EE2211 Tutorial 4

(Systems of Linear Equations)

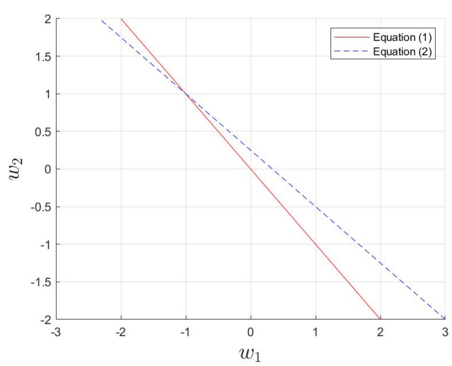
Question 1:

Given $\mathbf{Xw} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an even-determined system.
(b) det($\mathbf{X}$) = 1×4 – 1×3 = $1 \neq 0$. $\mathbf{X}^{-1} = \frac{1}{1}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.
(c) $\hat{\mathbf{w}} = \mathbf{X}^{-1}\mathbf{y} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.




```
import numpy as np

m_list = [[1, 1], [3, 4]]

X = np.array(m_list)

inv_X = np.linalg.inv(X)

y = np.array([0, 1])

w = inv_X.dot(y)

print(w)
```

(Systems of Linear Equations)

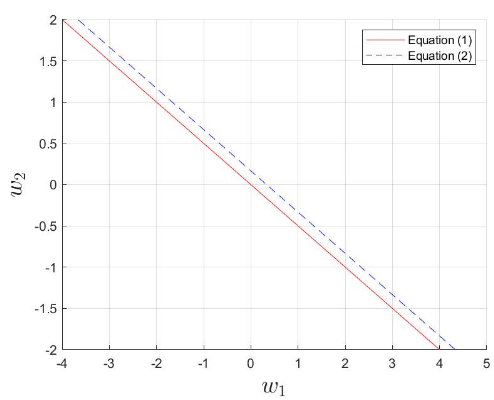
Question 2:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an even-determined system.
(b) $\mathbf{X}$ is NOT invertible since the determinant of $\mathbf{X}=1\times 6 - 2\times 3 = 0$.
(c) There is no solution for $\mathbf{w}$ since the rows/columns of $\mathbf{X}$ are inter-dependent. The two lines shown in the plot are parallel and has no intersection.



(Systems of Linear Equations)
Question 3:

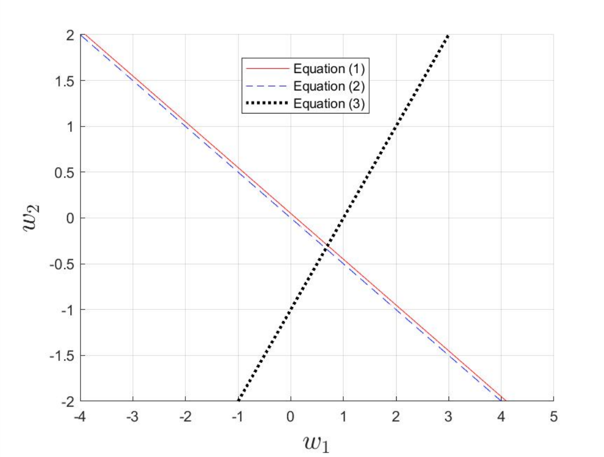
Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Find a solution for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an over-determined system.
(b) $\mathbf{X}$ is NOT invertible but $\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$ is. The determinant of $\mathbf{X}^T\mathbf{X} = 6\times 21 - 9\times 9 = 45$.
(c) An approximated solution is given by

$$\hat{\mathbf{w}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}.$$




```
# import numpy as np

# m_list = [[1, 2], [2, 4], [1, -1]]

# X = np.array(m_list)

# inv_XTX = np.linalg.inv(X.transpose().dot(X))

# pinv = inv_XTX.dot(X.transpose())

# y = np.array([0, 0.1, 1])

# w = pinv.dot(y)

# print(w)


import numpy as np

from numpy.linalg import inv

X = np.array([[1, 2], [2, 4], [1, -1]])

y = np.array([0, 0.1, 1])

w = inv(X.T @ X) @ X.T @ y

print(w)
```

(Systems of Linear Equations)

Question 4:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an under-determined system.
(b) $\mathbf{X}$ is NOT invertible but $\mathbf{X}\mathbf{X}^T$ is.

The determinant of $\mathbf{X}\mathbf{X}^T = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}\right) = 2\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 2\times 8 - 2\times 4 + (-4) = 4$.

(c) $\hat{\mathbf{w}} = \mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

(Systems of Linear Equations)
Question 5:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an even-determined system.
(b) $\mathbf{X}$ is NOT invertible since the determinant of $\mathbf{X} = 1\times 6 - 2\times 3 = 0$.
(c) There is no solution for $\mathbf{w}$ (two parallel lines).

(Systems of Linear Equations)
Question 6:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where

$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an under-determined system (there are 3 unknowns with 2 equations).
(b) $\mathbf{X}$ is NOT invertible but $\mathbf{X}^T\mathbf{X}$ is. The determinant of $\mathbf{X}^T\mathbf{X}$ $=6\times21-9\times9=45$.
(c) A constrained solution (exact) is given by

$\hat{\mathbf{w}}^T = (\mathbf{X}\mathbf{a})^T$ (The 3-dimensional vector $\mathbf{w}$ can be constrained by projecting $\mathbf{X}$ onto a 2-dimensional vector $\mathbf{a}$)

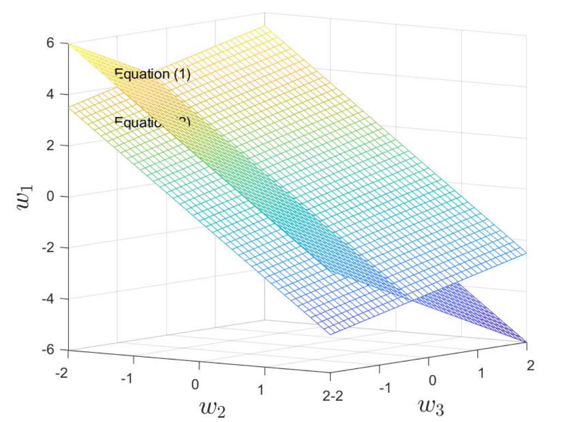
$$= \mathbf{a}^T\mathbf{X}^T$$

$$= \mathrm{y}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0667 & 0.1333 & -0.3333 \end{bmatrix}.$$

Note: dim($\mathbf{X}$) is 3×2, dim($\mathbf{a}$) is 2×1, estimation is done/constrained on/to the lower dimension of (3×2) and then projected back to the higher dimension 3.



(Systems of Linear Equations)
Question 7:
This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$$\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) True
b) False

Answer: b)

Solution: Left inverse is given by $(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ where $\mathbf{X}^T\mathbf{X}$ should be invertible. In this case, $\mathbf{X}^T\mathbf{X}$ is not invertible so the matrix does not have a left inverse.

(Systems of Linear Equations)

Question 8:

MCQ: Which of the following is/are true about matrix $\mathbf{A}$ below? **There could be more than one answer.**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

a) $\mathbf{A}$ is invertible
b) $\mathbf{A}$ is left invertible
c) $\mathbf{A}$ is right invertible
d) $\mathbf{A}$ has no determinant
e) None of the above

Answer: c and d.