THE USE OF THE FUNCTION CONCEPT IN FIRST YEAR ALGEBRA[1]

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We have all tried at various times in our lives to solve riddles and we have all observed others try to solve them too. Now when it comes to solving riddles folks may be classified roughly under two heads: those who give up if the answer does not come to their minds immediately and those who will try for hours. There are the easy givers up and those with the inquiring mind. The very fact that we have devoted a good part of our time to the study and teaching of mathematics must indicate that in our younger days, at least, we were not always with the easy givers up when occasionally a mathematical riddle was propounded to us from back of the teacher's desk. Of course, in these days, mathematics is taught quite differently and many of the problems of the riddle type have given place to a more rational kind of exercise, but even so some of us have had ample opportunity to observe how folks react when riddles are assigned. They may show themselves to be the easy givers up or they may show the inquiring attitude of mind.

Since the report was made by the National Committee on the Reorganization of Mathematics in Secondary Education, much publicity has been given to the use of the function concept as an organizing principle. Just how we can make use of this idea is a question which presents itself as something of a riddle to the members of our profession and so far as I know the answer is not yet forthcoming. But the national committee has succeeded in directing the attention of many folk who do possess the inquiring mind to the solution of this problem. If there is any satisfactory answer it will be found and if not, that fact too will be discovered. I believe that some time in the near future there is going to be an adequate answer, perhaps by one of you, in the form of a text book wherein the subject matter is so organized that the development of the habit of functional thinking is the main aim.

[1] Read Before the Mathematics Section of the High School Conference at University of Illinois, Nov. 19, 1925.

Since Leibnitz and Newton invented the calculus, the function concept has dominated mathematics and all exact science, but it is only within the last decade than anyone seems to have had even a remote idea that teachers of secondary mathematics could make much use of the notion. Although the development of this idea stands out as one of the highest and most significant achievements of man, that does not mean that it is so difficult that no possible use can be made of it in first year algebra. True, indeed, until the seventeenth century, the mathematicians groped about blindly to find a principle of such a fundamental nature that it could be used to unify all of their knowledge. But now that they have bequeathed it to us, we are beginning to wonder if it is not far reaching enough in its significance to simplify and clarify what would otherwise be a somewhat unrelated mass of meaningless processes and manipulations even in first year algebra.

Consider for a moment the role played by the theory of evolution in natural science. It is, I believe, a similar example of a unifying principle by means of which the phenomena of natural science are organized and explained even in the very elementary phases of the subject. It is a mode of thought not too difficult in its simpler aspects for even the young student to grasp. Since it is the method of thinking used by the natural scientist, we start to interpret the phenomena of natural science in the light of it. Because this theory was developed late in the history of man, it does not follow that it is fundamentally more difficult to understand than earlier notions concerning the subject. It is not necessary to know all about it to use it very effectively in interpreting many facts satisfactorily and in relating them all about a central core, and thus to simplify enormously the explanation of many phenomena. The function concept may become to mathematics what the theory of evolution is to natural science—a beautiful example of crude and blundering methods being pushed aside to be replaced by a method much more powerful and much more far reaching in its field of usefulness.

The National Committee on the Reorganization of Mathematics recommends: "The one great idea which is best adapted to unify the course is that of the functional relation. The primary and underlying principle of the course should be the idea of re-relationships between variables." But how are we to make use of this recommendation? This small detail is left entirely to us. In the amount of time allowed for this paper only a very limited treatment of this subject will be possible, so I would like to make some suggestions for the application of the function idea to the solution of verbal problems. It seems to me that the verbal problems should hold the most important place in the course and that symbolic manipulation should be a secondary part of the work. The pupil must understand the four fundamental operations with integers and fractions, a little of factoring—in short he must acquire various skills in handling symbols. But for what purpose? Chiefly that he may have the tools which make it possible for him to handle the equations which he sets up in the solutions of verbal problems. It is this type of problem which gives him an opportunity to think about quantities in their relationships to one another and to make some practical applications of the processes which he has learned. If more stress is placed upon verbal problems, there will be less time for the purely mechanical aspects of algebra, but even so it seems to me that there is much to gain. Some of us still spend time teaching complicated problems in complex fractions, long problems in multiplication, division, extraction of roots of polynomials, for all of which there is little or no chance for application in elementary algebra. The demands of the equation are rather simple but the pupil must be thoroughly familiar with all the symbolic manipulations which he will need to use in solving his equations. But if we spend more time on the verbal problem and teach some of this mechanical work in connection with the equations derived, the pupil will see its use and he will have more enthusiasm for the drill work which is necessary for its mastery. It is through the verbal problem, I believe, that the child is likely to get the most from algebra that is truly educative. It is here that his mind has the best opportunity to take hold of quantitative relations and to understand the "why" of the things that he does rather than just the "how" to follow blindly certain rules.

When it comes to the solution of verbal problems under our present method of teaching them, the pupil is likely to come to the conclusion that they are not very important since we have only a few of them now and then. Furthermore, he is likely to take the attitude that they are solved mainly by hit and miss or haphazard methods which consist largely of guessing and trying first one thing and then another; that there is no general method of analysis which applies to all such problems. If through a consideration of the idea of relationships some fairly general method of analysis could be found, it seems to me that this would be a great aid in developing the ability of our pupils to do orderly thinking. Perhaps we may be able to find a better way to relate and connect the work with verbal problems so that the pupil will feel more keenly that his skill at solving such problems is increasing with each problem solved.

I find that very few teachers who have written any thing at all on the use of the function concept have attempted to give any definite suggestions for its application. They prefer to play safe and make their remarks very general. However, Mr. Paul Ligda of the McClymonds High School, Oakland, California, has just written a book on the Teaching of Elementary Algebra which suggests such a plan. I found this quite interesting, so I shall review it briefly for you. Most of the verbal problems with which we deal in first year algebra are governed by a general formula $R = SxT$ which has also the other two aspects:

$$S = \frac{R}{T} \text{ or } T = \frac{R}{S}.$$

A few examples of these are as follows:

$D = RxT$ (uniform rate formula)
$S = AxB$ (surface area of rectangle, parallelogram, square)
$I = RxP$ (interest formula)
$C = NxP$ (cost formula)

Also each of these problems deals with either one or two situations, and each of these situations is governed by a general or characteristic formula connecting the unlike quantities involved.

Here is an illustration: "A man has 22 minutes to go to the station a distance of 2 miles. If he takes a car which travels at the rate of 1 mile in 8 minutes, at what distance from the station can he get out and walk, if he walks at the rate of 1 mile in 16 minutes?"

This problem contains two situations: the riding situation and the walking situation. The quantities entering each of these situations are time, rate and distance and they are governed by the uniform rate formula $D = RxT$. This formula gives the relationship among unlike quantities in each situation which is not given in the problem but it is a simple relationship governed by a general formula which the pupil knows. The problem contains statements which connect like quantities in the two situations.

Let us now exhibit the relationships existing among the quantities involved in this problem by means of a schematic problem. Let $D = RxT$ represent the quantities entering the walking situation and $d =$ rt (small letters) represent the quantities entering the riding situation. The implied relations connecting like quantities in the two situations are:

(1) $D + d = 2$ (The total distance which the man walks and rides is 2 miles)

(2) $T + t = 22$ min. (The total time spent in riding and walking is 22 minutes.)

(3) The problem gives explicitly the values $R = 1/16$ mile per minute; $r = 1/8$ mile per minute.

Write vertically the characteristic formula governing each situation. Then summarize all of the relationships between quantities thus:

Walking		Riding
D	$+$	$d = 2$
"		"
$1/16 = R$		$r = 1/8$
x		x
T	$+$	$t = 22$

Read horizontally this diagram shows the way in which like quantities in the two situations are related to each other. Read vertically it shows how unlike quantities in each of the two situations are related to each other. The problem is now translated into symbols. If any relationship were missing the diagram would automatically indicate that. Now that every quantity is either completely described or related we are ready for the symbolic solution by means of substitution and elimination. To do this eliminate R and r by substituting their numerical values.

$$
\begin{array}{rcl}
D/T = 1/16 & \qquad & d/t = 1/8 \\
16D = T & & 8d = t \\
16D + 8d = 22 & & \\
D + d = 2 & &
\end{array}
$$

Solve these two simultaneous linear equations for D and d.

Translate the result back into words and prove the solution is correct.

As here illustrated, most of the verbal problems solved in first year algebra are problems that contain two statements and thus can be expressed by means of two equations. The method of solution used by practical mathematicians in solving a problem is that of writing down the possible equations and substituting. Why then not introduce simultaneous equations early in the course and let the student obtain a thorough training in substitution, and evaluation; the methods used in work with formulae? This use of simultaneous equations is an essential feature of the method of solving verbal problems by the relationship method. In order to analyze the problem we need as many symbols as there are quantities involved. As long as we adhere to the method of expressing all the quantities in terms of one quantity, preferably the smallest, any general method for analyzing problems is not possible. But if we use a symbol for each quantity and set forth the relationships by means of the diagram, then we can proceed in a systematic manner to eliminate quantities as a part of our symbolic solution.

This type of analysis and solution will apply to most of the problems not in first year algebra. Let us summarize the steps:

I. Analysis (based on function concept, the relationship among quantities).

1. Division of the problem into situations.
2. Recognition of characteristic formula.
3. Write characteristic formula vertically.
4. Division of problem into statements.
   a. Completely describing a quantity.
   b. Explicitly stating a relation between like quantities in two situations.
   c. Recognition of implied statements.

II. Translation into symbols.

III. Symbolic Solution. (Based on equation law.)

IV. Translating Result back into words.

V. Check.

If instead of the usual academic verbal problems, we restate the problems giving them an industrial setting, the practical minded student will take more interest in them. The problem previously used in the illustration may be reworded as follows: "A contractor has 22 days to finish a contract to remove 2,000 tons of rock. If he rents a large steam shovel which can handle 1,000 tons in 8 days, how long should he keep it if his own shovel can handle 1,000 tons in 16 days?"

That this is exactly the same problem as the one which he have previously solved is seen by a glance at the relationship diagram.

Big Shovel		Little Shovel
A	$+$	$a = 2$
"		"
$1/8 = R$		$r = 1/16$
x		x
T	$+$	$t = 22$

The result is the same only the numbers have a different meaning.

Perhaps this method of analysis which is suggested and further elaborated by Mr. Ligda is sufficiently general to make it a valuable aid in the teaching of verbal problems. If it does train the pupil to be constantly watching for relationships among quantities, it is surely worthy of consideration. I believe that we are making a practical application of the function concept in first year algebra when our pupils take that attitude toward verbal problems.

A further application of this method to some problems chosen from a certain well-known first year algebra may prove interesting.

"Two automobiles started at the same time from the same point in opposite directions. The first traveled 5 miles more per hour than the second. At the end of 8 hours they were 360 miles apart. At what rate did each travel?"

Analysis: There are two situations:
Automobile A traveling one direction,
Automobile B traveling the opposite direction.

The characteristic formula governing both situations is uniform rate formula $d = rt$ since the related quantities involved are distance, rate, time.

Use capitals $D = R, T$ for situation A, and small letters $d = r_x t$ for situation B. Write these characteristic formulas vertically.

A		B
D	$+$	$d = 360$
"		"
R	$=$	$r + 5$
x		x
T	$=$	$t = 8$

Divide the problem into statements.

1. Sum of the distance traveled by the two autos is 360 miles. Then $D + d = 360$.

2. The rate of the first auto equals the rate of the second increased by 5 miles. Then $R = r + 5$.

3. Both autos traveled the same length of time—8 hours. Translate into symbols and locate the translated symbols in the diagram.

Symbolic solution: Eliminate T and t by substituting their values $8R = D$ $8r = d$.

In $D \quad d = 360$, substitute $8R$ for D and $8r$ for d and get

$$8R + 8r = 360$$
$$R = r + 5$$

Solve these two equations simultaneously for R and r. $r = 20$ $R = r + 5 = 25$.

Translate the result into words.

Check.

Second example: A boy, Albert, weighing 85 pounds, sits 7 feet from the fulcrum and balances a boy, John, who is sitting 6 feet from the fulcrum on the other side. What is the weight of the boy, John?

Analysis: There are two situations in this problem, Albert sitting on one side of the fulcrum and John sitting on the other.

The characteristic formula governing both situations is $L = WxD$. (The law of the lever.)

Use capitals $L = WxD$ for the situation created by Albert and $1 = wxd$ for the situation created by John. Write these characteristic formulas vertically.

Albert		John
L	$=$	1
"		"
$85 = W$		w
x		x
$7 = D$		$d = 6$

Divide the problem into statements.

Albert weighs 85 pounds.
Albert sits 7 feet from the fulcrum.
John sits 6 feet from the fulcrum.
$L = 1$ (law of leverage).

Translate into symbols and locate these symbols in the diagram.

Symbolic Solution:

$L = 1$
$WxD = wxd.$

Eliminate W, D, and d by substituting their values.

$85.7 = Wx6$
$w = 99\frac{1}{6}$ pounds.

Translate the result into words. Check.

If we are to make a practical application of the use of the function concept in first year algebra, we must aim constantly to develop in our pupils the power to understand the relationships that exist among quantities. A proper emphasis upon the study of verbal problems seems to offer the best possibility for realizing this aim. It is here that the pupil has presented the kind of situations which develop the "why" and the "what for" and the "what happens if" attitude toward the quantitative side of life. Here he cultivates the habit of expecting to understand rather than just the ability to use rules after he has been shown "how." Some points of view which I believe are important in helping us to make the best use of the verbal problem for developing this type of quantitative thinking are:

First: There must be no teaching of functional theory as such. The teacher must constantly bear in mind that the aim of the course is to develop skill in functional thinking, but in working toward this end, it will not be necessary even to mention the word "function." I am well aware of the fact that there is much disagreement on this point. Many of my co-workers believe that since the function concept is so fundamental to an understanding of mathematics in later courses, we should begin to use the word "function" even in first year algebra; to introduce even then the notion that one variable is a function of another variable. They claim that since a thorough appreciation of this notion is so necessary, it must be presented many times before the student is able to grasp it, and that it is advisable to teach some functional theory even in first year algebra. But since the word "function" has a much different meaning in common parlance than it has in mathematics, I believe that it is better to use such expressions as: "the value of y depends upon the value of x" rather than "y is a function of x." For example, it is better to say, "distance depends upon rate and time" rather than "distance is a function of rate and time." A very informal treatment of the function theory seems to me to be all that is either necessary or desirable in this first year of algebra.

Second: An early introduction of the study of the graph and tables which show relationships between numbers which change together is advisable. This is a type of work which the children enjoy—but that is not all, they work at it from an understanding point of view—just the attitude that we want them to get toward every phase of algebra. It may be noted here that not all of the work with graphs and tables which we usually do is functional in nature. Statistical graphs could not be considered as of much value in developing the idea of a continuous variation among variables, but such graphs as those showing the relationship between principal and interest, or cost and number are examples of the kind that do develop this idea.

The study of the graph and table makes it possible very soon to introduce the formula and simultaneous linear equations. This early introduction of simultaneous equations seems advisable if verbal problems are to be treated from the viewpoint of the relationships involved among quantities. Most problems contain two situations, so can be more easily expressed and solved by two equations than by only one, thus avoiding the necessity of expressing all of the unknown quantities in terms of one. Children occasionally discover almost unaided, the method of solving problems by simultaneous equations some time before the middle of the second semester when the subject is usually introduced. Just recently, I was trying to teach some verbal problems which are supposed to lead to fractional equations involving one unknown. After school, a boy whom I shall call John, came to me about a problem which he said he had not yet been able to solve. Here is the problem: "Separate 45 into two parts such that $5/9$ of the greater exceeds $1/2$ of the less by 6." John said: "There are two numbers to be found here, are there not? Why can't you use two different letters—one to represent each? If you add them you get 45. If you add 6 to half of the small one you get $5/9$ of the greater one. Isn't there any way to find both these numbers this way?"

It occurred to me that John was about as near ready to learn something about simultaneous linear equations as he would ever be. What should I have done with him?

Third: After the equation has been obtained it must be solved without juggling or artificial devices. The pupil should be able to justify every step with good reasons and the word "transpose" when applied to an equation should not be in his vocabulary. It is an enemy to an understanding attitude of mind. Let him look upon his equation as a question which asks him to find the value of some unknown number. He may write it thus until he is thoroughly familiar with its interrogative nature.

$$5x + 8 \stackrel{?}{=} 24 - 3x$$

For what value of x is this statement true? There is but one equation law which he need consider. "Whatever is done to one member of this equality must also be done to the other." Let him assume that he has two equal numbers $5x + 8$ and $24 - 3x$. If he wants to know what $5x$ equals he may find out by subtracting 8 from the left members, but in order to preserve the equality or balance, he must also subtract 8 from the right members. This gives $5x = 16 - 3x$.

In the right member he now has $3x$ less than 16. If $3x$ be added to the right member the result will be 16 and clearly now the right member contains no x term. If then $3x$ be added to both members of this equation, the balance will not be destroyed.

$$8x = 16.$$

Divide both members of this equation by 8, to find the value of $1x$.

It may appear to be more laborious to think of all of these additions, subtractions, multiplications, divisions and the like in all such cases, but when the pupil handles the equation this way he understands what he is doing. He is not just juggling symbols.

Fourth: We must ever bear in mind that the main purpose of algebra is to develop the powers of reason. The habit of approaching a problem from the viewpoint of studying the relationships among the quantities involved is a type of thinking which is directly useful to the pupil whether he goes to college or not. The ability to think in terms of relationships among quantities, space and variables is often required of all intelligent human beings—hence there is ever a field in real life into which this type of training can "transfer." In the affairs of real life, good citizens are constantly called upon to estimate, to compare or to see clearly the relations between variable quantities even when exact computations are not made. Hence it appears that the final computation in connections with equations is not so valuable a training as the thinking required to formulate the equation, graph or table from the situations set forth in the problem. A proper use of the verbal problem provides excellent training in just the type of functional thinking which the pupil is sure to find very valuable.

If then we wish to make use of this idea of studying mathematics from the viewpoint of the function concept it is necessary that our work be so organized that the analysis of verbal problems hold the most important place in the course. There should be no teaching of functional theory as such. The graph, the table and the formula are an inseperable trio for expressing relationships among quantities which should be introduced early. Simultaneous linear equations should be introduced as soon as their use makes it possible to solve problems more readily. Symbolic manipulation can be taught as we go along, chiefly as the pupil needs it to help him in the solution of the equations which he sets up. It must be possible for a pupil to give a satisfactory reason for every step which he takes in solving an equation—there must be no juggling. The pupil must develop his ability to see relationships among quantities for this is the very essence of intelligence.