Time Series Coursework 2

Question 1

Calculating the roots of the characteristic polynomial:

$$\begin{aligned} X_t &= 1.8X_{t-1} - 0.9X_{t-2} + \varepsilon_t \\ \Phi(z) &= 1 - 1.8z + 0.9z^2 \\ z &= \frac{1.8 \pm \sqrt{3.24 - 4 \times 0.9}}{2 \times 0.9} \\ &= 1 \pm i\frac{1}{3} \end{aligned}$$

The roots of the characteristic polynomial are $1 + i\frac{1}{3}$ and $1 - i\frac{1}{3}$. Since $\sqrt{1 + \frac{1}{9}} = \frac{\sqrt{10}}{3} > 1$, both of the roots are outside the unit circle, and therefore the process is stationary.

From lecture, the roots can be written in a different form: $z_1 = \frac{1}{r}e^{-i2\pi f'}, z_2 = \frac{1}{r}e^{i2\pi f'}$ and the spectrum will be at its largest when $f \approx \pm f'$ where f is its frequency. So taking the positive frequency when $f \approx \frac{1}{2\pi}tan^{-1}(\frac{1}{3}) \approx 0.0512$ there is a peak in the spectrum.

Question 2 (i)

First use the Yule-Walker equations $\gamma_{\mathbf{p}} = \mathbf{\Gamma_p}\phi_{\mathbf{p}}$.

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_0 \end{bmatrix} \begin{pmatrix} 1.8 \\ -0.9 \end{pmatrix}$$

$$s_0 = \phi_{1,p}s_1 + \phi_{2,p}s_2 + \sigma_\varepsilon^2$$

$$\Rightarrow \begin{cases} 1.9s_1 - 1.8s_0 = 0 \\ s_2 - 1.8s_1 + 0.9s_0 = 0 \\ s_0 - 1.8s_1 + 0.9s_2 - 1 = 0 \end{cases}$$

$$s_1 = \frac{18}{19}s_0 \Rightarrow \begin{cases} s_0 - 1.8 \times \frac{18}{19}s_0 + 0.9s_2 - 1 = 0 \\ s_2 - 1.8 \times \frac{18}{19}s_0 + 0.9s_0 = 0 \end{cases} \Rightarrow \begin{cases} s_2 - \frac{153}{190}s_0 = 0 \\ 0.9s_2 - \frac{67}{95}s_0 = 1 \end{cases}$$

$$\therefore s_0 = \frac{1900}{37}, s_1 = \frac{1800}{37}, s_2 = \frac{1530}{37}$$

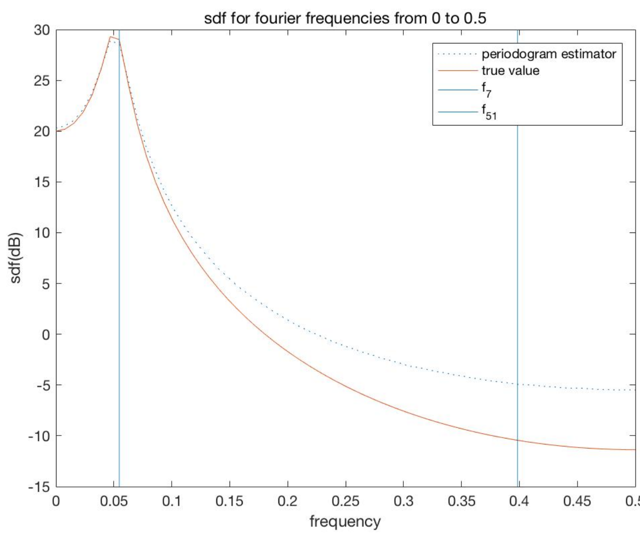
From coursework I, $D = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_0 \end{bmatrix}, C = \begin{bmatrix} \sqrt{s_0} & 0 \\ 0 & \sqrt{s_0 - \frac{s_1^2}{s_0}} \end{bmatrix} = \begin{bmatrix} \frac{10\sqrt{703}}{37} & 0 \\ 6.7888 & \frac{10\sqrt{19}}{19} \end{bmatrix}, Y = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$

$$\begin{aligned} \Rightarrow CY &= \begin{bmatrix} 7.1660\varepsilon_1 \\ 6.7888\varepsilon_1 + 2.2942\varepsilon_2 \end{bmatrix} \\ X_1 &= 7.1660\varepsilon_1 \\ X_2 &= 6.7888 \cdot \frac{X_1}{7.1660} + 2.2942\varepsilon_2 \\ &= 0.9474X_1 + 2.2942\varepsilon_2 \end{aligned}$$

Therefore, the forms of X_1 and X_2 are verified.

Question 2 (ii)

Plot of the true spectrum for the model and $\bar{S}^{(p)}(f_j)$ at the Fourier frequencies with the two vertical lines that mark f_7 and f_{51} are shown below.

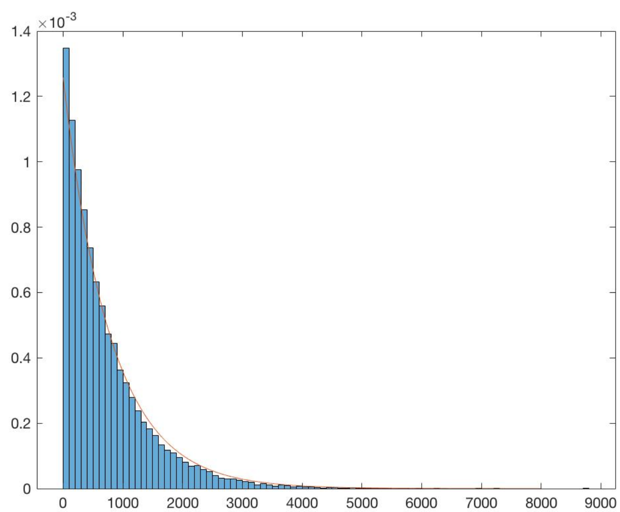


Notice that because the algorithm of fft that matlab gives is a bit different from what we want, an additional exponential $e^{-i2\pi f}$ needs to be used.

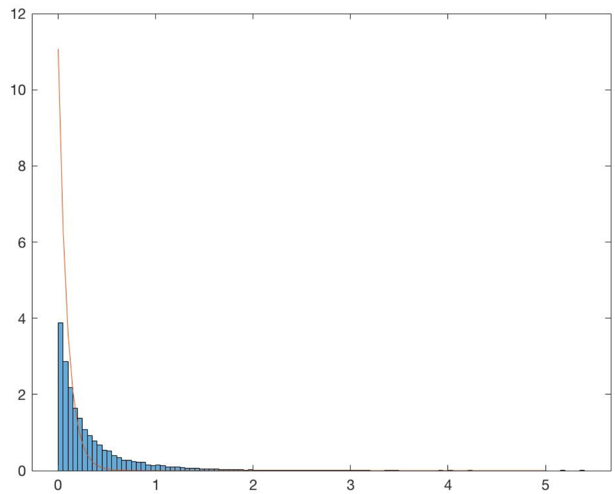
Question 2 (iii)

Compute and plot the histograms for $\left\{\hat{S}^{(p)}(f_7, k=1:N_r)\right\}$ and $\left\{\hat{S}^{(p)}(f_{51}, k=1:N_r)\right\}$. In addition, the pdfs suggested by the theory is plotted on the same graph.

For f_7:



For f_{51}:

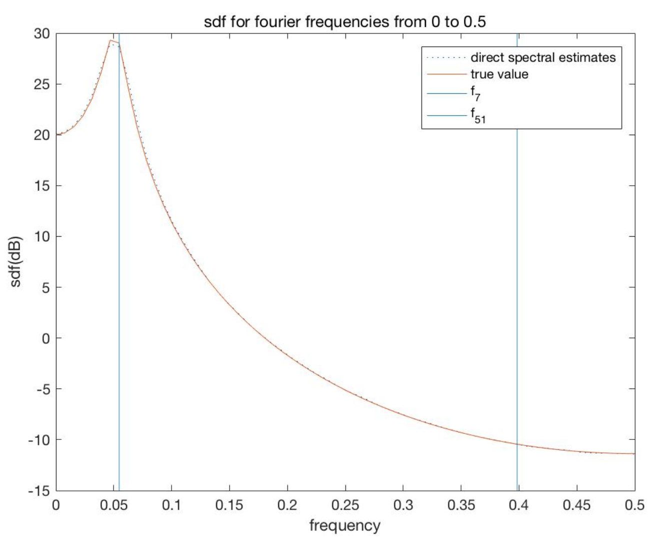


Comment:

1. In Question 2(ii), on the first vertical line (f_7) the estimated value and the true value are very close to each other while on the second vertical line (f_{51}) they are much apart.
2. In Question 2(iii), I observed that for f_7, the line of the pdfs suggested by the theory match the histogram much better than for f_{51}. In addition, they have very different scales.

Question 3(i)

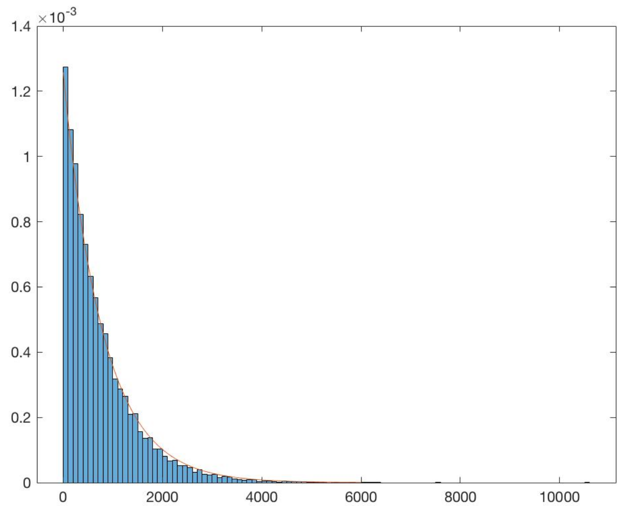
Repeat Question 2(ii) with tapering method. The plot is given below:



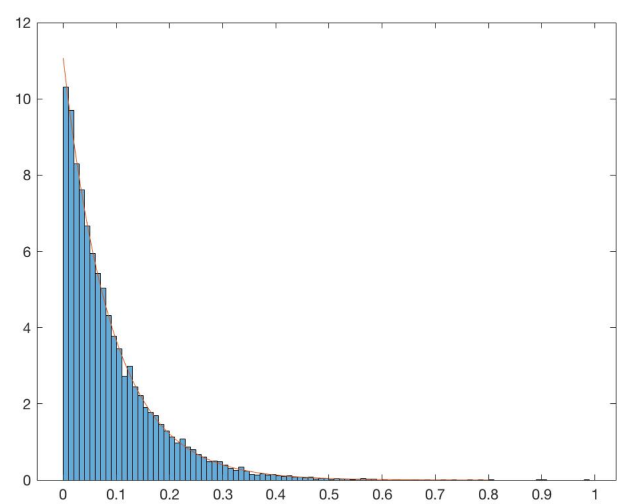
Question 3(ii)

Below is the plots of histograms for $\left\{\hat{S}^{(d)}(f_7, k = 1 : N_r)\right\}$ and $\left\{\hat{S}^{(d)}(f_{51}, k = 1 : N_r)\right\}$ with pdfs suggested by the theory plotted on the same graph.

For f_7:



For f_{51}:



Comment: For the plot in question 3(i), it can be seen that now the estimated spectrum line match the true spectrum line completely which is much better than in question 2(ii).
For the plot in question 3(ii), it can be seen that compare to the plots in question 2, the pdfs match the histogram much better, especially for f_{51}.

The effects of tapering:

1. The resolution of the spectrum decreased. (for f_{51} the range for the x-axis decreased approximately from 1.7 to 0.6)
2. The sidelobe leakage decreased. (for f_{51} it can be seen that for the spectral estimation by tapering, the pdfs match the histogram much better.)

Appendix

Codes for Question 2(ii): computing the periodogram and the plot.

```
N=128;                              % initializing
Nr=20000;
x=zeros(128,1);
freq=zeros(N/2+1,1);
cc=zeros(65,1);
y=zeros(65,Nr);                     % matrix to hold all the values ofs
avey=zeros(65,1);
strue=zeros(65,1);

for j=1:N/2+1                       % computing Fourier frequencies
    freq(j)=(j-1)/N;
end

for t= 1:65
    cc(t)=exp( (-1i*2*pi)*(freq(t)) );
end                                 % Because the algorithm of fft function in matlab is ...
    different from what the question require, an additional exponential needs to be used ...
    to time the fft function

for m=1:Nr                          % to repeat 20000 times
    e=normrnd(0, 1,[1 N]);          % computing different realization of epsilon each time
    x(1)=7.166*e(1);
    x(2)=0.9474*x(1)+2.2942*e(2);
    for i =3:N                      % computing x
        x(i)=1.8*x(i-1)-0.9*x(i-2)+e(i);
    end
xdft=fft(x);                        % using fast fourier transform
xdft = xdft(1:65);                  % taking only the first 65 terms of X
y(:,m)=(1/N).* ((xdft.*cc).*conj(xdft.*cc)) ;    % storing the periodogram for each realization
end


for n=1:N/2+1
    avey(n)=sum(y(n,:))/Nr;         % averaging the periodograms over realizations
end

for l=1:N/2+1                       % computing the true spectrum
    strue(l)= 1/ (abs (1 - 1.8 *exp(-1i*2*pi*freq(l)) + 0.9 *exp(-1i*2*pi*freq(l)*2)) .^2);
end

q=10*log10(avey);                   % using the decibel
plot(freq,q,':')                    % plotting the periodogram estimator of the spectrum
hold on;
plot(freq,10*log10(strue),'-')      % plotting the true value
xlabel('frequency')                 % x-axis label
ylabel('sdf(dB)')                   % y-axis label

line([freq(8) freq(8)], [-15 30] )% adding the vertical lines
line([freq(52) freq(52)], [-15 30])
legend('periodogram estimator','true value','f_7','f_{51}')
title('sdf for fourier frequencies from 0 to 0.5')
```

Codes for Question 2(iii): computing histogram of $\left\{\hat{S}^{(p)}(f_7, k = 1 : N_r)\right\}$ and corresponding pdfs suggested by the theory then plot them.

```
f7=y(8,:);                % using the 8th row
u=0:8000;                 % for scaling
chis7 = chi2pdf(2*u./strue(8),2);      % using chi-squared pdf with d.f.=2
ss7=2./(strue(8)) * chis7;             % pdf suggested by the theory
histogram(f7,'Normalization','pdf')    % required histogram (normalized)
hold on;
plot(u,ss7)                            % plot pdf suggested by the theory
```

Codes for Question 2(iii): computing histogram of $\left\{\hat{S}^{(p)}(f_{51}, k = 1 : N_r)\right\}$ and corresponding pdfs suggested by the theory then plot them.

```
f51=y(52,:);                  % using the 52nd row
u=linspace(0,5,100);          % for scaling
chis51 = chi2pdf(2*u./strue(52),2);   % using chi-squared pdf with d.f.=2
ss51=2./(strue(52)) * chis51;         % pdf suggested by the theory
histogram(f51,'Normalization','pdf')  % required histogram (normalized)
hold on;
plot(u,ss51)                          % plot pdf suggested by the theory
```

Codes for Question 3(i): repeating Question 2(ii) with tapering method.

```
N=128;                          % initializing
Nr=20000;
x=zeros(N,1);
freq=zeros(N/2+1,1);
cc2=zeros(65,1);
y2=zeros(65 ,Nr);               % matrix to hold all the values ofs
avey2=zeros(65 ,1);
strue2=zeros(65 ,1);

taper=zeros(N,1);
for t=1:N                       % computing the Hanning taper
    taper(t)= ( (2/(3*(N+1))  )^0.5) * ( 1 - cos( (2*pi*t) / (N+1) ) );
end


for j=1:N/2+1                   % computing Fourier frequencies
    freq(j)=(j-1)/N;
end

for t= 1:N/2+1
    cc2(t)=exp( (-1i*2*pi)*(freq(t)) );
end

for m=1:Nr                      % to repeat 20000 times
    e=normrnd(0, 1,[1 N]);      % computing epsilon
    x(1)=7.166*e(1);
    x(2)=0.9474*x(1)+2.2942*e(2);
    for i =3:N                  % computing x
        x(i)=1.8*x(i-1)-0.9*x(i-2)+e(i);
    end
hx=taper.*x;                    % X times the taper
hxdft=fft(hx);                  % using fast fourier transform
hxdft=hxdft(1:65);
y2(:,m)=((hxdft.*cc2).*conj(hxdft.*cc2)) ;   % computing the estimator
end

for n=1:65
    avey2(n)=sum(y2(n,:))/Nr;   % taking the average value
end
```

```

for l=1:65                          % computing the true spectrum
    strue2(l)= 1/ (abs (1 - 1.8 *exp(-1i*2*pi*freq(l)) + 0.9 *exp(-1i*2*pi*freq(l)*2)) .^2);
end

q2=10*log10(avey2);                 % using the decibel

plot(freq,q2,':')                   % plotting the estimator of the spectrum
hold on;
plot(freq,10*log10(strue2),'-')           % plotting the true value
line([freq(8) freq(8)], [-15 30])         % adding the vertical lines
line([freq(52) freq(52)], [-15 30])

xlabel('frequency')                 % x-axis label
ylabel('sdf(dB)')                   % y-axis label
legend('direct spectral estimates','true value','f_7','f_{51}')
title('sdf for fourier frequencies from 0 to 0.5')
```

Codes for Question 3(ii): computing histogram of $\left\{\hat{S}^{(d)}(f_7, k=1:N_r)\right\}$ and corresponding pdfs suggested by the theory then plot them.

```
f7_2=y2(8,:);                              % using the 8th row
u=0:6000;                                  % for scaling
chis7_2 = chi2pdf(2*u./strue2(8),2);       % using chi-squared pdf with d.f.=2
ss7_2=2./(strue(8)) * chis7_2;             % pdf suggested by the theory
histogram(f7_2,'Normalization','pdf')      % required histogram (normalized)
hold on;
plot(u,ss7_2)                              % plot pdf suggested by the theory
```

Codes for Question 3(ii): computing histogram of $\left\{\hat{S}^{(d)}(f_{51}, k=1:N_r)\right\}$ and corresponding pdfs suggested by the theory then plot them.

```
f51_2=y2(52,:);                            % using the 52nd row
u=linspace(0,0.8,100);                     % for scaling
chis51_2 = chi2pdf(2*u./strue2(52),2);     % using chi-squared pdf with d.f.=2
ss51_2=2./(strue2(52)) * chis51_2;         % pdf suggested by the theory
histogram(f51_2,'Normalization','pdf')     % required histogram (normalized)
hold on;
plot(u,ss51_2)                             % plot pdf suggested by the theory
```