation is usually written

$$y - y_0 = m(x - x_0) \tag{4}$$

In this equation, x_0, y_0, and m are parameters. In any actual application there will be numerical values for these parameters. Just as in the example $x_0 = 73$, $y_0 = 45$, and $m = 0.6$. In using the general form Eq. (4), think of the parameters as fixed numbers and of the variables x and y as representing many different points all on the line.

Like the slope–intercept form, the point–slope form can be used to visualize the graph given the equation. For example, in $y - 5 = .12(x - 4)$ we can recognize immediately that the point $(4, 5)$ is on the line, and the slope is .12. As before, we can generate a series of points by starting at $(4, 5)$ and repeatedly going up .12 and over 1.

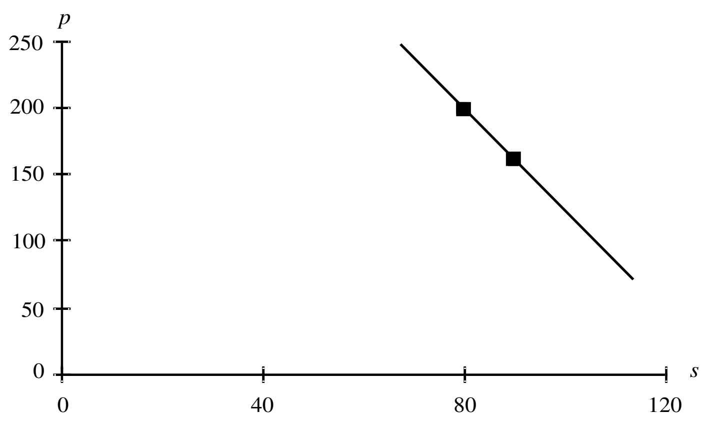

FIGURE 4.10
Sales vs. Price

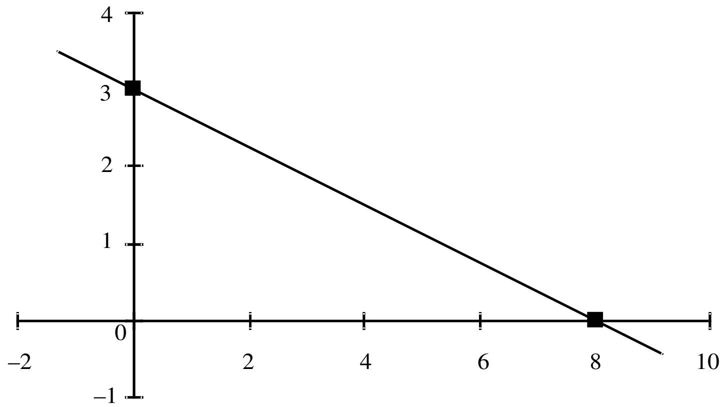

FIGURE 4.11
Using Both x and y Intercepts

The Two-Intercept Form. There is a third form of a linear equation that is especially useful if we know where the line crosses both axes. In Fig. 4.11 the line shown crosses the x axis at 8 and the y axis at 3. From this information the following equation can be written down immediately:

$$\frac{x}{8} + \frac{y}{3} = 1$$

The reasoning for this equation is a little different from the previous cases. Observe that this *is* a linear equation, so the graph must be a straight line. It is clear that the illustrated x intercept satisfies the equation, for at the x intercept, $x = 8$ and $y = 0$. By similar reasoning the illustrated y intercept also satisfies the equation. This shows that the graph of the line must go through the two intercepts shown, and so must be exactly the line that is illustrated. This example uses what is called the 2–intercept form of a linear equation. The general expression of this form is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5}$$

Here, the letters a and b are parameters. In using this form, a should be replaced by the x intercept of the line and b should be replaced by the y intercept of the line. Note: there is no 2–intercept form for a line that goes through the point $(0, 0)$. For such a line, the x and y intercepts are both 0. With $a = 0$ and $b = 0$, Eq. (5) would not make sense.

Activity. You have now read about three different forms for linear equations. Before proceeding with the reading, it would be a good idea to go back and review those three forms. In your notes for this section, write out the three forms using parameters (like m and b, or m, x_0, and y_0). For each equation, write out in words what each parameter tells you about the graph. Give a specific example of each equation, using numbers in place of the parameters, and show the graph of each example.

Fitting a Line to Data

It is almost always the case that a linear model only approximates the data that appear in a problem. In that case, the linear model is chosen to come as close as possible to the data points. How is this done? One approach is to draw the line that appears best to the eye, and then determine the equation using the methods of this chapter. The equation derived in this way then becomes part of the linear model.

As an example of this approach, consider again the oil consumption data used before, and repeated below in Table 4.1. The graph for the individual data points is shown in Fig. 4.12. The line shown in the graph was simply drawn by eye to fit the data. Likewise, the two points shown on the line were estimated by eye using the labels on the axes. Using these points, the slope of the line is $(67.78 - 66.6)/(3 - .5) = 1.18/2.5 = .472$. Then the equation of the line is $c - 66.6 = .472(n - .5)$ where c stands for consumption and n stands for number of years after 1991. The linear equation can be rearranged to $c = .472(n - .5) + 66.6 = .472n + 66.364$. Note that this is very similar in form to the first equation we used for consumption: $c = .3n + 66.6$. The slope and intercept are slightly different, and reflect our attempt to get the line close to all the data. The point

Year	1991	1992	1993	1994	1995
Oil Used	66.6	66.9	66.9	67.6	68.4

TABLE 4.1
World Oil Consumption in Millions of Barrels per Day

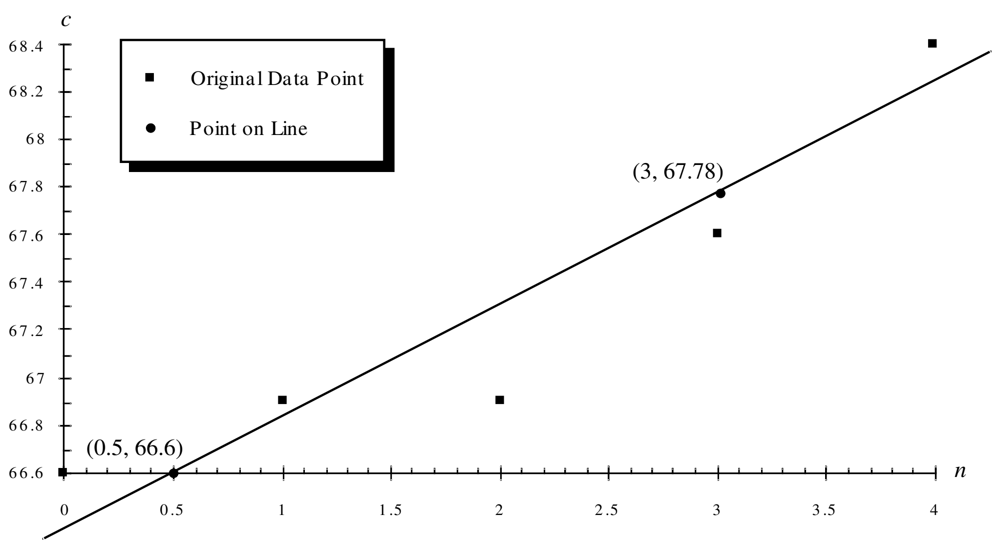

FIGURE 4.12
Best line by eye

of this example is to illustrate the process of starting with a graph and determining the corresponding equation. The new equation once again expresses c as a function of n, and can be used in the same way that the earlier equation was used. It can be thought of as the best linear model for the data.

In this discussion the line was picked by visual appearance. In a later chapter the idea of choosing the best line to fit a set of data will be reconsidered in a more analytical way.

Formulating Linear Models

The properties of linear equations and their graphs can be used to formulate linear models. Because there are several ways to write a linear equation, there is no one way to formulate a linear model. The key point is to recognize that a linear model is called for, and then to use the given information in an appropriate way. To illustrate this process, the example problem from Chapter 3 will be redone below.

Example Problem. A cellular phone company has equipment that can service 100 thousand customers. In 1990 they had 70 thousand customers. Over the last few years the number of customers has been increasing by about $4{,}500$ per year. Assuming that the growth continues at the same pace, what will the situation be in the year 2000? 2010? How long will it be before additional equipment will be needed?

Step 1: Recognize this as a linear model. In this case, the idea that the number of customers goes up each year by the same amount, $4{,}500$, identifies this as an arithmetic growth model, and we know that those always lead to linear models. That means we could formulate a difference equation and eventually find a linear equation, just as in the previous chapter. That is one effective approach. Here, we will use an alternative approach, one that does not involve difference equations.

What is it we wish to study? The number of customers for each year. So make up a name for the variables: *Let c be the number of customers, in thousands, and let y be the year.* Notice that this statement gives units for the variables, as well as defining their meanings.

Now suppose each year the number of customers increases by the same amount. That is an indication that this is a linear model. You can see this visually by graphing several data points, or can reason it out using the idea of slope. The number of customers increases at the rate of $4{,}500$ customers per year. A rate is a slope. Here the rate is the same each year, so on the graph the slope will be same between any two points. That means the graph is a straight line and indicates a linear model.

In some problems, the linear model will only approximately fit the given data. In that case, you might decide a linear model is appropriate by graphing the data and observing that the points are nearly arranged in a straight line.

Step 2: Identify the given information in terms of points, slopes, and intercepts. In 1990 there were 70 thousand customers. This gives a data point: $(y, c) = (1990, 70)$. Note that I have made y the first variable because I am envisioning a graph with years along the horizontal axis. The increase in customers is given as $4{,}500$ per year, or in the units we are using for c, as 4.5 thousands of customers per year. That is a slope. You can also find the slope by finding another data point. If the increase is 4.5 per year, then in 1991 there must be 74.5 thousand customers. That gives the point $(1991, 74.5)$. Now that two points have been found, the slope can be calculated as the rise over the run. By either method, the slope is 4.5.

Step 3: Draw a graph. The graph shows points and intercepts pictorially. It also gives you a visual portrayal of the way the variables are related. If you are a visual thinker, the graph makes using the slope–intercept or point–slope forms clearer.

Step 4: Find an equation for the variables. Now that we know one point $(1990, 70)$ and the slope (4.5), we can use the point–slope form of linear equation. Remember that we are using y and c as the variables instead of x and y. The equation is

$$c - 70 = 4.5(y - 1990)$$

In other problems, the given information might be in a different form. If two points are given, you can compute a slope. It may be that the y intercept is given. Then you can use the slope–intercept form. In some cases you will have several data points given, and will want to choose the best fitting straight line, as described earlier.

Step 5: Answer any specific questions. Restate the questions in terms of the model. What will the situation be in the year 2000? That is, how many customers will there be. In terms of the model, the question is this: when $y = 2000$, what is c? To find the answer, substitute 2000 for y in the linear equation, and solve for c. The equation you must solve is $c - 70 = 4.5(2000 - 1990)$, and the answer is $c = 115$. That is, in 2000 the model predicts 115 thousand customers. If we are going to do this for several different years, it is convenient to rewrite the equation so that c is expressed as a function of y. The new equation is $c = 4.5(y - 1990) + 70$ or, more simply, $c = 4.5y - 8885$. In 2010, we compute $c = 4.5 \cdot 2010 - 8885 = 160$. When will there be 100 thousand customers? That is, if $c = 100$ what is y? As before, we can substitute the given variable value, in this case c into the linear equation and solve for the unknown y. Which equation should you use? Either one is correct. But if there will be many questions of this type, it is best to get a new equation that expresses y as a function of c. That gives $y = (c + 8885)/4.5$. Then, with $c = 100$ we find immediately $y = 8985/4.5 = 1996.66667 = 1996$ and $2/3$. This should be interpreted to mean that the number of customers will reach 100 thousand two-thirds of the way through 1996.

Proportional Reasoning

A linear model embodies a concept referred to as *proportional reasoning*. This idea was discussed in the previous chapter in an example involving soda prices and sales. In that context, the statement is made that for every increase of 5 cents in the price there will be a decrease of 10 in soda sales. With what you have learned in this chapter, you should now recognize this as a description of a slope: on a graph of soda sales versus price, between any two points the slope is $-10/5 = -2$. Proportional reasoning always involves straight lines and slopes.

Proportional reasoning is a way of relating changes of two quantities. For the soda example, it says that the changes of price and sales are always in the proportion $-10/5$. Notice that this is another example of an invariant. Under proportional reasoning, it is assumed that dividing a change in one variable by the corresponding change in another variable always results in the same number. In a graph showing the first variable on the vertical axis and the second variable on the horizontal axis, dividing the change in the first variable by the corresponding change of the second variable is simply computing the rise over the run, that is, the slope. Now we can see that the underlying assumption in proportional reasoning is that there is a linear relation between the two variables. Put another way, using proportional reasoning in a model connecting two variables is precisely the same as adopting a linear model.

Remember the discussion on page 44 about how soda prices affect the number that can be sold? You were told that 120 cans of soda would sell at 40 cents per can, and that 90 cans would be sold at 55 cents per can. Then you were supposed to use common-sense to figure out how many cans would sell at 60 cents per can. The reasoning that was then presented (and which was supposed to be your common sense approach) is really just proportional reasoning. From 40 to 55 cents a can is a change of 15 cents in price. From 120 to 90 cans of soda is a change of -30 cans that will be sold. These changes form a ratio (or proportion) of $15/-30 = -.5$, and we assume that this is an invariant. That is, we assume that making a change in the price per can always leads to a change in the number that will be sold in the same proportion, a decrease of 2 cans for each one cent increase in price. So, what was called common sense earlier, is just another instance of proportional reasoning.

For most people, proportional reasoning is easy to use when the numbers work out nicely, but becomes more difficult with fractions or decimal figures. As you now know, a proportional reasoning problem can always be expressed in the form of a linear model. This chapter has presented a large amount of information about linear equations and models. When it has been mastered, these are powerful tools for formulating models. Unfortunately, just using common sense to formulate models can lead to dramatic errors. Here is an illustration.

A Common Error. Returning to an earlier example, let us again consider the connection between price and number of sales for sodas. The information we used previously was that 200 sodas could be sold at a price of 80 cents and 160 could be sold at a price of 90 cents. To predict how many could be sold at other prices, say 87 cents, it would be very handy to have an equation giving the number of sales s as a function of the price p. Using proportional reasoning, we would say that a 10 cent increase in price produced a 40 soda decrease in sales, so that for every penny the price is raised, the sales go down by 4. How can that be expressed as an equation?

Many students will write as an answer

$$p + 1 = s - 4$$

and will interpret it loosely to mean *an addition of* 1 *to price goes with a subtraction of* 4 *from sodas.* This is wrong. The problem is that = is being misinterpreted. It does not mean *goes with.* It means *is the same number as*. Reread the equation in that light. It says that the price, plus one more, is the same number as the number of sodas minus 4. This just isn't true. In the first place, the two sides of the equations are in different units, cents on the left and sodas on the right, so saying they are equal is like comparing apples and oranges. Second, we know that at a price of 80 cents we can sell 200 sodas. That is when $p = 80$ we know $s = 200$. Using proportional reasoning again, at a price of 81 the number of sales would be 196. Here the $p + 1$ is 81, and the $s - 4$ is 196. These numbers go together, but they are certainly not equal. To repeat: $p + 1 \neq s - 4$.

It is clearly not right to translate the idea *adding* 1 *to the price subtracts* 4 *from sales* into the equation $p + 1 = s - 4$. This is an example of incorrectly formulating a linear model. At first it seems plausible, but closer inspection shows that it really doesn't make sense. So what is the right approach? The answer is to use all your tools about linear equations. First, recognize that by using proportional reasoning we are really adopting a linear model, meaning a linear equation for p and s. We know two data points, because $s = 200$ when $p = 80$ and $s = 160$ when $p = 90$. To find the equation for the straight line joining these data points, we can calculate a slope and use the point-slope form for a line. That is just what was done earlier, leading up to the equation $s - 200 = -4(p - 80)$. This is the right equation. It is not easy to find in this equation the numbers mentioned in the statement:

> For each 10 cent increase in price there is a 40 cent decrease in sales.

Making direct use of the proportional reasoning idea to derive the right equation is a little bit complicated and difficult to understand. It is much easier to apply a knowledge of algebra and linear equations.

Summary

In this chapter we have discussed linear equations and the use of continuous variables was introduced. The algebraic process of solving linear equations was presented. Connections between three different standard forms of a linear equation and the graphs of the equations were described. The concepts of slope and intercepts were also presented. These ideas were used to determine the equation for a line fit by eye to a set of data. Finally, linear equations were related to proportional reasoning.

Exercises

Reading Comprehension. Write short essay answers to each question. After you write an answer, compare it to the explanation that is given in the reading.

1. Explain the difference between a linear function and a linear equation.
2. What is a discrete variable? What is a continuous variable? How do we use each type of variable in this course?
3. The four equations shown on page 57 are examples of nonlinear equations. For each equation, give a specific reason that shows it is nonlinear. Example: in the

second equation there appears an x^2, which means $x \cdot x$. Linear equations never have variables with exponents.

4. In the reading there are two different definitions of linear function, one that emphasizes the way the function appears, and one that concerns the operations that are used in computation. Write versions of these definitions in your own words.
5. Describe the process of solving an equation. Include in your answer a list of the operations that are permitted for simplifying an equation.
6. On page 60 there is a very detailed example of solving a linear equation. Explain how each equation after the first was obtained from the preceding equation.
7. Explain how a linear equation with one variable can have no solution. Explain how a linear equation can have an infinite number of solutions.
8. Explain what it means to invert a function. Give an example.
9. Suppose in the equation $T = -0.01h + 72$, the variable h stands for a height off the ground, and the variable T stands for the temperature at that height. Explain what it means to solve the equation for h in terms of T. Why might it be useful to do this?
10. What is slope? What is an intercept? How are these related to linear equations and their graphs?
11. Explain why a slope is a kind of rate. Give an example.
12. Describe three different forms for a linear equation. For each type, give an example and explain how the graph is related to the equation.

Mathematical Skills

1. Solve the following equations for x.
   a. $3x - 4 = 5 + 2x$
   b. $3(x - 5) + 3 = 9x + 6(2 - x)$
   c. $2(x - 3) = 3(x - 2)$
   d. $4(4 - x) + 5(x - 3) = x + 1$
2. For each equation, tell what form the equation is in (point–slope, slope–intercept, or 2–intercept form), and describe how to create a quick graph.
   a. $y = 3x - 4$ b. $y - 2 = .5(x - 3)$
   c. $y = x + 2$ d. $y - 5 = 2x$
   e. $x/2 + y/4 = 1$ f. $y = 2.3x$
3. Change each equation to slope–intercept form.
   a. $3x - 4 = 5 + 2y$
   b. $3(y - 5) + 3 = 9x + 6(2 - x)$
   c. $2(x - 3) = 3(y - 2)$
   d. $4(2 - x) + 5(y - 3) = x + 1$
4. For each part of the preceding problem, use algebra to express x as a function of y.
5. For each part of problem 3, express the line in the 2–intercept form, if possible.

Problems in Context

1. In the Fahrenheit temperature scale, water freezes at 32 degrees and boils at 212 degrees. The centigrade scale is defined so that water freezes at 0 and boils at 100 degrees. There is a linear equation that can be used to convert a centrigrade temperature to Fahrenheit. Find this equation. (Hint: Make a graph showing data points of the form (C, F), where C is the centrigrade temperature corresponding to a Fahrenheit temperature of F. This graph will be a straight line. You are looking for the equation of this line.) Is it possible for a temperature to be the same number using both Fahrenheit and centigrade?

2. Scuba divers are subject to the effects of increasing pressure as they go deeper and deeper into the ocean. As a rule of thumb, divers use the following linear model. At the surface, the water exerts a pressure of 15 pounds per square inch. For every 33 feet of depth, the pressure increases by 15 pounds per square inch, so that at a depth of 33 feet, say, the pressure is 30 pounds per square inch. Write an equation for pressure as a function of depth. What is the pressure at a depth of 100 feet? Suppose an underwater camera can withstand pressures of up to 1,000 pounds per square inch. How deep can the camera go into the ocean safely?

3. A company manufactures backpacks. The total cost to make each backpack, including materials and labor, is \$23. In addition, the company has expenses of \$12,000 per month for items such as rent, insurance, and power. These expenses do not depend on the number of backpacks made. Using this information, develop a linear model for monthly total costs as a function of the number of backpacks made each month. Write a short report about your model, defining the variables you use, showing the equation for costs, and explaining your reasoning.

4. In a large city, air pollution increases during the day, as auto emissions and other types of pollution enter the atmosphere. One day, the pollution level was 20 parts per million at 8 in the morning, and had increased to 80 parts per million by noon. Develop a linear model for the pollution level as a function of time. The Air Quality Management District is required to publish an unhealthy air alert on any day when the pollution level reaches 150 parts per million. If the linear model is valid from 8 in the morning to 6 at night, will it be necessary to publish an alert?

5. In Chapter 3 starting on page 42 two different variables are used in a model involving soda prices and demand. Starting from a base price of 40 cents per soda, the price is raised by a nickel several times. This gives a sequence of prices, 40, 45, 50, 55, and so on. These prices are represented by the variable p, with p_n the price after n increases. For each n there is also a demand, s_n. For example, s_1 is the number of sodas that can be sold when the price is p_1. As presented in the earlier discussion,

$$s_n = 141 - 12n$$

$$p_n = 40 + 5n$$

In this exercise, you will combine the equations to relate s_n and p_n directly. To simplify the algebra, just write s instead of s_n and p instead of p_n.

a. First, observe that the two equations can be used to find pairs of prices and demands that go together. For example, for $n = 1$ we can see that the price will be 45 and that there will be a demand for 129 sodas. This gives a data point, $(p_1, s_1) = (45, 129)$. Make a graph showing p on one axis and s on the other, and plot (p_1, s_1) as well as several other points. What does this graph suggest about the relationship between p and s?

b. Solve the second equation for n in terms of p. That is, rearrange the equation so that starting with a given price p, you can compute the corresponding value of n.

c. Use the equation for n in terms of p to replace the n in $s = 141 - 12n$. That is, you have a formula for n as a function of p: replace n in the s equation using that formula. [Note that this gives s as a function of p. With this equation, you can easily compute how many sodas can be sold at a given price.]

d. Use the preceding equation to find out how many sodas will be sold if the price is 75 cents.

e. Use your equation to solve for p in terms of s.

f. Use the equation from the preceding question to figure out what price should be charged in order to sell 200 sodas.

g. Simplify the equation for s in terms of p into the slope-intercept form, so that you can find the slope and intercept. Use this information to sketch the graph of p and s. Compare the result with the graph you made in part *a*.

Group Activities. Many scientists are interested in the subject of global warming. The basic issue concerns whether human activities are influencing the climate of the entire world. One widely held theory says that the amount of carbon dioxide in the atmosphere is increasing as a result of burning various kinds of fuel. According to this theory, the atmosphere will heat up as a result of the increased carbon dioxide levels. How much? And how soon? These questions are studied by developing models and making predictions. The models are very involved, and consider many variables. In this problem, we will look at just one of the variables: the number of automobiles in the US. This variable is used to predict how much gasoline is burned in the US, and that leads to predictions about the amount of carbon dioxide added to the atmosphere. In Table 4.2 are

Year	Automobiles
1940	27.5
1950	40.3
1960	61.7
1970	89.3
1980	121.6
1986	135.4

TABLE 4.2
Millions of Automobiles in the US

data on the number of automobiles in the US[1]. The figures are in units of one million, so that in 1940 there were 27.5 million automobiles. In this activity you will develop and use a linear model for the number of automobiles in the US.

1. Make a graph of the data. This can be done by hand or with a computer or calculator graphing tool. If you do it by hand, use graph paper and be careful to plot the points accurately. Do the points appear to be approximately in a straight line? A visual inspection of this type is an important first step in modeling. It would not be a good idea to use a linear model if the data do not have some appearance of lining up.
2. A second way to check if the data are approximately lined up is to compute the slope from each point to the next point. If these slopes are all identical, then the points must line up exactly. If the slopes are approximately equal, a linear model might be appropriate. Compute these slopes and see how close they all are.
3. Using a ruler, draw a line that comes as close as possible to the data points in your graph. One way to do this is to use a piece of string or thread. Pull the string tight and move it around until it seems to fit the data very well. Then have someone else use a pen to mark one point on the graph at each end of the string. Finally, use the ruler to draw the line on the graph.
4. Find the equation of your line.
5. Use the equation of the line to estimate how many autombiles were in the US in 1990. Also predict how many automobiles will be in the US in 2000.
6. How good was your estimate? Go to the reference desk in the library and ask how to find out the number of automobiles in the US in 1990. Compare the actual figure with the estimate from your model. Does the result give you any more or less confidence in your projection for the year 2000?
7. There is a theoretical way to choose the best line to fit a set of data. It is based on a carefully spelled out explanation of what is meant by *best* in this context. Use a calculator or computer to obtain the equation for the theoretically best fitting line for the automobile data in this problem. Your instructor can explain how to do this. On a graph, compare the calculator or computer's line with the one you came up with earlier. Does one seem better than the other?

Solutions to Selected Exercises

Mathematical Skills

1. a. $x = 9$ b. No solutions
   c. $x = 0$ d. Every number

[1] The data in this table were taken from *Earth Algebra,* preliminary edition, by Christopher Schaufele and Nancy Zumhoff. Harper Collins, 1993, page 89.

2. a. Slope–intercept. Start at -4 on the y axis, and plot several points going up 3 and over 1 each time.

   b. Point–Slope. Start at the point $(3, 2)$ and plot several more points going up 1 and over 2 each time. The slope here is .5 or $1/2$. A slope of 1 *over* 2 means you can find points by going up 1 and over 2.

   d. Point–Slope with point $(0, 5)$ and slope 2.

   e. Two–Intercept form. Mark 2 on the x axis and 4 on the y axis, then draw the straight line connecting these points.

   f. Slope–Intercept with slope 2.3 and y intercept 0. Also, Point–Slope with point $(0, 0)$ and slope 2.3.

3. a. $y = (3/2)x - 9/2$ or $y = 1.5x - 4.5$.

   c. $y = (2/3)x$.

4. a. $x = (2/3)(y + 9/2)$ or $x = (2/3)y + 3$.

   c. $x = (3/2)y$.

5. a. $\dfrac{x}{3} + \dfrac{y}{-4.5} = 1$.

   c. There is no Two–Intercept form for this line.

Problems in Context

1. Make a graph with the horizontal axis labeled C and the vertical axis labeled F. When it is 0 centigrade, it is 32 degrees Fahrenheit. This gives one data point $(C, F) = (0, 32)$. In fact, this is the F axis intercept for the line. Another data point is $(C, F) = (100, 212)$. Using the two data points, we can compute a slope. From $(0, 32)$ to $(100, 212)$, there is a rise of $212 - 32 = 180$ and a run of $100 - 0 = 100$. The slope is therefore $180/100 = 1.8$. This gives the equation $F = 1.8C + 32$. Suppose that there is a temperature that is the same in both temperature units. Call that temperature T. That means that (T, T) is one of the points on the graph of centigrade–Fahrenheit data points. Using the equation, that gives us $T = 1.8T + 32$. Solving that equation for T produces the answer: -40. That is, if the temperature is 40 degrees below 0 centigrade, it is also 40 below zero Fahrenheit.

2. The first thing to do is recognize this as a linear model. The pressure increases 15 pounds per square inch (psi) for each 33 feet of depth. If the depth is increased by 66 feet, there will be a 30-psi increase in pressure; if the depth is increased by 11 feet, there will be a 5-psi increase in pressure. This is proportional reasoning, and indicates that a linear model is called for. What variables should be used? Let p stand for pressure and d for depth below the surface, with p in psi and d in feet. Now express the given information in terms of points, slopes, and so on. At the surface, $d = 0$ and $p = 15$. That gives one point: $(d, p) = (0, 15)$. If you graph this with the d along the x axis and the p on the y axis, you will see that you now have a y intercept of 15. The pressure increases by 15 when the depth increases by 33. That is a slope of $15/33$. Or, if you prefer, get another data point by observing that at a depth of 33 feet the pressure must be 30. Then use the points $(0, 15)$ and $(33, 30)$

to compute the slope. With either of these two points and the slope, or using the slope and the intercept, an equation can be found: $p = (15/33)d + 15$. Notice that p is all by itself on one side, so the equation gives p as a function of d, as required. What is the pressure at a depth of 100 feet? That is, when $d = 100$, what is p? The answer is $p = (15/33)100 + 15 = 60.4545$. It is a good idea to leave the units in the calculation. Remember that the slope is in units of psi per foot. Then we have $p = (\frac{15}{33}$ psi per foot$) \cdot 100$ feet $+ 15$ psi. The answer comes out in units of psi, as it should for a pressure. Sometimes, when an error is made, it will be revealed by the fact that units come out wrong. We also want to know what the depth is if the pressure is 1,000 psi. That is, if $p = 1{,}000$ what is d? Using the equation for p and d and using 1,000 for p we obtain $1{,}000 = (15/33)d + 15$. Solve for d: $d = (1{,}000 - 15)(33/15) = 2{,}167$. This shows that the camera will be safe up to a depth of 2,167 feet.

3. In this problem you are told to use a linear model. The variables are c the total monthly costs and b the number of backpacks made each month. Suppose that $b = 10$. What is the total monthly cost? The backpacks cost \$23 each, for a total of \$230. Together with the other expenses of \$12,000 per month the total is \$12,230. This gives one data point. Using similar reasoning you can find another data point, and then the slope, and then the equation of the line for c and b. The final result will be $c = 23b + 12{,}000$.

4. Let p be the pollution level, in parts per million, t hours after 8 A.M. We have two data points $(0, 20)$ and $(4, 80)$. These can be used to find the equation $p = 15t + 20$. What is the highest the pollution gets? At 6 P.M. $t = 10$ so $p = 170$. Since that is above the 150 level, it will be necessary to publish an alert.

5. a. Plot points like these: $(45, 129)$, $(50, 117)$, $(55, 105)$, $(60, 93)$, etc.

   b. $n = (p - 40)/5$

   c. $s = 141 - 12(p - 40)/5$.

   d. $s = 141 - 12(75 - 40)/5 = 57$.

   e. $p = -5(s - 141)/12 + 40$

   f. $p = -5(200 - 141)/12 + 40 = 15.416666$. Of course, we can only set the price to a whole number of cents. Make the price 15 cents and we will actually sell 201 sodas.

   g. $s = (-12/5)p + 237$ The slope is $-12/5$ or -2.4, and the s intercept is 237. If this graph is drawn on the same axes as in part *a*, the line should go through all the points that were originally plotted.