Robust Optimization of Portfolio Selection

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Measuring Porfolio Risk is a crucial process in Smartfolios' robo-advisor when determining the optimal asset allocation corresponding to an individual's risk profile. Since we all use models such as Markowits Mean-Variance(MV) model or Mean-CVaR model to get the optimal asset allocation results, the Portfolio Risk, however, by its single term in the background of modelling has two meaning: one is value loss risk, the other is estimation risk of model input parameters(like mean returns and covariance matrix of returns). In particular, small differences in the estimates of returns can result in large variations in the portfolio compositions. For this reason, this white paper will detail our appropriate ways to take this estimation risk into account when using the Mean-Variance model and Mean-CVaR model.

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Chapter 1

Introduction

1.1 Why Robust Portfolio Allocation

Half a century ago, Markowitz provided a framework for the selection of assets for portfolio construction to obtain an optimal portfolio. In that model, Markowitz measured an asset's return by its mean return, risk by its variance of returns and the co-movement of assets by all the asset's covariance matrix of returns. The portfolio's expected return and risk are then determined by the weights of each asset included in the portfolio. For this reason, the Markowitz framework is commonly referred to as Mean-Variance model.

However, the variance is an inadequate risk measure, considering that it rewards the desirable upside movements as much as it punishes the undesirable downside movements. Moreover, it did not captured the tail risk. Hence, another method is becoming more and more popular. Conditional Value at Risk (CVaR) also known as Expected Shortfall, is a probabilistic risk measure that allows us to account for the fat left tails and extreme events of an asset's returns distribution. The CVaR at $\alpha\%$ is defined as the average return in the worse $\alpha\%$ of cases. The smaller the CVaR of an asset, the less risky it is. CVaR provides us with an intuitive definition of risk and does not penalize upside volatility.

However, all the models employing Variance and CVaR above face a stubborn problem, i.e. the portfolio sensitivity. Portfolio sensitivity indicates that the resulting portfolio constructed using these models and its performance is heavily dependent on the inputs of the model. Hence, provided that the estimated input values are even slightly different from their true values, the estimated optimal portfolio will actually be far away from the best choice. Hence, there has been increased interest in the construction of a portfolio that offer more robust performance even during most uncertain market periods. Although various techniques have been applied to improve the stability of portfolios, one of the approaches that has received much attention is robust portfolio optimization. In this white paper, we will detail various methodology in this field.

1.2 Data Used in This White Paper

We will be using Exchange Traded Funds (ETFs) to represent different major asset classes for our investigation and as our portfolio components (see Table 1.1). An Exchange Traded Fund is marketable security which tracks a specific index, commodity or bonds (e.g Total US Stocks, Foreign Bonds). The data we use to model our returns from 2007/07/26 to 2018/01/22. (Note that the period of Great Recession is within this time interval, for which many extreme negative returns come from)

Table 1.1: Details of ETFs Used in This White Paper

Symbol	Name	Asset Class
VEA	Vanguard FTSE Developed Markets ETF	Foreign stocks (Developed)
VTI	Vanguard Total Stock Market ETF	US stocks
VWO	Vanguard FTSE Emerging Markets ETF	Foreign stocks (Emerging)
EMB	iShares J.P. Morgan USD Emerging Markets Bond ETF	Emerging Market Bonds
LQD	iShares Investment Grade Corporate Bond ETF	Investment Grade Corporate Bonds
IEF	iShares 7-10 Year Treasury Bond ETF	US Treasury Bonds (7-10 Years)
BWZ	SPDR Barclays Capital Short Term International Treasury Bond	Foreign Treasury Bonds(1-3 year)
SHV	iShares Short Treasury Bond	US Treasury Bonds (1-12 Months)
DBC	PowerShares DB Commodity Index Tracking 14 commodities	Commodities Futures

Besides, CAPM fitted returns are used as the estimated return values in this white paper. This is different from most optimization works as they use mean returns of past data as the estimated return values.

Chapter 2

Robust Optimization

As we have introduced in chapter 1, from a practical point of view, it is important to make the portfolio selection process robust to different source of risk – including estimation risk and model risk. There are, indeed, several methods to to increase the level of robustness in the portfolio selection process.

One of the pioneer work is from Michaud[8]. In 2008, he introduced the portfolio re-sampling techniques. The methodology is to employ Monte Carlo simulation to generate various re-sampled datasets from estimated inputs and use them to compute all individual frontiers. The final frontier will be the average of all individual frontiers. However, Fabozzi[2] indicated that due to the averaging in the calculation of the re-sampled portfolios, all assets will most likely obtain a non-zero weight. This may lead to a resulting portfolio that violates the imposed constraints. Another problem of the portfolio based on this method is that it is suboptimal in terms of expected utility maximization. Moreover, the computation for large portfolios may become very cumbersome using this method.

Other ways of optimization under uncertainty are stochastic programming and dynamic programming. However, Fabozzi[2] indicated that a major problem with dynamic and stochastic programming formulations is that in practice it is often difficult to obtain detailed information about the probability distributions of the uncertainties in the model. At the same time, depending on the number of scenarios involved, the computation cost of dynamic and stochastic programming methods can be prohibitively high.

Hence here, we introduce robust optimization. Robust optimization is a recently developed technique that addresses the same type of problems as stochastic programming does; however, it typically makes relatively wide assumptions on the probability distributions of the uncertain parameters for fear that the problem formulation may be intractable. In Robust Optimization, the true values of the model's parameters are not known with certainty, but the bounds are assumed to be known. Hence, uncertainty in the inputs can be modelled by taking it into consideration directly in the optimization process.This is an intuitive and efficient way to model this form of uncertainty.

There are two method to implement robust optimization. The first one is called min-max method, the second one is called conditional value-at-risk(CVaR) robust method. In this white paper, we will introduce both two methods and do a thorough analysis on both.

2.1 Min-Max Method

Kim et all[7] has introduced a concept of worst-case decision making. In the example he illustrated, the optimal choice for a an uncertainty-averse decision-maker was made by observing the worst state. In the same manner, optimizing the worst case to minimize risk in an uncertain situation can be applied to portfolio selection. Since future stock returns are not known with certainty, the best portfolio for the worst possible returns can be selected to reduce future losses in case the actual returns turn out to be as low as the predicted worst values.

It can be further demonstrated in a math form. Suppose in the below example, we have a optimization problem:

$$\begin{aligned} &\underset{\omega}{min}\ f_0(\omega) \\ &\quad \omega \in \Omega \end{aligned} \tag{2.1}$$

where ω is the decision variable and f_0 is the objective function. However f_0 contains uncertain parameters u and u belongs to a uncertainty set U. Hence, by the approach of min-max, we try to optimize the worst case because on condition that a solution satisfies a constraints for its worst case, it will surely meet the constraint for other cases. Hence the problem in (2.1) can be written as:

$$\begin{aligned} &\underset{\omega}{min}\ \underset{u \in U}{max}\ f_0(\omega) \\ &\qquad\quad \omega \in \Omega \end{aligned} \tag{2.2}$$

From this point, the min-max method can also be viewed as worst-case optimization. To compute the solution, we need to define the uncertainty set. In a typical Mean-Variance Optimization Problem, the inputs used — the vector of mean returns and the covariance matrix of returns — become the uncertain parameters for finding the optimal portfolio. As the min-max method depends heavily on specification of uncertainty sets, we need to define the uncertain parameters set U.

There are two most popular method to define uncertainty set: interval(box) uncertainty set and ellipsoidal uncertainty set. In the next two sub-sections, these two methods will be briefly explained.

2.1.1 Box Uncertainty Set

Tütüncü and Koenig [14] presented an uncertainty sets S_μ and S_Σ to define the boundaries. The definition is as follows:

$$\begin{aligned} S_\mu &= \{\mu | \mu^L \leq \mu \leq \mu^U\} \\ S_\Sigma &= \{\Sigma | \Sigma^L \leq \Sigma \leq \Sigma^U, \Sigma \succeq 0\} \end{aligned} \tag{2.3}$$

where μ^L and μ^U are the the boundary values of estimated return uncertainty set interval and Σ^L and Σ^U are the boundary values of co-variance uncertainty set intervals. $\Sigma \succeq 0$ indicates that Σ is a symmetric positive semi-definite matrix. The min-max problem of equation 2.3 with this uncertainty set is given by:

$$\underset{\omega}{min}\ \underset{\mu \in\ S_\mu, \Sigma \in\ S_\Sigma}{max}\ f_0(x) \tag{2.4}$$

Halldórsson and Tütüncü transformed the above problem as a saddle-point problems [5]. To solve such problem, an interior-point method is developed by them. In this paper, since we did not used this box uncertainty sets, the details will not be explained here.

2.1.2 Ellipsoidal Uncertainty Set

Goldfarb and Iyengar[4] first introduced this concept. As shown below, the ellipsoidal uncertainty set for estimated return is a combined uncertainty rather than a stock-wise construction of uncertainty as in the box definition:

$$S_\mu = \{\mu | (\mu - \hat{\mu})' \Sigma_\mu^{-1} (\mu - \hat{\mu}) \leq \delta^2\} \tag{2.5}$$

where $\Sigma_\mu^{-1} \in \mathbb{R}^{N \times N}$ is the covariance matrix of estimation errors, and $\delta \in \mathbb{R}$ controls the size of the ellipsoid. μ is the random realized return and $\hat{\mu}$ is the input estimated return value. The above equation denotes a uncertainty set of all random realized returns within the imposed ellipsoid.

The intuition behind this uncertainty set is as follows. Random return estimates $\hat{\mu}$ are assumed to be close to the real value μ, but they may deviate. They are more likely to deviate if their variability (measured by their standard deviation) is higher. So deviations from the mean are scaled by the inverse of the covariance matrix of the estimation errors. In this context, this form of deviation is also denoted as Mahlanobis distance. We use this form of distance to to measure the distance between the estimated value and real value. The benefit of Mahlanobis distance is that it has already considered the multi-variate distribution estimated error.

The Mahlanobis distance is the Euclidean distance of the estimated value and real value then be divided by the covariance matrix of estimation errors. In this sense, provided that the Euclidean distance between estimated value and real value, together with their covariance are both high, the final Mahlanobis distance will not be as high as the sole Euclidean distance as Mahlanobis distance is the quotient of high Euclidean distance and high covariance value. The parameter δ corresponds to the overall amount of scaled deviations of the estimated returns from the real returns.

Due to the Mahlanobis feature, the final uncertainty set appear like a ellipsoidal set as the major axis of the ellipse denotes those extreme uncertainty values either bearing too much deviations or having too little correlations with the real value while the minor axis's uncertainty values behave the opposite way.

The min-max problem of equation 2.2 with this uncertainty is given by:

$$\underset{\omega}{min} \quad \underset{\mu \in \{\mu | (\mu - \hat{\mu})' \Sigma_\mu^{-1} (\mu - \hat{\mu}) \leq \delta^2\}}{max} \quad f_0(x) \tag{2.6}$$

δ is a single value that controls the combined deviation. Garlappi et all[3] fixed the value of δ by assuming the asset returns have a joint normal distribution. With this assumption, $(\mu - \hat{\mu})' \Sigma_\mu^{-1} (\mu - \hat{\mu})$ is estimated as a chi-squared distribution with N degrees of freedom. Consequently, the value of δ^2 becomes the $(100 \times \beta)^{th}$ percentile of the chi-squared distribution with N degrees of freedom for a confidence level of β. In this paper, we use the CAPM model-fitted estimated returns to represent $\hat{\mu}$.

Regarding to the detailed methodology to solve such problem we will detail it in the following chapters.

2.1.3 Estimation Error Matrix

Estimation error matrix Σ_μ in equation 2.5 is the covariance matrix of the estimation error. There are several methods to approximate this matrix. The simplest method is to use Σ to be divided by T, where Σ is the covariance matrix of ETF returns, and T is the sample size. However, in practice, this method is not recommended[13]. This is because this method need to assume the return process is stationary and we are computing our estimate of $\hat{\mu}$ by the average of a time-series of realized returns of length T. In practice, this is obviously not the case.

In this paper, since we use CAPM fitted estimated returns of each asset as parameters together with decision variable – weights for each assets, to optimize final portfolio returns. One of the estimation error source is from this CAPM fitted estimated returns for each asset. Hence, we compute the prediction error by using the historical realized returns and CAPM model predicted returns.

Suppose we have CAPM model predicted data in the form of α^t and historical realized data in the form of r^t for each security for the past T time periods. For each time period t, we estimate the error in the alpha estimate for security i as $(\alpha_i^t - r_i^t)$. If we assume that the estimation errors are normally distributed over the time horizon with constant mean and variance, we can compute the variance of the sampling distribution for security i as :

$$\hat{\sigma_i}^2 = \frac{\sum_{t=1}^{T}[(\alpha_i^t - r_i^t) - \hat{\mu_i}]^2}{T(T-1)} \tag{2.7}$$

where

$$\hat{\mu_i} = \frac{1}{T}\sum_{t=1}^{T}(\alpha_i^t - r_i^t) \tag{2.8}$$

Covariances for each pair of assets' estimation error can be computed similarly.

In this white paper, we use past 360 days data to get one specific day's α^t.

2.2 CVaR Method

Zhu[15] indicated that since an unknown expected return may have infinite number of possible scenarios, its uncertainty set typically corresponds to some confidence level $p \in [0, 1]$ with respect to an assumed distribution. In this regards, min-max robust optimization is a quantile-based approach, with the boundaries of an uncertainty set equal to certain quantile values for p.

One drawback with min-max robust model is that, it entirely ignores the severity of the tail scenarios which occur with a probability of 1-p. Instead, it determines a min-max robust portfolio solely based on the single quantile value which corresponds to the worst sample scenarios of the expected return. Thus, the dependence on a single worst sample scenario makes a min-max robust portfolio quite sensitive to the initial data used to generate uncertainty set. In particular, inappropriate boundaries of uncertainty sets can cause min-max robust optimization to be either too conservative or not conservative enough. In practice, it is difficult to choose appropriate uncertainty sets.

Zhu[16] also mentioned that with an ellipsoidal uncertainty set based on the statistics of sample mean estimates, the robust portfolio from the min-max robust Mean-Variance Model equals to the optimal portfolio from the standard MV model based on the nominal mean estimate but with a larger risk aversion parameter(We will show this later). If the uncertainty set for mean return contains the worst sample scenario, the min-max robust Mean-Variance model often produces portfolios with very low returns. Portfolios with higher returns can be generated in the model by choosing the uncertainty interval to correspond to a smaller confidence interval. Unfortunately, this is at the expense of ignoring worse sample scenarios and runs counter to the principle of min-max robustness.

Zhu[16] hence proposed a new method to overcome this drawback of min-max method. The CVaR Robust method uses the Conditional Value-at-Risk (CVaR) to measure the estimation risk in estimated return, and control the conservatism level of CVaR robust portfolio with respect to estimation risk.

Before we touch on how CVaR is applied in estimation risk control. Let's take a look at how CVaR is applied in traditional return-risk analysis. In traditional return-risk analysis, CVaR is used to quantify the portfolio loss due to the volatility of asset returns. To show it more conveniently, this CVaR is denoted as:

$$\underset{\omega}{CVaR^{\mu}_{\beta}} = \underset{\alpha_1, L_1}{min}(\alpha_1 + (1-\beta)^{-1}E([L_1 - \alpha_1]^{+})) \tag{2.9}$$

where

$\underset{\omega}{CVaR^{\mu}_{\beta}}$ denotes the portfolio loss due to return volatility by the confidence level $\beta 1$
α_1 defines the VaR^{μ}
E means expectation
$[z]^{+}$ is defined as max(z,0)
L_1 is portfolio return loss scenario, here it equals $f_1(\omega, \mu)$ where ω is the decision weight variable and r constitute a joint distribution of returns. The portfolio return loss scenario at time is as follows

$$L_1 = f_1(\omega, \mu) = -\omega^{'}\mu = -[\mu_1\omega_1 + \mu_2\omega_2 + ... + \mu_n\omega_n] \tag{2.10}$$

The equation 2.9 is from a transformation of the CVaR definition proposed by Rockafellar and Uryasev [9]. The purpose of the transformation is to facilitate the portfolio optimization for $CVaR^{\mu}_{\beta}$ based model. It introduced a linear programming approach to solve such problem of minimizing $CVaR^{\mu}_{\beta}$ under a constraint on the expected return.

In the estimation risk analysis addressed in this white paper, CVaR is also used to quantify the estimation risk of the estimated returns. The estimation risk is defined as the tail of the portfolio's estimated loss scenarios. How to understand this estimated loss scenarios? Firstly, we need to understand how we get the portfolio estimated return.

There are various methods to generate the portfolio estimated return uncertainty set, like boostrapping or resampling. Zhu[16] also introduced another method called Chi-squre method. In this whitepaper, resampling method(RS) from Michaud[8] is employed.

The process of sampling the mean using RS techniques will be briefly introduced here. Suppose we have a specific distribution of random return for an asset. From this distribution, we can generate 100 return samples and from these 100 return samples we can calculate the mean return value. This mean return value will be regarded as this asset's estimated return. We repeat this process for 10000 times and apply it to all the assets in the portfolio. Finally, we have 10000 vector of simulated portfolio estimated returns or portfolio mean returns. We denote one random vector of portfolio estimated returns or portfolio mean returns as $\bar{\mu}$.

Let's define L_2 as the portfolio mean loss scenarios. It equals $f_2(\omega, \bar{\mu})$ where ω is the decision weight variable and $\bar{\mu} \in R$ is the random vector of the portfolio mean returns we mentioned above. The portfolio mean loss is defined as following:

$$L_2 = f_2(\omega, \bar{\mu}) = -\omega^{'}\bar{\mu} = -[\bar{\mu}_1\omega_1 + \bar{\mu}_2\omega_2 + ... + \bar{\mu}_n\omega_n] \tag{2.11}$$

In the previous min-max method, we use worst case sample scenario in the uncertainty set of estimated return to get the robust optimal portfolio. In CVaR method, we use the tail of the portfolio's mean loss scenarios specified by the confidence level $\beta 2$ to set the optimization condition. Here is the definition for this CVaR:

$$\underset{\omega}{CVaR_{\beta}^{\bar{\mu}}} = \underset{\alpha_2, L_2}{min}(\alpha_2 + (1-\beta)^{-1}E([L_2 - \alpha_2]^+)) \tag{2.12}$$

where

$\underset{\omega}{CVaR_{\beta}^{\bar{\bar{\mu}}}}$ denote the portfolio estimated loss due to return uncertainty by the confidence level $\beta 2$
α_2 defines the $VaR^{\bar{\bar{\mu}}}$
E means expectation
$[z]^+$ is defined as max(z,0)
L_2 is portfolio estimated loss scenario, here it equals $f_2(\omega, \bar{\mu})$ where ω is the decision weight variable and $\bar{\mu} \in R$ is the random vector of the portfolio mean return

The CVaR Robust model control the conservatism level of a estimation risk by adjusting the confidence level of CVaR, $\beta 2 \in [0, 1)$. CVaR is a coherent risk measure, hence it can be used to quantify the risk of a portfolio under a specified distribution assumption.

We want to emphasis that these two CVaR are different, $CVaR_{\beta}^{\mu}(x)$ denote the $CVaR$ of the portfolio loss scenarios due to return volatility while $CVaR_{\beta}^{\bar{\mu}}(x)$ denote the $CVaR$ of the portfolio mean loss scenarios due to estimated return uncertainty.

β in $CVaR_{\beta}^{\bar{\mu}}(x)$, as acquired as the conservatism level coefficient for estimation risk, has a very straight explanation. As β approaches 1, the CVaR robust MV model considers the worst mean loss scenario and the resulting portfolio is the most conservative, As the value of β decreases, better mean loss scenarios are included for consideration and the dependency on the worst case is decreased. Thus the resulting portfolio is less conservative. When $\beta = 0$, all sample mean loss scenarios are considered in the model; this may be appropriate when an investor has complete tolerance to estimation risk. Thus the confidence level β can be interpreted as an estimation risk coefficient.

In practice, we will incorporate the non-normality behaviour of estimated returns into estimated return uncertainty set to pursue a more justifiable assumption. That means we do not use the general normal distribution assumption when we build the portfolio estimated return uncertainty set. This will be explained later in next chapter.

Before that, lets take a look at a basic example to illustrate how CVaR is applied in practice

2.2.1 CVaR Robust Mean-Variance Model

The CVaR Robust Mean-Variance Model can be defined as below:

$$\begin{aligned} \underset{\omega}{min}\ \omega'\Sigma\omega + CVaR_{\beta}^{\bar{\mu}}(-\hat{\mu}'\omega) \\ s.t.\ \omega^{'}\hat{\mu} = u_{required} \\ \omega' t = 1 \\ \omega \geq 0 \end{aligned} \tag{2.13}$$

where

$\hat{\mu} \in \mathbb{R}^N$ is the random mean return scenario
$\Sigma \in \mathbb{R}^{N \times N}$ is the covariance matrix of the past returns
$\omega \in \mathbb{R}^N$ is the portfolio weights
$t \in \mathbb{R}^N$ is a vector of ones
$N \in \mathbb{N}$ is the number of assets
$\hat{\mu}$ is fixed estimated expected return and $u_{required}$ is the desired return

Here, $CVaR_{\beta}^{\bar{\mu}}(-\bar{\mu}'\omega)$ replaced the $-\lambda\mu'\omega$ in the Mean-Variance Model. The former represents the uncertainty risk of estimated mean-loss while the later represents estimated mean loss of the portfolio.

Chapter 3

Incorporating Non-Normality of Markets Returns

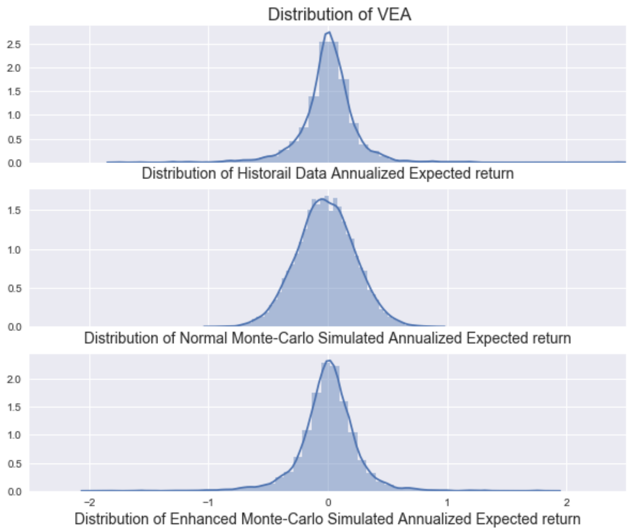
3.1 Shortfalls of Normal Distribution

In most portfolio optimization works, "normality" is a frequently used assumption for estimated returns distribution. However, in real-world situation, distribution of returns often display negative skew and excess kurtosis. To solve such problem, Sheikh incorportaed the non-normality into his simulation[11]. In white paper "Mean-CVaR Allocation Framework"[1], Smartfolios employed a similar framework in which non-normality behaviours were considered. In particular, it has two parts:

1. Modelling the tail distribution of an asset's return using extreme value analysis's Peak Over Threshold method. The overall distribution is divided into 3 portions, two marginal portion and one middle portion. The middle portion is modelled by kernel density estimation and the two marginal portions are estimated by Generalized Pareto Distribution. The final complete distribution of the asset's returns is a hybrid model which combines all the 3 portions into 1 distribution.

2. Modelling varying correlation using student t copula theory. The simple correlations often used in traditional asset allocation models assume a linear relationship between asset classes, i.e. they assume that the relationship between variables at the extremes is similar to their relationship at less extreme conditions. However, in real-world situations, many correlations under extreme conditions are quite different compared under normal conditions. Hence, the copula theory is introduced to solve such correlation break-down issue.

In this whitepaper, we will directly employ the method described in that whitepaer[1] to incorporate the non-normality behaviour into our simulation. We denote this method as Enhanced Monte-Carlo Simulation. In next page, we exhibit distribution comparisons for each ETFs. Within each comparison, the first is historical return distribution, the second is distribution from Normal Monte-Carlo Simulation and the last is from our Enhanced Monte-Carlo Simulation. From these graphs, we can find that using the method of Smartfolios will returns a more realistic distribution to be optimized.



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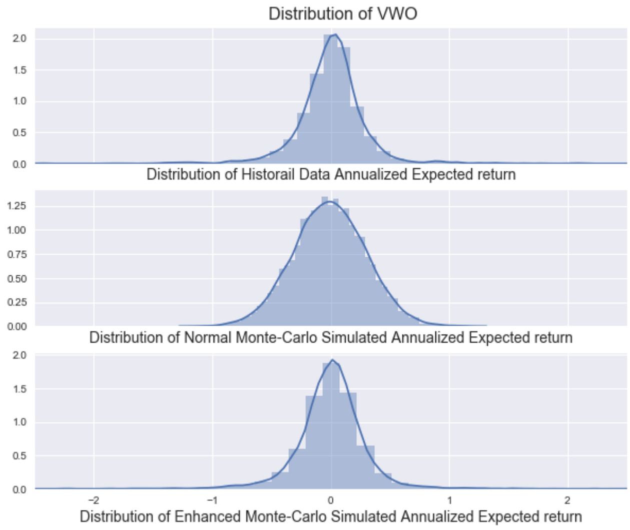

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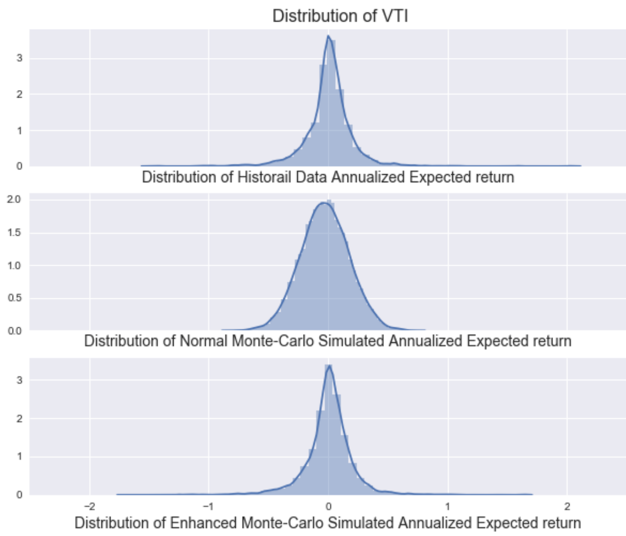

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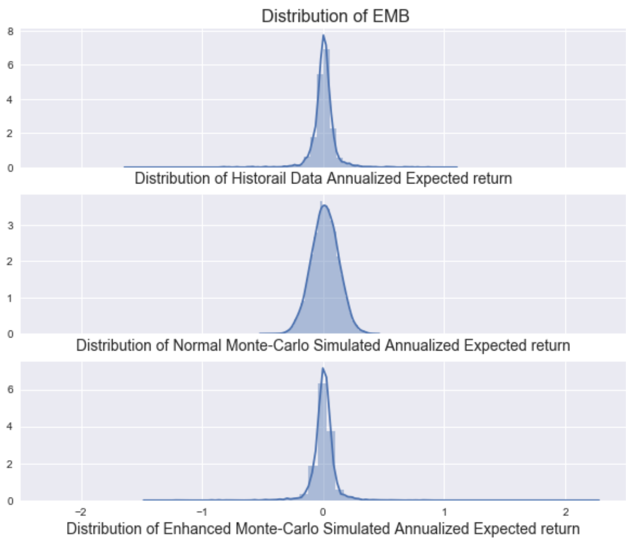

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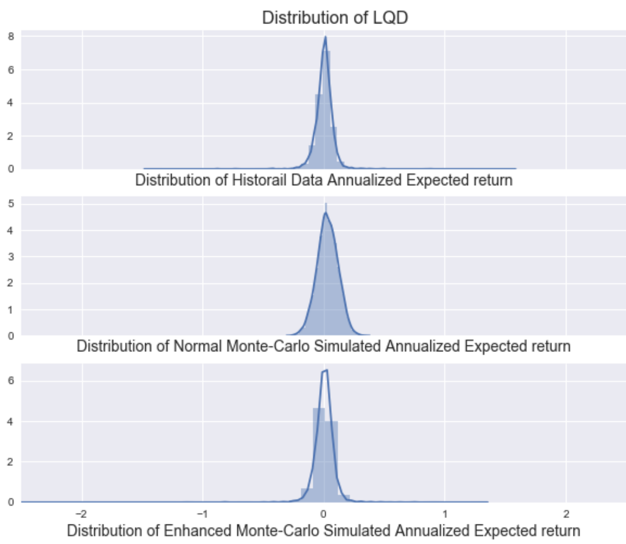

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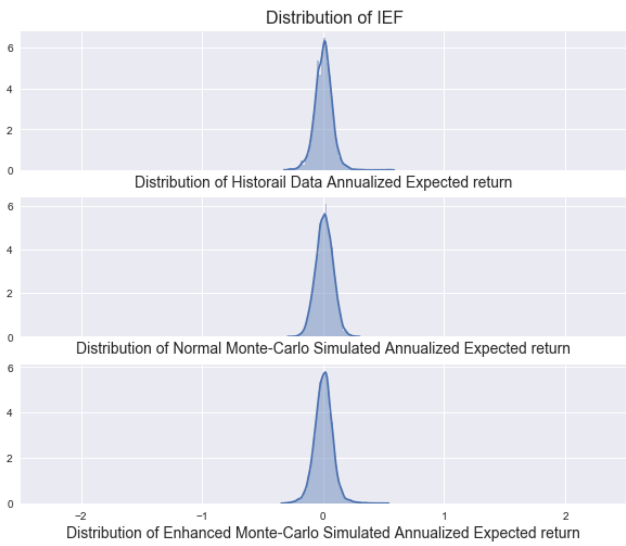

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Figure 3.1: Distribution Comparison

3.2 Application of Enhanced Monte Carlo Simulation

In this white paper, the following simulation data will be simulated by Enhance Monte Carlo Simulation method:

Table 3.1: Applications of Enhanced Monte Carlo Simulation

Section	Data
3.3	RS sampling data used for 50 robust mean-variance frontiers computation
4.3	RS sampling data used for 50 robust mean-cvar frontiers computation
5.1	100000 RS sampling data used for random mean return scenarios
5.3	RS sampling data used for 50 cvar robust mean-cvar frontiers computation

Chapter 4

Construction of Robust Optimization for Mean-Variance Model — Min-Max Method

4.1 Model Explanation

In this chapter, we will introduce the most basic version of min-max robust optimization method based on Markowitz Mean-Variance Model. The Min-Max Mean-Variance Model can be expressed as the following:

$$\begin{aligned} \underset{\omega}{min}\ & \omega'\Sigma\omega - \lambda\hat{\mu}'\omega \\ s.t.\ & \omega' t = 1 \\ & \omega \geq 0 \end{aligned} \tag{4.1}$$

where

$\omega \in \mathbb{R}^N$ is the vector of portfolio weights
$\Sigma \in \mathbb{R}^{N\times N}$ is the covariance matrix of the past returns
$\lambda \in \mathbb{R}$ is the risk-seeking coefficient
$\hat{\mu} \in \mathbb{R}^N$ is the vector fixed expected return, in this white paper, we use CAPM model fitted
returns for this vector
$t \in \mathbb{R}^N$ is a vector of ones
$N \in \mathbb{N}$ is the number of assets

In this white paper, we use the ellipsoidal uncertainty set for estimated return value. Hence the problem can be transformed as:

$$\underset{\omega}{min} \underset{\mu \in \{\mu | (\mu-\hat{\mu})'\Sigma_\mu^{-1}(\mu-\hat{\mu}) \leq \delta^2\}}{max} \omega'\Sigma\omega - \lambda\hat{\mu}'\omega \tag{4.2}$$

As it has been mentioned before, μ is the random estimated return value, Σ_μ^{-1} is the estimation error matrix and δ is a single value that controls the uncertainty set boundary. The value of δ^2

becomes the $(100 \times \alpha)^{th}$ percentile of the chi-squared distribution with N degrees of freedom for a confidence level of β.

From the book Robust Equity Portfolio Management[7], the problem defined above can be transformed to an SOCP(Second Order Cone Problem) of below form:

$$\underset{\omega}{min}\ \omega'\Sigma\omega - \lambda(\hat{\mu}'\omega - \delta\sqrt{\omega'\Sigma_\mu\omega}) \tag{4.3}$$

The optimal solution of this optimization problem is acquired by satisfying the condition that the overall amount of scaled deviations of the realized returns from the estimates is up to δ.

The problem of 4.3 need to be solved with an interior-point optimizer. The implementation of this optimization model in python environment needs cvxpy package. By solving problem 4.3 for all possible values of λ from 0 to ∞, we can get the efficient frontier. In this white paper, we solve 100 problems of 4.3 for 100 different λ values to get an approximated frontier.

4.2 Numerical Results

In Figure 4.1, the allocation results of standard mean-variance model and robust mean-variance model with $\frac{\Sigma}{T}$ estimation error matrix and 95% confidence interval of ellipsoidal uncertainty set are compared. The allocation results for robust model lacks the part for annualized CVaR greater than 63 %. Other than that, these two are exactly the same. This means the portfolio from the min-max robust mean-variance model equals the portfolio from the standard Mean-Variance model but with a larger risk aversion parameter. Higher risk aversion coefficient will lead lower maximum returns thereby decreasing the maximum CVaR.

This coincide with Zhu's finding[15]. We can also do a brief transformation of equation 4.3:

$$\underset{\omega}{min}\ -\lambda\hat{\mu}'\omega + (1 + \frac{\delta}{\sqrt{\omega'\Sigma_\mu\omega T}})\omega'\Sigma\omega \tag{4.4}$$

The formulation in 4.4 shows that a min-max robust mean-variance optimization based on ellipsoidal uncertainty set and $\frac{\Sigma}{T}$ estimation error matrix can be computed as solving a standard Mean-Variance portfolio optimization problem, but with a larger risk aversion coefficient $(1 + \frac{\delta}{\sqrt{\omega'\Sigma_\mu\omega T}})$ instead of 1.

Hence, from this point of view, the min-max robust model with $\frac{\Sigma}{T}$ error matrix adds robustness for estimation risk by increasing the risk aversion parameter. Thus the min-max robust actual frontiers from problem 4.4 are squeezed segments of nominal actual frontiers from problem 4.3. Figure 4.2 justifies this reasoning.

This depreciates the robust optimization as a disastrous estimated mean will only give us a disastrous robust frontiers since it is only a part of the normal disastrous frontier. Hence, in practice, such method of employing min-max method, ellipsoidal uncertainty set and $\frac{\Sigma}{T}$ estimation error matrix is not recommended.

In this white paper, we also explored the CAPM model fitted estimation error matrix for min-max method. The rationale of this error matrix was explained in section 2.1.3. Figure 4.3 displays the allocation results using such matrix under different risk confidence intervals.

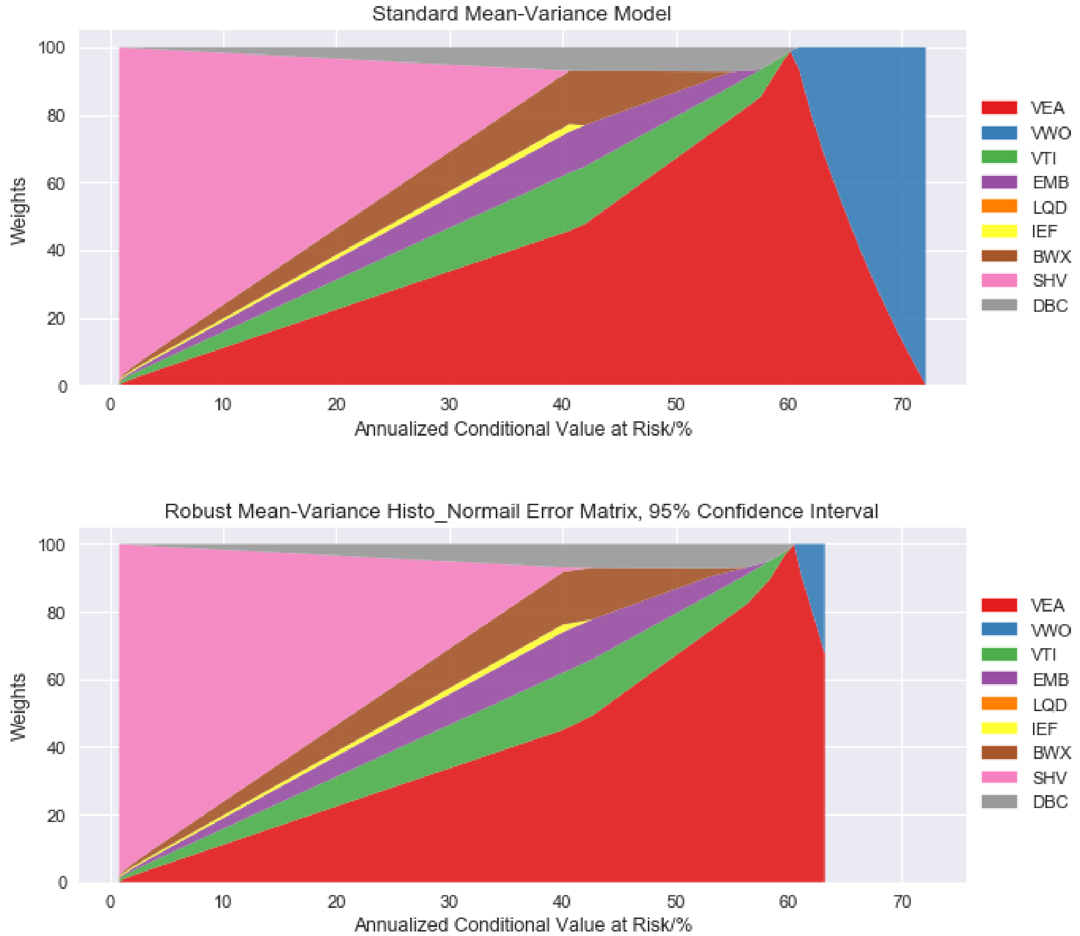

Figure 4.1: Comparison of Normal Mean-Variance Model and Robust Mean-Variance Model

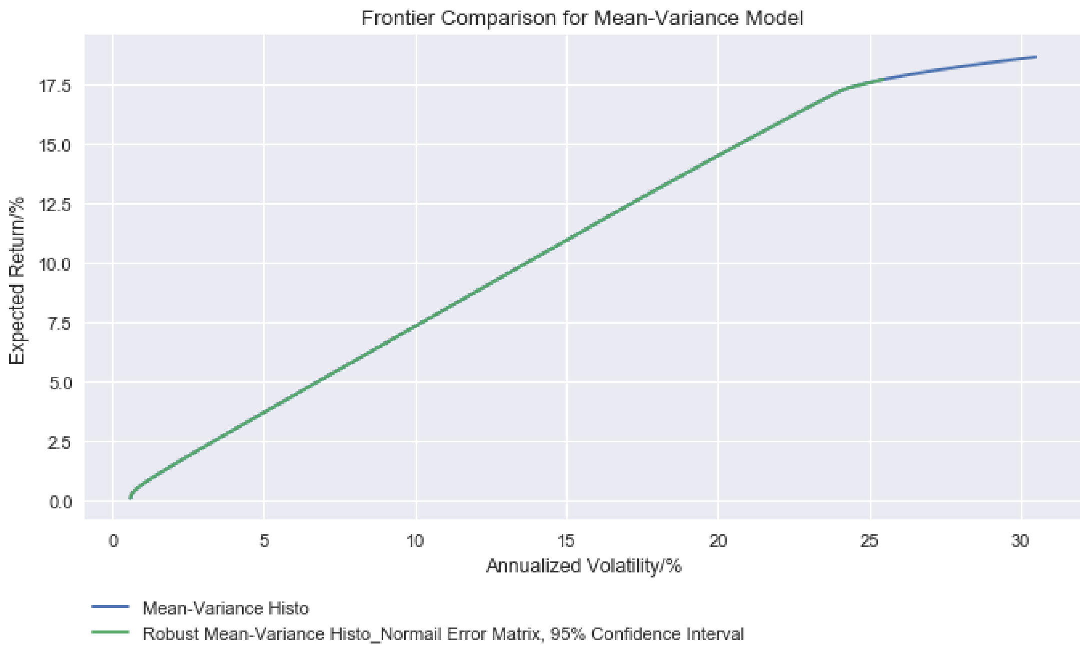

Figure 4.2: Mean-Variance Model Frontiers

If we look at the estimation error matrix (Table 3.1) we used in this white paper, we could find that VEA, VWO are those ETFs who have high estimation error volatility within their asset

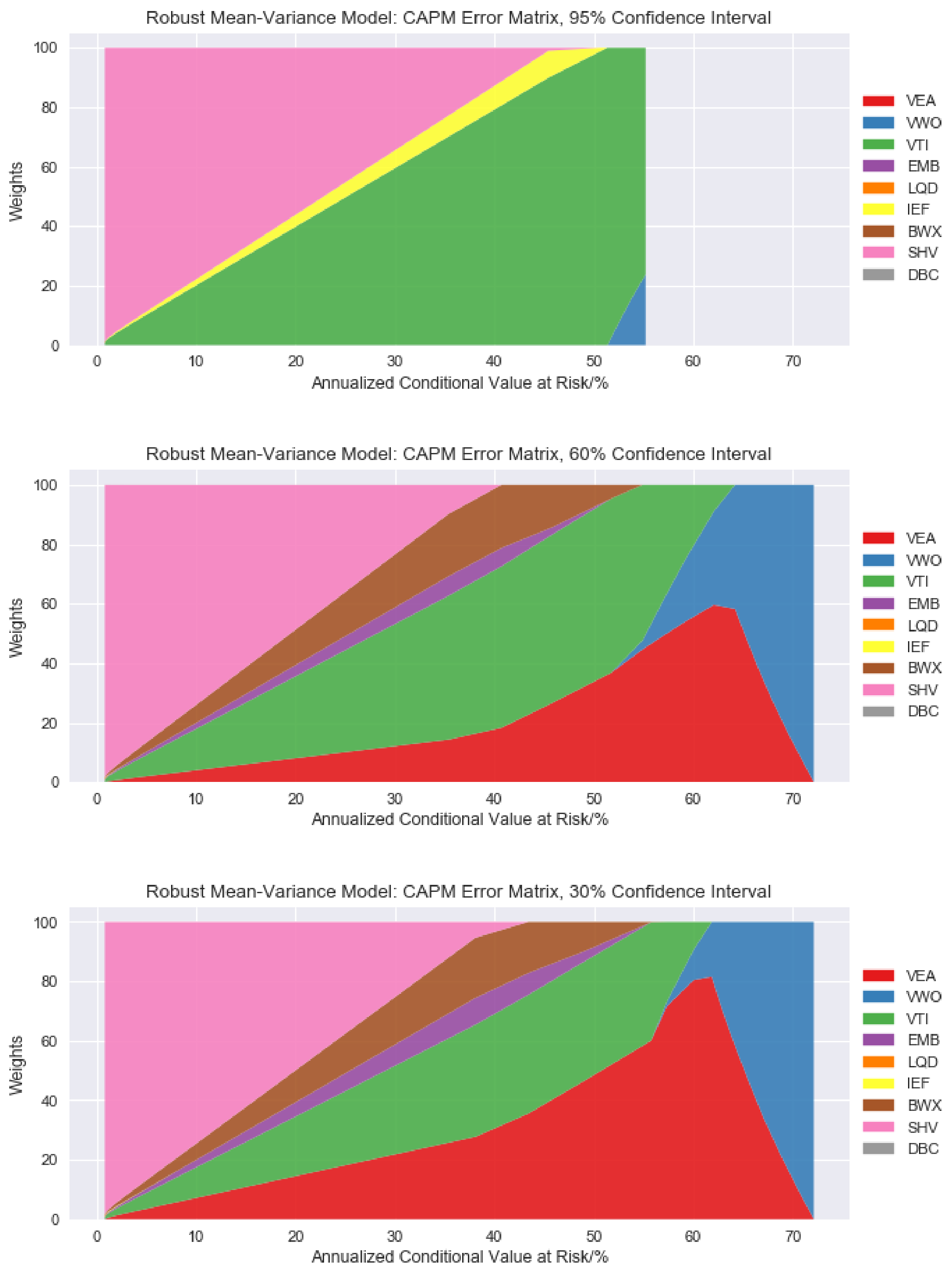

Figure 4.3: Robust Mean-Variance Model, Various Confidence Intervals

group. Compared to VTI, VEA and VWO exhibit high estimation error volatility within high return stocks group. The allocation results indicates this point as well. We notice that when our confidence level is 95%, which is quite high, those ETF with high estimation error volatility, like VEA, VWO, DBC, EMB are not chosen. The model only select those exhibiting low estimation error volatility for robustness purpose.

Table 4.1: Estimation Error Matrix Based on CAPM Model

	VEA	VWO	VTI	EMB	LQD	IEF	BWX	SHV	DBC
VEA	0.00172								
VWO	0.00181	0.00194							
VTI	0.00130	0.00138	0.00099.8						
EMB	0.000306	0.000344	0.00023	0.0000846					
LQD	-0.0000751	0.0000941	0.0000579	0.0000287	0.0000159				
IEF	-0.000168	-0.000157	-0.000127	-0.0000075	0.000008	0.0000337			
BWX	0.000358	0.000373	0.000261	0.0000719	0.0000214	-0.0000261	0.0000809		
SHV	-0000004	-0000005	-0000003	-0000001	0	0	-0000001	0	
DBC	0.000985	0.00101	0.000742	0.000154	0.0000332	-0.000112	0.0001929	-0000003	0.000605

Regarding to frontiers, Figure 4.4 shows that under the same specification of 95% risk confidence interval, the frontier of the robust mean-variance model with CAPM fitted error matrix also deviate from normal frontiers. This is reasonable as robust optimization result would have lower expected return due to employing the worst case scenario.

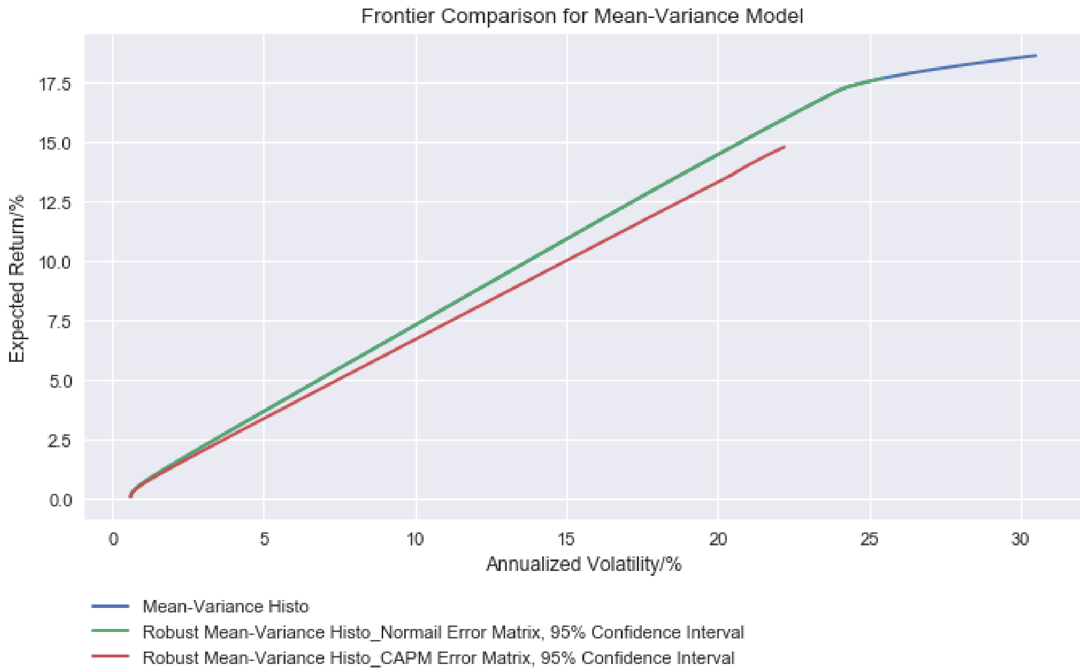

Figure 4.4: Frontiers of Robust Mean-Variance Model

We could also do a comparison of different risk confidence intervals' effect. From Figure 4.5, we find that with higher risk confidence interval, the expected return tends to be more conservative. This comply with the principle of robust optimization.

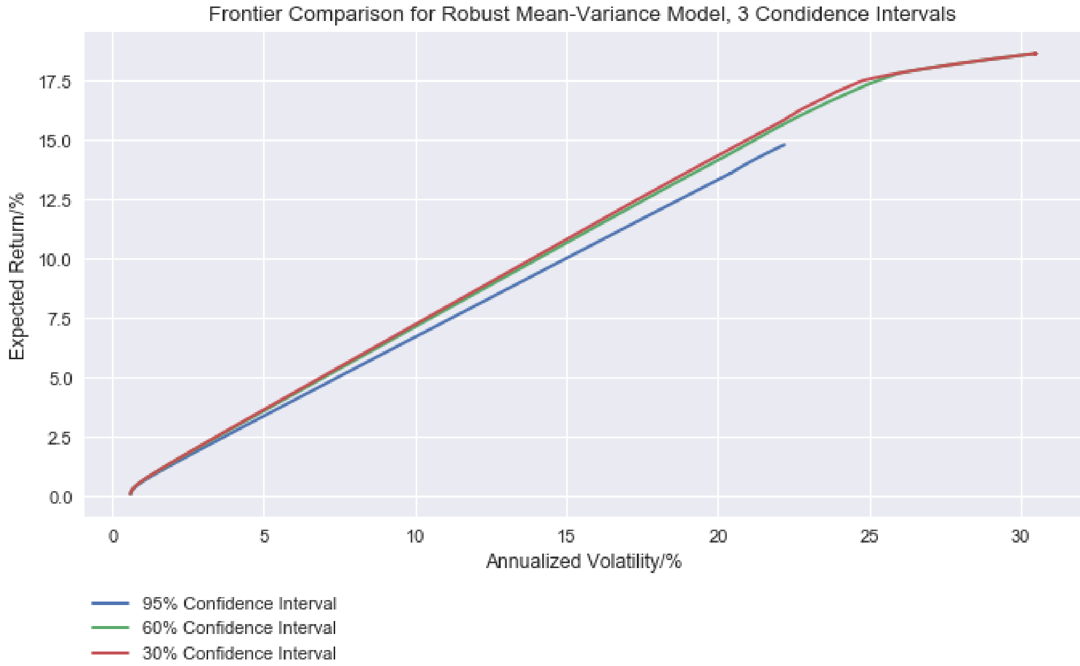

Figure 4.5: Frontiers of Robust Mean-Variance Model, Various Confidence Interval

4.3 Sensitivity to Initial Data

One way of assessing the robustness of a portfolio optimization technique is to measure the variation of its actual frontiers calculated from different estimated parameters. Less variation in the actual frontiers can be considered more robust. Hence, we repeat RS sampling technique 50 times. Each time we get a sample set of 200 data. From each 200 data, we use the method stated above to get a frontier. The variation within this 50 frontiers will display how robust this model is. Do take note that the RS sampling incorporated the non-normality behaviour of the distribution of estimated returns.

Figure 4.6 shows the results. Except for the first figure displaying the standard Mean-Variance Model results, each below figure display 50 Robust Mean-Variance actual frontiers for risk confidence interval $\beta = 30\%, 60\%, 95\%$ respectively.

As can be observed from these figures, the robust mean-variance actual frontiers change with initial sample data. As β increase, the variation become smaller, however, the frontier also become shorter. This comply with the reason stated in section 4.2.

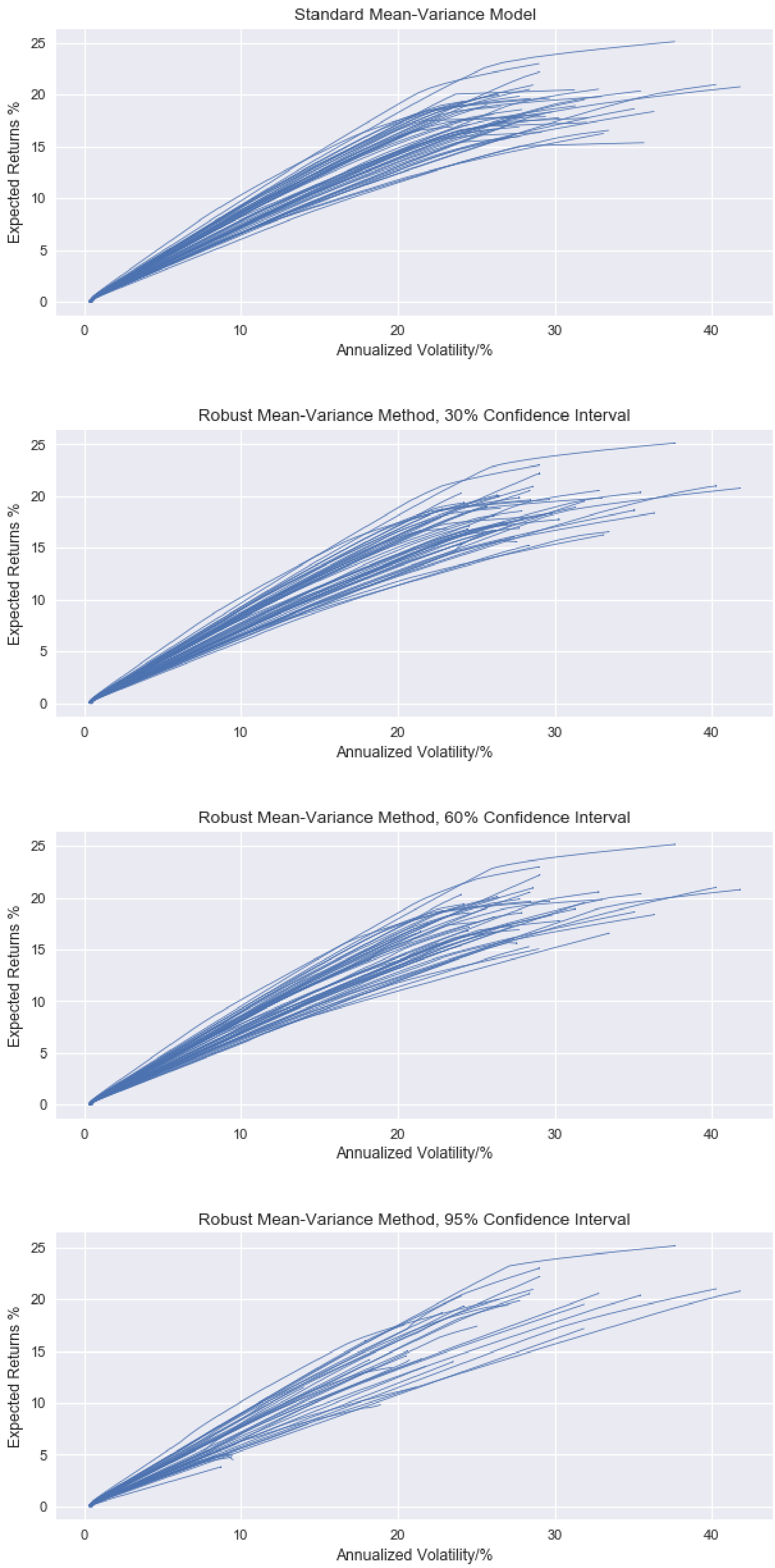

Figure 4.6: Sensitivity to Initial Data, Robust Mean-Variance Model

Chapter 5

Construction of Robust Optimization for Mean-CVaR Model — Min-Max Method

5.1 Model Explanation

In the CVaR framework for min-max method, we try to minimize our portfolio risk(CVaR) given a fixed estimated portfolio expected return and confidence interval for the estimated returns' ellipsoidal uncertainty set. We obtain the constrained Mean-CVaR optimization problem from Rockafellar[9] and it is expressed in Equation 5.1 where the objective function is convex.

$$
\begin{aligned}
\min_{\alpha,\omega}\ \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [-\mu_i^{'}\omega - \alpha]^+ \\
s.t.\ \omega' t = 1 \\
\omega \geq 0 \\
\hat{\mu}^{'}\omega = u_{required}
\end{aligned}
\tag{5.1}
$$

where

α represents VaR^{μ}
ω is the weight vector
β is the specified $CVaR^{\mu}$ confidence interval
$\mu_i^{'}$ is the vector of simulated random return at time
m is the total number of scenarios
$\hat{\mu} \in \mathbb{R}^N$ is the vector of fixed expected returns, in this white paper, we use CAPM model fitted
returns
$u_{required}$ is the required return of the whole portfolio

In this robust CVaR framework, we assume that the random simulated returns lie within a ellipsoidal uncertainty set. The uncertainty set follows the same format of equation 4.2. Then, the problem becomes:

$$\underset{\omega}{min} \underset{\mu \in \{\mu | (\mu - \hat{\mu})' \Sigma_\mu^{-1} (\mu - \hat{\mu}) \leq \delta^2\}}{max} \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [-(\mu)_i' x - \alpha]^+ \tag{5.2}$$

In 2016, Katrik[12] introduced a transformation of above problem. He transformed the above problem into the below second-order cone program(SOCP):

$$\underset{\alpha,\omega}{min} \ \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [-\mu_i' x - \alpha]^+ + \delta \sqrt{\omega' \Sigma_\mu^{-1} \omega} \tag{5.3}$$

From the above equation, we note that the only difference between equation 5.3 and equation 5.1 is the the risk term $\sqrt{\omega' \Sigma_\mu^{-1} \omega}$. However, find a solution for such SOCP problem using interior-point algorithm is time-consuming and challenging[6].

This formulation need to be solved with an interior-point optimizer. The implementation of this optimization model in python environment needs cvxpy package.

5.2 Numerical Results

In Figure 5.1, the allocation results of standard Mean-CVaR model and Robust Mean-Cvar Models with 30%, 60%, 95% risk confidence intervals are compared. The allocation results do not exhibit much difference, but there is one subtle change among them. That is the allocation weight of VTI increases along with the increase of estimation risk aversion. This is reasonable and is justified by the table 4.1 that VTI exhibit lower estimation error among the high return stocks group(VEA, VWO and VTI). When estimation risk appetite is high, the frontiers will become shorter and conservative, this can be observed from the 95% risk confidence interval plot.

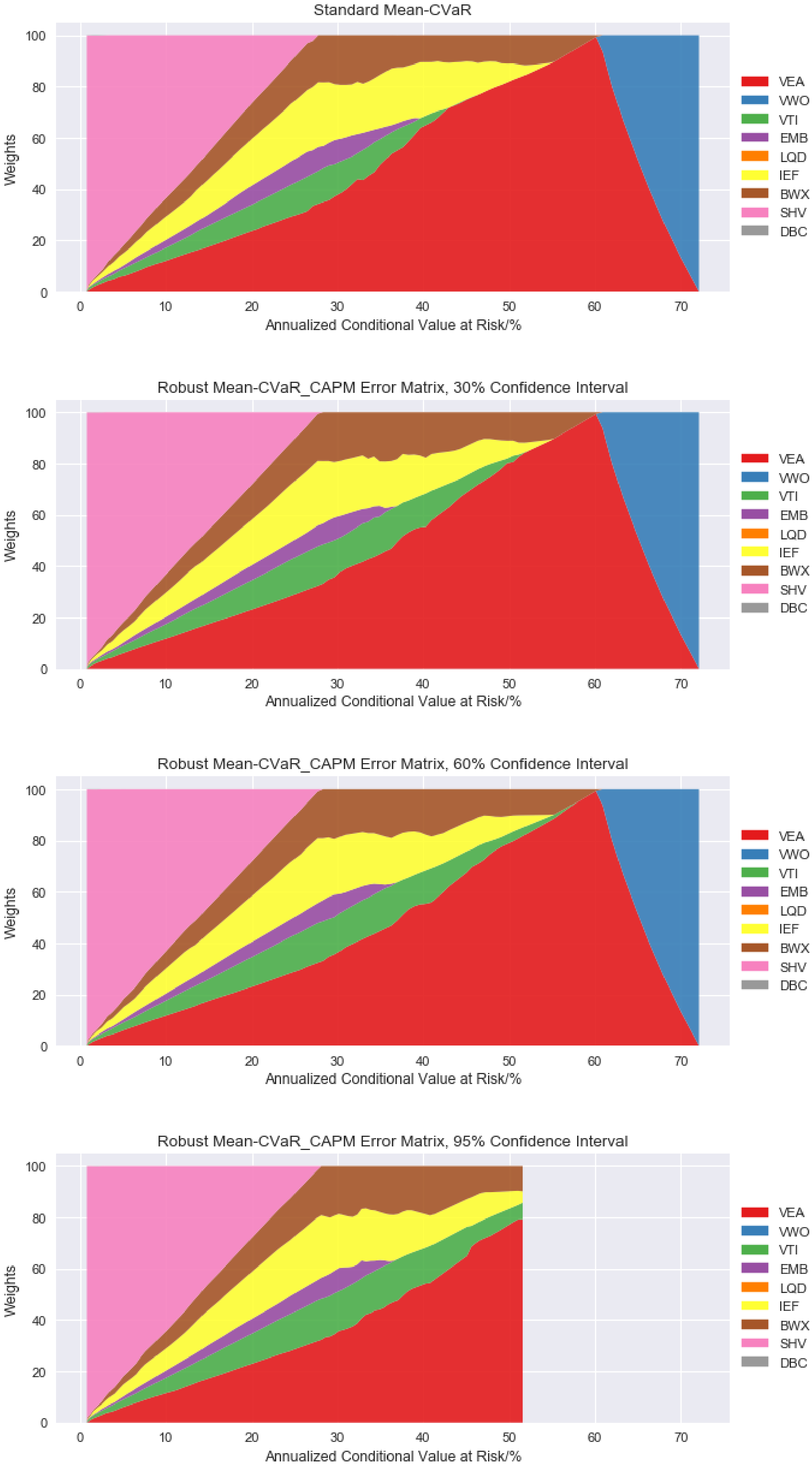

Figure 5.1: Robust Mean-CVaR Model Allocation Results

5.3 Sensitivity to Initial Data

Using the same method in section 4.3, we could access the effectiveness of the min-max Robust Mean-Cvar model. Here is the results. From the plot, we find that the robustness of this method is better than Robust Mean-Variance method stated in chapter 4.

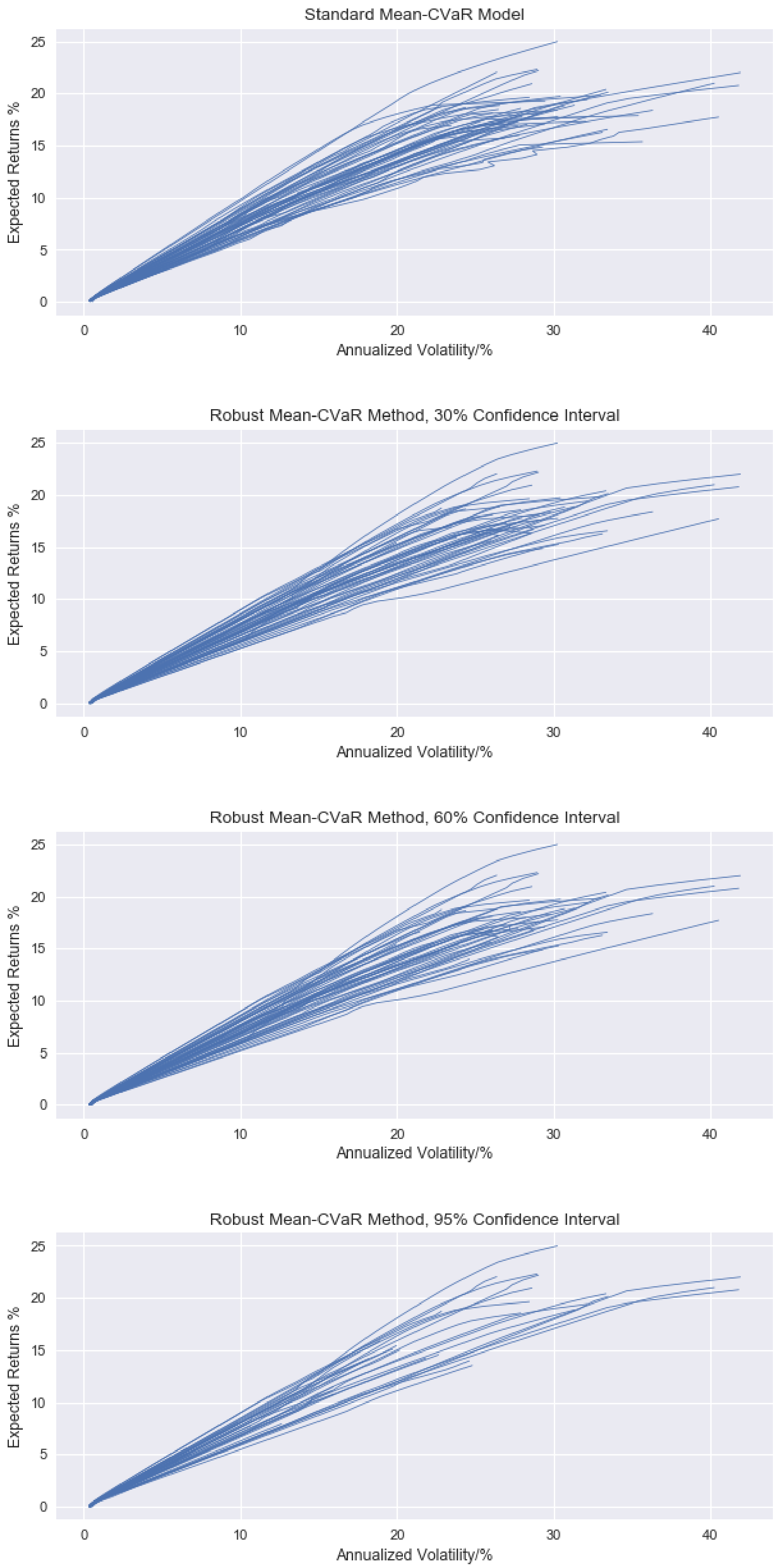

Figure 5.2: Sensitivity to Initial Data, Robust Mean-CVaR Model

Chapter 6

Construction of Robust Optimization for Mean-CVaR Model — CVaR Method

The CVaR Robust Mean-CVaR method was proposed by Salahi[10] in 2003. The idea is very simple. It replace the term $\omega'\Sigma\omega$ in equation 2.13 with $CVaR^{\mu}_{\beta 1}(-\mu^{'}\omega)$. Here is the formal definition:

$$\begin{aligned} \underset{\omega}{min}\ \lambda CVaR^{\mu}_{\beta 1}(-\mu^{'}\omega) + CVaR^{\bar{\mu}}_{\beta_2}(-\bar{\mu}^{'}\omega) \\ s.t.\ \omega' t = 1 \\ \omega \geq 0 \end{aligned} \tag{6.1}$$

where

λ is the risk aversion coefficient
β_1 is the confidence interval for loss scenario
$\mu \in \mathbb{R}^N$ is the vector of simulated random return scenario
β_2 is the confidence interval for estimation uncertainty
$\bar{\mu} \in \mathbb{R}^N$ is the vector of simulated random mean return scenario
$\omega \in \mathbb{R}^N$ is the vector portfolio weights
$t \in \mathbb{R}^N$ is a vector of ones
$N \in \mathbb{N}$ is the number of assets

If we discrete this problem by the method of Rockfellar[9], the problem become:

$$\underset{\omega,\alpha_2,\alpha_2}{min}\ \lambda(\alpha_1 + \frac{1}{(1-\beta_1)m}\sum_{i=1}^{m}[v_i]^+) + \alpha_2 + \frac{1}{(1-\beta_2)T}\sum_{i=1}^{T}[z_i]^+ \tag{6.2}$$

$$\begin{aligned} s.t.\ v_i = -\mu_i^{'}\omega - \alpha_1 \\ z_i = -\bar{\mu}_i^{'}\omega - \alpha_2 \\ \omega' t = 1 \\ \omega \geq 0 \end{aligned} \tag{6.3}$$

where

α_1 is the VaR for $CVaR^{\mu}_{\beta}$

α_2 is the VaR for $CVaR_{\beta}^{\bar{\mu}}$
$\bar{\mu}_i^m \in \mathbb{R}^N$ is the vector of simulated random mean return scenarios
$\mu_i^m \in \mathbb{R}^N$ is the vector of simulated random return scenarios
β_1 is the confidence interval for loss scenario
β_2 is the confidence interval for estimation uncertainty
m is the total number of scenarios for returns
T is the total number of scenarios for mean loss

We use the RS sampling technique to simulate the random mean return scenarios. Here we also incorporated the non-normality behaviours of the estimated returns. The technique was discussed in section 2.2. We will use 10000 simulated random mean return scenarios in optimization.

6.1 Numerical Results

Figure 6.1 shows the allocation results for different risk confidence intervals. From the figure we observe that this method behave in the same way of min-max method. Higher risk confidence interval will lead to more allocation of ETFs with low error volatility. We can also observe that an larger estimation risk coefficient β to lead to a more diversified portfolio.

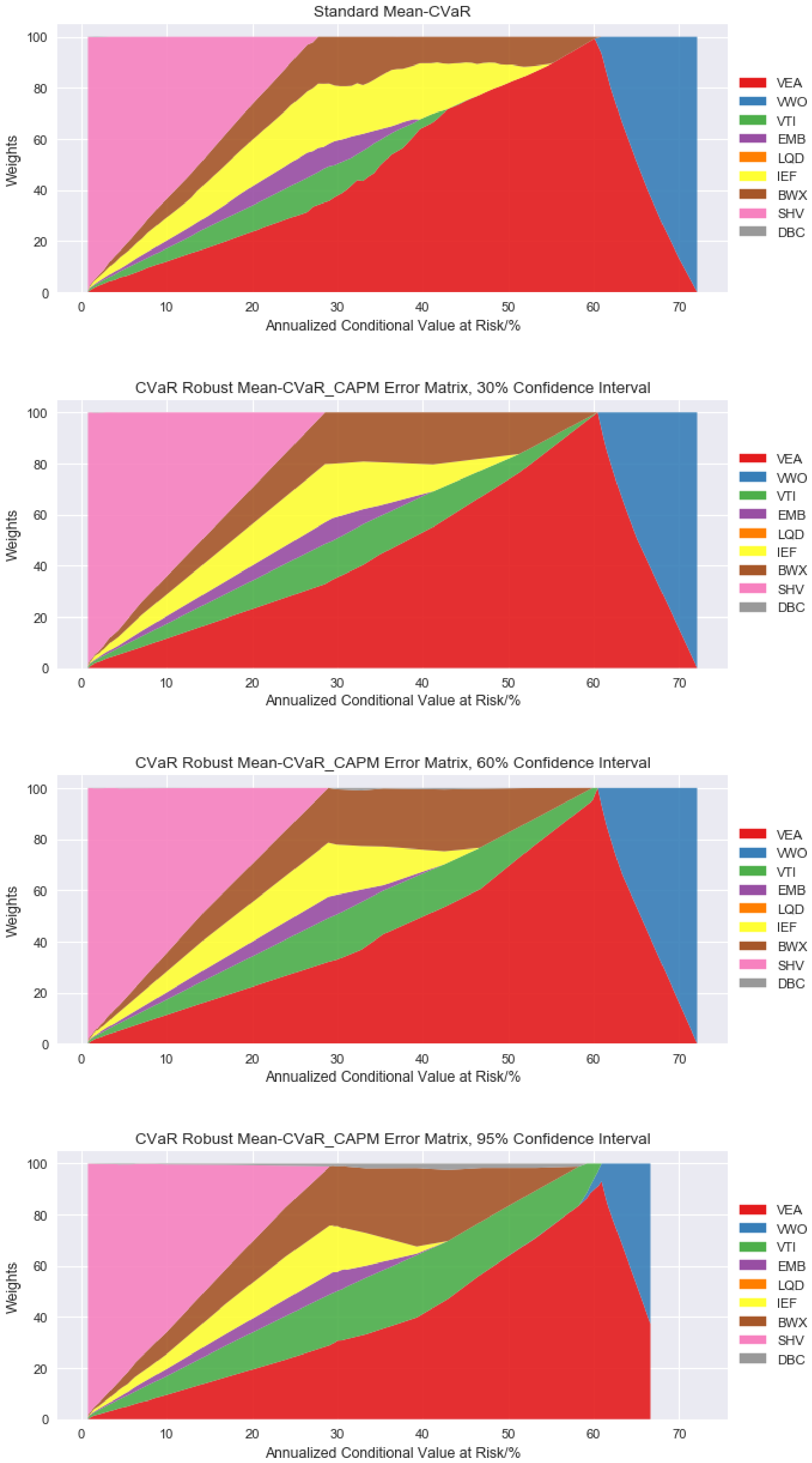

Figure 6.1: CVaR Robust Mean-CVaR Model Allocation Results

The figure below shows the frontiers of standard Mean-CVaR and 60% risk confidence interval respectively. This shows that with higher estimation risk aversion in this model, the frontier will become more conservative.

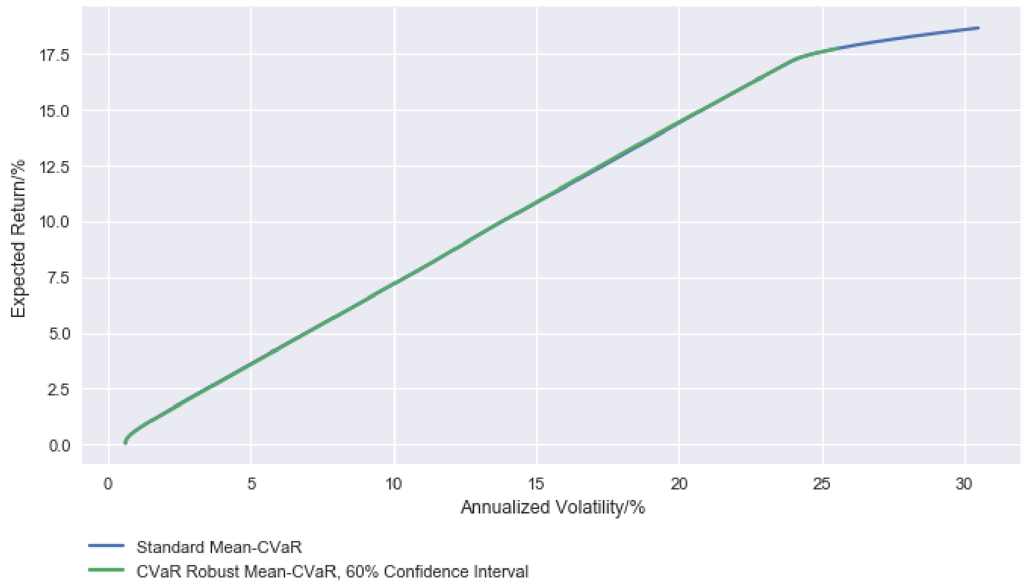

Figure 6.2: Frontiers of CVaR Robust Mean-CVaR Model

6.2 Sensitivity to Initial Data

Using the same method in section 4.3 and 5.3, we could access the effectiveness of the CVaR Robust Mean-CVaR Model.

From Figure 6.3, we observes the same phenomenon as in chapter 4 and chapter 5. When the estimation risk appetite become higher, The frontiers will become more stable. In the real investment practice, this means that our final allocation value is more robust to the fluctuations of markets.

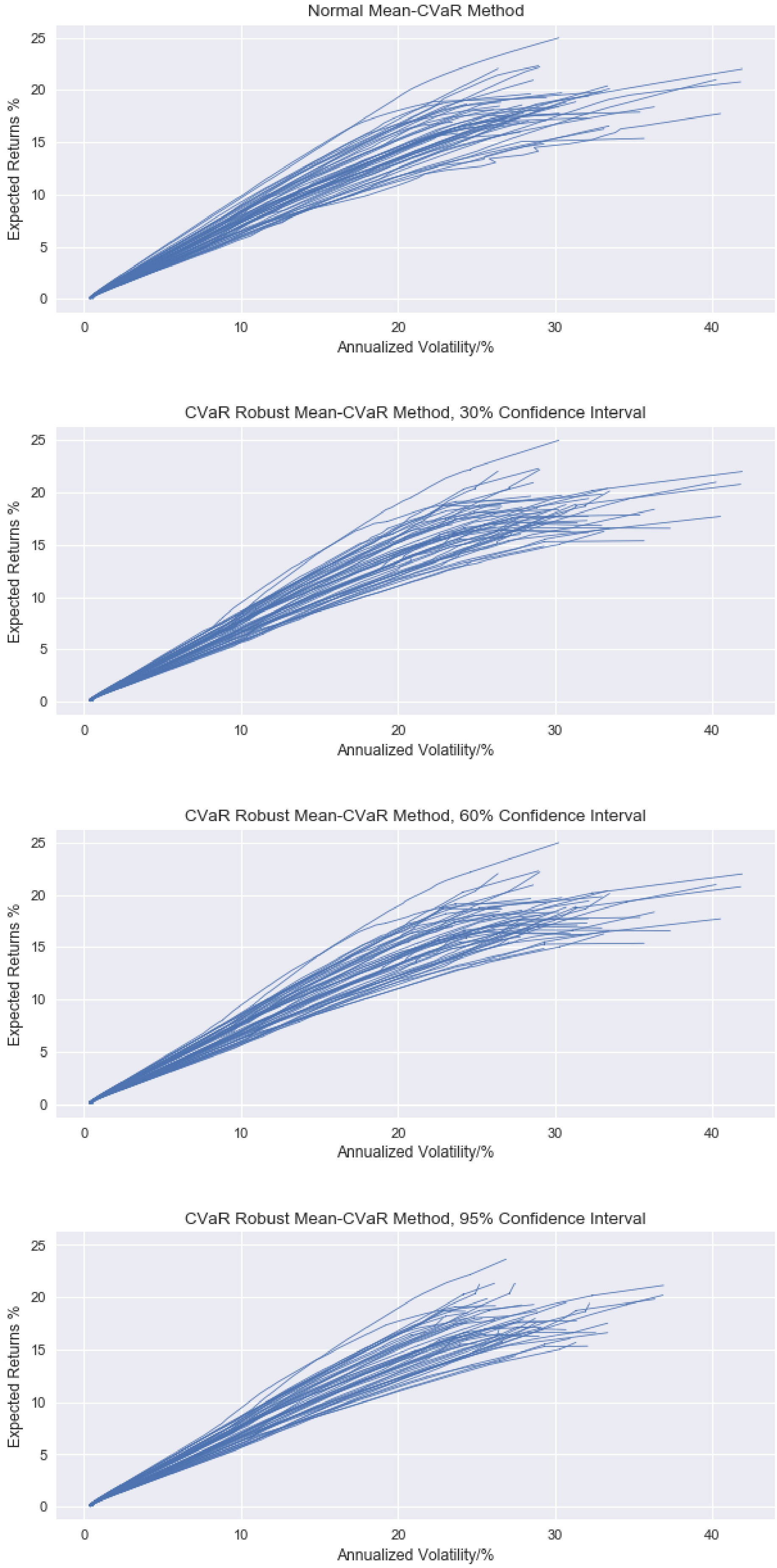

Figure 6.3: Sensitivity to Initial Data, CVaR Robust Mean-CVaR Model

Chapter 7

Final Remarks

In this white paper, three methodologies were introduced for robust portfolio allocation. They all show reasonable performance in our proposed framework. However, they do have some respective situations to apply.

The most important advantage of min-max Robust Mean-Variance Model is the calculation speed. However, since it employs variance as the risk measure, it have some inherent deficiency in real-world risk management. Then min-max Robust Mean-CVaR method was developed to overcome its drawbacks. However, its long computation time is also a concern. The CVaR Method is a new developed method to overcome the sensitivity to initial data issue of min-max method. It also requires less computation resources. But in Zhu's finding[16] this method was also affected by the sampling technique for sample mean. Choosing a proper sampling technique, either RS, chi-squre, bootstrap can be very critical for the model.

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