(Systems of Linear Equations)
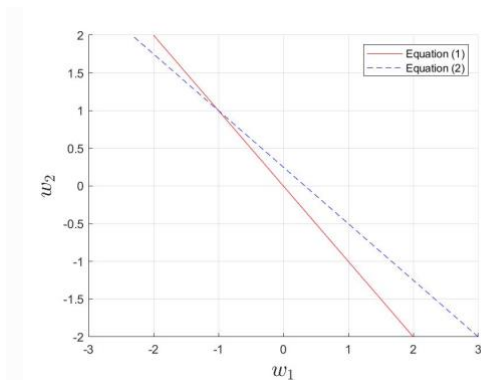**Question 1:**

Given $\mathbf{Xw} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    (a)   What kind of system is this? (even-, over- or under-determined?)
    (b)   Is $\mathbf{X}$ invertible? Why?
    (c)   Solve for $\mathbf{w}$ if it is solvable.

Answer:

    (a)   This is an even-determined system.

    (b) $\det(\mathbf{X}) = 1 \times 4 - 1 \times 3 = 1 \neq 0.$ $\mathbf{X}^{-1} = \frac{1}{1}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}.$

    (c) $\hat{\mathbf{w}} = \mathbf{X}^{-1}\mathbf{y} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$


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import numpy as np

m_list = [[1, 1], [3, 4]]

X = np.array(m_list)

inv_X = np.linalg.inv(X)

y = np.array([0, 1])

w = inv_X.dot(y)

print(w)
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(Systems of Linear Equations)
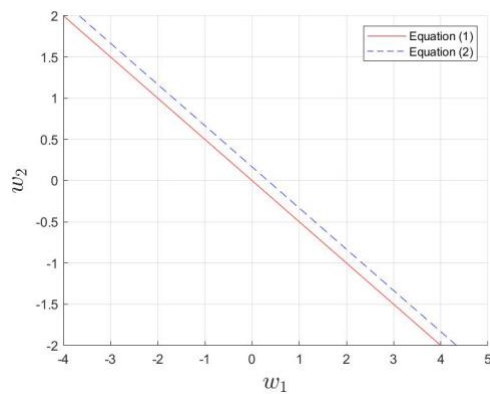**Question 2:**

Given $\mathbf{Xw} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an even-determined system.
(b) $\mathbf{X}$ is NOT invertible since the determinant of $\mathbf{X} = 1 \times 6 - 2 \times 3 = 0$.
(c) There is no solution for $\mathbf{w}$ since the rows/columns of $\mathbf{X}$ are inter-dependent. The two lines shown in the plot are parallel and has no intersection.
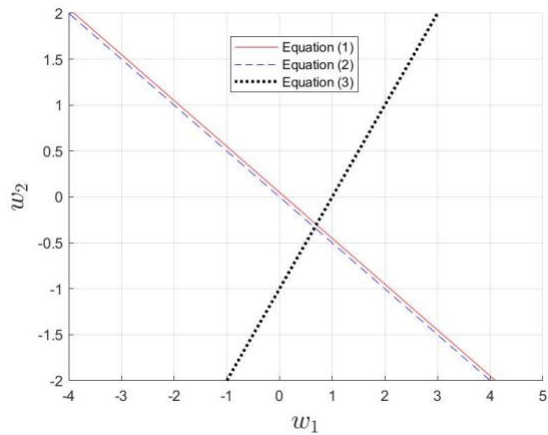

(Systems of Linear Equations)
Question 3:

Given $\mathbf{Xw} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why?
(c) Find a solution for $\mathbf{w}$ if it is solvable.

Answer:

(a) This is an over-determined system.
(b) $\mathbf{X}$ is NOT invertible but $\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$ is. The determinant of $\mathbf{X}^T\mathbf{X} = 6 \times 21 - 9 \times 9 = 45$.
(c) An approximated solution is given by

$$\widehat{\mathbf{w}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}.$$

import numpy as np

m_list = [[1, 2], [2, 4], [1, -1]]

X = np.array(m_list)

inv_XTX = np.linalg.inv(X.transpose().dot(X))

pinv = inv_XTX.dot(X.transpose())

y = np.array([0, 0.1, 1])

w = pinv.dot(y)

print(w)

import numpy as np

from numpy.linalg import inv

X = np.array([[1, 2], [2, 4], [1, -1]])

y = np.array([0, 0.1, 1])

w = inv(X.T @ X) @ X.T @ y

print(w)

(Systems of Linear Equations)
Question 4:

Given $\mathbf{Xw} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

    (a) What kind of system is this? (even-, over- or under-determined?)
    (b) Is $\mathbf{X}$ invertible? Why?
    (c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

    (a) This is an under-determined system.
    (b) $\mathbf{X}$ is NOT invertible but $\mathbf{XX}^T$ is.

The determinant of $\mathbf{XX}^T = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}\right) = 2\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 2\times 8 - 2\times 4 + (-4) = 4$.

    (c) $\hat{\mathbf{w}} = \mathbf{X}^T(\mathbf{XX}^T)^{-1}\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

(Systems of Linear Equations)
Question 5:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    (a) What kind of system is this? (even-, over- or under-determined?)
    (b) Is $\mathbf{X}$ invertible? Why?
    (c) Solve for $\mathbf{w}$ if it is solvable.

Answer:

    (a) This is an even-determined system.
    (b) $\mathbf{X}$ is NOT invertible since the determinant of $\mathbf{X} = 1\times 6 - 2\times 3 = 0$.
    (c) There is no solution for $\mathbf{w}$ (two parallel lines).

(Systems of Linear Equations)
Question 6:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

    (a) What kind of system is this? (even-, over- or under-determined?)
    (b) Is $\mathbf{X}$ invertible? Why?

(c) Solve for **w** if it is solvable.

Answer:

(a) This is an under-determined system (there are 3 unknowns with 2 equations).
(b) **X** is NOT invertible but $\mathbf{X}^T\mathbf{X}$ is. The determinant of $\mathbf{X}^T\mathbf{X}$ $= 6 \times 21 - 9 \times 9 = 45$.
(c) A constrained solution (exact) is given by

$$\widehat{\mathbf{w}}^T = (\mathbf{X}\mathbf{a})^T \quad \text{(The 3-dimensional vector } \mathbf{w} \text{ can be constrained by projecting } \mathbf{X} \text{ onto a 2-dimensional vector } \mathbf{a})$$

$$= \mathbf{a}^T\mathbf{X}^T$$

$$= \mathbf{y}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$$

$$= [0 \quad 1]\begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= [0.0667 \quad 0.1333 \quad -0.3333\,].$$

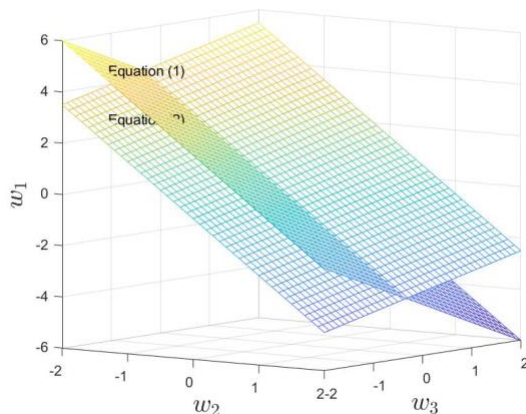Note: If we solve w (asked exactly from the question):

We can do a Transpose at first on both sides, say

$$(\mathbf{w}^T\mathbf{X})^T = \mathbf{y}^T \quad => \quad \mathbf{X}^T\mathbf{w} = \mathbf{y}$$

Assume that we have a new notation $\boldsymbol{\theta} = \mathbf{X}^T$ and we can then use the formula: $\mathbf{w} = \mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}\mathbf{y}$

$$=> \quad \mathbf{w} = \boldsymbol{\theta}^T(\boldsymbol{\theta}\,\boldsymbol{\theta}^T)^{-1}\mathbf{y} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{y}$$

Note: dim(**X**) is 3×2, dim(**a**) is 2×1, estimation is done/constrained on/to the lower dimension of (3×2) and then projected back to the higher dimension 3.

(Systems of Linear Equations)
Question 7:
This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) True
b) False

Answer: b)

Solution: Left inverse is given by $(X^TX)^{-1}X^T$ where X^TX should be invertible. In this case, X^TX is not invertible so the matrix does not have a left inverse.

(Systems of Linear Equations)
Question 8:
MCQ: Which of the following is/are true about matrix **A** below? **There could be more than one answer.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

a) **A** is invertible
b) **A** is left invertible
c) **A** is right invertible
d) **A** has no determinant
e) None of the above

Answer: c and d.