Голубович 953501 ЛР6 (вариант 2)

Лабораторная работа №6

Обыкновенные дифференциальные уравнения высших порядков (2)

Задание 1. Решите уравнения и сравните с результатами, полученными в Maple. Построить в одной системе координат несколько интегральных кривых.

+ 1.2.1) $x = y'' + \ln y''$

$$\begin{cases} x = t + \ln t \\ y'' = t \end{cases} \qquad \begin{cases} dy' = t\,dx \\ dx = \left(1 + \frac{1}{t}\right)dt \end{cases}$$

$$dy' = t\left(1 + \frac{1}{t}\right)dt = (t+1)dt$$

$$y' = \left(\frac{t^2}{2} + t + C_1\right)$$

$$dy = \left(\frac{t^2}{2} + t + C_1\right)dx = \left(\frac{t^2}{2} + t + C_1\right)\left(1 + \frac{1}{t}\right)dt$$

$$dy = \left(\frac{t^2}{2} + t + C_1 + \frac{t}{2} + 1 + \frac{C_1}{t}\right)dt$$

$$y = \frac{t^3}{6} + \frac{t^2}{2} + C_1 t + \frac{t^2}{4} + t + C_1 \ln|t| + C_2$$

$$\begin{cases} y = \frac{t^3}{6} + \frac{3}{4}t^2 + C_1 t + t + C_1 \ln(t) + C_2 \\ x = t + \ln t \end{cases}$$

Решение в системе Maple:

$$t = y'' = \frac{dy'}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{ddy \cdot dx - ddx \cdot dy}{(dx)^3}$$

$x := t \to t + \ln(t):$

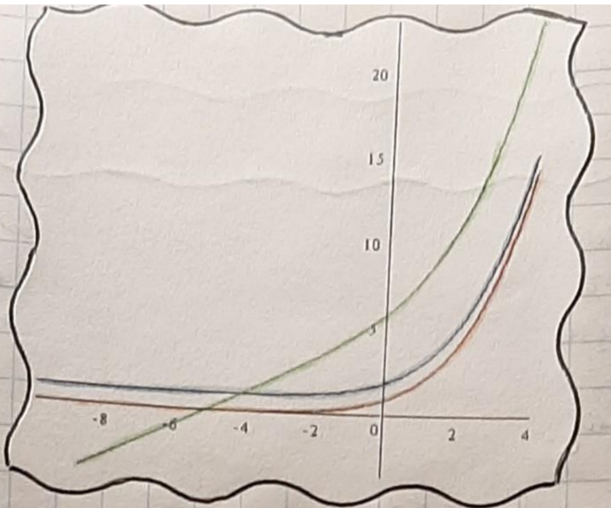
$$d := rhs\left(dsolve\left(t = \frac{\frac{d^2}{dt^2}y(t)\cdot\frac{d}{dt}x(t) - \frac{d}{dt}y(t)\cdot\frac{d^2}{dt^2}x(t)}{\left(\frac{d}{dt}x(t)\right)^3}\right)\right):$$

$y := subs(\_C1 = C_1, \_C2 = C_2, d);$

$$y := \frac{1}{6}t^3 + \frac{3}{4}t^2 + C_1 t + t + C_1 \ln(t) + C_2$$

$g := (a, b) \to plot([x(t), subs(\_C1 = a, \_C2 = b, d), t = -3..3]):$

$plots[display]([g(0, 0), g(0, 1), g(1, 5)], color = [red, blue, green]);$



Ответ: $\begin{cases} x = t + \ln t \\ y = \frac{t^3}{6} + \frac{3}{4}t^2 + C_1 t + t + C_1 \ln(t) + C_2. \end{cases}$

+ 2) $(x^2+1)(yy'' - y'^2) = 2xyy' \quad | \div y^2$ $\quad (y \overset{?}{=} 0)$ – входит в общее при $C_2 = 0$

$$(x^2+1)\left(\frac{yy'' - y'^2}{y^2}\right) = \frac{2xy'}{y}$$

• $z = \frac{y'}{y}$, $z = z(x)$, $z' = \left(\frac{y'}{y}\right)' = \frac{y''y - y'^2}{y^2}$

$$(x^2+1)z' = 2xz$$

$$\frac{dz}{z} = \frac{2x\,dx}{x^2+1} \left(= \frac{dx^2}{x^2+1} \right.$$

$(z \overset{?}{=} 0)$ $\quad \frac{y'}{y} = 0 \quad y = C$

входит в общее при $C_1 = 0$

$$\ln|z| = \ln|x^2+1| + \ln|C_1|$$

$$z = (x^2+1)C_1$$

• $z = \frac{y'}{y} = (x^2+1)C_1$

$$\frac{dy}{y} = (x^2C_1 + C_1)\,dx \quad \Rightarrow \quad \ln|y| = \frac{x^3}{3}C_1 + C_1 x + C_2$$

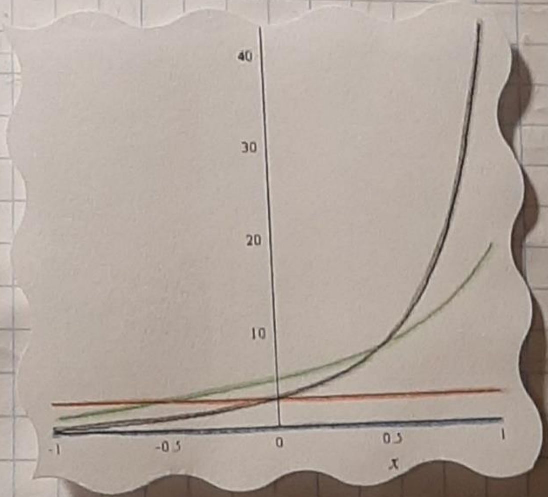
$$y = e^{\frac{x^3}{3}C_1} \cdot e^{C_1 x} \cdot C_2$$

Решение в системе Maple:

$d := dsolve\left(\left(x^2+1\right)\cdot\left(y(x)\cdot diff(y(x), x\$2) - diff(y(x), x)^2\right) = 2\cdot x\cdot y(x)\cdot diff(y(x), x), y(x)\right) :$
$y := subs\left(\_C1 = C_1, \_C2 = C_2, rhs(d)\right)$

$$y := e^{\frac{1}{3} C_1 x^3}\, e^{C_1 x}\, C_2$$

$g := (a, b) \rightarrow plot(subs(\_C1 = a, \_C2 = b, rhs(d)), x = -1..1) :$
$plots[display]([g(0, 3), g(3, 0), g(1, 5), g(2, 3)],\ color = [red, blue, green, black]);$



Ответ: $y = e^{\frac{x^3}{3} C_1} \cdot e^{C_1 x} \cdot C_2.$

+ 3) $y' = xy'' - e^{y''}$

• $y' = z, \quad z = z(x), \quad y'' = z' \quad \Rightarrow z = xz' - e^{z'}$ – уравнение Клеро

$\begin{cases} z' = t \\ z = xt - e^t \end{cases} \qquad \begin{cases} dz = t\,dx \\ dz = t\,dx + (x - e^t)\,dt \end{cases}$

$t\,dx = t\,dx + (x - e^t)\,dt$

$\left[\begin{array}{l} dt = 0 \\ x - e^t = 0 \end{array}\right. \qquad \left[\begin{array}{ll} t = C_1 & -(\mathrm{I}) \\ x = e^t & -(\mathrm{II}) \end{array}\right.$

(I) $z = xC_1 - e^{C_1}$
$y' = xC_1 - e^{C_1} \Rightarrow dy = (xC_1 - e^{C_1})\,dx \Rightarrow y = \frac{C_1 x^2}{2} - e^{C_1} x + C_2$

(II) $\begin{cases} x = e^t \\ z = e^t t - e^t = e^t(t-1) \end{cases}$

$\begin{cases} \ln x = t \\ z = x(\ln x - 1) \end{cases}$

2-ое уравнение системы: $z = y' \Rightarrow y' = x(\ln x - 1)$

$dy = (x\ln x - x)dx$

$$\boxed{\int x\ln x\,dx} = \left| \begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = x\,dx & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2\ln x}{2} - \int \frac{x}{2}dx = \frac{x^2\ln x}{2} - \frac{x^2}{4}$$

$$y = \frac{x^2\ln x}{2} - \frac{x^2}{4} - \frac{x^2}{2} + C \left(= \frac{x^2\ln x}{2} - \frac{3x^2}{4} + C \right.$$

$$y = \frac{x^2}{2}\left(\ln x - \frac{3}{2}\right) + C$$

```
y := subs(_C1 = C, rhs(dsolve(diff(y(x), x) = x·(ln(x) − 1), y(x))));
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$$y := \frac{1}{2}x^2\ln(x) - \frac{3}{4}x^2 + C$$

Решение в системе Maple:

```
d := rhs(dsolve(diff(y(x), x) = x·diff(y(x), x$2) − e^(diff(y(x), x$2)), y(x)))
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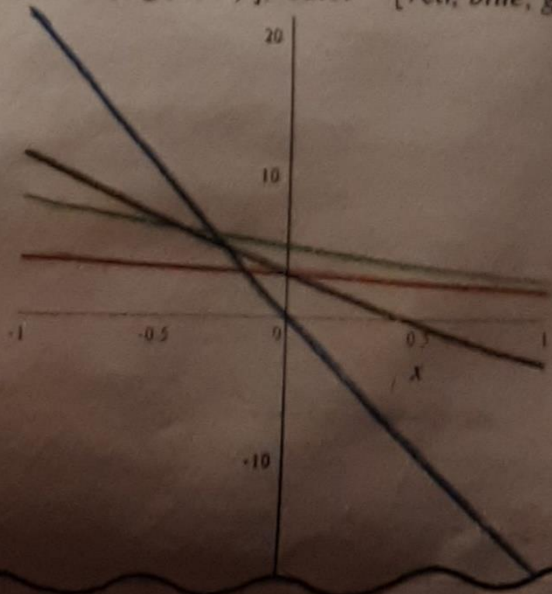
$$d := \frac{1}{2}\_C1\,x^2 + RootOf\left(-\mathrm{LambertW}\left(-e^{-\_C1}e^{\_Z}\right) + \_Z\right)x + \_C2$$

```
y := subs(_C1 = C[1], _C2 = C[2], allvalues(d));
```

$$y := \frac{1}{2}C_1 x^2 - e^{C_1}x + C_2$$

```
g := (a, b) → plot(subs(_C1 = a, _C2 = b, allvalues(d)), x = −1..1) :
plots[display]([g(0, 3), g(3, 0), g(1, 5), g(2, 3)], color = [red, blue, green, black]);
```



Ответ: $y = \frac{C_1 x^2}{2} - e^{C_1} x + C_2$, $y = \frac{x^2}{2}(\ln x - \frac{3}{2}) + C$

+ 4) $y'' = 2\left(\frac{y'}{x} - \frac{y}{x^2}\right) + \frac{1}{x^2} e^{\frac{1}{x}}$

$y'' = \frac{2y'}{x} - \frac{2y}{x^2} + \frac{1}{x^2} e^{\frac{1}{x}}$ — линейное неоднородное ДУ

$\frac{x^2}{4}$

- Решим однородное ЛДУ:

$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$

Заметим: $y_1 = x$ — решение. Найдем второе частное решение y_2 по формуле Лиувилля – Остроградского:

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = C_1 e^{-\int p_1(x)\,dx}$ $\qquad p_1(x) = \frac{a_1(x)}{a_2(x)}$

$y_1 y_2' - y_2 y_1' = C_1 e^{-\int -\frac{2}{x}dx} = C_1 e^{\int \frac{2}{x}dx} = C_1 e^{2\ln|x|} = C_1 e^{\ln x^2} = C_1 x^2$

$\frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{C_1 x^2}{y_1^2} = \frac{C_1 x^2}{x^2} = C_1 \Rightarrow \left(\frac{y_2}{y_1}\right)' = C_1$

$\frac{y_2}{y_1} = C_1 x + C_2 \Rightarrow y_2 = y_1(C_1 x + C_2) = x(C_1 x + C_2) = C_1 x^2 + C_2 x$

Общее решение ОЛДУ: $y_0 = C_1 x^2 + C_2 x$

- Решим неоднородное ЛДУ методом Лагранжа:

$$\begin{cases} y = C_1(x) x^2 + C_2(x) x \quad (*) \\ y' = \underline{C_1'(x) x^2} + C_1(x)\cdot 2x + \underline{C_2'(x) x} + C_2(x) \quad \Rightarrow \text{ в условие} \\ y'' = C_1'(x)\cdot 2x + 2C_1(x) + C_2'(x) \end{cases}$$

$2xC_1'(x) + 2C_1(x) + C_2'(x) = \frac{2\cdot 2x C_1(x)}{x} + \frac{2C_2(x)}{x} - \frac{2C_1(x)x^2}{x^2} - \frac{2C_2(x)x}{x^2} + \frac{1}{x^2} e^{\frac{1}{x}}$

$$\begin{cases} 2xC_1'(x) + C_2'(x) = \frac{1}{x^2} e^{\frac{1}{x}} \\ C_1'(x)\cdot x^2 + C_2'(x) x = 0 \end{cases}$$

$$\begin{cases} x^2C_1' + xC_2' = 0 \\ 2xC_1' + C_2' = \dfrac{e^{\frac{1}{x}}}{x^2} \end{cases} \qquad \begin{cases} C_2' = \dfrac{e^{\frac{1}{x}}}{x^2} - 2xC_1' \\ x^2C_1' + \dfrac{e^{\frac{1}{x}}}{x} - 2x^2C_1' = 0 \end{cases}$$

Рассмотрим 2-ое уравнение системы:

$$C_1'(x^2 - 2x^2) + \frac{e^{\frac{1}{x}}}{x} = 0$$

$$C_1'x^2 = \frac{e^{\frac{1}{x}}}{x} \Rightarrow C_1' = \frac{e^{\frac{1}{x}}}{x^3} \Rightarrow C_1(x) = \int \frac{e^{\frac{1}{x}}}{x^3}dx$$

$$\int \frac{e^{\frac{1}{x}}}{x^3}dx = \left| \begin{matrix} t = \frac{1}{x} & x = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt & \end{matrix} \right| = -\int \frac{e^t t^3}{t^2}dt = -\int te^t dt =$$

$$= \left| \begin{matrix} u = t & du = dt \\ dv = e^t dt & v = e^t \end{matrix} \right| = -te^t + \int e^t dt = -te^t + e^t = e^t(1-t) =$$

$$= e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = \frac{(x-1)e^{\frac{1}{x}}}{x}$$

$$C_1(x) = \frac{(x-1)e^{\frac{1}{x}}}{x} + C_1$$

Рассмотрим 1-ое уравнение системы:

$$C_2' = \frac{e^{\frac{1}{x}}}{x^2} - \frac{2xe^{\frac{1}{x}}}{x^3} \Big(= \frac{e^{\frac{1}{x}}}{x^2} - \frac{2e^{\frac{1}{x}}}{x^2} = -\frac{e^{\frac{1}{x}}}{x^2} \Rightarrow C_2(x) = -\int \frac{e^{\frac{1}{x}}}{x^2}dx$$

$$-\int \frac{e^{\frac{1}{x}}}{x^2}dx = \left| \begin{matrix} t = \frac{1}{x} & x = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt & \end{matrix} \right| = \int \frac{e^t t^2}{t^2}dt = \int e^t dt = e^t = e^{\frac{1}{x}}$$

$$C_2(x) = e^{\frac{1}{x}} + C_2$$

Подставим $C_1(x)$ и $C_2(x)$ в $(*)$:

$$y = x^2\left(\frac{(x-1)e^{\frac{1}{x}}}{x} + C_1\right) + x\left(e^{\frac{1}{x}} + C_2\right)$$

$$y = x^2e^{\frac{1}{x}} - xe^{\frac{1}{x}} + C_1x^2 + xe^{\frac{1}{x}} + C_2x$$

$$y = x^2e^{\frac{1}{x}} + C_1x^2 + C_2x$$

- Решение в системе Maple:

$$yo1 := rhs\left(dsolve\left(diff(y(x), x\$2) = \frac{2\cdot diff(y(x), x)}{x} - \frac{2\cdot y(x)}{x^2}, y(x)\right)\right):$$

$$yo := subs(\_C1 = C_1, \_C2 = C_2, yo1);$$

$$yo := x^2 C_2 + x C_1$$

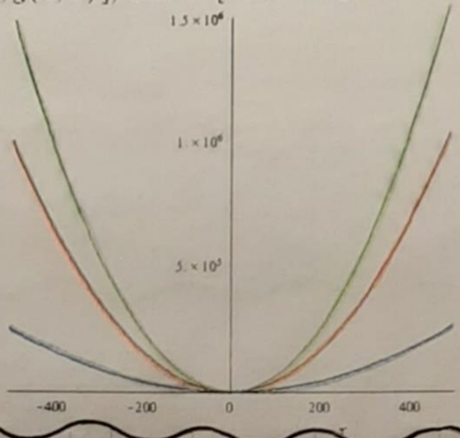
$$yn1 := rhs\left(dsolve\left(diff(y(x), x\$2) = \frac{2\cdot diff(y(x), x)}{x} - \frac{2\cdot y(x)}{x^2} + \frac{1}{x^2} e^{\frac{1}{x}}, y(x)\right)\right):$$

$$yn := subs(\_C1 = C_1, \_C2 = C_2, yn1);$$

$$yn := x^2 e^{\frac{1}{x}} + C_2 x^2 + C_1 x$$

$$g := (a, b) \rightarrow plot(subs(\_C1 = a, \_C2 = b, yn1), x = -500..500):$$

$$plots[display]([g(0, 3), g(3, 0), g(1, 5)],\ color = [red, blue, green]);$$



Ответ: $y = x^2 e^{\frac{1}{x}} + C_1 x^2 + C_2 x$.

Задание 2. Найдите общее решение уравнения и сравните с результатом, полученным в Maple.

2.2. $xy''' + y'' = 1$

• $y'' = z$, $z = z(x)$, $y''' = z'$

$xz' + z = 1 \Rightarrow z' = \frac{1-z}{x} \Rightarrow \frac{dz}{z-1} = -\frac{dx}{x}$ $\quad (z - 1 \overset{?}{=} 0)$

$y'' = 1 \quad y' = x + \tilde{C}_1$

$y = \frac{x^2}{2} + \tilde{C}_1 x + \tilde{C}_2$

входит в общее

$\ln|z-1| = -\ln|x| + \ln|C_1| \Rightarrow z - 1 = \frac{C_1}{x}$

$z = \frac{C_1}{x} + 1$

- $y'' = z \Rightarrow y'' = \frac{C_1}{x} + 1$

$dy' = \left(\frac{C_1}{x} + 1\right) dx$

$y' = C_1 \ln|x| + x + C_2$

$y = C_1 \int \ln(x) dx + \frac{x^2}{2} + C_2 x + C_3$

$\boxed{\int \ln x dx} = \begin{vmatrix} u = \ln x & du = \frac{1}{x} dx \\ dv = dx & v = x \end{vmatrix} = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - x$

$y = C_1 x (\ln x - 1) + \frac{x^2}{2} + C_2 x + C_3$

- Решение в системе Maple:

```
y := rhs(dsolve(x·diff(y(x), x$3) + diff(y(x), x$2) = 1, y(x))):
y := subs(_C1 = C1, _C2 = C2, _C3 = C3, y);
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$$y := \ln(x)\, x\, C_1 - x\, C_1 + \frac{1}{2} x^2 + C_2 x + C_3$$

Ответ: $y = C_1 x (\ln x - 1) + \frac{x^2}{2} + C_2 x + C_3$.

Задание 3. Найти общее решение дифф. уравнения

3.2. $y'' - 4y' + 4y = -e^{2x} \sin 6x$ — НЛДУ с постоянными коэффициентами

- Решим ОЛДУ:

$y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda_{1,2} = 2$

$y_0 = e^{2x} C_1 + x e^{2x} C_2$ — общее решение ОЛДУ

- Решим НЛДУ:

$f(x) = -e^{2x} \sin 6x \Rightarrow y^* = -e^{2x}(A \cos 6x + B \sin 6x)$

$y^{*\prime} = -2e^{2x}(A\cos 6x + B\sin 6x) - e^{2x}(-6A\sin 6x + 6B\cos 6x) =$

$= -e^{2x}((2A+6B)\cos 6x + (2B-6A)\sin 6x)$

$y^{*\prime\prime} = -2e^{2x}((2A+6B)\cos 6x + (2B-6A)\sin 6x) - e^{2x}((-12A-36B)\sin 6x + (12B-36A)\cdot$

$\cdot\cos 6x) = -e^{2x}(((4A+12B+12B-36A)\cos 6x + (4B-12A-12A-36B)\sin 6x)$

Подставим $y^*, y^{*\prime}, y^{*\prime\prime}$ в условие (начальное уравнение):

$-e^{2x}((24B-32A)\cos 6x + (-24A-32B)\sin 6x) - (-4e^{2x})((2A+6B)\cos 6x + (2B-6A)\sin 6x) -$

$-4e^{2x}(A\cos 6x + B\sin 6x) = -e^{2x}\sin 6x$

$$\begin{cases} (24B - 32A - 8A - 24B + 4A)\cos 6x = 0 \\ (-24A - 32B - 8B + 24A + 4B)\sin 6x = \sin 6x \end{cases}$$

$$\begin{cases} -36A = 0 \\ -36B = 1 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = -\frac{1}{36} \end{cases}$$

$y^* = +e^{2x}\cdot\frac{1}{36}\sin 6x$ – частное решение НЛДУ

Общее решение НЛДУ:

$y = e^{2x}C_1 + xe^{2x}C_2 + e^{2x}\cdot\frac{1}{36}\sin 6x.$

- Решение в системе Maple:

$y := rhs(dsolve(diff(y(x), x\$2) - 4\cdot diff(y(x), x) + 4\cdot y(x) = -e^{2\cdot x}\cdot\sin(6\cdot x), y(x)))$:

$y := subs(\_C1 = C_1, \_C2 = C_2, y)$;

$$y := e^{2x}C_2 + e^{2x}xC_1 + \frac{1}{36}e^{2x}\sin(6x)$$

Ответ: $y = e^{2x}C_1 + xe^{2x}C_2 + \frac{1}{36}e^{2x}\sin 6x.$

> #Task 1

> #1)

> restart;

> with(DEtools) :

> $x := t \rightarrow t + \ln(t)$:

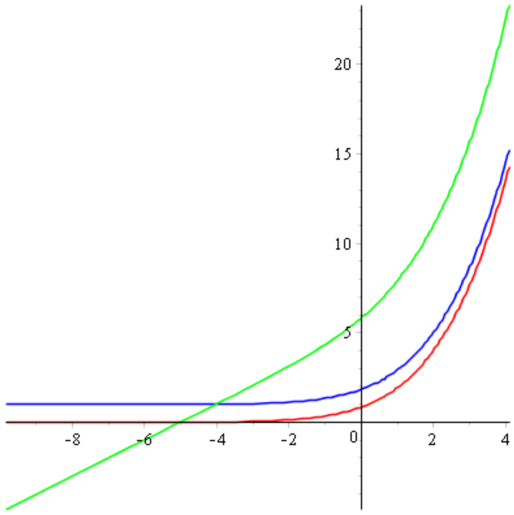
> $d := rhs\left(dsolve\left(t = \dfrac{\frac{d^2}{dt^2} y(t) \cdot \frac{d}{dt} x(t) - \frac{d}{dt} y(t) \cdot \frac{d^2}{dt^2} x(t)}{\left(\frac{d}{dt} x(t)\right)^3}\right)\right)$:

> $y := subs(\_C1 = C_1, \_C2 = C_2, d)$;

$$y := \frac{1}{6} t^3 + \frac{3}{4} t^2 + C_1 t + t + C_1 \ln(t) + C_2 \tag{1}$$

> $g := (a, b) \rightarrow plot([x(t), subs(\_C1 = a, \_C2 = b, d), t = -3..3])$:

> plots[display]([g(0, 0), g(0, 1), g(1, 5)], color = [red, blue, green]);



> #2)

> restart;

> with(DEtools) :

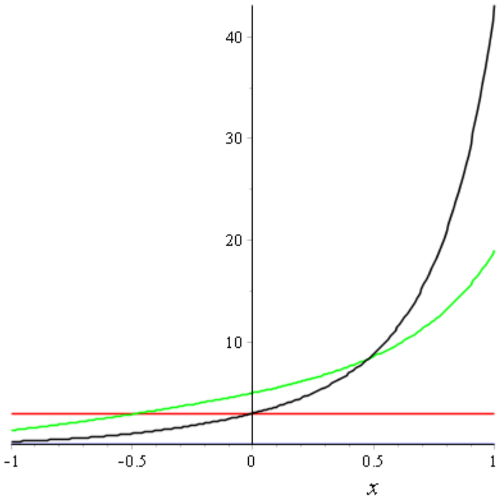
> $d := dsolve((x^2 + 1) \cdot (y(x) \cdot diff(y(x), x\$2) - diff(y(x), x)^2) = 2 \cdot x \cdot y(x) \cdot diff(y(x), x), y(x))$:

> $y := subs(\_C1 = C_1, \_C2 = C_2, rhs(d))$

$$y := e^{\frac{1}{3} C_1 x^3} e^{C_1 x} C_2 \tag{2}$$

> $g := (a, b) \rightarrow plot(subs(\_C1 = a, \_C2 = b, rhs(d)), x = -1..1)$:

> plots[display]([g(0, 3), g(3, 0), g(1, 5), g(2, 3)], color = [red, blue, green, black]);



> #3)

> restart;

> with(DEtools) :

> $d := rhs\left(dsolve\left(diff(y(x), x) = x \cdot diff(y(x), x\$2) - e^{diff(y(x), x\$2)}, y(x)\right)\right)$

$$d := \frac{1}{2}\_C1\, x^2 + RootOf\left(-\mathrm{LambertW}\left(-e^{-\_C1}\, e^{-\_Z}\right) + \_Z\right) x + \_C2 \quad \mathbf{(3)}$$

> $y := subs\left(\_C1 = C_1, \_C2 = C_2, allvalues(d)\right);$

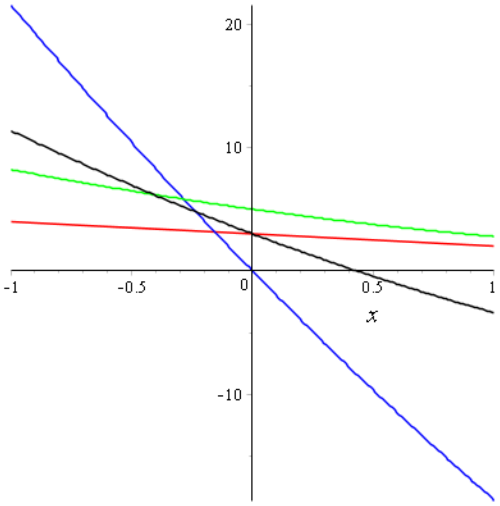
$$y := \frac{1}{2} C_1 x^2 - e^{C_1} x + C_2 \quad \mathbf{(4)}$$

> $y := subs(\_C1 = C, rhs(dsolve(diff(y(x), x) = x \cdot (\ln(x) - 1), y(x))));$

$$y := \frac{1}{2} x^2 \ln(x) - \frac{3}{4} x^2 + C \quad \mathbf{(5)}$$

> $g := (a, b) \rightarrow plot(subs(\_C1 = a, \_C2 = b, allvalues(d)), x = -1 ..1) :$

> $plots[display]([g(0, 3), g(3, 0), g(1, 5), g(2, 3)],\ color = [red, blue, green, black]);$



> #4)

> restart;

> with(DEtools) :

> $yo1 := rhs\left(dsolve\left(diff(y(x), x\$2) = \frac{2 \cdot diff(y(x), x)}{x} - \frac{2 \cdot y(x)}{x^2}, y(x)\right)\right) :$

> $yo := subs\left(\_C1 = C_1, \_C2 = C_2, yo1\right);$

$$yo := x^2 C_2 + x C_1 \quad \mathbf{(6)}$$

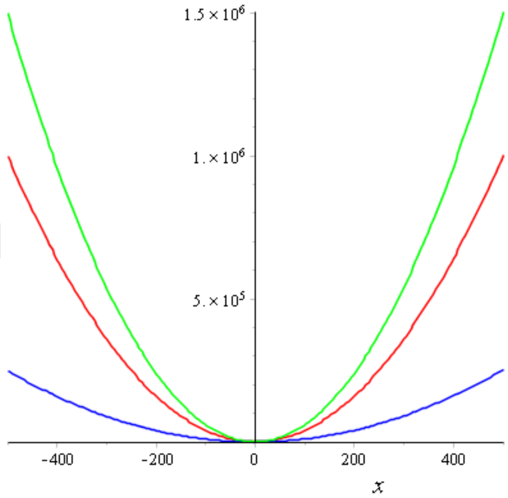
> $yn1 := rhs\left(dsolve\left(diff(y(x), x\$2) = \frac{2\cdot diff(y(x), x)}{x} - \frac{2\cdot y(x)}{x^2} + \frac{1}{x^2}e^{\frac{1}{x}}, y(x)\right)\right):$

> $yn := subs(\_C1 = C_1, \_C2 = C_2, yn1);$

$$yn := x^2\,e^{\frac{1}{x}} + C_2\,x^2 + C_1\,x \qquad (7)$$

> $g := (a, b) \rightarrow plot(subs(\_C1 = a, \_C2 = b, yn1), x = -500..500):$

> $plots[display]([g(0, 3), g(3, 0), g(1, 5)],\ color = [red, blue, green]);$



> #Task2

> restart;

> with(DEtools) :

> $y := rhs(dsolve(x\cdot diff(y(x), x\$3) + diff(y(x), x\$2) = 1, y(x))) :$

> $y := subs(\_C1 = C_1, \_C2 = C_2, \_C3 = C_3, y);$

$$y := \ln(x)\,x\,C_1 - x\,C_1 + \frac{1}{2}x^2 + C_2\,x + C_3 \qquad (8)$$

> #Task3

> restart;

> with(DEtools) :

> $y := rhs\left(dsolve\left(diff(y(x), x\$2) - 4\cdot diff(y(x), x) + 4\cdot y(x) = -e^{2\cdot x}\cdot\sin(6\cdot x), y(x)\right)\right):$

> $y := subs(\_C1 = C_1, \_C2 = C_2, y);$

$$y := e^{2x}\,C_2 + e^{2x}\,x\,C_1 + \frac{1}{36}\,e^{2x}\sin(6\,x) \qquad (9)$$