Continuous Probability function / Probability Distribution / Prob. Density function.

The function $f(x)$ is called probability density function for a continuous random variable X if $f(x)$ satisfies the following properties :-

1) $\boxed{f(x) \geq 0}$, $\forall\ x \in R$

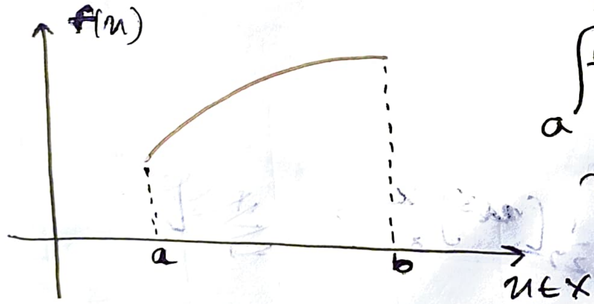
2) $\boxed{\int_{-\infty}^{\infty} f(x)}$

Continuous Distribution Function

The distribution function $F(x)$ for continuous random variable X.

$$\boxed{F(x) = P(X \leq x) = P(-\infty < x \leq x) = \int_{-\infty}^{x} f(u)\,du}$$

→ Graphical illustration



$$\int_a^b f(x)\,dx \cong \sum f(x)$$

$\int_a^b f(x)\,dx$ → x is Continuous RV

$\sum f(x)$ → x is Discrete RV

$$P(a < X \leq b) = P(a \leq x < b) = P(a < X < b) = P(a \leq X \leq b)$$

Q1. Find the constant 'c' such that the function $f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a density function & compute $P(1 < X < 2)$ hence find distribution function $F(x)$

A. $f(x) \geq 0$, $\forall\ x \in R$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} 0 + \int_{0}^{3} cx^2\,dx + \int_{3}^{\infty} 0 = 1$$

$$c\left[\frac{x^3}{3}\right]_0^3 = 1 \rightarrow c\,\frac{27}{3} = 1 \rightarrow 9c = 1$$

$$\boxed{c = \frac{1}{9}}$$

$$\Rightarrow P(1 < x < 2) = \int_1^2 \frac{x^2}{9}\,dx \rightarrow \left[\frac{x^3}{9\times 3}\right]_1^2 \rightarrow$$

$$\Rightarrow \frac{1}{27}[8-1] \rightarrow \frac{7}{27}$$

$\Rightarrow$ Distribution function $\rightarrow$ Case 1 if $\boxed{x \leq 0}$

$$F(x) = P(X \leq x \leq 0) \rightarrow \int_{-\infty}^{0} 0 \rightarrow 0$$

Case 2 if $\boxed{0 < x < 3}$

$$F(x) = P(0 \leq X \leq x) \rightarrow \int_0^x \frac{u^2}{9} \rightarrow \frac{1}{27}\left[u^3\right]_0^x \rightarrow$$

$$\Rightarrow \int_0^x f(u)\,du$$

$$\Rightarrow \frac{1}{27}\left[u^3\right]_0^x \rightarrow \frac{1}{27}\left[\frac{x^3}{1}\right] = \frac{x^3}{27}$$ ans.

Case 3 if $\boxed{x \geq 3}$

$$F(x) = P(X \geq 3) = \int_{-\infty}^{0} 0 + \int_0^3 \frac{x^2\,dx}{9} + \int_3^x 0 \Rightarrow \left[\frac{x^3}{27}\right]_0^3$$

$$\Rightarrow 1$$

Mean & Variance of a Distribution

1.) $\boxed{\mu = \sum x_i f(x_i) = \sum x_i p_i}$ if x is Discrete RV

2.) $\boxed{\mu = \int_{-\infty}^{\infty} x f(x)\,dx}$ if x is continuous R.V

3.) $\boxed{\sigma^2 = \sum (x_i - \mu)^2 f(x_i)}$ if x is DRV

4.) $\boxed{\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx}$ if x is CRV

Q1. Find the mean & variance of the RV x where
x = no. of heads in a single toss of fair coin

A.

X	0 (x_1)	1 (x_2)
$f(x)$	1/2 ($f(x_1)$)	1/2 ($f(x_2)$)

Discrete RV

$$\mu = \sum x_i f(x_i) \rightarrow x_1 f(x_1) + x_2 f(x_2)$$

$$\Rightarrow 0(1/2) + 1(1/2) \rightarrow 1/2$$

$\boxed{\mu = 1/2}$

$$\sigma^2 = \sum (x_i - \mu)^2 f(x_i) \rightarrow (x_1 - 1/2)^2 f(x_1) + (x_2 - 1/2)^2 f(x_2)$$

$$\Rightarrow (0 - 1/2)^2 (1/2) + (1 - 1/2)^2 (1/2)$$

$$\Rightarrow \frac{1}{8} + \frac{1}{8} \rightarrow \frac{1}{4}$$

$\boxed{\sigma^2 = \frac{1}{4}}$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

A. $\cancel{\mu = \int_{-\infty}^{\infty} f(x)\,dx \rightarrow \int_{-\infty}^{0} 0 + \int_{a}^{b} \frac{1}{b-a} + \int_{0}^{\infty} 0}$

$\cancel{\Rightarrow \frac{1}{b-a}[x]_a^b \rightarrow \frac{b-a}{b-a} \rightarrow \underline{\underline{1}}}$

$\cancel{\boxed{\mu = 1}}$

$\cancel{\sigma^2 = \int_a^b (x-1)^2 \frac{1}{b-a} \rightarrow \int_a^b (x^2+1-2x)\frac{1}{b-a}}$

$\cancel{\Rightarrow \left[\frac{x^3}{3} + x - x^2\right]_a^b \frac{1}{b-a} \rightarrow \left[\frac{b^3}{3} + b - b^2\right] - \left[\frac{a^3}{3} + a - a^2\right] \frac{1}{b-a}}$

$\Rightarrow$

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \rightarrow \int_a^b \frac{x}{b-a}\,dx \quad \frac{1}{b-a} \frac{(b^2-a^2)}{2} \rightarrow \boxed{\frac{b+a}{2} = \mu}$$

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx \rightarrow \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx$$

$$\Rightarrow \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \Rightarrow \frac{1}{b-a}\left[\frac{\left(x - \frac{a+b}{2}\right)^3}{3}\right]_a^b = \frac{1}{3(b-a)}\left[\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3\right.$$

$$\Rightarrow \frac{1}{24(b-a)}\left[(b-a)^3 + (b-a)^3\right) \rightarrow \frac{\cancel{2}(b-a^{\cancel{3}})^{\to 2}}{\underset{12}{\cancel{24}}\cancel{(b-a)}} \rightarrow \frac{b-a}{12}$$