Nash equilibria of quasisupermodular games

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Abstract

We prove three results on the existence and the structure of Nash equilibria for quasisupermodular games. They all require certain continuity as assumption. The first result is purely order-theoretic, where we assume a continuity introduced by Shannon. The second continuity hypothesis is a mixture in the following sense. We require another order-theoretic continuity defined by Milgrom and Roberts, which we show to be only a half of Shannon's condition. As compensation, we also require a weak topological continuity proposed by Tian and Zhou. The last continuity supposition is purely topological, which is a slightly stronger continuity due to Tian and Zhou. The last theorem simultaneously generalizes Zhou's theorem on supermodular games and Calciano's theorem on quasisupermodular games.

Keywords— Quasisupermodular game, Nash equilibrium, Tarski fixed point theorem, complete lattice

1 Introduction

Quasisupermodular games, defined by Shannon and Milgrom in [18, 15], are generalizations of supermodular games introduced by Topkis [21]. Supermodular games exhibit "strategic complementarities", which roughly means that when one player takes a higher action, the others want to do the same.

The defining properties of supermodular games, namely supermodularity and increasing differences property for *real-valued* payoff functions, rely essentially on the additive group structure of the *codomain* $\mathbb{R}$. Milgrom and Shannon [15, p.160, p.162] generalize the two conditions to *quasisupermodularity* and *single crossing property*. We review the conditions in Definitions 4.1 and 6.1 respectively. As [22, p.59] remarks, these conditions require no longer any additive structure on the codomain. Thus, one can naturally extend both quasisupermodularity and single crossing property to chain-valued functions.

The study on the structure of Nash equilibria of supermodular games under *topological* hypotheses is initialized by Topkis [21, Thm. 3.1]. He shows that if the strategy set S is a compact sublattice of an Euclidean space, and if the payoff functions are upper

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semicontinuous, then the set of Nash equilibria is nonempty and contains a greatest and a least element. Vives strengths Topkis's theorem as follows. In [23, Thm. 4.2 (i)], Vives proves that Topkis's theorem generalizes to the case where S is a lattice compact in a topology finer than its interval topology, which may not embed into any Euclidean space. Under assumptions stronger than supermodularity and increasing differences property, Vives [23, Thm. 4.2 (ii)] gets a finer result: the set of Nash equilibria with the inherited order structure is a nonempty complete lattice. Vives's stronger assumptions are removed by Zhou [25, Thm. 2]. Zhou shows that for every supermodular game, the set of Nash equilibria forms a nonempty complete lattice. This generalizes simultaneously [23, Thm. 4.2 (i) and (ii)].

For quasisupermodular games with topologically upper semicontinuous payoff functions, Milgrom and Shannon [15, Theorem 12] give a result similar to [23, Thm. 4.2 (i)]. Calciano [2, Theorem 3] proves that the Nash equilibria of such a game form a nonempty complete lattice, generalizing Zhou's theorem and the Milgrom-Shannon theorem.

Assuming *order upper semicontinuity* on payoff functions, Milgrom and Roberts [14, p.1266] show that a supermodular game has a least and a largest Nash equilibrium. The same conclusion holds even if the continuity restriction is relaxed and the payoff functions are allowed to be more general poset-valued functions, as [3, Proposition 8.49] proves. Here and below, a set equipped with a partial order is called a *poset*.

The main objective of this paper is twofold, and both concern the structure of the set of Nash equilibria of quasisupermodular games. First, we provide a *purely order-theoretic* result (Theorem 6.8). Compared to Nash's theorem on the existence of Nash equilibria for normal form games, it does not require any topological hypothesis. The proof of Theorem 6.8 needs an existence result of maximum (Theorem 5.4). Second, in the spirit of [17, Theorem 1], we relax the conventional topological upper semicontinuity of the payoff functions to *transfer upper continuity* (recalled in Definition 3.1) in Theorems 6.11 and 6.13.

The text is organized as follows. Section 2 summarizes two subcompleteness properties for lattices and shows their equivalence. Section 3 recalls several conditions analogous to continuity, both in the topological and the order-theoretic sense. Section 4 reviews quasisupermodularity. Section 5 gives existence results about maximum for functions. Finally in Section 6, results from the previous sections are used to establish the existence of Nash equilibria for quasisupermodular games. The order structure of the set of all Nash equilibria is also discussed.

2 Characterization of subcompleteness

In Section 2, we give a self-contained proof of a characterization of subcompleteness using chain conditions, Lemma 2.3 2. It is used in the proof of Theorem 6.8, and we cannot find a published proof in the literature.

A poset L is called a *lattice* if for any $x, y \in L$, both $\sup_L\{x, y\}$ and $\inf_L\{x, y\}$ exist in L, classically denoted by $x \vee y$ and $x \wedge y$ respectively. A lattice L is called *complete* if for every nonempty subset A of L, both $\sup_L(A)$ and $\inf_L(A)$ exist. A poset is called a *chain* (resp. an *anti-chain*) if any two distinct elements are comparable (resp. incomparable). In an anti-chain with more than one element, every element is minimal but not the least element. For a lattice, Lemma 2.1 shows the two notions agree.

Lemma 2.1. *A minimal (resp. maximal) element of a lattice is the least (resp. largest) element.*

Proof. Let $s \in L$ be a minimal element of a lattice L. For every $s' \in L$, the element $s \wedge s'$ exists in L. Since s is minimal and $s \wedge s' \leq s$, one has $s = s \wedge s' \leq s'$, which shows that s is the least element. The statement in the parentheses is similar. $\square$

Definition 2.2. Let P be a poset. Let S be a subset of P. If for any $x, y \in S$, one has $\inf_P\{x, y\}, \sup_P\{x, y\} \in S$, then S is called a *sublattice* of P. If for every nonempty chain (resp. nonempty subset) $C \subset S$, both $\sup_P(C)$ and $\inf_P(C)$ exist and lie in S, then S is called *chain-subcomplete* (resp. *subcomplete*) in P.

For every subcomplete sublattice S of P, the poset S is a complete lattice. Still, as [14, p.1260] remarks, the sublattice $[0, 1) \cup \{2\}$ of the complete chain $[0, 2]$ is itself a complete lattice, but not a subcomplete sublattice of $[0, 2]$.

In [12, Theorem B], Lemma 2.3 2 is stated without proof and referred to Veinott's unpublished work. We provide a new proof for the reader's convenience. By [9, p.115], for every set X, there is an object $|X|$, called the *cardinal number* or *cardinality* of X, with the property that for any two sets X and Y, there is a bijection $X \to Y$ exactly when $|X| = |Y|$.

Lemma 2.3. *Let S be a nonempty subset of a poset P.*

1. *Assume that*

   (I) for any $s \neq s' \in S$, the join $\sup_P\{s, s'\}$ exists in S;

   (II) for every nonempty chain C in S, the supremum $\sup_P C$ exists in S.

   Then for every nonempty subset A of S, the supremum $\sup_P A$ exists in S.

2. *Suppose that S is a sublattice of P. Then S is subcomplete if and only if it is chain-subcomplete in P.*

Proof. 1. Let $\mathcal{P}$ be the set of all nonempty subsets A' of S such that $\sup_P A'$ does not exist in S. We prove $\mathcal{P} = \emptyset$.

   Otherwise, $\mathcal{P}$ is nonempty. From [8, the first sentence], there is $A \in \mathcal{P}$ with $|A| \leq |A'|$ for all $A' \in \mathcal{P}$. By Assumption (I), the set A is infinite. Then by Markowsky's sharpened version [13, Thm. 1] of Iwamura's lemma, there is a chain I and a nonempty subset A_α of A for every $\alpha \in I$, such that:

   (a) For all $\alpha \in I$, one has $|A_\alpha| < |A|$;

   (b) One has $A = \cup_{\alpha \in I} A_\alpha$;

   (c) For any $\alpha \leq \beta$ in I, one has $A_\alpha \subset A_\beta$.

   By 1a, one has $A_\alpha \notin \mathcal{P}$ for all $\alpha \in I$. Therefore, $a_\alpha := \sup_P(A_\alpha)$ exists in S. By 1c, for any $\alpha \leq \beta$ in I, one has $a_\alpha \leq a_\beta$. Thus, $\{a_\alpha\}_{\alpha \in I}$ is a nonempty chain in S. By Assumption (II), the element $u := \sup_P\{a_\alpha\}_{\alpha \in I}$ exists in S.

   We claim that $u = \sup_P A$. By 1b, for every $a \in A$, there is $\alpha_0 \in I$ with $a \in A_{\alpha_0}$. Then $a \leq a_{\alpha_0} \leq u$. If $u' \in P$ is another upper bound on A, then $u' \geq a_\alpha$ for all $\alpha \in I$. Therefore, $u' \geq u$. The claim is proved. However, the claim contradicts $A \in \mathcal{P}$.

2. The "only if" part follows from the definition. The"if" part follows from Point 1.

$\square$

Remark 2.4. Lemma 2.3 2 implies [4, Thm. 2.41 (iii)]: a lattice having no infinite chains is complete.

Remark 2.5. Let γ be a cardinal number. Let P be a poset. Markowsky [13, Cor. 1] shows that if P is chain γ-complete in the sense of [13, p.53], then for every directed subset D of P with $|D| \leq \gamma$, $\sup_P D$ exists. It is an intrinsic property of posets, and Lemma 2.3 2 is an extrinsic variant of it. By Markowsky's definition, every nonempty, chain γ-complete poset has a least element. By contrast, a chain-subcomplete subset of a poset may not have least element. Lemma 2.3 2 is not covered by [13, Cor. 1].

Remark 2.6. Let P be a poset. Markowsky [13, Cor. 5] shows that if P has a least element, and if all nonempty chains in P as well as all nonempty finite subsets of P have supremum, then P is a complete lattice. We generalize Markowsky's result to a characterization of subcompleteness. Compared to Lemma 2.3 2, we do not assume that S is a sublattice of P.

Let S be a nonempty subset of a poset P. Then the following are equivalent.

1. S is a subcomplete sublattice of P;
2. Assumptions (I) and (II) in Lemma 2.3 are satisfied, and
   - (i) S has a least element;
   - (ii) for every $p \in P$, if the set $\{x \in S | x \geq p\}$ is nonempty, then it has a least element.

Assume Condition 1 and we prove (ii). By subcompleteness, $s_p := \inf_P\{x \in S | x \geq p\}$ exists in S. One has $s_p \geq p$ and for $x \geq s_p$ every $x \in S$ with $x \geq p$. Then $s_p = \min\{x \in S | x \geq p\}$. Conversely, we show Condition 2 implies 1. It remains to prove that for every nonempty subset B of S, the supremum $\inf_P B$ exists in S. Let B' be the set of $x \in P$ with $x \leq b$ for all $b \in B$. By Assumption (i), B' contains $\min S$. So it is nonempty. For every $x \in B'$, the set $\{s \in S | s \geq x\}$ contains B, so it is nonempty. By Assumption (ii), it has a least element s_x. By Lemma 2.3 1, $s_0 := \sup_P\{s_x\}_{x \in B'}$ exists in S. For every $b \in B$ and every $x \in B'$, one has $s_x \leq b$, so $s_0 \leq b$. It implies $s_0 \in B'$. For every $x \in B'$, one has $x \leq s_x \leq s_0$, so $s_0 = \max B' = \inf_P B$.

When $S = P$, Assumption (ii) is automatic, so the equivalence specializes to Markowsky's result.

3 Continuity

In Section 3, we review some variants of continuity, in the context of topology and order theory. They are different from the conventional upper semicontinuity, but suffice to ensure the existence of a maximum. They are part of the hypotheses of our main results in Section 6.

Transfer continuity conditions are introduced by Tian and Zhou [20, Definitions 1, 2]. They are used to establish the existence of Nash equilibria in discontinuous games in [16].

Definition 3.1. Let X be a topological space. A function $f : X \to \mathbb{R}$ is called *transfer* (resp. *weakly transfer*) *upper continuous* if for any $x, y \in X$ with $f(y) < f(x)$, there exists a point $x' \in X$ and an open neighborhood U of y in X, such that $f(z) < f(x')$ (resp. $f(z) \leq f(x')$) for every $z \in U$.

An upper semicontinuous function is transfer upper continuous, and a transfer upper continuous is transfer weakly upper continuous. As [20, Examples 1, 2] show, the converses fail.

Let $(L, \leq)$ be a lattice. The interval topology in the sense of [7, p.570] of L is defined to be the smallest topology such that $\{x \in L : x \geq a\}$ and $\{x \in L : x \leq a\}$ are closed for every $a \in L$. It is a topology canonically determined by the order structure.

We recall several sorts of continuity involving order. Let $f : L \to \mathscr{C}$ be a function from a lattice to a chain. It is called *topologically upper semicontinuous*, if for every $a \in \mathscr{C}$, the set $[f \geq a] := \{x \in L : f(x) \geq a\}$ is closed in the interval topology of L. (When $\mathscr{C} = \mathbb{R}$, this reduces to the upper semicontinuity of real-valued functions.)

Assume that L is a complete lattice. Definition 3.2 is an order-theoretic counterpart of the topological semicontinuity. We need the notation from [14, p.1261]. For a nonempty chain $C \subset L$, the pair $(C, \leq)$ is naturally a directed set. Then the inclusion $C \to L$ is a net in L, which is denoted by $x \in C, x \uparrow \sup_L(C)$. Define the reversed order $\preceq$ on C, by $x \preceq y$ if and only if $x \geq y$ for all $x, y \in C$. Then $(C, \preceq)$ is another directed set. The inclusion $C \to L$ gives another net in L, which is denote by $x \in C, x \downarrow \inf_L(C)$. For every net $x_\bullet : (I, \leq) \to L$, define the limit superior along the net by $\limsup_{i \in I} x_i := \inf_{i \in I} \sup_{j \geq i} x_j$. It is well-defined by completeness of L.

Definition 3.2 ([14, p.1260], [17, Definition 1])**.** Let $f : L \to \mathscr{C}$ be a function from a complete lattice to a complete chain. It is *upward* (resp. *downward*) *upper semicontinuous*, if for every nonempty chain $C \subset L$, one has

$$\limsup_{x \in C, x \uparrow \sup_L(C)} f(x) \leq f(\sup_L C)$$

$$(\text{resp.} \quad \limsup_{x \in C, x \downarrow \inf_L(C)} f(x) \leq f(\inf_L C)).$$

If f is both upward and downward upper semicontinuous, then it is called *order upper semicontinuous.*

Remark 3.3. For a finite chain C, the corresponding inequalities in Definition 3.2 are automatic. In particular, if every chain in L is finite, then every function on L is order upper semicontinuous. The sum of two upward (resp. downward) upper semicontinuous functions on L is also upward (resp. downward) upper semicontinuous.

Definition 3.4 is an extension of Definition 3.2 to non-complete chains. This notion is used in Theorem 6.8.

Definition 3.4 ([18, p.8])**.** Let $f : L \to \mathscr{C}$ be a function from a complete lattice to a chain. It is *upper chain subcomplete*, if for every $a \in \mathscr{C}$ and every nonempty chain C in $[f \geq a]$, both $\sup_L(C)$ and $\inf_L(C)$ are in $[f \geq a]$.

Proposition 3.5. *Let $f : L \to \mathscr{C}$ be a function from a complete lattice to a complete chain. Then f is order upper semicontinuous if and only if f is upper chain subcomplete.*

Proof. We prove that f is upward upper semicontinuous if and only if for every $a \in \mathscr{C}$ and every nonempty chain C in $[f \geq a]$, one has $\sup_L C \in [f \geq a]$.

Assume that f is upward upper semicontinuous. Then for every $a \in \mathscr{C}$ and every nonempty chain C in $[f \geq a]$, one has

$$f(\sup_L C) \geq \limsup_{x \in C, x \uparrow \sup_L C} f(x) \geq a,$$

so $\sup_L C \in [f \geq a]$.

Conversely, assume that for every $a \in \mathscr{C}$ and every nonempty chain C' in $[f \geq a]$, one has $\sup_L(C') \in [f \geq a]$. Fix a nonempty chain C contained in L. Set $b_2 := \limsup_{x \in C, x \uparrow \sup_L C} f(x)$. We need to prove

$$f(\sup_L C) \geq b_2. \tag{1}$$

Assume, by contradiction, $f(\sup_L C) < b_2$. Because $\mathscr{C}$ is a chain, there are exactly two cases.

1. There is $b_1 \in \mathscr{C}$ with $f(\sup_L C) < b_1 < b_2$. In this case, for every $x \in C$, there is $y_x \in C$ with $y_x \geq x$ and $f(y_x) \geq b_1$.
2. For every $\beta \in \mathscr{C}$ with $\beta > f(\sup_L C)$, one has $\beta \geq b_2$. From $f(\sup_L C) < b_2$, for every $x \in C$, there is $y_x \in C$ with $y_x \geq x$ and $f(y_x) > f(\sup_L C)$. In this case, one has $f(y_x) \geq b_2$.

In Case 1 (resp. 2), set b to be b_1 (resp. b_2). Then in both cases, for every $x \in C$ one may fix a choice of $y_x \in C$ with $f(y_x) \geq b$. Let H be the subset of C consisting of the y_x. Then $H \subset [f \geq b]$ and $\sup_L C \geq \sup_L H$. For every $x \in C$, one has $x \leq y_x \leq \sup_L H$, so $\sup_L C \leq \sup_L H$ and hence $\sup_L C = \sup_L H$.

As H is a chain in $[f \geq b]$, one has $\sup_L C = \sup_L H \in [f \geq b]$, i.e.,

$$f(\sup_L C) \geq b > f(\sup_L C).$$

This contradiction proves (1). Then f is upward upper semicontinuous.

Similarly, f is downward upper semicontinuous if and only if for every $a \in \mathscr{C}$ and every chain C in $[f \geq a]$, one has $\inf_L C \in [f \geq a]$. The two parts finish the proof. $\square$

Lemma 3.6 is used in the proof of Lemma 3.7.

Lemma 3.6. *Let W, V be two nonempty subsets of a complete lattice L such that $C := W \cup V$ is a chain. Then $\sup_L(C) = \sup_L(W)$ or $\sup_L(C) = \sup_L(V)$.*

Proof. Since $W \subset C$ (resp. $V \subset C$), one has $\sup_L(W) \leq \sup_L(C)$ (resp. $\sup_L(V) \leq \sup_L(C)$). Hence $\sup_L(W) \vee \sup_L(V) \leq \sup_L(C)$. Since $\sup_L(W) \vee \sup_L(V)$ is an upper bound on C, one has $\sup_L(W) \vee \sup_L(V) = \sup_L(C)$. If for every $u \in W$, there is $v \in V$ with $u \leq v$, then $\sup_L(W) \vee \sup_L(V) = \sup_L(V)$. Otherwise, as C is a chain, there is $u_0 \in W$ such that $u_0 \geq v$ for all $v \in V$. Hence $u_0 \geq \sup_L(V)$ and $\sup_L(W) \vee \sup_L(V) = \sup_L(W)$. $\square$

Roughly speaking, Lemma 3.7 means that a subset closed in the interval topology is also closed under certain order theoretic operations.

Lemma 3.7. *Let A be a subset of a complete lattice L. If A is closed in the interval topology of L, then A is chain-subcomplete in L.*

Proof. By the definition of interval topology, one can write $A = \cap_{\alpha \in I} A_\alpha$, where A_α is a finite union of closed intervals for each index $\alpha \in I$. More precisely, there is an integer $n(\alpha) \geq 1$ and closed intervals I_α^i $(i = 1, \ldots, n(\alpha))$ in L with $A_\alpha = \cup_{i=1}^{n(\alpha)} I_\alpha^i$. Let C be a nonempty chain in A. For every $1 \leq i \leq n(\alpha)$, one has $\sup_L(I_\alpha^i \cap C) \in I_\alpha^i$. For every $\alpha \in I$, one has $C \subset A_\alpha$, so $C = \cup_{i=1}^{n(\alpha)} (I_\alpha^i \cap C)$. By Lemma 3.6, there is $1 \leq i_\alpha \leq n(\alpha)$ with $\sup_L(C) = \sup_L(I_\alpha^{i_\alpha} \cap C)$. For every $\alpha \in I$, one has $\sup_L(C) \in I_\alpha^{i_\alpha} \subset A_\alpha$. Hence $\sup_L C \in A$. Similarly, one has $\inf_L C \in A$. $\square$

Remark 3.8. In Example 4.4, the subset X is chain-subcomplete in the complete lattice L, but not closed in the interval topology. Whence, the converse of Lemma 3.7 fails. Topkis [22, Thm. 2.3.1] shows that for every integer $n \geq 0$, every chain-subcomplete *sublattice* of $\mathbb{R}^n$ is closed in the *Euclidean* topology. Still, the *subset* $\{(t, 1-t) | 0 < t < 1\}$ of $\mathbb{R}^2$ is chain-subcomplete, but not closed in the Euclidean topology.

Proposition 3.9 shows that the topological continuity is stronger than the order continuity.

Proposition 3.9. *Let $f : L \to \mathscr{C}$ be a topologically upper semicontinuous function from a complete lattice to a chain. Then f is upper chain subcomplete. If further $\mathscr{C}$ is complete, then f is order upper semicontinuous.*

Proof. For every $a \in \mathscr{C}$, by the topological upper semicontinuity of f, the subset $[f \geq a]$ of L is closed. By Lemma 3.7, for every chain C in $[f \geq a]$, one has $\sup_L(C), \inf_L(C) \in [f \geq a]$. Therefore, f is upper chain subcomplete. When $\mathscr{C}$ is complete, by Proposition 3.5, f is order upper semicontinuous. $\square$

4 Quasisupermodular functions

Let L be a lattice. A function $g : L \to \mathbb{R}$ is called *supermodular* if for any $x, y \in L$, one has $g(x) + g(y) \leq g(x \wedge y) + g(x \vee y)$. The condition uses the additive group structure of the codomain $\mathbb{R}$. As [22, Example 2.6.5] shows, supermodularity may not be preserved by a strictly increasing transformation. Quasisupermodularity is a generalization of supermodularity that is preserved by such transformations. Strictly increasing transformations are called monotone transformations in economics. Quasisupermodularity is an ordinal notion of complementarities.

Definition 4.1. [15, p.162] A function $f : L \to C$ from a lattice to a chain C is called *quasisupermodular* if for any $x, y \in L$, the condition $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$, and the condition $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$.

Every supermodular function is quasisupermodular. Example 4.2 shows that unlike supermodular functions, the sum of two quasisupermodular function may not be quasisupermodular.

Example 4.2. Consider the sublattice $L = \{(1,1), (2,3), (3,2), (4,5)\}$ of $\mathbb{R}^2$. Define functions $f, g : L \to \mathbb{R}$ by $f(x,y) = x$ and $g(x,y) = -y$. Then f is supermodular and g is quasisupermodular. But the sum $h := f + g$ is not quasisupermodular, since $h((2,3) \wedge (3,2)) = h(1,1) < h(3,2)$ and $h(2,3) = h(4,5) = h((2,3) \vee (3,2))$.

Statement 4.3 stated as [15, Theorem A3] shall be compared with Proposition 3.9.

Statement 4.3. Let L be a complete lattice. Let $f : L \to \mathbb{R}$ be a quasisupermodular and order upper semicontinuous function. Then f is upper semicontinuous in the interval topology of L.

However, it is false even if f is supermodular, as Example 4.4 shows. The mistake in the proof of [15, Theorem A3] originates from [15, Theorem A2], to which Kukushkin [11, Example 3.2] provides a counterexample. These incorrect results in [15] are minor.

Example 4.4. Let X be an infinite anti-chain. Let L be the set obtained by adding two elements $m \neq M$ to X. We extend the partial order from X to L by requiring $m \leq x \leq M$ for every $x \in X$. Then L is a complete lattice. The interval topology of L is T_1 and compact, but not Hausdorff.

Fix $x_0 \in X$. Define a function $f : L \to \mathbb{R}$ by setting $f(x) = 1$ for every $x \neq x_0$ and $f(x_0) = 0$. By Remark 3.3, since every chain in L is finite, f is supermodular and order upper semicontinuous. Moreover, $\arg\max f$ is compact but not closed in the interval topology of L. In particular, f is not upper semicontinuous in the interval topology. Thus, the converse of Proposition 3.9 fails.

Take an element $x'(\neq x_0)$ of L. Then $f(x_0) < f(x')$ but f does not satisfy the defining property of transfer upper continuity. This appears to contradict the literal statement of [20, Theorem 2]. The reason is that the topology in [20, Theorem 2] is implicitly assumed to be Hausdorff.

5 Existence of maximum

In a normal form game, every player wants to maximize his/her own payoff function within the set of available strategies. In this sense, the study of existence of a maximum of a given function is meaningful. The main objective of Section 5 is Theorem 5.4, which is used in Section 6.

Fact 5.1 is [20, Theorem 2], except that the authors implicitly assume X to be Hausdorff, as Example 4.4 explains. Since the interval topology of a complete lattice may not be Hausdorff, we state the following version. The original proof still works.

Fact 5.1 (Tian-Zhou)**.** *Let X be a compact (but not necessarily Hausdorff) topological space. Let $f : X \to \mathbb{R}$ be a transfer upper continuous function. Then* $\arg\max f$ *is a nonempty closed subset of X, hence compact.*

Lemma 5.2 generalizes [18, Propostion 3], which only asserts the completeness of $\arg\max f$.

Lemma 5.2. *Let $f : L \to C$ be an upper chain subcomplete function from a complete lattice to a chain. If* $\arg\max f$ *is a nonempty sublattice of L, then it is* subcomplete *in L. In particular,* $\arg\max f$ *is compact in the interval topology of L.*

Proof. For every nonempty chain $\mathscr{C} \subset \arg\max f$, since f is upper chain subcomplete and $\arg\max f = \{s \in L : f(s) \geq \max f\}$, one has $\sup_L \mathscr{C}, \inf_L \mathscr{C} \in \arg\max f$. Therefore, $\arg\max f$ is chain-subcomplete in L. By Lemma 2.3 2, the sublattice $\arg\max f$ is subcomplete in L. The compactness follows from [24, Theorem 2.9]. $\square$

Remark 5.3. In Lemma 5.2, the compactness is not clear *a priori*. If the condition "upper chain subcomplete" is changed to "upper semicontinuity in the interval topology of L", then $\arg\max f$ is *compact* in L. *Given* the compactness, from completeness of L and [5, Footnote d, p.187], this sublattice is subcomplete. Still, Example 4.4 and Proposition 3.5 show that an upper chain subcomplete function may *not* be upper semicontinuous in the interval topology.

Theorem 5.4 generalizes simultaneously [14, Theorems 1, 2] and [15, Thm. A4]. (Although the proof of [15, Thm. A4] uses the wrong Statement 4.3, the conclusion itself is correct.) It relaxes the supermodularity condition in [14, Theorems 1, 2] to quasisupermodularity, and strengthens the completeness result in [14, Thm. 2] and [15, Thm. A4] to subcompleteness. It gives a purely order-theoretic sufficient condition for the existence of maximum of functions on lattices. A related condition is given in [12, Corollaries 3.1, 3.2].

Theorem 5.4. *Let $f : L \to C$ be a quasisupermodular, upper chain subcomplete function from a complete lattice to a chain. Then* $\arg\max f$ *is a nonempty* subcomplete *sublattice of L.*

Proof. Define a correspondence $F : f(L) \to 2^L$ by $F(c) = \{x \in L : f(x) \geq c\}$. Since f is upper chain subcomplete, for every $c \in f(L)$, the value $F(c)$ is chain-subcomplete in L.

For any $c < c'$ in C, every $x \in F(c)$ and every $x' \in F(c')$ with $x \wedge x' \notin F(c)$, one has $f(x \wedge x') < c \leq f(x)$. Since f is quasisupermodular, one has $c' \leq f(x') < f(x \vee x')$. Thus, $x \vee x' \in F(c')$. Therefore, the correspondence F is increasing with respect to the relation $\geq^{\mathrm{wV}}$ defined in [10, (2d)]. By [10, Theorem 2.2], there is an increasing selection $r : f(L) \to L$ of F.

For any $t \leq t'$ in $f(L)$, we have $r(t') \in F(t') \subset F(t)$. Hence, $\{r(t')\}_{t' \geq t}$ is a chain in $F(t)$. Since $F(t)$ is chain-subcomplete in L, the element $m := \sup_L r(f(L)) = \sup_L \{r(t')\}_{t' \geq t}$ exists and is in $F(t)$. Then $f(m) \geq t$ for every $t \in f(L)$. Hence $m \in \arg\max f$.

For any $u, v \in \arg\max f$, since $f(u \vee v) \leq \max f = f(u)$ (resp. $f(u \wedge v) \leq \max f = f(u)$) and f is quasisupermodular, one has $\max f = f(v) \leq f(u \wedge v)$ (resp. $\max f = f(v) \leq f(u \vee v)$). Therefore, $u \wedge v$ (resp. $u \vee v$) is in $\arg\max f$. So, $\arg\max f$ is a sublattice of L. Then by Lemma 5.2, the nonempty sublattice $\arg\max f$ is subcomplete. $\square$

Remark 5.5. Zhou's theorem [25, Thm. 1] is stated for correspondences whose values are *subcomplete* sublattices. That is why we need Theorem 5.4 in the proof of Theorem 6.8.

6 Nash equilibria of quasisupermodular games

In Section 6, we recall the definition of quasisupermodular games and give several results about the existence and the structure of Nash equilibria. Quasisupermodular games retain the main feature of supermodular games, i.e., complementarities among actions. The scope of quasisupermodular games surpasses that of supermodular games, broadening the spectrum of economic scenarios to which it can be applied.

Definition 6.1. [15, p.160] Let P, Q be two posets. Let C be a chain. A function $f : P \times Q \to C$ satisfies the *single crossing property* relative to (P, Q) if for any $p \leq p'$ in P and any $q \leq q'$ in Q, the condition $f(p, q) \leq f(p', q)$ implies $f(p, q') \leq f(p', q')$, and the condition $f(p, q) < f(p', q)$ implies $f(p, q') < f(p', q')$.

A function having increasing differences relative to $P \times Q$ (in the sense of [22, p. 42]) satisfies the single crossing property relative to (P, Q). However, unlike the increasing differences property, single crossing property is not symmetric in the two variables, as shown by [22, Example 2.6.9].

Lemma 6.2 compares the hypotheses of Theorem 6.8 with those in Veinott's theorem on parameterized games briefly recalled in Remark 6.9.

Lemma 6.2. *Let L, L' be two lattices. Let C be a chain. Let $f : L \times L' \to C$ be a function. Suppose that for every chain $\mathscr{C} \subset L'$, the restriction $f|_{L\times\mathscr{C}} : L \times \mathscr{C} \to C$ is quasisupermodular. Then f satisfies the single crossing property relative to (L, L'), and for every $t \in L'$, the function $f(\cdot, t) : L \to C$ is quasisupermodular.*

Proof. Consider $x \le x'$ in L and $t \lneq t'$ in L'. Then $\{t, t'\}$ is a chain in L'. Thus, $f|_{L\times\{t,t'\}}$ is quasisupermodular. Since $(x, t') \wedge (x', t) = (x, t)$ and $(x, t') \vee (x', t) = (x', t')$, the condition $f(x, t) \le f(x', t)$ (resp. $f(x, t) < f(x', t)$) implies $f(x, t') \le f(x', t')$ (resp. $f(x, t') < f(x', t')$). Thus, f satisfies the single crossing property relative to (L, L'). Since $\{t\}$ is a chain in L', the function $f|_{L\times\{t\}} : L \times \{t\} \to C$ is quasisupermodular, i.e., $f(\cdot, t) : L \to C$ is quasisupermodular. □

Example 6.3. Let $L = \{0, 1, a, b\}$ be a lattice, where $0 = \min L$, $1 = \max L$ and a, b are incomparable. Then $a \wedge b = 0$ and $a \vee b = 1$. Let $C = \{0, 1\}$ be a *chain*. Define a function $f : L \times C \to \mathbb{R}$ by

$$f(0, 0) = f(b, 1) = 2, \quad f(a, 0) = f(b, 0) = 1,$$
$$f(1, 0) = f(1, 1) = 0, \quad f(a, 1) = 9,\ f(0, 1) = 10.$$

Then f satisfies the single crossing property relative to (L, C). For every $t \in C$, the function $f(\cdot, t) : L \to \mathbb{R}$ is quasisupermodular. However, f is not quasisupermodular. Indeed, one has $f((a, 0)\wedge(b, 1)) = f(0, 0) = 2 = f(b, 1)$ and $f((a, 0)\vee(b, 1)) = f(1, 1) = 0 < 1 = f(a, 0)$.

Definition 6.4. A *quasisupermodular game*

$$(N, \{S_i\}_{i\in N}, \{f_i\}_{i\in N}, \{C_i\}_{i\in N}) \tag{2}$$

is the following data:

1. a nonempty set of players N;
2. for every $i \in N$, a nonempty lattice S_i of the strategies of player i. Write $S = \prod_i S_i$ for the set of joint strategies and $S_{-i} := \prod_{j\neq i} S_j$;
3. for every $i \in N$, a nonempty chain C_i representing the possible gains of player i;
4. for each player $i \in N$, a payoff function $f_i : S \to C_i$ such that for every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to C_i$ is quasisupermodular;
5. for every $i \in N$, the function f_i satisfies the single crossing property relative to (S_i, S_{-i}).

Remark 6.5. Such a game bears various names in the literature. It is called a game with the single crossing property in [18, Sec. 3] and is said to have (ordinal) strategic complementarities in [15, p.175]. The definition on [22, p.179] requires further each S_i to be a sublattice of some Euclidean space.

From now to the end of the paper, fix a quasisupermodular game (2).

Definition 6.6. A joint strategy $x \in S$ is a Nash equilibrium of the game (2) if for every $i \in N$ and every $y_i \in S_i$, one has $f_i(y_i, x_{-i}) \le f_i(x)$. The subset of S comprised of all Nash equilibria is denoted by E.

Remark 6.7. We cling to the analysis of pure-strategy Nash equilibria. Mixed-strategy Nash equilibria are also discussed in [6] for quasisupermodular games.

Classically, the best response correspondence is an essential tool to study Nash equilibria. For each player $i \in N$, the (individual) best response correspondence $R_i : S \to 2^{S_i}$ is defined by

$$R_i(x) = \operatorname*{arg\,max}_{y_i \in S_i} f_i(y_i, x_{-i}).$$

The joint best response $R : S \to 2^S$ is defined as $R(x) = \prod_{i \in N} R_i(x_{-i})$. Then E coincides with the set of fixed points of R.

Theorem 6.8 is purely order theoretic. For supermodular games, a purely order-theoretic result about the existence of the largest and the least Nash equilibria is in [14, p.1266].

Theorem 6.8. *Assume that for every $i \in N$, the lattice S_i is complete, and for every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to C_i$ is upper chain subcomplete. Then the set of Nash equilibria E (with the order structure inherited from S) is a nonempty complete lattice.*

Proof. By Theorem 5.4, for every $x \in S$ and every $i \in N$, the value $R_i(x)$ is a nonempty subcomplete sublattice of S_i. Therefore, $R(x)$ is also a nonempty subcomplete sublattice of S. Moreover, by [22, Theorem 2.8.6], the correspondence R_i is increasing in the sense of [22, p.33]. So $R : S \to 2^S$ is also increasing. Because every S_i is complete, the product lattice S is complete. Then by Zhou's fixed point theorem [25, Thm. 1], E is a nonempty complete lattice. $\square$

Remark 6.9. By Proposition 3.9, the continuity hypothesis of Theorem 6.8 is weaker than the topological upper semicontinuity.

Veinott gives a similar result for parameterized games. When restricted to normal form games, Veinott's theorem requires that for every $i \in N$ and every chain $\mathscr{C} \subset S_{-i}$, the function $f_i|_{S_i \times \mathscr{C}} : S_i \times \mathscr{C} \to C$ is quasisupermodular. By Lemma 6.2, this hypothesis implies that the game is a quasisupermodular game in the sense of Definition 6.4. However, as Example 6.3 shows, a quasisupermodular game may not satisfy Veinott's hypothesis. Furthermore, Veinott's theorem requires every C_i to be complete. In this sense, Theorem 6.8 is not covered by Veinott's theorem.

Example 6.10. Let $N = \{1, 2\}$. Let $C_i = \mathbb{R}$ for $i = 1, 2$. Let S_1 be the complete lattice L constructed in Example 4.4. Let S_2 be the finite lattice X in Example 6.3. Let x_0, f be as in Example 4.4. Define a function $f_1 : S_1 \times S_2 \to \mathbb{R}, \quad (s_1, s_2) \mapsto f(s_1)$. Then $f_1(\cdot, 0) : S_1 \to \mathbb{R}$ is not upper semicontinuous in the interval topology of S_1. Define a function $f_2 : S_1 \times S_2 \to \mathbb{R}$ by $f_2(s_1, 0) = 0$, $f_2(s_1, a) = f_2(s_1, b) = 1$ and $f_2(s_1, 1) = 1.5$ for every $s_1 \in S_1$. Then $f_2(s_1, \cdot) : S_2 \to \mathbb{R}$ is not supermodular. Still, the corresponding game satisfies the hypotheses of Theorem 6.8. In this case, one has $E = (L \setminus \{x_0\}) \times \{1\}$.

For every $i \in N$, assume $C_i = \mathbb{R}$. Let τ_i be a *compact* topology on S_i finer than the interval topology of S_i. By the Frink-Birkhoff theorem [1, Theorem 20, p.250], the lattice S_i is complete. Consequently, their product S is also a complete lattice.

Theorem 6.11 below is similar to [17, Theorem 1]. In [17, Theorem 1], each player's strategy set S_i is a chain, the set of players N is finite, and only the existence of Nash equilibria is proved. On the one hand, we allow the S_i to be lattices, the set N to be infinite and the existence of the greatest Nash equilibrium is established. On the other hand, [17] weakens the single crossing condition to upward or downward transfer single-crossing condition in the sense of [17, Definition 4].

Theorem 6.11. *Assume that for every $i \in N$ and every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to \mathbb{R}$ is transfer weakly upper continuous relative to τ_i, and upward (resp. downward) upper semicontinuous. Then there is a largest (resp. least) Nash equilibrium.*

Proof. By symmetry, it suffices to prove the statement without parentheses. By [20, Theorem 1], for every $i \in N$ and every $x_{-i} \in S_{-i}$, the compactness of τ_i and the transfer weak upper continuity of $f_i(\cdot, x_{-i})$ imply that $R_i(x)$ is nonempty. Therefore, by [22, Thm. 2.8.6], the correspondence $R_i : S \to 2^{S_i}$ is increasing. So, the correspondence $R : S \to 2^S$ is increasing. Whence, $R(x)$ is a nonempty sublattice of S for every $x \in S$.

We claim that for every $x \in S$, the largest element of $R(x)$ exists.

For every $i \in N$ and every chain C in $R_i(x)$, the upward upper semicontinuity implies

$$f(\sup_{S_i} C, x_{-i}) \geq \limsup_{t \in C, t \uparrow \sup_{S_i} C} f(t, x_{-i}) = \max_{S_i} f(\cdot, x_{-i}),$$

so $\sup_{S_i} C \in R_i(x)$. Now that every chain in the nonempty poset $R_i(x)$ has an upper bound, by Zorn's lemma, $R_i(x)$ has a maximal element m_i. Then $(m_i)_{i \in N} \in R(x)$ is a maximal element of $R(x)$. By Lemma 2.1, it is the largest element of $R(x)$. The claim is proved.

As R is increasing, the single-valued function $\max R : S \to S$ is increasing. As S is a complete lattice, $\max S$ exists. Then by Tarski's fixed point theorem [19, Thm. 1], the function $\max R : S \to S$ has a fixed point $x^* := \max\{y \in S : y \leq \max R(y)\}$. Then x^* is a Nash equilibrium.

We show that x^* is the largest Nash equilibrium. For every $s \in E$, one has $s \in R(s) \leq \max R(s)$. By definition of x^*, we have $s \leq x^*$. □

Example 6.12. Let $N = \{1, 2\}$. Let $S_1 = [0, 1]$ and $S_2 = [0, 1]^2$. Define a function

$$f_1 : S_1 \times S_2 = [0,1]^3 \to \mathbb{R}, \quad (s_1, a, b) \mapsto \begin{cases} 1 & s_1 > 0, \\ 0 & s_1 = 0. \end{cases}$$

Then $f_1(\cdot, 0, 0) : S_1 \to \mathbb{R}$ is not order upper semicontinuous. Define a function $f_2 : S_1 \times S_2 = [0,1]^3 \to \mathbb{R}, \quad (s_1, a, b) \mapsto s_1 + a + b$. The data define a quasisupermodular game satisfying the conditions of Theorem 6.11. The set E is $(0, 1] \times \{(1, 1)\}$. The largest Nash equilibrium is $(1, 1, 1)$, but there is no least Nash equilibrium. As S_2 is not a chain, one cannot apply [17, Theorem 1] in this case.

With a topological hypothesis on the payoff functions strictly weaker than continuity, we get a result on the order structure of Nash equilibria similar to Theorem 6.8.

Theorem 6.13. *Assume that for every $i \in N$ and every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to \mathbb{R}$ is transfer upper continuous relative to τ_i. Then the set of Nash equilibria E is a nonempty complete lattice.*

Proof. For every $i \in N$ and every $x \in S$, by Fact 5.1 and the compactness of τ_i, the value $R_i(x)$ is nonempty and compact in (S_i, τ_i). As τ_i is finer than the interval topology of S_i, the subset $R_i(x)$ is compact in the interval topology of S_i. By Topkis's monotonicity theorem [22, Theorem 2.8.6], the correspondence $R_i : S \to 2^{S_i}$ is increasing. Therefore, the correspondence $R : S \to 2^S$ is increasing and $R_i(x)$ is a sublattice of S_i. By [24, Theorem 2.9], the sublattice $R_i(x)$ is subcomplete in S_i. Therefore, $R(x)$ is a nonempty subcomplete sublattice of S. By Zhou's fixed point theorem [25, Thm. 1], E is a nonempty complete lattice. □

Corollary 6.14 follows from Theorem 6.13. When N is finite, it specializes to Calciano's result in [2, Sec. 4.1] about the structure of the set of Nash equilibria.

Corollary 6.14. *Suppose that for every $i \in N$ and every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to \mathbb{R}$ is upper semicontinuous relative to τ_i. Then E is a nonempty complete lattice.*

Remark 6.15. Under an extra continuity hypothesis (namely, for every $i \in N$ and every $x_i \in S_i$, the function $f_i(x_i, \cdot) : S_{-i} \to \mathbb{R}$ is continuous), the existence of both least and largest Nash equilibria is showed in [15, Theroem 12], extending a previous result [18, Proposition 4].

Example 6.16. Let $N = \{1, 2\}$. Let $S_1 = S_2 = [0, 1]$. Define functions

$$f_1 : S_1 \times S_2 \to \mathbb{R}, \quad (s_1, s_2) \mapsto \begin{cases} 0 & \text{if } s_1 \leq 1/3, \\ 1 & \text{if } 1/3 < s_1 < 2/3, \\ 2 & \text{if } 2/3 \leq s_1. \end{cases}$$

$$f_2 : S_1 \times S_2 \to \mathbb{R}, \quad (s_1, s_2) \mapsto (s_1 + 1)(s_2 - s_2^2).$$

Then $f_1(\cdot, 0) : S_1 \to \mathbb{R}$ is not upper semicontinuous. The data define a quasisupermodular game satisfying the conditions of Theorem 6.13 but not those of Corollary 6.14. The set $E = [2/3, 1] \times \{1/2\}$. Thus, Theorem 6.13 is strictly stronger than Corollary 6.14.

Corollary 6.17. *If for every $i \in N$, the set S_i is finite, then E is a nonempty complete lattice.*

Proof. For every $i \in N$, take τ_i to be the discrete topology on S_i. Then τ_i is compact, and for every $x_{-i} \in S_{-i}$, the function $f_i(\cdot, x_{-i}) : S_i \to \mathbb{R}$ is upper semicontinuous relative to τ_i. The result follows from Corollary 6.14. $\square$

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References

[1] Garrett Birkhoff. *Lattice theory*. 3rd. Vol. 25. American Mathematical Soc., 1973.

[2] Filippo L. Calciano. *Games with complementarities*. `https://sites.uclouvain.be/econ/DP/IRES/2007-11.pdf`. 2007.

[3] Siegfried Carl and Seppo Heikkilä. *Fixed point theory in ordered sets and applications: from differential and integral equations to game theory*. Springer Science & Business Media, 2010.

[4] Brian A. Davey and Hilary A. Priestley. *Introduction to lattices and order*. 2nd. Cambridge university press, 2002.

[5] Jacques Durieu, Hans Haller, Nicolas Quérou, and Philippe Solal. “Ordinal games”. In: *International Game Theory Review* 10.02 (2008), pp. 177–194.

[6] Federico Echenique. “Mixed equilibria in games of strategic complementarities”. In: *Economic Theory* 22.1 (2003), pp. 33–44.

[7] Orrin Frink. “Topology in lattices”. In: *Transactions of the American Mathematical Society* 51 (1942), pp. 569–582. DOI: 10.2307/1990078.

[8] Chaim Samuel Hönig. “Proof of the well-ordering of cardinal numbers”. In: *Proceedings of the American Mathematical Society* 5.2 (1954), p. 312.

[9] David Lawrence Johnson. *Elements of logic via numbers and sets*. Springer Science & Business Media, 1998.

[10] Nikolai S. Kukushkin. “Increasing selections from increasing multifunctions”. In: *Order* 30.2 (2013), pp. 541–555.

[11] Nikolai S. Kukushkin. *On the existence of monotone selections*. https://mpra.ub.uni-muenchen.de/15845/1/MPRA_paper_15845.pdf. 2009.

[12] Nikolai S. Kukushkin. “On the existence of optima in complete chains and lattices”. In: *Journal of Optimization Theory and Applications* 154.3 (2012), pp. 759–767.

[13] George Markowsky. “Chain-complete posets and directed sets with applications”. In: *Algebra universalis* 6.1 (1976), pp. 53–68.

[14] Paul Milgrom and John Roberts. “Rationalizability, learning, and equilibrium in games with strategic complementarities”. In: *Econometrica: Journal of the Econometric Society* 58.6 (1990), pp. 1255–1277. DOI: 10.2307/2938316.

[15] Paul Milgrom and Chris Shannon. “Monotone comparative statics”. In: *Econometrica: Journal of the Econometric Society* 62.1 (1994), pp. 157–180.

[16] Rabia Nessah and Guoqiang Tian. “On the existence of Nash equilibrium in discontinuous games”. In: *Economic Theory* 61 (2016), pp. 515–540.

[17] Pavlo Prokopovych and Nicholas C. Yannelis. “On strategic complementarities in discontinuous games with totally ordered strategies”. In: *Journal of Mathematical Economics* 70 (2017), pp. 147–153.

[18] Chris Shannon. *An ordinal theory of games with strategic complementarities*. https://eml.berkeley.edu/~cshannon/wp/qsm-org.pdf. 1990.

[19] Alfred Tarski. “A lattice-theoretical fixpoint theorem and its applications”. In: *Pacific journal of Mathematics* 5.2 (1955), pp. 285–309.

[20] Guoqiang Tian and Jianxin Zhou. “Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization”. In: *Journal of Mathematical Economics* 24.3 (1995), pp. 281–303.

[21] Donald M. Topkis. “Equilibrium points in nonzero-sum n-person submodular games”. In: *Siam Journal on Control and Optimization* 17.6 (1979), pp. 773–787. DOI: 10.1137/0317054.

[22] Donald M. Topkis. *Supermodularity and complementarity.* Princeton university press, 1998.

[23] Xavier Vives. "Nash equilibrium with strategic complementarities". In: *Journal of Mathematical Economics* 19.3 (1990), pp. 305–321.

[24] Lu Yu. "Existence and structure of Nash equilibria for supermodular games". In: *Order* (2024). DOI: 10.1007/s11083-024-09686-6.

[25] Lin Zhou. "The set of Nash equilibria of a supermodular game is a complete lattice". In: *Games and Economic Behavior* 7.2 (1994), pp. 295–300.