NOTIONS OF RELATIVE INTERIOR FOR COMPACT CONVEX SETS

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Abstract. We give an explicit example of a compact convex set without relative interior. Moreover, we show that a Banach space in which every nonempty compact convex subset has nonempty relative interior is finite dimensional. The relation between several close variants of the notion of relative interior is discussed. Finally, we generalize the Borwein-Lewis theorem on the existence of quasi relative interior.

1. Introduction

Relative interior is a fundamental concept in convex optimization. The relative interior (Definition 2.2) of a subset of a topological vector space (TVS) is a refinement of the concept of interior.

In Euclidean spaces, the relative interior of a nonempty convex subset is always nonempty, making it a valuable tool for studying low-dimensional sets in higher-dimensional vector spaces. In the infinite-dimensional case, we give an explicit example to illustrate that relative interior may be empty.

Proposition (Proposition 2.3)**.** *There is a nonempty, convex, compact subset of a Hilbert space without relative interior.*

More generally, we prove that for a class of TVS, if the relative interior always exists, then the space is finite dimensional.

Theorem (Corollary 3.5)**.** *A quasi-complete, locally convex and metrizable TVS is finite dimensional if and only if the relative interior of every nonempty convex compact subset is nonempty.*

As relative interior does not always exist in infinite dimension, different alternatives for relative interior are proposed in the literature. We compare several of them in Section 4. Among such variants, quasi

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relative interior always exists for certain class of TVS, as proved in Section 5. This generalizes the Borwein-Lewis theorem.

2. A compact convex subset without relative interior

The Hilbert cube is an explicit example of a compact convex subset of a Hilbert space. We prove that its relative interior is empty.

Definition 2.1. Let X be a real vector space. Let $A \subset X$ be a subset.

- ([AO16, Definition 3.2]) The subset
$$\mathrm{aff}(A) = \{\sum_{i=1}^{n} a_i x_i | n \in \mathbb{N}_{>0}, \sum_{i=1}^{n} a_i = 1, x_i \in A\}$$
is called the affine hull of A. Then $\mathrm{span}(A) := \mathrm{aff}(\{0\} \cup A)$ is the vector subspace of X generated by A.
- ([AB06, Lemma 7.32]) The dimension $\dim A$ of A is defined to be that of the affine space $\mathrm{aff}(A)$.

Definition 2.2 ([AB06, p.278])**.** Let E be a TVS, and let $C \subset E$ be a subset. The interior of C relative to $\mathrm{aff}(C)$ is called the *relative interior* of C and is denoted by $\mathrm{Ri}(C)$.

Proposition 2.3. *Let l^2 denote the real Hilbert space*
$$\{(a_i)_{i=1}^{\infty} \in \mathbb{R}^{\infty} : \sum_{i\geq 1} a_i^2 < +\infty\}.$$
Let $C = \{(a_i)_{i=1}^{\infty} \in l^2 : \forall i \geq 1, |a_i| \leq 1/i\}$ be the Hilbert cube. Then C is nonempty convex and compact in l^2, and $\mathrm{Ri}(C)$ is empty.

Proof. The subset C is nonempty and convex. By [Kas71, Thm. 71.3], C is compact in l^2. Since $0 \in C$, one has $\mathrm{aff}(C) = \mathrm{span}(C)$. For every integer $n \geq 1$, we have $(0, \dots, 0, 1/n, 0, \dots) \in C$, where the only nonzero element is placed at the n-th place. Then $\mathrm{span}(C)$ contains
$$X_0 := \{(a_i)_{i=1}^{\infty} : a_i \neq 0 \text{ for only finitely many } i\}.$$

Assume to the contrary that $\mathrm{Ri}(C)$ is nonempty. Take $x \in \mathrm{Ri}(C)$. There is $r > 0$ with $B(x, r) \cap \mathrm{span}(C) \subset C$, so $B(x, r) \cap X_0 \subset C$. Since $\lim_{n\to\infty}(x_{n+1} - 1/n)^2 + \sum_{i>n+1} x_i^2 \to 0$, there is an integer $n_0 > 1$ with $(x_{n_0+1} - 1/n_0)^2 + \sum_{i>n_0+1} x_i^2 < r^2$. Set $x' := (x_1, \dots, x_{n_0}, 1/n_0, 0, 0, \dots)$ which is in X_0. Then
$$\|x - x'\|^2 = (x_{n_0+1} - 1/n_0)^2 + \sum_{i>n_0+1} x_i^2 < r^2,$$

so $x' \in B(x,r) \cap X_0$. Then $x' \in C$. However, the (n_0+1)-th coordinate $1/n_0$ of x' is larger than $1/(n_0+1)$, contradicting the definition of C! Therefore, $\mathrm{Ri}(C)$ is empty. $\square$

Remark 2.4. A non-compact nonempty convex subset with empty relative interior is given in [AO16, Example 3.18]. Proposition 2.3 provides a compact example.

3. When does relative interior always exist?

The main purpose of Section 3 is Corollary 3.5. It shows that a Banach space in which every nonempty convex compact subset has relative interior must be finite dimensional.

Lemma 3.1. *Let X be a TVS. Let $A \subset X$ be a compact subset with nonempty relative interior* $\mathrm{Ri}(A)$*. Then A is finite dimensional.*

Proof. By translation, we may assume $0 \in \mathrm{Ri}(A)$. Then $\mathrm{aff}(A) = \mathrm{span}(A)$. Endow it with the topology inherited from X. By definition, there is an open subset U of $\mathrm{span}(A)$ with $0 \in U \subset A$. Since A is compact and X is Hausdorff, A is closed in $\mathrm{span}(A)$. Then A contains the closure V of U in $\mathrm{span}(A)$. Since V is closed in a compact space, V is also compact. By [Rud91, Theorem 1.22], $\dim A = \dim \mathrm{span}(A)$ is finite. $\square$

Definition 3.2 ([SW99, p.27])**.** A TVS is *quasi-complete* if every bounded, closed subset is complete.

A complete TVS is quasi-complete. Thus, every Banach space is quasi-complete, locally convex and metrizable.

Lemma 3.3. *Every infinite dimensional, quasi-complete, locally convex and metrizable TVS has an infinite dimensional convex compact subset.*

Proof. Let X be such a TVS. Choose a metric d on X compatible with the topology of X. Then the function $d(0,\cdot) : X \to [0,+\infty)$ is continuous. As X is infinite dimensional, there is a sequence $\{e_i\}_{i\geq 1}$ of linearly independent vectors in X.

For every integer $i \geq 1$, one has $\lim_{a\to 0^+} ae_i = 0$, so $\lim_{a\to 0^+} d(0, ae_i) = 0$. Thus, there is $a_i > 0$ with $d(0, a_ie_i) < 1/i$. Then $\lim_{i\to+\infty} d(0, a_ie_i) = 0$ and hence $\lim_{i\to+\infty} a_ie_i = 0$. Therefore, the set $A := \{0\} \cup \{a_ie_i\}_{i\geq 1}$ is compact in X. By [SW99, Corollary, p.50], since X is quasi-complete, the closure $B := \overline{\mathrm{co}}(A)$ of the convex hull of A in X is convex and compact. As the subfamily $\{a_ie_i\}_{i\geq 1}$ of B is linearly independent, B is infinite dimensional. $\square$

Remark 3.4. The metrizability in Lemma 3.3 is necessary. Indeed, as [SW99, Exercise 7, p.69] shows, there is an infinite dimensional, complete, locally convex TVS X, where every bounded subset is finite dimensional. In particular, every compact subset of X is finite dimensional.

Corollary 3.5. *Let X be a quasi-complete, locally convex and metrizable TVS. Then X is finite dimensional, if and only if the relative interior of every nonempty convex compact subset of X is nonempty.*

Proof. The "only if" part is classical (see, *e.g.*, [AO16, Theorem 3.17]). The "if" part follows from Lemmas 3.1 and 3.3. □

Example 3.6 is similar to but not covered by Corollary 3.5. It shows that the weak dual of an infinite dimensional normed space has a nonempty convex compact subset without relative interior.

Example 3.6. Let $(E, \|\cdot\|)$ be a normed space. Let X be the topological dual of E under the weak*-topology. Assume that the relative interior of every nonempty convex compact subset of X is nonempty. Then X and E are finite dimensional.

Proof. Set $B = \{x \in E : \|x\| < 1\}$. By the Banach-Alaoglu theorem, $K := \{\Lambda \in X : |\Lambda(x)| \leq 1, \forall x \in B\}$ is convex compact in X. By assumption, $\mathrm{Ri}(K)$ is nonempty. For every $\Lambda \in X$, since $\Lambda : E \to \mathbb{R}$ is a bounded operator, there is $r > 0$ with $r\Lambda \in K$. One obtains $\operatorname{aff} K = X$. From Lemma 3.1, the dimension $\dim X = \dim K$ is finite. Then the dual X^* of X is finite dimensional. As the canonical map $E \to X^*$ is injective, $\dim E$ is finite. □

4. Comparison of several notions of relative interior

There are several different notions of relative interior for convex compact sets in the literature. We study the relation between them.

Definition 4.1 ([XZ11, p.1739])**.** For a subset A of a linear normed space X, a point $x \in A$ is defined to be an *interior point with respect to relative topology* of A if there exists a real number $r > 0$ such that $B(x, r) \cap \operatorname{span}(A) \subset A$. Let $\mathrm{ri}(A)$ denote the set of interior points with respect to relative topology of A.

If $x \in \mathrm{ri}(A)$, then

$$B(x, r) \cap \operatorname{aff}(A) \subset B(x, r) \cap \operatorname{span}(A) \subset A,$$

so $x \in \mathrm{Ri}(A)$. Hence $\mathrm{ri}(A) \subset \mathrm{Ri}(A)$. The inclusion can be strict.

Example 4.2. If $X = \mathbb{R}$ and if $A = \{1\}$, then $\mathrm{ri}(A)$ is empty while $\mathrm{Ri}(A) = A$.

Remark 4.3. By [ET21, Lemma 2.1 (ii)], the set C in Proposition 2.3 is s-convex (in the sense of [XZ11, Definition 1.3]) for every $s \in (0, 1]$. However, one has $\mathrm{ri}(C) = \mathrm{Ri}(C) = \emptyset$. This contradicts the existence of interior point with respect to relative topology in [XZ11, Remark 2.1], which is used in the proof of [XZ11, Theorem 2.13]. See also [Yu23, Corollary 2.5].

Definition 4.4 ([BG03, Definition 2.1])**.** Let C be a nonempty subset of a TVS X. The *topological relative interior* of C, denoted by $\mathrm{tri}(C)$, is the interior of C relative to $\overline{\mathrm{aff}}(C)$ (the closure of $\mathrm{aff}(C)$ in X).

Lemma 4.5 shows that Definition 4.4 coincides with Definition 2.2 for compact subsets.

Lemma 4.5. *Let C be a nonempty subset of a TVS X.*

(1) *One has* $\mathrm{tri}(C) \subset \mathrm{Ri}(C)$.
(2) *If C is compact, then* $\mathrm{tri}(C) = \mathrm{Ri}(C)$.

Proof. (1) For every $x \in \mathrm{tri}(C)$, there is an open neighborhood U of x in $\overline{\mathrm{aff}}(C)$ with $U \subset C$. Then $U \cap \mathrm{aff}(C)$ is an open neighborhood of x in $\mathrm{aff}(C)$. This neighborhood is contained in C, so $x \in \mathrm{Ri}(C)$. Therefore, $\mathrm{tri}(C) \subset \mathrm{Ri}(C)$.

(2) Assume to the contrary $\mathrm{tri}(C) \neq \mathrm{Ri}(C)$. By Part (1), the set $\mathrm{Ri}(C)$ is nonempty. By compactness of C and Lemma 3.1, the affine space $\mathrm{aff}(C)$ is finite dimensional. From [Rud91, Theorem 1.21 (b)], one has $\overline{\mathrm{aff}}(C) = \mathrm{aff}(C)$. By definition, one has $\mathrm{tri}(C) = \mathrm{Ri}(C)$, a contradiction.

□

Remark 4.6. The inclusion in Lemma 4.5 (1) can be strict. Indeed, [BG03, Example 3.7] gives an example of a non-closed vector subspace S of the Hilbert space l^2, for which $\mathrm{tri}(S)$ is empty while $\mathrm{Ri}(S) = S$.

Corollary 4.7 recovers the equivalence between statements (a) and (e) in [BG03, Theorem 4.2].

Corollary 4.7. *Let X be a Banach space. Then X is finite dimensional if and only if for every nonempty compact convex subset $C \subset X$, the set* $\mathrm{tri}(C)$ *is nonempty.*

Proof. If X is finite dimensional, then $\mathrm{aff}(C)$ is closed in X. By Corollary 3.5, the set $\mathrm{tri}(C) = \mathrm{Ri}(C)$ is nonempty. Conversely, if $\mathrm{tri}(C)$ is nonempty for every compact convex set $C \subset X$, then by Lemma 4.5

(1), the set $\mathrm{Ri}(C)$ is nonempty. By Corollary 3.5, the dimension of X is finite. □

Another kind of "relative interior" turns out to be larger. For a subset S of a vector space E, let $\mathrm{cone}(S) := \{r \cdot s | r \in [0, +\infty), s \in S\}$ be the cone generated by S.

Definition 4.8 ([BL92, Def. 2.3], [BG03, Definition 2.2 (a)])**.** Let C be a convex subset of a TVS E. The *pseudo relative interior* (resp. *quasi relative interior*) of C is the set $\mathrm{pri}(C)$ (resp. $\mathrm{qri}(C)$) of those $x \in C$ for which $\mathrm{cone}(C - x)$ (resp. $\overline{\mathrm{cone}}(C - x)$) is a vector subspace of E.

Lemma 4.9 is a slight refinement of [BG03, Theorem 2.12], which proves the inclusion $\mathrm{tri}(C) \subset \mathrm{pri}(C) \subset \mathrm{qri}(C)$ in Banach spaces X.

Lemma 4.9. *Let C be a nonempty convex subset of a TVS X. Then* $\mathrm{tri}(C) \subset \mathrm{Ri}(C) \subset \mathrm{pri}(C) \subset \mathrm{qri}(C)$.

Proof. By Lemma 4.5 (1), one has $\mathrm{tri}(C) \subset \mathrm{Ri}(C)$. By [MN22, Thm. 2.174], one has $\mathrm{pri}(C) \subset \mathrm{qri}(C)$. It remains to prove $\mathrm{Ri}(C) \subset \mathrm{pri}(C)$. Take an arbitrary $x \in \mathrm{Ri}(C)$. There is an open neighborhood U of x in $\mathrm{aff}(C)$ with $U \subset C$. One has

$$\mathrm{span}(C - x) \supset \mathrm{cone}(C - x) \supset \mathrm{cone}(U - x).$$

Since $U - x$ is an open neighborhood of 0 in

$$\mathrm{aff}(C) - x = \mathrm{aff}(C - x) = \mathrm{span}(C - x),$$

one has $\mathrm{cone}(U-x) = \mathrm{span}(C-x)$. Therefore, $\mathrm{cone}(C-x) = \mathrm{span}(C-x)$ is a vector subspace of X. By definition, $x \in \mathrm{pri}(C)$. □

For a subset C of a TVS X and $\hat{x} \in C$, define the *normal cone* to C at $\hat{x}$ by $N_C(\hat{x}) := \{\phi \in X^* | \phi(x - \hat{x}) \leq 0, \forall x \in C\}$.

We compute the pseudo relative interior of the Hilbert cube, to show that the inclusions $\mathrm{Ri}(C) \subset \mathrm{pri}(C)$ and $\mathrm{pri}(C) \subset \mathrm{qri}(C)$ in Lemma 4.9 can also be strict even if C is compact.

Example 4.10. Let C be as in Proposition 2.3. Then

(1) $\mathrm{pri}(C)$ is the set of $x \in l^2$ for which there is $a > 0$ such that $|x_i| \leq (1 - a)/i$ for all integers $i > 0$.
(2) $\mathrm{qri}(C)$ is the set of $x \in l^2$ with $|x_i| < 1/i$ for all integers $i > 0$.

Proof. (1) For $x \in l^2$ such that there is $a > 0$ with $|x_i| \leq (1 - a)/i$ for all $i > 0$, we prove $x \in \mathrm{pri}(C)$. Indeed, for $y \in C$, set $z := (1 + a/2)x - ay/2$. For every $i > 0$, one has $z_i = (1 + a/2)x_i - ay_i/2$, so

$$|z_i| \leq (1 + a/2)|x_i| + a|y_i|/2 \leq (1 + a/2)(1 - a)/i + a/(2i) \leq 1/i.$$

Therefore, $z \in C$. Consequently, x is a relative absorbing point of C. By [BG03, Lem. 2.3], one has $x \in \mathrm{pri}(C)$.

Conversely, fix $x \in \mathrm{pri}(C)$. Then $y := -(\mathrm{sgn}(x_i)/i)_{i>0}$ is in C. By [BG03, Lem. 2.3], there is $\lambda > 0$ such that $z := (1+\lambda)x - \lambda y$ is in C. Let $a := \frac{2\lambda}{1+\lambda}$. For every integer $i > 0$, since $z_i = (1+\lambda)x_i + \lambda\,\mathrm{sgn}(x_i)/i$, one has $1/i \geq |z_i| = (1+\lambda)|x_i| + \lambda/i$. Then $0 < a \leq 1 - i|x_i|$.

(2) For $\hat{x} \in l^2$ with $|\hat{x}_i| < 1/i$ for all $i > 0$, we prove $\hat{x} \in \mathrm{qri}(C)$. By [BL92, Prop. 2.8], it suffices to prove that every $y \in N_C(\hat{x})$ is 0. Assume to the contrary $y \neq 0$. Then there is a positive integer i_0 with $y_{i_0} \neq 0$. Set $x := (x_i)_{i>0}$, with

$$x_i = \begin{cases} \hat{x}_i & i \neq i_0, \\ \mathrm{sgn}(y_{i_0})/i_0 & i = i_0. \end{cases}$$

Then $x \in C$. One has $\langle y, x - \hat{x} \rangle = y_{i_0}(\mathrm{sgn}(y_{i_0})/i_0 - \hat{x}_{i_0}) > 0$, which is a contradiction.

Conversely, for every $\hat{x} \in \mathrm{qri}(C)$, we prove $|\hat{x}_i| < 1/i$ for all $i > 0$. Assume to the contrary that there is $i_0 > 0$ with $|\hat{x}_{i_0}| = 1/i_0$. Let $y = (0, \dots, 0, \mathrm{sgn}(\hat{x}_{i_0}), 0, 0, \dots)$, where the unique nonzero coordinate is in the i_0-th component. Then $y \in l^2$. For every $x \in C$, one has $\langle y, x \rangle = \mathrm{sgn}(\hat{x}_{i_0})x_{i_0} \leq 1/i_0 = \langle y, \hat{x} \rangle$. Thus, one has $y \in N_C(\hat{x})$. Since $\langle -y, 0 \rangle = 0 > -1/i_0 = \langle -y, \hat{x} \rangle$, one has $-y \notin N_C(\hat{x})$, which contradicts [BL92, Prop. 2.8].

□

5. Existence of quasi relative interior

By [MN22, Example 2.176], the pseudo relative interior of a nonempty closed convex subset of a separable Hilbert space may be empty. By contrast, the quasi relative interior of a convex subset is frequently nonempty.

The existence of quasi relative interior is important in applications. For instance, separation theorems in duality theory [CDB05, Thm. 2.1], [CDB07, Thm. 2.5] need the quasi relative interior to be nonempty. Optimization theorems in [ZY11, Sec. 3] also require this condition.

Definition 5.1. [Jam72, p.37] A subset C of a TVS E is called *CS-closed*, if for every sequence $\{x_n\}_{n\geq 1}$ in C and every sequence $\{a_n\}_{n\geq 1}$ in $[0, +\infty)$ with $\sum_{n=1}^N a_n x_n \overset{N\to\infty}{\to} y$ in E and $\sum_{n\geq 1} a_n = 1$, one has $y \in C$.

Every CS-closed subset is convex. By [Jam72, Prop. 1], every convex sequentially closed subset is CS-closed. A TVS whose topology is induced by a complete invariant metric is called an *F-space*. A locally convex F-space is called a *Fréchet space*.

Fact 5.2 (Borwein-Lewis). *[BL92, Thm. 2.19] Suppose that (X,τ) is a TVS with either*

(a) *(X,τ) a separable Fréchet space, or*
(b) *$X = Y^*$ with Y a separable normed space, and $\tau = \sigma(Y^*, Y)$ is the weak*-topology.*

Let C be a nonempty CS-closed subset of X. Then $\mathrm{qri}(C) \neq \emptyset$.

Definition 5.3 proposes a class of TVSs that contains the two cases in Fact 5.2.

Definition 5.3. Let E be a TVS. If for every sequence $\{x_n\}_{n\geq 1}$ in E, there is a sequence $\{a_n\}_{n\geq 1}$ in $(0,+\infty)$ with $\sum_{n\geq 1} a_n < \infty$, such that

(1) $\sum_{n\geq 1} a_n x_n$ converges in E, then E is called *CS-summable*;
(2) for every real sequence $\{b_n\}_{n\geq 1}$ with $0 < b_n \leq a_n$ for all n, the series $\sum_{n\geq 1} b_n x_n$ converges in E, then E is called *strongly CS-summable*.

A strongly CS-summable TVS is CS-summable. Every closed vector subspace of a (strongly) CS-summable TVS is (strongly) CS-summable.

Remark 5.4. Let E be a locally convex, sequentially complete, CS-summable TVS. Then E is strongly CS-summable. Indeed, for every sequence $\{x_k\}_{k\geq 1}$ in E, there is a sequence $\{a_k\}_{k\geq 1}$ in $(0,+\infty)$ with $\sum_{k\geq 1} a_k < \infty$, such that $\sum_{k\geq 1} a_k x_k$ converges in E. As E is locally convex, for every open neighborhood U of 0 in E, there is a convex open neighborhood V of 0 in U. There is an integer $N > 1$ with $a_k x_k \in V$ for all $k \geq N$. Let $\{b_k\}_{k\geq 1}$ be a sequence with $0 < b_k \leq a_k/2^k$ for all $k \geq 1$. For any integers $m \geq n \geq N$, the sum $s := \sum_{k=n}^m b_k/a_k \leq \sum_{k=n}^m 1/2^k$, so $0 < s \leq 1$. The convex combination $\sum_{k=n}^m \frac{b_k/a_k}{s} \cdot a_k x_k$ is in V, so $\sum_{k=n}^m b_k x_k \in sV \subset V$. Therefore, the sequence $\{\sum_{k=1}^n b_k x_k\}_{n\geq 1}$ is a Cauchy sequence. As E is sequentially complete, the sequence converges. Therefore, E is strongly CS-summable.

Lemma 5.5. *Every F-space is strongly CS-summable.*

Proof. Let X be an F-space, whose topology is induced by a complete invariant metric d. Let $\{x_i\}_{i\geq 1}$ be a sequence in X. Since $d(0,\cdot) : X \to [0,+\infty)$ is continuous, there is $a_i \in (0, 2^{-i})$ with $d(0, a_i x_i) < 2^{-i}$. Then the series $\sum_{i\geq 1} a_i$ converges. For every $\epsilon > 0$, there is an integer $N \geq 1$, such that for any integers $m > n \geq N$, one has $\sum_{i=n}^m 2^{-i} < \epsilon$. Let $\{b_n\}_{n\geq 1}$ be a sequence with $0 < b_n \leq a_n$ for all n. Because d is translation invariant and satisfies the triangle inequality, one has $d(0, \sum_{i=n}^m b_i x_i) \leq \sum_{i=n}^m d(0, b_i x_i)$. By [Rud91, Thm. 1.24], one may assume that the open balls centered at 0 are balanced. Then $d(0, b_i x_i) \leq d(0, a_i x_i)$ for all i. One has $d(0, \sum_{i=n}^m b_i x_i) \leq \sum_{i=n}^m 2^{-i} < \epsilon$. Therefore, $(\sum_{i=1}^n b_i x_i)_{n\geq 1}$ is a Cauchy sequence in X. Since X is complete, this sequence converges. □

Lemma 5.6 shows that in some sense, the image of a CS-summable space is still CS-summable.

Lemma 5.6. *Let $f : X \to Y$ be a continuous linear surjective map of TVSs. If X is (strongly) CS-summable, then so is Y.*

Proof. Assume that X is CS-summable. Let $\{y_n\}_{n\geq 1}$ be a sequence in Y. As f is surjective, for every $n \geq 1$, there is $x_n \in X$ with $f(x_n) = y_n$. Then there is $x_0 \in X$ and a sequence $\{a_n\}_{n\geq 1}$ in $(0, +\infty)$, such that $\sum_{i=1}^n a_i x_i \to x_0$ in X as $n \to +\infty$, and $\sum_{i\geq 1} a_i$ converges. By linearity of f, for every $n \geq 1$, one has $f(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i y_i$. By continuity of f, one has $\sum_{i=1}^n a_i y_i \to f(x_0)$ in Y. Then Y is CS-summable. The proof for the strongly CS-summable case is similar. □

Countable product preserves strong CS-summability.

Lemma 5.7. *Let $\{X_i\}_{i\geq 1}$ be a sequence of strongly CS-summable TVSs. Then $\prod_{i\geq 1} X_i$ with the product topology is strongly CS-summable.*

Proof. Let $\{x^j\}_{j\geq 1}$ be a sequence in $\prod_{i\geq 1} X_i$. For any integers $i, j \geq 1$, let $x_i^j \in X_i$ be the i-th coordinate of x^j. For every $i \geq 1$, since X_i is strongly CS-summable and $\{x_i^j\}_{j\geq 1}$ is a sequence in X_i, one may take a sequence $\{a_i^j\}_{j\geq 1}$ in $(0, +\infty)$ as in Definition 5.3 (2). For every $j \geq 1$, let $a^j := \min_{1\leq i\leq j} a_i^j$. Then $a^j > 0$. It suffices to prove that for every sequence $\{b^j\}_{j\geq 1}$ with $0 < b^j \leq a^j$ for all j, the series $\sum_{j\geq 1} b^j x^j$ converges in $\prod_{i\geq 1} X_i$. For every integer $i \geq 1$ and every integer $j > i$, one has $b^j \leq a^j \leq a_i^j$. By the choice of $\{a_i^j\}_{j\geq 1}$, the series $\sum_{j\geq 1} b^j x_i^j$ converges in X_i. It holds for all integers $i \geq 1$, so $\sum_{j\geq 1} b^j x^j$ converges. □

Theorem 5.8 unifies the two cases in Fact 5.2.

Theorem 5.8. *Let X be a CS-summable, locally convex TVS. Then every nonempty, separable, CS-closed subset of X has nonempty quasi relative interior.*

Proof. Let $C \subset X$ be such a subset. As C is separable and nonempty, it has a dense sequence $\{x_n\}_{n\geq 1}$. As X is CS-summable, there is a sequence $\{a_n\}_{n\geq 1}$ in $(0, +\infty)$ such that $\sum_{i\geq 1} a_i x_i$ converges to some

$x_0 \in X$, and $\sum_{i\geq 1} a_i$ converges to some $a \in (0, +\infty)$. Replacing $\{a_n\}_{n\geq 1}$ with $\{a_n/a\}_{n\geq 1}$ and x_0 with x_0/a, one may assume $a = 1$. Since C is CS-closed, one has $x_0 \in C$.

We prove $x_0 \in \mathrm{qri}(C)$. As C is CS-closed, it is convex. Recall that $N_C(x_0) := \{\phi \in X^* | \phi(x - x_0) \leq 0, \forall x \in C\}$ is a convex cone in X^*. By [BL92, Prop. 2.8], since X is locally convex, it remains to prove $-\phi \in N_C(x_0)$ for every $\phi \in N_C(x_0)$. In fact, by [BL92, the second paragraph of the proof of Thm. 2.19], ϕ is constant on C, hence $-\phi \in N_C(x_0)$. □

When X_2 (resp. X_1) is zero, Example 5.9 reduces to case (a) (resp. (b)) of Fact 5.2. Thus, Example 5.9 shows that Theorem 5.8 is a proper generalization of Fact 5.2.

Example 5.9. Let X_1 be a separable Fréchet space. Let Y be a separable normed space. Let X_2 be the topological dual of Y with the weak*-topology. Then $\mathrm{qri}(C) \neq \emptyset$ for every nonempty CS-closed subset C of $X_1 \times X_2$.

Indeed, let X_3 be the norm dual of Y, which is a Banach space. By Lemma 5.5, both X_1 and X_3 are strongly CS-summable. The TVS X_2 is locally convex, and the identity map $X_3 \to X_2$ is continuous linear surjective. From Lemma 5.6, the TVS X_2 is strongly CS-summable. From Lemma 5.7, the product space $X_1 \times X_2$ is strongly CS-summable. It is also locally convex.

By [BL92, Lem. 2.17], X_2 is hereditarily separable. The separable metric space X_1 is cosmic in the sense of [Mic66, p.993], so by [LT18, Prop. 1.2], $X_1 \times X_2$ is also hereditarily separable. Then C is separable. From Theorem 5.8, it has nonempty quasi relative interior.

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