Recap

$$Xw = y$$

X			solution w
square	even	$\det(X) \neq 0$	$\hat{w} = X^{-1}y$
tall	over	$\det(X^TX) \neq 0$	$\hat{w} = (X^TX)^{-1}X^Ty$
wide	under	$\det(XX^T) \neq 0$	$\hat{w} = X^T(XX^T)^{-1}y$

Python

`np.linalg.inv()`	inverse	X^{-1}
`np.transpose()`	transpose	X^T
`.dot()`	matrix multiplication	AB
`np.linalg.det()`	determinant	$\det(X)$

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1. even

$$\det(X) = 1 \times 4 - 1 \times 3 = 1 \neq 0$$

X is invertible

$$\hat{w} = X^{-1} y$$

$$= \underset{2\times 2}{\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times 1}$$

2. even

$$\det(X) = 1 \times 6 - 2 \times 3 = 0$$

3. over

$\det(X^T X)$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$$

symmetric — $X^T X$, XX^T

$\det(X^T X) = 6 \times 21 - 9 \times 9$

$= 45 \neq 0$

$\hat{w} = (X^T X)^{-1} X^T y$

$$= \frac{1}{45} \begin{bmatrix} 21 & -9 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$$

2×2 $\quad 2 \times 3$ $\quad 3 \times 1$

$$= \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix} \quad 2 \times 1$$

4. under

$$\det(XX^T) \neq 0$$

$$XX^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\det(XX^T)$$

$$= 2\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 2 \times 8 - 2 \times 4 + (-4)$$

$$= 4 \neq 0$$

$$\hat{w} = X^T(XX^T)^{-1}y$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

4×3 $\quad$ 3×3 $\quad$ 3×1

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

4×1

5. $Xw = y$

$(Xw)^T = y^T$

$w^T X^T = y^T$

Let $X = X^T$

even

$\det(X) = 0$

no solution

6.

$$w^T X = y^T$$

let $X = X^T$, $X^T = X$

$$\det(XX^T)$$

$$\det(X^T X) = 45 \neq 0$$

$$\hat{w}' = X^T (XX^T)^{-1} y$$

$$\hat{w} = X (X^T X)^{-1} y$$

$$(\hat{w})^T = \left(X (X^T X)^{-1} y\right)^T$$

$$w^T = y^T \left(X (X^T X)^{-1}\right)^T$$

$$= y^T \left((X^T X)^{-1}\right)^T X^T$$

$$= y^T (X^T X)^{-1} X^T$$

$$\hat{w}^T = y^T (X^T X)^{-1} X^T$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

1×2 $\quad 2 \times 2$ $\quad 2 \times 3$

$$= \begin{bmatrix} 0.0667 & 0.1333 & -0.3333 \end{bmatrix} \quad 1 \times 3$$