MATH PRIMER / NOTATIONS

Sets: Collection of objects, usually denoted with a symbol such as X

e.g. $X = \{\text{apple, orange, fish}\}$

$\mathbb{Z} \triangleq \{x : x \text{ is an integer}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

"defined to be"

$\mathbb{R} \triangleq \{x : x \text{ is a real number}\}$

Functions: (roughly speaking) data of $(X, Y, x \mapsto f(x))$

where X, Y are sets and $x \mapsto f(x)$ assigns $x \in X$ to some $f(x) \in Y$.

"in"

X: domain

Y: target / codomain

Notation $f : X \to Y$

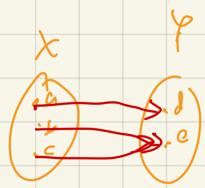
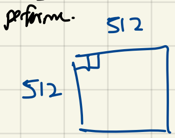


image classifi.

$f : [0, 256]^{512 \times 512} \to \{0, 1, \ldots, 9\}$.

$\|f_{true} - f_{pred}\|$ → measures perform.



(real-valued)

e.g. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$ — e.g. $f(x) = x+1$

$f : \mathbb{Z} \to \mathbb{R}$, $n \mapsto e^n$ — e.g. $f(x) = e^x$ but x is integer

$f : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$ — take largest integer less than or equal to x. "Floor fn"

Vector Spaces: roughly speaking is a set V with addition & scalar multiplication-by-$\mathbb{R}$ structure.

$(V, \mathbb{R}, +, \cdot, 0, 1)$.

key examples:

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}$$

2×3 matrices →
$$\mathbb{R}^{2\times 3} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}$$

Linear maps / operators: functions $f : \mathbb{R}^d \to \mathbb{R}^m$ preserving v.sp. structure

i.e. writing $x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{R}^d$, $c \in \mathbb{R}$

- $f(x+y) = f(x) + f(y)$
- $f(c \cdot x) = c \cdot f(x)$

matrices give linear maps!

$$\vec{v} \mapsto M\vec{v}.$$

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1\cdot x + 0\cdot y + 1\cdot z \\ 0\cdot x + 2\cdot y + 3\cdot z \end{pmatrix} = \begin{pmatrix} x+z \\ 2y+3z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\underset{M}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\cdot M\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\cdot M\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\, M\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

e.g. $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathbb{R}^3$.

Tut 2

RECAP (w. Extra stuff)

- $\mathbb{R}^n = \left\{ \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} : x_i \text{ is real number} \right\}$ (n-tuples) (abbrev. $x_i \in \mathbb{R}$) is a vector space over real numbers.

 unpacked: we can add: $\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} + \begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix} = \begin{pmatrix}x_1+y_1\\\vdots\\x_n+y_n\end{pmatrix}$

 scalar multiply: for $c \in \mathbb{R}$, we have $c \cdot \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}c\cdot x_1\\\vdots\\c\cdot x_n\end{pmatrix}$

 vector space.

- Elements of $\mathbb{R}^n$ are called vectors (of n-tuples)

- For vectors $v_1 = \begin{pmatrix}x_{11}\\x_{21}\\\vdots\\x_{n1}\end{pmatrix}$, $v_2 = \begin{pmatrix}x_{12}\\\vdots\\x_{n2}\end{pmatrix}$, ..., $v_m = \begin{pmatrix}x_{1m}\\\vdots\\x_{nm}\end{pmatrix}$

 in $\mathbb{R}^n$, we say that $\{v_1, \dots, v_m\}$ are linearly dependent if we can write one of the other vectors as a linear sum of the other vectors

 e.g. $v_1 = \underset{\mathbb{R}}{c_2} \cdot v_2 + \underset{\mathbb{R}}{c_3} \cdot v_3 + \dots + \underset{\mathbb{R}}{c_m} \cdot v_m$

Intuition: v_1 is in the subspace spanned by $v_2, \dots, v_n$.

- The opposite is linear independence (intuition: every vector adds one direction)

 show: $\sum_{i=1}^{n} c_i v_i = 0 \Rightarrow c_i = 0$ for all $1 \le i \le m$

- Linear Maps are functions of vector spaces that are linear

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

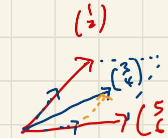
Impt E.g. matrices

$$\boxed{\begin{aligned} L(x_1 + x_2) &= L(x_1) + L(x_2) \\ L(c \cdot x) &= c \cdot L(x) \end{aligned}}$$

$\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}$.

e.g. $\left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}5\\6\end{pmatrix} \right\}$ are linearly dependent

since $\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}5\\6\end{pmatrix}$



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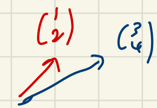
e.g. $\left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \right\}$ is linearly independent

$\begin{pmatrix}1\\2\end{pmatrix} \neq c \begin{pmatrix}3\\4\end{pmatrix}$

$\begin{pmatrix}3\\4\end{pmatrix} \neq c \begin{pmatrix}1\\2\end{pmatrix}$ for any $c \in \mathbb{R}$



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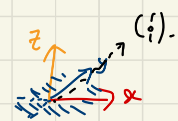
e.g. $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ are linearly independent since

if $c_1, c_2, c_3 \in \mathbb{R}$ satisfy: $c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$,

then, we have $\left.\begin{aligned} c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}\right\}$ $c_i = 0$ for all $i = 1, 2, 3$



Matrices are just linear maps are just matrices and

E.g. 2×3 matrix $M = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 3 & 4 \end{pmatrix}$.

As a fn., we have $M : \mathbb{R}^3 \to \mathbb{R}^2$

columns encode M. standard's coordinates

Then, $M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + (-1)y + 0 \cdot z \\ 5x + 3y + 4z \end{pmatrix}$

what is $M \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$?

$M \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$?

$M \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$?

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a linear sum $= x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\, M \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \cdot M\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot M\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Suppose we have linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ and

$L \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$L \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ just like M

$L \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$L \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

by linearity

$$= x \cdot L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \cdot L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x \\ 5x \end{pmatrix} + \begin{pmatrix} -y \\ +3y \end{pmatrix} + \begin{pmatrix} 0z \\ 4z \end{pmatrix}$$

$$= \begin{pmatrix} 2x - y + 0 \cdot z \\ 5x + 3y + 4z \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Future application: differentiation

what is the derivative of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$?

"instantaneous slope" $f'(x) = 2x$ is the slope at x.

what is the derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix}$?

what is a derivative?

$f(x + \varepsilon) - f(x) \approx f'(x)\varepsilon$.

Linear map here is $\varepsilon \mapsto \underbrace{f'(x)}_{\text{num.}} \cdot \varepsilon$.

Best linear approximation at $\begin{pmatrix} x \\ y \end{pmatrix}$

$f\begin{pmatrix} x + \varepsilon \\ y + \delta \end{pmatrix} - f\begin{pmatrix} x \\ y \end{pmatrix} \approx L\begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$, where L is a linear map aka matrix!

$\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$C^\infty(\mathbb{R}) \triangleq \{$ infinitely differentiable fns on $\mathbb{R} \}$

usp. $(f(\cdot) + g(\cdot))(x) = f(x) + g(x)$.

$(c \cdot f)(x) = c \cdot f(x)$.

$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$.

$\frac{d(c \cdot f)}{dx} = c \cdot \frac{df}{dx}$.

(new concept): Invertibility of a linear map

L looks like a $(n \times m)$ matrix

$$n\left\{\underbrace{\begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1m} \\ L_{21} & \ddots & & \\ \vdots & & \ddots & \\ L_{n1} & & & L_{nm} \end{pmatrix}}_{m}\right.$$

DEFINITION: Let $L: \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. We say that L is invertible if and only if there exist another linear map $F: \mathbb{R}^n \to \mathbb{R}^m$ such that

(HAS right-inverse) $L \circ F : \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{L} \mathbb{R}^n = id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ ← what is the matrix of this?

AND

(HAS left-inverse) $F \circ L : \mathbb{R}^m \xrightarrow{L} \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m = id_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$

Right-Inverse : $LF =$

Left-Inverse : $FL =$

THEOREMS: Fix $L: \mathbb{R}^m \to \mathbb{R}^n$ to be a linear map.

① L has Left Inverse if and only if L is injective (one-one)

② L has Right Inverse if and only if L is surjective (onto)

③ L is invertible if and only if $m=n$ & L is injective

④ L is invertible if and only if $m=n$ & L is surjective

⑤ L is invertible if and only if $\frac{1}{\det X}$ exist if and only if $\det X \neq 0$

⑥ L is invertible if and only if we have equality $\underbrace{rk}_{\text{rank}}L$ $= \dim \mathbb{R}^n = n$ rank(L)

DEFINITIONS

Let $f: X \to Y$ be a function.

- f is injective (one-one) if for all $x_1, x_2 \in X$, f has the property that $f(x_1) = f(x_2) \underset{\text{implies}}{\Longrightarrow} x_1 = x_2$

- f is surjective (onto) if for all $y \in Y$, there exist some $x \in X$ s.t. $f(x) = y$

Rank of a matrix / linear map

$L(\mathbb{R}^m) = \{ y \in \mathbb{R}^n : Lx = y$ for some $x \in \mathbb{R}^m \}$

Defn: Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map.

image of L

We define $\text{rank } L := \dim L(\mathbb{R}^m) := \dim \operatorname{Im} L$

to be the dimension of the image of L.

Consequence: write L as a matrix

$$L = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & \ddots & \vdots \\ \vdots & & & \\ a_{m1} & - & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} L_1 & L_2 & \cdots & L_n \end{pmatrix}$$

where $L_1, \cdots L_n$ are the columns

then, the image of L in $\mathbb{R}^m$

is just all the possible linear sum of L_is

$$L(\mathbb{R}^m) = \{ c_1 L_1 + \cdots + c_n L_n \in \mathbb{R}^m \mid c_1, \ldots, c_n \in \mathbb{R} \}$$

e.g. if $L_1, \ldots, L_n$ are lin. independent, then imm. have: $\text{rank } L = \dim L(\mathbb{R}^n) = n$.

Fact: if among $\{L_1, \ldots, L_n\}$, we can choose just k of them, say $L_1, L_2, \ldots, L_k$ s.t. $L_1, \ldots, L_k$ are linearly independent, then $\dim L(\mathbb{R}^n) \geq k$

e.g. $n = 5$
$k = 3 \leq n$.

Fact: if for all choices of ℓ vectors from $\{L_1, \ldots, L_n\}$, $L_{j_1}, \ldots, L_{j_\ell}$, it happens that $\{L_{j_1}, \ldots, L_{j_\ell}\}$ is linearly dependent, then $\text{rank } L < \ell$.

e.g. $\ell = 4$

Row Echelon Form

How to get the dimension of the image?

e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 6 \\ 2 & 3 & 6 \end{pmatrix}$?

Fact: the following operations do not change the dimension of the image

E_1: Add a scalar multiple of one row to another row

e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 1 & 5 & 7 \\ 2 & 3 & 6 \end{pmatrix} \overset{R_2 + (-1)R_1}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 2 \\ 2 & 3 & 6 \end{pmatrix}$

E_2: scalar multiply one row

e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 2 \\ 2 & 3 & 6 \end{pmatrix} \overset{R_2 \to \frac{1}{2}R_2}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & & 6 \end{pmatrix}$

E_3: exchange two rows

e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 3 & 6 \end{pmatrix} \overset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 3 & 6 \\ 0 & 1 & 1 \end{pmatrix}$

Use this to get (reduced) row echelon form

e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 1 & 5 & 7 \\ 2 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 3 & 6 \end{pmatrix} \overset{R_3 + 3R_2}{\sim} \begin{pmatrix} 1 & \cancel{3}\,0 & \cancel{5}\,0 \\ 0 & 1 & \cancel{1}\,0 \\ 0 & 0 & -1 \end{pmatrix}$

What is the dim. of v.sp. spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \right\}$?

Hint: each vector adds a new direction!

$$f : X \to Y.$$

surjective: $f(X) = Y$.

for every $y \in Y$, there exist $x \in X$ s.t. $f(x) = y$.

injective: if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

to be invertible

$\Downarrow$

exist L^{-1} s.t.

$L \circ L^{-1} = I$

$L^{-1} \circ L = I$,

need surjective + injective.

$m = n$. $\Leftarrow$ ① $m \geq n$ ① $m \leq n$.

$L : \mathbb{R}^2 \to \mathbb{R}^2$.

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$,

$\mathbb{R}^2 = 2 \neq 1 = \dim L(\mathbb{R}^2)$.

fails to be surjective