A Brouwer-Tarski fixed-point theorem

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Abstract

New n-dimensional fixed-point theorems for possibly discontinuous functions are established, extending Brouwer's fixed-point theorem and encompassing interesting cases of Tarski's fixed-point theorem. The main idea is to require that at any discontinuity point where the graph of the function "crosses the diagonal", the function satisfies a specific assumption (similar to upward jumps). For this purpose, we classify some types of discontinuities, and use them to prove our new fixed-point theorems, with some applications in game theory.

Keywords: fixed-point, Brouwer's fixed-point theorem, Tarski's fixed-point theorem, discontinuity, Nash equilibrium, supermodular game.

1 Introduction

There are two main approaches to proving the existence of a Nash equilibrium in normal form games:

1. The first approach, initiated by Nash and Glicksgberg's theorems ([7],[11]), is primarily topological, and requires continuity and some quasi-concavity assumption on the payoff functions. Since then, many discontinuous extensions have been proposed ([2],[3],[12],[**?**],[4],[9],[10],[13],[14]), and most of them still rely on topological methods (i.e. they are derived from topological fixed-point theorems as Brouwer's theorem or Kakutani's theorem).

2. The second approach, initiated by Topkis' theorem ([16]), relies on order theoretic methods, and requires some supermodularity and increasing difference assumptions on the payoff functions (see also [1],[17],[18],[19]).

The following table summarizes the main differences between Nash-Glicksberg's theorem and Topkis' theorem:

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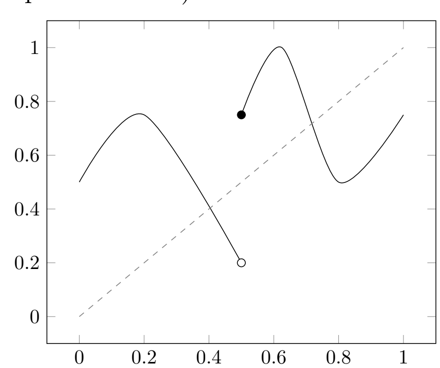
Topkis (order theoretic)	*Nash-Glicksberg (topologic and geometric)*
No convexity assumption on the strategy spaces	strategy spaces are convex
Strategy spaces are complete lattices	No lattice assumption on the strategy spaces
payoffs are own-strategy upper semicontinuous	payoffs are continuous w.r.t. strategy profiles
No quasi-concavity assumption	payoff functions are quasi-concave
payoff functions have increasing differences	No increasing difference assumption
payoff functions are supermodular	No supermodularity assumption
Application: Bertrand Oligopoly	Application: mixed extension of finite games
Existence proof uses Tarski's theorem applied to well-chosen selections from the best-response correspondence	Existence proof uses Kakutani's theorem (a consequence of Brouwer's fixed-point theorem) applied to the best-response correspondence
1-dimensional version of Tarski's theorem: "Every non decreasing function $f : [0, 1] \to [0, 1]$ has a fixed-point"	1-dimensional version of Brouwer's theorem: "Every continuous function $f : [0, 1] \to [0, 1]$ has a fixed-point"

The main drawback of these 2 approaches is that they require global assumptions: the payoff functions have to be continuous *everywhere*, or the payoff functions must have increasing differences *everywhere*. In particular, it does not consider cases where the payoff functions are continuous in some regions, and have increasing differences in other regions. The main objective of this paper is to unify, as far as we can, the two approaches mentioned above.

To understand our approach, consider the unidimensional case. An intuition that partially captures Tarski's theorem and Brouwer's theorem is that if $f : [0, 1] \to [0, 1]$ is continuous except at some interior point $\bar{x}$, and if at that point, the graph of f crosses the diagonal in an upward direction, then there should still exist a fixed-point (see picture below).



This intuition is further contained in another result of Tarski, Tarski's intersection theorem (see [15]), which states[1] that a function $f : [0, 1] \to [0, 1]$ which is quasi-increasing (with no downward jumps, see Definition 5) has a fixed-point. This theorem reconciles the two previous theorems of Brouwer and of Tarski (at least on $[0, 1]$) as both continuous functions and non decreasing functions $f : [0, 1] \to [0, 1]$ may have only upward jumps (if any).

[1] Here, we present a specific version where the domain of the function is $[0, 1]$. More generally, Tarski considers domains that are complete and densely ordered chain.

In this paper, we propose a new approach which allows us to unify Brouwer's theorem and many non pathological cases of Tarski's theorem beyond dimension 1. The idea is to study the discontinuities of any function $f : C \to C$ locally at points $x \in C$ where the graph of f "crosses the diagonal $y = x$": in the unidimensional case, such points x are those for which x is a limit point of the sets $\{y \in C \setminus x : f(y) \leq y\}$ and $\{y \in C \setminus x : f(y) \geq y\}$ (if non empty). In this paper, we call these points "generalized" fixed-points of f. This is straightforward to obtain the following fixed-point theorem which generalizes the theorems of Brouwer and Tarski on $[0, 1]$.

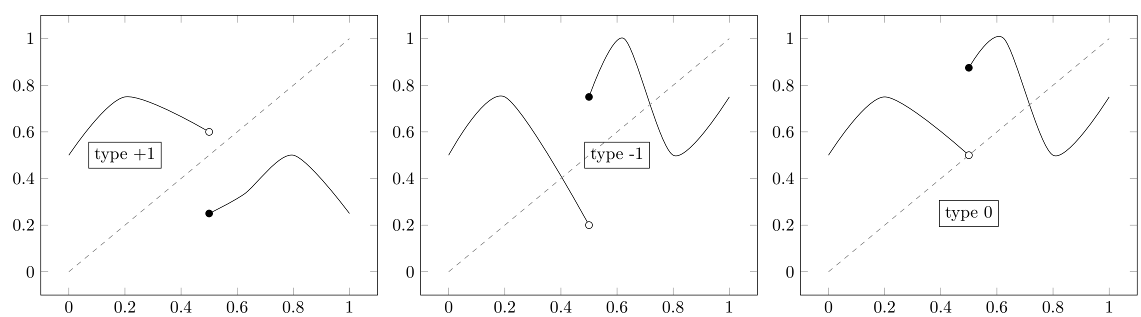
Theorem *(Local Tarski-Brouwer's theorem)*

Let $f : [0, 1] \to [0, 1]$ be quasi-increasing (with no downward jumps, see Definition 5) at every generalized fixed-point x. Then f has a fixed-point.

To prove this theorem, consider $\bar{x}$ to be the supremum of the set $\{x \in [0, 1] : f(x) \geq x\}$. Then it is straightforward to see that if $\bar{x}$ is not a fixed-point, then $\bar{x}$ is a generalized fixed-point. However, due to quasi-increasing assumption at $\bar{x}$, there should be a downward jump at $\bar{x}$, which leads to a contradiction with the definition of $\bar{x}$.

The main objective of this paper is to explore the extent to which this can be generalized in any dimension. Note that for $n = 1$, the difficulty is hidden, because on $\mathbf{R}$, the order topology (i.e. the topology generated by the sets $\{x < a\}$ and $\{x > b\}$) coincides with the usual topology. An extension of this theorem to higher dimension raises questions about the appropriate definition to use: if defining a generalized fixed-point in any dimension is easy (see Definition 1), extending quasi-increasing concept to dimension n seems more problematic.

To adress this problem, one of the main ideas of this paper is to introduce a new topological notion, called the type of discontinuity of a function, which describes some local behavior of the discontinuity: intuitively, type -1 (resp. +1) at some point x corresponds to the case where the graph of f crosses the diagonal in an upward direction (resp. a downward direction) around (x, x). Type 0 corresponds to the case where the graph of f does not cross the diagonal. This is illustrated by the following pictures:



Our main theorem states that any function f from a convex and compact subset of a Euclidean space into itself has a fixed-point when every generalized fixed-point of f is interior to C and has a type 0 or -1 (intuitively, the graph of f cannot cross the diagonal $y = x$ in a downward direction). Our theorem generalizes Brouwer's fixed-point theorem, but also includes many interesting cases of Tarski's theorem.

We illustrate our theorem with some by-products in dimension 2 or higher. The main advantage of our approach is that proving the existence of a fixed-point amounts to studying the type of discontinuity around particular points (the generalized points), which are very few in general. This allows us to localize the difficulty since it is easy to discard many points which cannot be generalized fixed-points.

We also provide applications to game theory, connecting our approach to supermodular games. They are a class of games for which the existence of a Nash equilibrium can be proven by applying Tarski's theorem to a well chosen selection of the best-response correspondance. Topkis' theorem is the most well-known theorem following this approach to establish the existence of Nash equilibria.

In our primary game theoretic illustration, we explain how we can obtain results in the spirit of Topkis' theorem, but where we can localize increasing difference assumption (an assumption inherent in the definition of supermodular games). Essentially, we need to assume increasing difference assumption only in neighborhoods of "generalized Nash equilibria" - profiles of actions x which are candidates to be Nash equilibria. This discards many profiles of actions for which no assumption is required.

We give examples satisfying our local assumption, despite not satisfying supermodularity or recent extensions such as those proposed by Amir and Castro (see [1]). In [1], the authors propose a new general class of strategic games and develop an associated existence result for pure-strategy Nash equilibrium for 2-player games. Quoting the authors:"our result imposes different requirements on the two players' reaction curves. For one player, this curve must be both continuous and increasing, while for the other player all that is needed is that his reaction curve do not possess any downward jump discontinuities." A by-product of our approach generalizes their result, by pinpointing the possible jumps of the reaction curves at particular profiles of actions.

We believe our approach is novel and valuable for its own merit. Nevertheless, finding applications is challenging, as it is often the case when attempting to use the fixed-point approach to prove the existence of Nash equillibria. Indeed, connecting the topological properties of selections of the best-response correspondence to the properties of the payoff functions is generally difficult.[2] In the standard case of Nash-Glicksberg's theorem, where the payoff functions are continuous and own-link quasiconcave, it is easy to demonstrate that the best-response correspondence satisfies Kakutani's fixed-point theorem. In the case of Topkis' theorem, where the payoff functions are supermodular and satisfy the increasing difference assumption together with some mild topological assumptions, it can be proven that well chosen selections of the best-response correspondence satisfies Tarski's fixed-point theorem. The recent paper of Amir and Castro [1] has the notable merit of introducing a new assumption (called shifted single-crossing), under which it's possible to prove the existence of a Nash equilibrium using Tarski's intersection theorem (replacing monotonicty assumption by quasi-increasing assumption). It's worth noting that their result holds for $n = 2$ (two players with 1 dimensional strategy set) mainly because they have to apply a one dimensional Tarski intersection theorem to $f \circ g$ where f and g are some selections of the best-response correspondences of each player. Our approach not only allows us to obtain a local version of their result but also extends beyond the 2-player case. Further exploration of applications to game theory (and Economics) is deferred to another paper.

The organization of the paper is as follows. We begin in Section 2 by defining the main notion of discontinuity type, with some examples. In Section 3, we provide our main fixed-point theorem, along

[2] This is probably why Reny's theorem or its by-products are the most applicable theorems to prove the existence of a Nash equilibrium when discontinuities arise. Indeed, the proof of Reny's theorem does not attempt to work at the level of the best-response correspondences (by trying to apply new fixed-point results) but rather at the level of payoff functions (attempting to "approximate" them with continuous ones).

with an extension to the multivalued case. Some illustrations and examples in dimension 1, 2 or above are given in Section 4. Last, Section 5 is devoted to our applications in game theory.

2 A Classification of discontinuities: generalized fixed-point and type

Throughout this paper, $N \geq 1$ is an integer, C is a convex subset of $\mathbb{R}^N$ with nonempty interior, and $f : C \to C$ is a mapping. The main objective of the paper is to obtain new fixed-point results for discontinuous functions. To achieve this, we first establish a classification of different types of discontinuities. The idea is to formalize and extend the intuition of $f : \mathbf{R} \to \mathbf{R}$ being discontinuous at $\bar{x}$, where the function locally jumps from one side to another side of the "diagonal" $y = x$ (this corresponds to type +1 or -1 below) or remains "far enough" from this diagonal (this corresponds to type 0). As a matter of fact, discontinuities of f will really matter only at "generalized fixed-point", which we define below. First define the correspondence from C to C, denoted $\mathrm{co}\overline{f}$, such that for every $x \in C$,

$$co\overline{f}(x) := co\{y : \exists (x_n)_{n\in\mathbb{N}} \in C^{\mathbb{N}} \text{converging to } x \text{ such that } \lim_{n\to+\infty} f(x_n) = y\}.$$

Definition 1. *We say that $\bar{x} \in C$ is a generalized fixed-point of f if $\bar{x} \in co\overline{f}(\bar{x})$. We denote by $GFix(f)$ (resp. by $Fix(f)$) the set of generalized fixed-points of f (resp. of fixed-points of f.)*

Proposition 1. *The set of generalized fixed-points of f is always nonempty.*

Proof. This is a immediate application of Kakutani's theorem to the multivalued function $co\overline{f}$ which has a closed graph and has nonempty and convex values.

In particular, when f is continuous at $\bar{x}$, then $\bar{x}$ is a generalized fixed-point if and only if it is a fixed-point of f. Now, the intuitive property that the graph of f "crosses the diagonal at $(\bar{x}, \bar{x})$" implies formally $\bar{x} \in GFix(f)$. Moreover, when $f : \mathbf{R} \to \mathbf{R}$, the previous condition together with the following one

$$\langle x - f(x), \bar{x} - x \rangle > 0 \text{ for every } x \text{ in a neighborhood of } \bar{x} \tag{1}$$

captures that the graph of f crosses the diagonal in an upward direction at $(\bar{x}, \bar{x})$. The intuition is that $id - f$ is "positively correlated" to the affine decreasing function $x \mapsto \bar{x} - x$ in some neighborhood of $\bar{x}$. In a similar way, the intuition that f crosses the diagonal in a downward direction corresponds to $id - f$ being "positively correlated" to the affine increasing function $x - \bar{x}$. Being of type -1 or of type +1 is kind of (reinforced) generalization of the previous conditions to n-dimensional spaces:

Definition 2. *A generalized fixed-point $\bar{x} \in \mathring{C}$ of f has type -1 (resp. type $+1$) if there exists a linear mapping $\ell : \mathbb{R}^N \to \mathbb{R}^N$ whose sign of jacobian determinant is -1 (resp. is $+1$) and there exist an open neighborhood U of $\bar{x}$ and a continuous function $h : U \setminus \{\bar{x}\} \to (0, +\infty)$ such that for every $x \in U \setminus \{\bar{x}\}$*

$$\langle x - f(x), \ell(x - \bar{x}) \rangle \geq h(x). \tag{2}$$

A generalized fixed-point $\bar{x} \in C$ of f has type 0 if there exist $p \neq 0$ in $\mathbb{R}^N$ and a continuous function $h : U \setminus \{\bar{x}\} \to (0, +\infty)$ such that: $\forall x \in U \setminus \{\bar{x}\}$,

$$\langle x - f(x), p\rangle \geq h(x). \tag{3}$$

The intuition of $\bar{x}$ being of type 0 is that the graph of f does not cross the diagonal around $\bar{x}$. The three pictures in the introduction illustrate the three possible types.

To compute the type in the first picture, you can simply consider $l(x) = x$, in the second, $l(x) = -x$ and in the last one, $p = -1$, where $h(x) = (x - \bar{x})^2$ in each situation.

Example 1. Let $f : (a, b) \to \mathbf{R}$. Consider a generalized fixed-point $\bar{x} \in (a, b)$ of f. Assume that there exists $n \geq 0$ such that

$$\limsup_{x \to \bar{x}^+} \frac{f(x) - x}{|x - \bar{x}|^n} < 0 \text{ and } \liminf_{x \to \bar{x}^-} \frac{f(x) - x}{|x - \bar{x}|^n} > 0$$

then x_0 has a type $+1$. Similarly, if there exists $n \geq 0$ such that

$$\limsup_{x \to \bar{x}^-} \frac{f(x) - x}{|x - \bar{x}|^n} < 0 \text{ and } \liminf_{x \to \bar{x}^+} \frac{f(x) - x}{|x - \bar{x}|^n} > 0$$

then x_0 has a type -1.

Proof. We only prove the first statement, the other one being similar. If the first property is true, then there exist $\alpha > 0$ and $\varepsilon > 0$ such that for every $x \in]\bar{x}, \bar{x} + \varepsilon[$, $f(x) - x \leq -\alpha|x - \bar{x}|^n$, and for every $x \in]\bar{x} - \varepsilon, \bar{x}[$, $f(x) - x \geq \alpha|x - \bar{x}|^n$. Take $\ell(x) = x$. Then for every $x \in]\bar{x}, \bar{x} + \varepsilon[$, $\langle x - f(x), \ell(x - \bar{x})\rangle \geq \alpha|x - \bar{x}|^{n+1}$. A similar proof provides the same inequality for every $x \in]\bar{x} - \varepsilon, \bar{x}[$. Thus, defining $h(x) = \alpha|x - \bar{x}|^{n+1}$, we get that x_0 is a generalized fixed-point of f of type $+1$ (since $\ell'(x_0) > 1$).

The following proposition states that our definition of type is "consistent", i.e. a generalized fixed-point $\bar{x}$ cannot have several different types:

Proposition 2. *A generalized fixed-point cannot have distinct types.*

See Appendix 6.2 for a proof. Actually, some generalized fixed-points can have neither type +1,-1 nor 0. This is a straightforward consequence of the following proposition, which states an important property of generalized fixed-points with a type +1,-1 or 0:

Proposition 3. *If $\bar{x} \in GFix(f)$ has a type -1, 0 or $+1$, then there exists a neighborhood U of $\bar{x}$ such that f has no generalized fixed-point in $U \setminus \bar{x}$. In particular, if $f : C \to C$ is continuous and C is compact, then f has a finite number of fixed-points with types -1, 0 or $+1$.*

Proof. The proof can be found in Appendix 6.2: see Lemma 1.

The following proposition relates the notion of type to the notion of stability, and also helps to understand the role of ℓ in our main definition. Recall that if $x'(t) = f(x(t)) - x(t)$ is a dynamical system (where $f : \mathbf{R} \to \mathbf{R}$ is smooth) and $f(x^*) = x^*$, then x^* is called an equilibrium. When $f'(x^*) < 1$, the equilibrium is said to be stable, and when $f'(x^*) > 1$, it is unstable.

Proposition 4. *Consider $f : C \to C$, $\bar{x} \in \mathring{C}$ be a fixed-point of f, and assume that f is C^1 in some neighborhood of $\bar{x}$ and that $D(id - f)(\bar{x})$ is invertible. The point $\bar{x}$ has a type -1 (resp. $+1$) if and only* *if the determinant of the jacobian of $id - f$ at $\bar{x}$ is negative (resp. positive). Moreover, the corresponding ℓ in Definition 2 can be taken equal to $D(id-f)(\bar{x})$. For $n = 1$, it means that if $\bar{x}$ has a type -1 (resp. $+1$) then it is unstable (resp. stable) with respect to the dynamical system $x'(t) = f(x(t)) - x(t)$.*

Proof. For a first implication, first assume that the determinant of the jacobian of $id - f$ at $\bar{x}$ is negative. Define

$$h(x) = \langle x - f(x), x - \bar{x} - Df(\bar{x})(x - \bar{x})\rangle =$$

$$\langle x - [f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + o(x - \bar{x})], x - \bar{x} - Df(\bar{x})(x - \bar{x})\rangle =$$

$$\langle x - \bar{x} - [Df(\bar{x})(x - \bar{x}) + o(x - \bar{x})], x - \bar{x} - Df(\bar{x})(x - \bar{x})\rangle =$$

$$\|x - \bar{x} - Df(\bar{x})(x - \bar{x})\|^2[1 + \frac{\langle o(x - \bar{x}), x - \bar{x} - Df(\bar{x})(x - \bar{x})\rangle}{\|x - \bar{x} - Df(\bar{x})(x - \bar{x})\|^2}].$$

Remark that we can divide by $\|x - \bar{x} - Df(\bar{x})(x - \bar{x})\|^2 = \|D(id - f)(\bar{x})(x - \bar{x})\|^2$ for x close enough to $\bar{x}$ (and different from it), because, from the invertibility of $D(id - f)(\bar{x})$, there exists $\alpha > 0$ such that $\|D(id - f)(\bar{x})(x - \bar{x})\|^2 \geq \alpha\|x - \bar{x}\|^2$.

Finally, from Cauchy-Schwarz inequality, we get

$$h(x) \geq \|x - \bar{x} - Df(\bar{x})(x - \bar{x})\|^2[1 - \frac{\|o(x - \bar{x})\|}{\|x - \bar{x} - Df(\bar{x})(x - \bar{x})\|}]$$

$$\geq \alpha\|x - \bar{x}\|^2[1 - \frac{\|o(x - \bar{x})\|}{\alpha\|x - \bar{x}\|}]$$

which is strictly positive for x close enough to $\bar{x}$ (and different from it). Thus (taking $\ell = D(id - f)$ in the definition) we get that $\bar{x}$ has a type -1.

Conversely, assume $\bar{x}$ has a type -1. By translation, one may assume that $\bar{x} = 0$. By assumption, there exists an open neighborhood U of $\bar{x} = 0$ and a continuous function $h : U \setminus \{0\} \to (0, +\infty)$ such that

$$\forall x \in U \setminus \{0\}, \langle x - f(x), \ell(x)\rangle \geq h(x) > 0, \tag{4}$$

where $\ell : \mathbb{R}^N \to \mathbb{R}^N$ is a linear mapping whose sign of jacobian determinant is -1. By Taylor's expansion, we get, for every x in $U \setminus \{0\}$,

$$\langle D(\mathrm{id} - f)(0)(x) + o(x), \ell(x)\rangle > 0. \tag{5}$$

Replacing x by $\frac{x}{m}$ and taking the limit when m converges to infinity, we get for every $x \neq 0$,

$$\langle D(\mathrm{id} - f)(0)(x), \ell(x)\rangle = \langle {}^t\ell \circ D(\mathrm{id} - f)(0)(x), x\rangle > 0, \tag{6}$$

where ${}^t\ell$ denotes the transposition of ℓ. From this inequality, we get that ${}^t\ell \circ D(\mathrm{id} - f)(0)$ has either strictly positive real eigenvalues or non zero complex-conjugate pairs of eigenvalues $\{z, \bar{z}\}$. Consequently, the determinant of ${}^t\ell \circ D(\mathrm{id} - f)(0)$ is positive, and since the determinant of ℓ is -1, the determinant of $D(\mathrm{id} - f)(0)$ is negative.

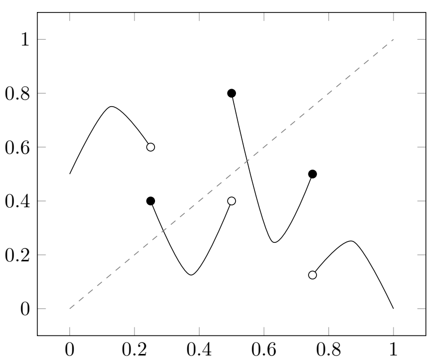
3 The main fixed-point theorem

3.1 Statement of the theorem

We now state our main existence result, proved in Appendix 6 :

Theorem 1. *Let $f : C \to C$, where C is a convex compact subset of $\mathbb{R}^N$ with a nonempty interior. Assume that:*
(Regularity assumption). *for every $\bar{x} \in C$ which is a generalized fixed-point and not a fixed-point of f, one has $\bar{x} \in \mathring{C}$ and $\bar{x}$ has a type 0 or -1.*
Then f has a fixed-point.

Throughout this paper, a function satisfying regularity assumption will be called regular. The following picture illustrates our theorem for $n = 1$:



As a corollary we get:

Corollary 1. *(Brouwer's theorem)*
Let $f : C \to C$ continuous, where C is a convex compact subset of $\mathbb{R}^N$. Then f has a fixed-point.

The following obvious 1-dimensional illustration shows why our main theorem may capture both topological fixed-point methods, but also situations where Tarski's theorem is usually invoked (our game theoretical applications will push further this remark):

Illustration 1. Consider a function $f : (a, b) \to (a, b)$ with a finite number of fixed-points and a finite number of discontinuities, and assume that f is non decreasing on some neighborhood of each discontinuity point. Then it is regular.

Proof. Indeed, let $\bar{x} \in (a, b)$ be a generalized fixed-point of f which is not a fixed-point of f. Since the set of fixed-points and of discontinuity points of f is finite, there exists V some open neighborhood of $\bar{x}$ such that f has no fixed-points on V, and is continuous on $V \setminus \{\bar{x}\}$. From f non decreasing on some neighborhood of $\bar{x}$, we have finally three cases:

1. $id - f < 0$ on $V \setminus \{\bar{x}\}$. In that case, for $p = -1$ and $h(x) = f(x) - x$ (continuous on $V \setminus \{\bar{x}\}$), we get $\langle x - f(x), p\rangle = h(x) > 0$ on $V \setminus \{\bar{x}\}$, thus $\bar{x}$ has a type 0.

2. $id - f > 0$ on $V \setminus \{\bar{x}\}$. In that case, taking $p = +1$, we get similarly that $\bar{x}$ has a type 0.

3. $id - f > 0$ on $V \cap (a, \bar{x})$ and $id - f < 0$ on $V \cap (\bar{x}, b)$. In that case, for $\ell(x) = -x$ and $h(x) = (x - f(x))(\bar{x} - x)$ (continuous on $V \setminus \{\bar{x}\}$), we get $\langle x - f(x), \ell(x - \bar{x})\rangle = h(x) > 0$ on $V \setminus \{\bar{x}\}$, thus $\bar{x}$ has a type -1 since $\ell' = -1$.

The illustration above shows that continuity (required in Brouwer's fixed-point theorem) and monotonicty (required in Tarski's fixed-point theorem) can be actually mixed in some property requiring either local regularity or monotonicity. Obviously, in the 1-dimensional case, it would be easy to find direct and simple methods to prove the existence of a fixed-point of any function statisfying such assumptions, but in this paper we will push further (in any finite dimension) this idea.

3.2 Extension to the multivalued case

Our main theorem has a straightforward generalization to correspondences, i.e. to multivalued functions. In the following, if $F : C \twoheadrightarrow C$ is a correspondence, then we note $\overline{F}$ the correspondence obtained from F by closing its graph, i.e.

$$\overline{F}(x) := \{y : \exists (x_n)_{n\in\mathbb{N}} \in C^{\mathbb{N}},\ \exists y_n \in F(x_n) \text{ converging to } x \text{ such that } \lim_{n\to+\infty} y_n = y\}.$$

Definition 3. *We say that $\bar{x} \in C$ is a generalized fixed-point of $F : C \twoheadrightarrow C$ if $\bar{x} \in co\overline{F}(\bar{x})$.*

In particular, when F has a closed graph and has convex values, then $\bar{x}$ is a generalized fixed-point of F if and only if it is a fixed-point of F. As in the second section, we get easily from Kakutani's fixed-point theorem that F always has some generalized fixed-point.

Definition 4. *A generalized fixed-point $\bar{x} \in \mathring{C}$ of F has type -1 (resp. type $+1$) if there exists a linear mapping $\ell : \mathbb{R}^N \to \mathbb{R}^N$ whose sign of jacobian determinant is -1 (resp. is $+1$), there exists an open neighborhood U of $\bar{x}$ and a continuous function $h : U \setminus \{\bar{x}\} \to (0, +\infty)$ such that for every $x \in U \setminus \{\bar{x}\}$ and every $y \in F(x)$,*

$$\langle x - y, \ell(x - \bar{x})\rangle \geq h(x). \tag{7}$$

We say that $\bar{x}$ is a discontinuity point of F of type 0 if there exists $p \neq 0$ in $\mathbb{R}^N$ and a continuous function $h : U \setminus \{\bar{x}\} \to (0, +\infty)$ such that: for every x in some neighborhood U of $\bar{x}$ and every $y \in F(x)$, we have

$$\langle x - y, p\rangle \geq h(x). \tag{8}$$

Corollary 2. *Let $F : C \twoheadrightarrow C$ be a correspondence with nonempty values, where C is a convex and compact subset of $\mathbb{R}^N$. Assume the following regularity assumption:*

Regularity: *for every $\bar{x} \in C$ which is a generalized fixed-point and not a fixed-point of F, one has $\bar{x} \in \mathring{C}$ and it has a type 0 or -1.*

Then F has a fixed-point.

Proof. If $f : C \to C$ is any selection of F, then f satisfies regularity assumption in Theorem 1. This provides the existence of a fixed-point of f, thus the existence of a fixed-point of F.

Remark 1. *In the above theorem, if we only require that there exists a particular regular selection of F, then we also obtain the existence of a fixed-point of F. This requirement is weaker than the one above, but seems highly untestable.*

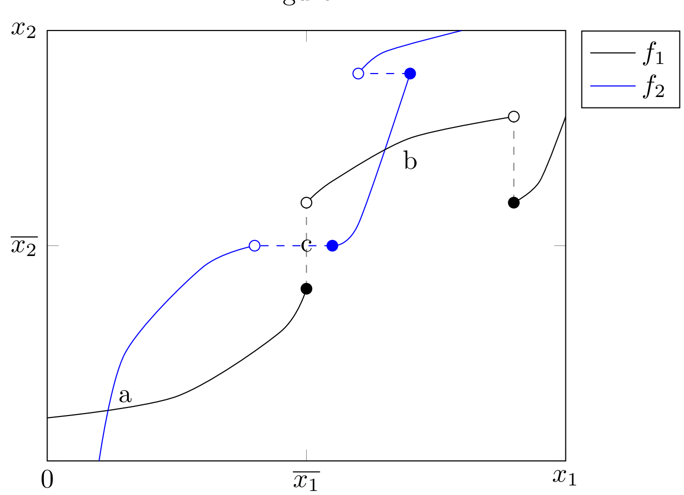
4 Some applications to fixed-point theory

4.1 A discontinuous fixed-point theorem in dimension n

Nash equilibria can be written as fixed-points of any selection f of the best-response correspondence. The particularity of such a selection f is that each component f_i of f does not depend on the strategy of player i. In this subsection, we present fixed-point results for mappings of this form, with some applications to Nash equilibrium existence in the next section. In the following, for every $1 \leq i \leq n$ and every $x = (x_i)_{1 \leq i \leq n}$, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ denotes the vector obtained from x by deleting component i.

Theorem 2. *Let $f : [0,1]^n \to [0,1]^n$ be a function of the form $f(x) = (f_i(x_{-i}))_{i=1}^n$. If for every generalized fixed-point x of f which is not a fixed-point, $x \in (0,1)^n$ and there exist $\epsilon > 0$ and V_x an open neighborhood of x such that for every $i \in N$, for every $y \in V_x$ such that $\sum_{k \neq i} y_k > \sum_{k \neq i} x_k$ (resp. $\sum_{k \neq i} y_k < \sum_{k \neq i} x_k$), $f_i(y_{-i}) \geq x_i + \epsilon$ (resp. $f_i(y_{-i}) \leq x_i - \epsilon$), then f admits a fixed-point.*

Figure 1



Remark 2. *The above corollary still holds true when n is even and if we modify the condition by reversing the inequalities as follows: for $x \in (0,1)^n$, there are $\epsilon > 0$ and V_x an open neighborhood of x such that for every $i \in N$, for every $y \in V_x$ such that $\sum_{k \neq i} y_k > \sum_{k \neq i} x_k$ (resp. $\sum_{k \neq i} y_k < \sum_{k \neq i} x_k$), $f_i(y_{-i}) \leq x_i + \epsilon$ (resp. $f_i(y_{-i}) \geq x_i - \epsilon$).*

When $n = 2$, a pair (x_1, x_2) is a fixed-point of $f = (f_1, f_2)$ if and only if $x_1 = f_1(x_2)$ and $x_2 = f_2(x_1)$. In particular, we can obtain fixed-points of f by intersecting the graph of f_1 and f_2. In the above figure, we give an example where we have 3 generalized fixed-points a, b, c. Among them, 2 are fixed-points (a and b) and $c = (\bar{x}_1, \bar{x}_2)$ is not a fixed-point. The assumption in Theorem 2 is true (intuitively, because

c does not belong to the boundary of the graph of both functions f_1 and f_2). Remark that we do not need f_1 or f_2 to be monotonic. Intuitively, we only require that when both graphs cross each other with a discontinuity (without intersecting), then (1) the "crossing-point" c avoids the boundary of the graphs, and (2) the jumps around c are above.

Proof. We prove Theorem 2 by contradiction. Assume that $f : [0,1]^n \to [0,1]^n$ satisfies the assumption in Theorem 2, but has no fixed-point. We will prove that f satisfies the assumptions in Theorem 1, contradicting the non-existence of a fixed-point. Consider $x \in [0,1]^n$ a generalized fixed-point of f which is not a fixed-point. From the assumption in Theorem 2, x is interior to $[0,1]^n$. We will prove that x has a type -1. To simplify notations, for every vector $x \in \mathbf{R}^n$, we denote by $s(x) = (s_i(x))_{i\in N} = (\sum_{k\neq i} x_k)_{i\in N}$. For every $y \in [0,1]^n$, we have:

$$\begin{aligned}
&\langle y - (f_i(y_{-i}))_{i\in N}, s(x-y)\rangle \\
&= \langle x - (f_i(y_{-i}))_{i\in N}, s(x-y)\rangle + \langle y - x, s(x-y)\rangle \\
&= \sum_{i\in N}(x_i - f_i(y_{-i}))s_i(x-y) + \sum_{i\in N}(y_i - x_i)s_i(x-y).
\end{aligned}$$

By assumption, there exist $\varepsilon > 0$ and V_x open neighborhood of x such that for every $i \in N$, for every $y \in V_x$,

$$\sum_{i\in N}(x_i - f_i(y_{-i}))(s_i(x-y)) \geq \varepsilon \sum_{i\in N} \mid s_i(x-y) \mid . \tag{9}$$

Moreover, we can take V_x small enough so that for every $i \in N$ and every $y \in V_x$, $\mid y_i - x_i \mid \leq \frac{\varepsilon}{2}$, thus

$$\sum_{i\in N}(y_i - x_i)(s_i(x-y)) \geq -\frac{\varepsilon}{2} \sum_{i\in N} \mid s_i(x-y) \mid . \tag{10}$$

Summing these two equations, we finally get

$$\langle y - (f_i(y_{-i})_{i\in N}), s(x-y)\rangle \geq \frac{\varepsilon}{2} \sum_{i\in N} \mid s_i(x-y) \mid .$$

Defining $\ell = -s$ and

$$h(y) = \frac{\varepsilon}{2} \sum_{i\in N} \mid s_i(x-y) \mid$$

,

we get that x has a type -1: indeed, the determinant[3] of ℓ is $1-n$ where $n \geq 2$ and $h(y) > 0$ when $y \neq x$.

Finally, f satisfies the assumptions in Theorem 1, contradicting the non-existence of a fixed-point, which ends the proof. To prove the assertion in Remark 2, we can simply define $\ell(y) = s(y)$ in the proof instead of $\ell(y) = -s(y)$, and the determinant of ℓ is unchanged from n even.

□

[3] Indeed, this is easy to compute the determinant $f(x)$ of the $n \times n$ matrix (a_{ij}) where $a_{ij} = a + x$ if $i > j$, $a_{ij} = b + x$ if $i > j$ and $a_{ij} = x$ if $i = j$: for $a \neq b$, just use that $f(-a) = (-a)^n$ and $f(-b) = (-b)^n$, and that $f(x)$ is affine in x. We get the case $a = b$ by passing to the limit when $a \to b$.

4.2 An improvement of the previous fixed-point theorem

In this section, we consider the 2 dimensional case for simplicity of the presentation, but it could be generalized to dimension n, as in the previous section. In Theorem 2, we have required a condition which implies that a generalized fixed-point $(\bar{x}, \bar{y})$ which is not a fixed-point does not belong to the closure of the graphs of f_1 and f_2 (see Figure 1). In the following theorem, we will see that even when this is not the case, we can get the existence of a fixed-point under the assumption that the graphs of f and g are not too "flat" (on the right and on the left) at $(\bar{x}, \bar{y})$ in that case.

Theorem 3. *Let $f : [0,1]^2 \to [0,1]^2$ be a function of the form $f(x,y) = (f_1(y), f_2(x))$. If for every generalized fixed-point $(\bar{x}, \bar{y})$ of f which is not a fixed-point, $(\bar{x}, \bar{y})$ is an interior point and $min(\alpha, \alpha') \cdot min(\beta, \beta') > 1$ where*

$$\beta := \lim_{x \to \bar{x}^+} \inf \frac{f_2(x) - \bar{y}}{x - \bar{x}} > 0, \;\; \beta' := \lim_{x \to \bar{x}^-} \inf \frac{\bar{y} - f_2(x)}{\bar{x} - x} > 0$$

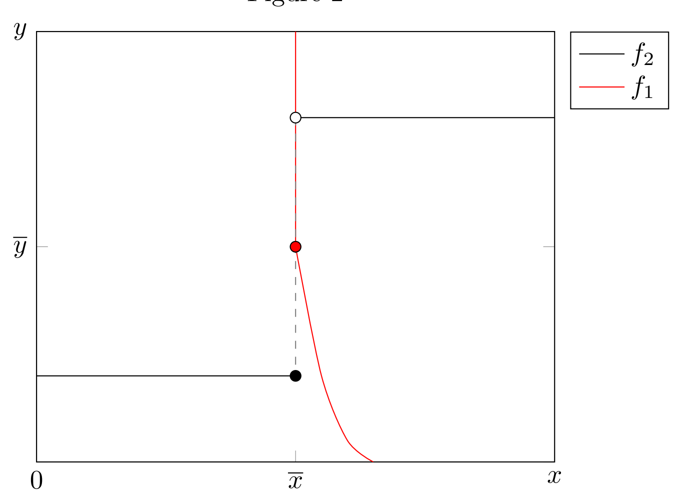
and

$$\alpha := \lim_{y \to \bar{y}^+} \inf \frac{f_1(y) - \bar{x}}{y - \bar{y}} > 0, \;\; \alpha' := \lim_{y \to \bar{y}^-} \inf \frac{\bar{x} - f_1(y)}{\bar{y} - y} > 0.$$

Then f admits a fixed-point.

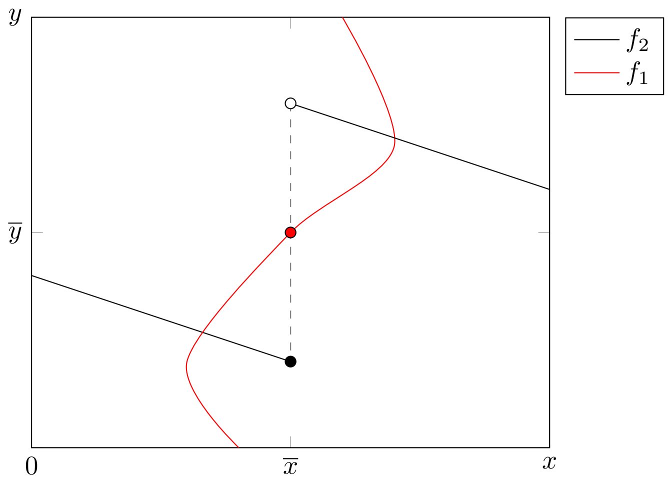
Remark that in Figure 1, the assumption in Theorem 3 is satisfied, and in that case, $\beta = \beta' = \alpha = \alpha' = +\infty$.

Figure 2



In Figure 2, $(\bar{x}, \bar{y})$ is a generalized fixed-point of $(x, y) \mapsto (f_2(x), f_1(y))$, and the associated values in the theorem are $\beta = \beta' = +\infty$ and $\alpha = \alpha' = 0$ (since the graph of f_1 is flat at $\bar{y}$). In this case, the theorem does not apply, and remark here that there is no fixed-point (since the two graphs do not intersect). This proves that the previous theorem is tight: in this example, if we perturb a little bit f_1 to get $\alpha > 0$ and $\alpha' > 0$ (keeping the same generalized fixed-point) we get the existence of a fixed-point.

Figure 3



In Figure 3, $(\bar{x}, \bar{y})$ is a generalized fixed-point of $(x, y) \mapsto (f_2(x), f_1(y))$, the associated values in the theorem are $\beta = \beta' = +\infty$ and $\alpha = \alpha' > 0$. Thus, the theorem applies.

Proof. We check that f satisfies the conditions in Theorem 1. Fix $\gamma \in [0, 1[$. Since $\beta := \liminf_{x \to \bar{x}^+} \frac{f_2(x) - \bar{y}}{x - \bar{x}} > 0$, there exists a right neighborhood $V_{\bar{x}}^+$ of $\bar{x}$ such that for every $x \in V_{\bar{x}}^+$, $f_2(x) - \bar{y} > \gamma\beta(x - \bar{x})$. Similarly, from $\beta' := \liminf_{x \to \bar{x}^-} \frac{\bar{y} - f_2(x)}{\bar{x} - x} > 0$, there exists a left neighborhood $V_{\bar{x}}^-$ of $\bar{x}$ such that for every $x \in V_{\bar{x}}^-$, $\bar{y} - f_2(x) > \gamma\beta'(\bar{x} - x)$.

Thus there exists some neighborhood $V_{\bar{x}}$ of $\bar{x}$ such that for every $x \in V_{\bar{x}}$,

$$(\bar{y} - f_2(x))(\bar{x} - x) \geq \min(\gamma\beta, \gamma\beta')(\bar{x} - x)^2.$$

Similarly, there exists some neighborhood $V_{\bar{y}}$ of $\bar{y}$ such that for every $y \in V_{\bar{y}}$,

$$(\bar{x} - f_1(y))(\bar{y} - y) \geq \min(\gamma\alpha, \gamma\alpha')(\bar{y} - y)^2.$$

Consequently, defining $g(x, y) = (\bar{y} - y, \bar{x} - x)$ for $(x, y) \in V_{\bar{x}} \times V_{\bar{y}}$, we have

$$\begin{aligned}
&\langle (x, y) - (f_1(y), f_2(x)), g(x, y) \rangle \\
&= \langle (\bar{x}, \bar{y}) - (f_1(y), f_2(x)), g(x, y) \rangle + \langle (x - \bar{x}, y - \bar{y}), g(x, y) \rangle \\
&\geq \min(\gamma\alpha, \gamma\alpha')(\bar{y} - y)^2 + \min(\gamma\beta, \gamma\beta')(\bar{x} - x)^2 + 2(x - \bar{x})(\bar{y} - y) := q(x, y).
\end{aligned}$$

The quadratic form $q(x, y)$ is positive definite for γ closed enough to 1, since we have $\min(\gamma\alpha, \gamma\alpha') > 0$, $\min(\gamma\beta, \gamma\beta') > 0$ and $\min(\gamma\beta, \gamma\beta') \min(\gamma\alpha, \gamma\alpha') > 1$. Finally, defining $\ell(x, y) = (-y, -x)$ and $h(x, y) = q(x, y)$, we get that $(\bar{x}, \bar{y})$ has a type -1 (since the determinant of ℓ is -1).

$\square$

4.3 A discontinuous fixed-point theorem in dimension $n = 1, 2$

The main message of this section is that for $n = 1, 2$, we can sometimes get extra results by replacing the topological degree technique (with our classification of types) by order arguments.

We state the main theorem of this section (it will be applied in Chapter 5 to provide new existence results of Nash equilibria in 2-player games):

Theorem 4. *Let $[a, b]$ and $[c, d]$ be closed intervals. Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to [a, b]$ be two functions. We define*

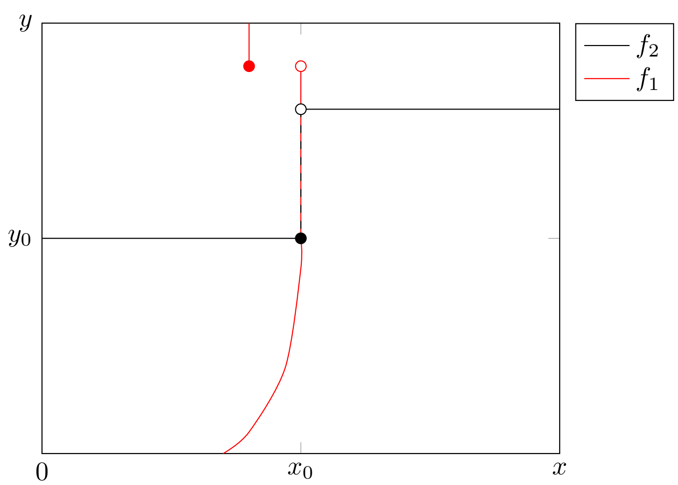
$$h : [a, b] \times [c, d] \to [a, b] \times [c, d]$$

$$(x, y) \mapsto h(x, y) = (g(y), f(x)).$$

Assume that at every generalized fixed-point (x_0, y_0) of h, f is increasing in some neighborhood of x_0 and g is increasing in some neighborhood of y_0. Then h admits a fixed-point.

In Figure 3, we have an example for which Theorem 2 applies, but not Theorem 4, because f_2 is not locally increasing at x_0. On the other side, in the following Figure 4, we give an example for which Theorem 4 applies, but neither Theorem 2 nor Theorem 3 can be applied (in the the first theorem because (x_0, y_0) belongs to the graph of f_2, in the second theorem because f_1 is flat at y_0).

Figure 4



Proof. Let $x_0 \in [a, b]$ be a generalized fixed-point of $g \circ f$, that is to say

$$x_0 \in \mathrm{co}\overline{g \circ f}(x_0). \tag{11}$$

We will prove that $g \circ f(x_0) = x_0$, which implies the theorem, since then $h(x_0, f(x_0)) = (x_0, f(x_0))$, i.e. $(x_0, f(x_0))$ is a fixed-point of h.

Step 1 : There exists $y_0 \in [c, d]$ such that (x_0, y_0) is a generalized fixed-point of h.

We prove Step 1 by contradiction: assume that for every $y \in [c, d]$, the point (x_0, y) is not a generalized fixed-point of h. It is straightforward that $\mathrm{co}\bar{h}(x_0, y) = \mathrm{co}\bar{g}(y) \times \mathrm{co}\bar{f}(x_0)$, thus we get

$$\forall y \in \mathrm{co}\bar{f}(x_0),\ x_0 \notin \mathrm{co}\bar{g}(y). \tag{12}$$

The set $\mathrm{co}\bar{f}(x_0)$ is a closed interval, on which the correspondence $\mathrm{co}\bar{g}$ has a closed graph. Define $U = \{y \in \mathrm{co}\bar{f}(x_0) : \mathrm{co}\bar{g}(y) \subset (-\infty, x_0)\}$ and $V = \{y \in \mathrm{co}\bar{f}(x_0) : \mathrm{co}\bar{g}(y) \subset (x_0, +\infty)\}$. Since $\mathrm{co}\bar{g} : \mathrm{co}\bar{f}(x_0) \to 2^{[a,b]}$ is upper semicontinuous, the sets U and V are open in $\mathrm{co}\bar{f}(x_0)$, and $U \cap V = \emptyset$. By (12), $U \cup V = \mathrm{co}\bar{f}(x_0)$. By connectedness of $\mathrm{co}\bar{f}(x_0)$, $U = \mathrm{co}\bar{f}(x_0)$ or $V = \mathrm{co}\bar{f}(x_0)$, i.e., either $x_0 > \mathrm{co}\bar{g}(y)$ for every $y \in \mathrm{co}\bar{f}(x_0)$ or $x_0 < \mathrm{co}\bar{g}(y)$ for every $y \in \mathrm{co}\bar{f}(x_0)$. Assume the second case (the proof of the first case being similar). From (11), there is a sequence $x_n \to x_0$ in $[a, b]$ such that $\lim_{n\to\infty}(g \circ f)(x_n) \le x_0$ exists. Taking a subsequence if necessary, we can assume that $f(x_n)$ converges to some y_1 in $[c, d]$. In particuler, $y_1 \in \mathrm{co}\bar{f}(x_0)$ and $\lim_{n\to\infty}(g \circ f)(x_n) \in \mathrm{co}\bar{g}(y_1)$, so we get from the assumption above $\lim_{n\to\infty} g \circ f(x_n) > x_0$, which gives a contradiction with the construction of x_n, and finishes the proof of Step 1.

From Step 1, and from the assumption in the theorem, f is increasing in some neighborhood of x_0 and g is increasing in some neighborhood of $f(x_0)$.

Step 2: We prove $g \circ f(x_0) = x_0$ by contradiction.
If $g \circ f(x_0) \ne x_0$, by symmetry, we can assume that $g \circ f(x_0) < x_0$. From (11), there exist two sequences $x_n^- \to x_0$ and $x_n^+ \to x_0$ such that $x_n^- \le x_0 \le x_n^+$, $g \circ f(x_n^-) \ge x_n^-$, $g \circ f(x_n^+) \le x_n^+$ for every $n \ge 1$. By taking subsequence if necessary, we may assume that $\lim_{n\to\infty} g \circ f(x_n^-)$ exists (and from the inequality above, it is greater than or equal to x_0). From f being increasing in some neighborhood of x_0, we get that $\alpha = \lim_{x\to x_0^-} f(x)$ exists and $\alpha \le \beta := f(x_0)$. Then $\mathrm{co}\bar{f}(x_0) = [\alpha, \lim_{x\to x_0^+} f(x)]$, and from $f(x_n^-) \to \alpha$, we get $\lim_{n\to\infty} g \circ f(x_n^-) \in \mathrm{co}\bar{g}(\alpha)$. So $\mathrm{co}\bar{g}(\alpha)$ has an element greater than or equal to x_0.

First, we prove $\alpha < \beta$. Otherwise, if $\alpha = \beta$, then from above, $\mathrm{co}\bar{g}(\alpha) = \mathrm{co}\bar{g}(\beta)$ has an element greater than or equal to x_0. But we also have $g(\beta) = g(f(x_0)) < x_0$ belongs to $\mathrm{co}\bar{g}(\beta)$, i.e. $\mathrm{co}\bar{g}(\beta)$ has an element smaller than or equal to x_0, so from convexity we get $x_0 \in \mathrm{co}\bar{g}(\beta)$. From $\beta = f(x_0) \in \mathrm{co}\bar{f}(x_0)$, we finally get that (x_0, β) is a generalized fixed-point of h (by definition). In particular, from the assumption in the theorem, g is increasing in some neighborhood of β, so $x_n^- \le g \circ f(x_n^-) \le g(\beta)$ for n large enough. Taking the limit, we get $x_0 \le g(\beta) = g(f(x_0))$, which is a contradiction with the assumption that $g \circ f(x_0) < x_0$. Thus we have proved $\alpha < \beta$.

Now, set

$$S = \{y \in [\alpha, \beta] : \forall y' \in [\alpha, y],\ \mathrm{co}\bar{g}(y') \text{ has an element greater than or equal to } x_0\}.$$

From the beginning of the proof of Step 2, we have $\alpha \in S$, in particular, $S \ne \emptyset$. Defining $\gamma = \sup S$, we easily get that for every $y' \in [\alpha, \gamma)$, $\mathrm{co}\bar{g}(y')$ has an element greater than or equal to x_0. Since the correspondence $\mathrm{co}\bar{g} : [\alpha, \gamma] \to 2^{[a,b]}$ has a closed graph, this implies that $\gamma \in S$.

If $\gamma < \beta$, then in particular $\mathrm{co}\bar{g}(\gamma)$ has an element smaller than or equal to x_0 (still from $\mathrm{co}\bar{g}$ having a closed graph and from $\gamma + \varepsilon \notin S$ and $\gamma + \varepsilon \in [\alpha, \beta]$ for ε small enough), so from convexity $x_0 \in \mathrm{co}\bar{g}(\gamma)$, and finally (x_0, γ) is a generalized fixed-point of h (since $\gamma \in [\alpha, \beta] \subset [\alpha, \lim_{x\to x_0^+} f(x)] = \mathrm{co}\bar{f}(x_0)$). Thus, from the assumption in the theorem, g is increasing in some neighborhood of γ, and from $x_0 \in \mathrm{co}\bar{g}(\gamma)$

this implies that $g \geq x_0$ on a right neighborhood of x_0, which contradicts that γ is the supremum of S.

If $\gamma = \beta$, then $\text{co}\bar{g}(\beta)$ has an element larger than or equal to x_0, but since $g(\beta) = g(f(x_0)) < x_0$ and from $g(\beta) \in \text{co}\bar{g}(\beta)$, we get $x_0 \in \text{co}\bar{g}(\beta)$. So (x_0, β) is a generalized fixed-point of h. From the assumption in the theorem, g is increasing in some neighborhood of β, and in particular $\text{co}\bar{g}(y') \leq g(\beta) < x_0$ on a left neighborhood of β (so it cannot have any element greater than or equal to x_0 on this neighborhood). This provides a contradiction with the definition of γ, and ends the proof of the theorem. $\square$

4.4 Relationship with existing works in fixed-point theory

In this section, we prove that our fixed-point theorem is different from the fixed-point theorems of Amir and Castro [1], Herings et al. [8] and Tarski [15].

We first recall the definition of quasi-increasing and quasi-decreasing functions.[4]

Definition 5. *Let X and Y be intervals of $\mathbf{R}$. A function $f : X \to Y$ is quasi-increasing if:*

$$\forall x \in X,\ \limsup_{y \to x-} f(y) \leq f(x) \leq \liminf_{y \to x+} f(y). \tag{13}$$

A function $f : X \to Y$ is quasi-decreasing if

$$\forall x \in X,\ \liminf_{y \to x-} f(y) \geq f(x) \geq \limsup_{y \to x+} f(y). \tag{14}$$

As a particular case of [15], it can be seen that any function $f : [a, b] \to [a, b]$ which is quasi-increasing has a fixed-point. Ricci remarks that it can be weakened as follows (see Remark 3 in [6]): every function $f : [a, b] \to [a, b]$ satisfying

$$\forall x \in X,\ \lim \sup_{y \to x-} f(y) \leq f(x) \leq \lim \sup_{y \to x+} f(y). \tag{15}$$

has a fixed-point.

A version of Herings et al.'s theorem [8] is the following:

Definition 6. *A function $f : [a, b] \to [a, b]$ is locally gross direction preserving if for every $x \in (a, b)$ which is not a fixed-point of f, there exists $U \subset [a, b]$ open neighborhood of x such that for every x_1 and $x_2 \in [a, b]$ which are not fixed-points of f, we have*

$$\langle f(x_1) - x_1, f(x_2) - x_2 \rangle \geq 0.$$

The one dimensional version of the main fixed-point existence result in [8] is that every locally gross direction preserving $f : [a, b] \to [a, b]$ has a fixed-point.

Now, if we define $f : [0, 1] \to [0, 1]$ as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{8}[, \\ \frac{1}{4} & \text{if } x \in [\frac{1}{8}, \frac{1}{2}], \\ \frac{3}{4} & \text{otherwise}, \end{cases}$$

[4] In [15], Tarski defines these concepts in a more general framework (i.e. for lattices). Here, we write these notions in the particular case of $\mathbf{R}$.

then the unique generalized fixed-point which is not a fixed-point is $\bar{x} = \frac{1}{2}$, whose type is -1 (see Example 1). Thus, this function satisfies the assumptions of Theorem 1. But, f is not quasi-increasing, and it does not satisfy (15). Last, it is not locally gross direction preserving (take $x = \frac{1}{2}$ in the definition).

5 Some applications to Game theory

In this section, we propose two applications to game theory of Theorem 2 and Theorem 4. We recall that a game is defined by a finite set of players N, some strategy spaces X_i for every player $i \in N$ and some payoff functions $u_i : \Pi_{i\in N} X_i \to \mathbf{R}$ for every player $i \in N$. Recall that $\bar{x} \in \Pi_{i\in N} X_i$ is a Nash equilibrium if $u_i(\bar{x}) = \max_{x_i \in X_i} u_i(x_i, \bar{x}_{-i})$ for every $i \in N$, where $u_i(x_i, \bar{x}_{-i})$ is a standard notation for $u_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots)$. More precisely, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)$ denotes the profile of strategies in x of all players except the strategy of player i, and X_{-i} denotes the set of such profiles. We also recall that for every $x_{-i} \in X_{-i}$, the set of best-responses of player i to x_{-i} is, by definition, $\arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$.

Throughout all the section, we consider games where $X_i = [0, 1]$ for every $i \in N$. Thus, we will simply denote such games by $G = (u_i)_{i\in N}$, but we will sometimes keep the notation X_i instead of $[0, 1]$ to specify that this is player i's strategy set. We note $|N|$ the cardinal of the set of players N.

5.1 Application of Theorem 2 to Nash equilibrium existence

Hereafter, we recall standard definitions (or generalization of standard definitions) useful for our main application:

Definition 7. *1. A Game $G = (u_i)_{i\in N}$ is quasi-concave if for every $i \in N$ and for every $x_{-i} \in X_{-i}$, $x_i \mapsto u_i(x_i, x_{-i})$ is quasi-concave.*

2. A Game $G = (u_i)_{i\in N}$ is weakly upper semicontinuous (w.u.s.c.) if for every $i \in N$ and every $x_{-i} \in X_{-i}$, $\max \arg\max_{d_i \in [0,1]} u_i(d_i, x_{-i})$ is well defined.[5]

3. A Game $G = (u_i)_{i\in N}$ is said to be additive if for every $i \in N$, there exists $\tilde{u}_i : [0, 1] \times [0, |N|-1] \to \mathbf{R}$ such that $u_i(x_i, x_{-i}) = \tilde{u}_i(x_i, \sum_{k\neq i} x_k)$. In that case, we note the additive game $G = (u_i, \tilde{u}_i)_{i\in N}$ to emphasize both payoff functions and the functions $\tilde{u}_i$ defining u_i.

Weak upper semicontinuity is new, it weakens slightly the standard assumption in the literature that $d_i \mapsto u_i(d_i, x_{-i})$ is u.s.c for every $x_{-i} \in X_{-i}$ (which guarantees that the set of best-responses of every player to every profile $x_{-i} \in X_{-i}$ is compact). Additivity says that each payoff function u_i depends only on player i's strategy and on the sum of other players' strategies. In particular, every 2-player game is additive.

We now provide an application of Theorem 2 to some Nash equilibrium existence result, in the spirit of Topkis [16], but where we authorize some *local* increasing differences assumption (instead of global increasing differences). Our application requires the following definition:

[5] Here, for every $E \subset \mathbf{R}$, $\max E$ denotes the maximum of E for the usual order on $\mathbf{R}$. In particular, if $d_i \mapsto u_i(d_i, x_{-i})$ is u.s.c for every $i \in N$ and for every $x_{-i} \in X_{-i}$, and if X_i is compact, then G is w.u.s.c..

Definition 8. *A profile $x = (x_i)_{i\in N}$ is said to be a generalized Nash equilibrium of $G = (u_i)_{i\in N}$ if x belongs to the convex hull of the set of profiles y for which there exist two sequences of strategy profiles x^n and y^n converging to x and y with y_i^n best-response to x_{-i}^n (for every player i), i.e. if*

$$x \in co\{y : \exists x^n \to x, \exists y^n \to y, \forall i \in N, \forall n \geq 0,\ u_i(y_i^n, x_{-i}^n) \geq \sup_{d_i \in X_i} u_i(d_i, x_{-i}^n)\}.$$

Clearly, if the game has continuous payoff functions and is quasi-concave, then a generalized Nash equilibrium is a Nash equilibrium. Indeed, in that case, for a given y in the above set, we can pass to the limit in the above inequality to get that y_i is a best-response to x_{-i}. Then, the condition above implies that for every $i \in N$, x_i is in the convex hull of the set of best-response to x_{-i}, so is itself a best-response to x_{-i} (from quasi-concavity assumption).

Recall that the best-response correspondence associates to every profile of strategies x the set $BR(x) = \Pi_{i\in N} BR_i(x_{-i})$, where $BR_i(x_{-i})$ is the set of best-responses of player i to x_{-i}. Roughly, the set of generalized Nash equilibria is a kind of convex regularization of BR. When $x = (x_i)_{i\in N}$ is a generalized fixed-point of some selection f of the best-response correspondence (if it exists), then x is a generalized Nash equilibrium of the game. Indeed, denoting $x = (x_i)_{i\in N}$ some generalized fixed-point of f, we get by definition $x \in co\overline{f}(x)$ where $\overline{f}(x) := \{y : \exists x_n \to x$ such that $\lim_{n\to+\infty} f(x_n) = y\}$. In particular, defining $y^n = f(x_n)$, we get by definition of f that $u_i(y_i^n, x_{-i}^n) \geq \sup_{d_i\in X_i} u_i(d_i, x_{-i}^n)$, thus x is a generalized Nash equilibrium.

Definition 9. *The function $\tilde{u}_i : [0,1] \times [0, |N|-1] \to \mathbf{R}$ has increasing differences* **locally at** *$s_{-i} \in [0, |N|-1]$ if there exists $V_{s_{-i}}$ an open neighborhood of s_{-i} such that for every $s'_{-i} \leq s''_{-i}$ in $V_{s_{-i}}$, $\tilde{u}_i(x_i, s''_{-i}) - \tilde{u}_i(x_i, s'_{-i})$ is increasing with respect to x_i on $[0,1]$ (we also say that $\tilde{u}_i$ has increasing differences on $V_{s_{-i}}$).*

In the above definition, s_{-i} will usually play the role of the sum of strategies of all players except player i, which explains the notation. If we consider an additive game $G = (u_i, \tilde{u}_i)_{i\in N}$, then for 2-player games, the previous definition generalizes the standard definition of increasing differences of Topkis , i.e. our definition may be seen as a local version of Topkis's definition for $N = 2$.

Proposition 5. *Let $V_{\bar{s}_{-i}}$ be an open neighborhood of $\bar{s}_{-i} \in [0,1]$, and assume that $\tilde{u}_i : [0,1]\times[0, |N|-1] \to \mathbf{R}$ has increasing differences on $V_{\bar{s}_{-i}}$. If $x_i \mapsto \tilde{u}_i(x_i, s_{-i})$ is w.u.s.c., then the mapping $s_{-i} \in V_{\bar{s}_{-i}} \mapsto \max\arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s_{-i})$ is increasing.*

Proof. For simplicity, we give a self-contained proof, even if the proof could be adapted from Topkis ([17], Theorem 2.8.1). Given w.u.s.c. assumption, the mapping $s_{-i} \in V_{\bar{s}_{-i}} \mapsto \max\arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s_{-i})$ is well defined. We do a proof by contradiction: assume that there exist $s'_{-i} < s''_{-i}$ in $V_{\bar{s}_{-i}}$ and $x'_i > x''_i$ with $x'_i = \max\arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s'_{-i})$ and $x''_i = \max\arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s''_{-i})$. Since $\tilde{u}_i$ has increasing differences on $V_{\bar{s}_{-i}}$, we have $\tilde{u}_i(x'_i, s''_{-i}) + \tilde{u}_i(x''_i, s'_{-i}) \geq \tilde{u}_i(x'_i, s'_{-i}) + \tilde{u}_i(x''_i, s''_{-i})$. But by definition of x'_i and x''_i, we have $\tilde{u}_i(x'_i, s'_{-i}) \geq \tilde{u}_i(x''_i, s'_{-i})$ and $\tilde{u}_i(x''_i, s''_{-i}) \geq \tilde{u}_i(x'_i, s''_{-i})$, and summing these inequalitites, we get $\tilde{u}_i(x'_i, s''_{-i}) + \tilde{u}_i(x''_i, s'_{-i}) \leq \tilde{u}_i(x'_i, s'_{-i}) + \tilde{u}_i(x''_i, s''_{-i})$, thus finally we should have the equaliy. And in particular, we should have $\tilde{u}_i(x'_i, s'_{-i}) = \tilde{u}_i(x''_i, s'_{-i})$ and $\tilde{u}_i(x''_i, s''_{-i}) = \tilde{u}_i(x'_i, s''_{-i})$. But this implies that $x'_i \in \arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s''_{-i})$ and we should have $x''_i = \max\arg\max_{x_i\in[0,1]} \tilde{u}_i(x_i, s''_{-i}) \geq x'_i$, a contradiction with the assumption.

□

Theorem 5. *Consider an additive and w.u.s.c. game $G = (u_i, \tilde{u}_i)_{i \in N}$. Assume that for every profile of strategies x which is a generalized Nash equilibrium and not a Nash equilibrium of G, we have:*

1. *Each $\tilde{u}_i$ has increasing differences* **locally** *at $\sum_{k \neq i} x_k$.*
2. *For every $i \in N$, there are an open neighborhood V_{x_i} of x_i and $\varepsilon > 0$ such that for every $y_i \in V_{x_i}$ and every $s_{-i} \in [0, |N|-1]$ such that $| s_{-i} - \sum_{k \neq i} x_k | \leq \varepsilon$, there is $y_i' \in [0,1]$ such that $\tilde{u}_i(y_i', s_{-i}) > \tilde{u}_i(y_i, s_{-i})$.*

Then there exists a Nash equilibrium.

Interestingly, we will see that for $N = 2$, we do not need the second assumption of Theorem 5 (see Theorem 6). As already discussed, the first assumption in the theorem is a local version of "increasing difference" assumption of Topkis. The second assumption guarantees that for every profile x which is a generalized Nash equilibrium but not a Nash equilibrium, then any small perturbation of x_i is not a best-response to a small perturbation of x_{-i} (for every $i \in N$).

Proof. We prove it by contradiction. Assume that there doesn't exist a Nash equilibrium. Since G is w.u.s.c., for every $i \in I$ and every profile of strategies x, we can define $f_i(x) \in X_i$ to be the maximum (for the standard order in $\mathbf{R}$) in the set of best-responses of player i against x_{-i}. Remark that since the game is additive, $f_i(x)$ depends only on $\sum_{k \neq i} x_k$. We note $f = (f_i)_{i \in N}$ (in particular, $f : X \to X$ is a selection of the best-response correspondence). We will prove that f has a fixed-point (a contradiction with the assumption that there is no Nash equilibrium) by applying Theorem 2 to f. Let F=co$\overline{f}$ be the convex hull of the closure of f.

To prove that f satisfies the property in Theorem 2, let x be a generalized fixed-point of f which is not a fixed-point of f. In particular, x is a generalized Nash equilibrium of G (see the remark after Definition 8), and it is not a Nash equilibrium (since we have assumed that there is no Nash equilibrium). Thus, the two properties in Theorem 5 at x have to be true.

To prove that we can apply Theorem 2, first, we have to prove that x cannot be on the boundary of X. By contradiction, for example, assume $x_1 = 0$. Since x is a generalized fixed-point of f, x is in the convex hull of the set of profiles y for which there exist 2 sequences of profiles y^n and z^n with $y_1^n = f_1(z_{-1}^n)$, y^n converging to y and z^n converging to x. In particular, since $x_1 = 0$, there should exist such a profile y for which $y_1 = 0$ (i.e. the corresponding sequence y_1^n converges to 0). But then, from assumption 2 in Theorem 5, we get that y_1^n cannot be a best-response of player 1 against z_{-1}^n for n large enough, a contradiction with $y_1^n = f_1(z_{-1}^n)$. So actually, x cannot be on the boundary point of X. The proof is similar if $x_1 = 1$, and at every other $x_i \in \{0, 1\}$.

Now, we can assume that x is interior to X. Applying the first assumption in Theorem 5, we get that for every $i \in N$, $\tilde{u}_i$ has increasing differences **locally at** $\sum_{k \neq i} x_k$. By Proposition 5, there exists $\delta > 0$ such that for every $i \in N$, $f_i(x)$ is locally increasing with respect to $\sum_{k \neq i} x_k$ in some neighborhood of $\sum_{k \neq i} x_k$.

We claim that there are $\epsilon > 0$ and V_x an open neighborhood of x such that for every $i \in N$, for every $y \in V_x$ such that $\sum_{k \neq i} y_k > \sum_{k \neq i} x_k$ (resp. $\sum_{k \neq i} y_k < \sum_{k \neq i} x_k$), $f_i(y_{-i}) \geq x_i + \epsilon$ (resp. $f_i(y_{-i}) \leq x_i - \epsilon$). Otherwise, for example, there exist $i \in N$ and a sequence y^n converging to x such that $\sum_{k \neq i} y_k^n > \sum_{k \neq i} x_k$ and $f_i(y_{-i}^n)$ converges to a value less or equal to x_i. But the fact that f_i is locally increasing with respect to $\sum_{k\neq i} x_k$ implies actually that $f_i(y^n_{-i})$ converges to x_i (indeed, from x generalized fixed-point of f and f_i locally increasing with respect to $\sum_{k\neq i} x_k$ in some neighborhood of $\sum_{k\neq i} x_k$, we have that $x_i \in [\lim_{z\to(\sum_{k\neq i} x_k)^-} f_i(z), \lim_{z\to(\sum_{k\neq i} x_k)^+} f_i(z)]$).

But this implies, from the second assumption in the theorem, that $f_i(y^n_{-i})$ cannot be a best-response of player i against y^n_{-i} for n large enough, a contradiction. So we have proved that f satisfies the conditions in Theorem 2. Thus, f has a fixed-point, a contradiction with the assumption that there is no Nash equilibrium, and we get the theorem by contradiction.

□

Example 2. An illustration

In practice, it is important to be able to localize the generalized Nash equilibria which are not Nash equilibria. The following simple example illustrates how it is possible, even when we cannot compute explicitly the set of Nash equilibria. The example will also illustrate that the class of games that are covered by our result is distinct from the previous results of Nash equilibrium existence in discontinuous games (see Section 5.4).

Consider, for each player $i = 1, 2, 3$, a function $l_i : [0, 1] \to [0, \frac{1}{2}[$ continuous on $[0, 1]$ and increasing in a left neighborhood of 1, and a function $r_i : [1, 2] \to]\frac{1}{2}, 1]$ continuous on $[1, 2]$ and increasing in a right neighborhood of 1.

Define

$$u_1(x_1, x_2, x_3) = \begin{cases} -|x_1 - l_1(x_2 + x_3)| & \text{if } x_2 + x_3 < 1, \\ -|x_1 - r_1(x_2 + x_3)| & \text{if } x_2 + x_3 \geq 1, \end{cases}$$

for every $x = (x_1, x_2, x_3)$ in $[0, 1]^3$, and define similarly u_2, u_3 (permuting the variables for each player).

First, this is easy to see that the unique generalized fixed-point which is not a Nash equilibrium is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. To prove that, we use mainly that the best-response of each player i is continuous except at $x_{-i} = (a, b)$ such that $a + b = 1$. Since $x = (x_1, x_2, x_3)$ is not a Nash equilibrium, up to a permutation of x_1, x_2 and x_3, we can assume $x_1 + x_2 = 1$ (otherwise, if $x_i + x_j \neq 1$ for every player $i \neq j$, then the best-reply of player 3 would be continuous at $x_1 + x_2$, and similarly for other players, and x generalized Nash equilibrium would be a Nash equilibrium, a contradiction). If $x_1 = x_2 = x_3 = \frac{1}{2}$ is false, up to a permutation of x_1 and x_2, we can treat two cases:

(1) either $x_1 + x_3 > 1$. In that case, since x is a generalized Nash equilibrium and since the best-response of player 2 is continuous at (x_1, x_3), we should have $x_2 = r_2(x_1 + x_3) > \frac{1}{2}$, but then $x_1 + x_2 = 1$ implies $x_1 < \frac{1}{2}$, thus $x_3 > \frac{1}{2}$, thus $x_2 + x_3 > 1$ and finally $x_1 = r_1(x_2 + x_3) > \frac{1}{2}$ (since the best-response of player 1 is continuous at (x_1, x_3)) a contradiction.

(2) Or $x_1 + x_3 < 1$, and we get a similar contradiction, by replacing r_i by l_i and by reversing inequality in the above argument. So $x_1 = x_2 = x_3 = \frac{1}{2}$.

To prove the first assumption in Theorem 5, we have to prove that each $\tilde{u}_i$ has increasing differences locally at 1. For example, if $s'_{-1} \leq s''_{-1} < 1$ are close enough to 1, then

$$\tilde{u}_1(x_1, s''_{-1}) - \tilde{u}_1(x_1, s'_{-1}) = \begin{cases} l_1(s'_{-1}) - l_1(s''_{-1}) \leq 0 & \text{if } x_1 \leq l(s'_{-1}), \\ l_1(s'_{-1}) + l_1(s''_{-1}) - 2x_1 & \text{if } x_1 \in [l(s'_{-1}), l(s''_{-1})], \\ l_1(s''_{-1}) - l_1(s'_{-1}) \geq 0 & \text{if } x_1 \geq l(s''_{-1}), \end{cases}$$

is increasing, and if $s'_{-1} < 1 < s''_{-1}$ are close enough to 1, then

$$\tilde{u}_1(x_1, s''_{-1}) - \tilde{u}_1(x_1, s'_{-1}) = \begin{cases} l_1(s'_{-1}) - r_1(s''_{-1}) \leq 0 & \text{if } x_1 \leq l(s'_{-1}), \\ l_1(s'_{-1}) + r_1(s''_{-1}) - 2x_1 & \text{if } x_1 \in [l(s'_{-1}), r(s''_{-1})], \\ r_1(s''_{-1}) - l_1(s'_{-1}) \geq 0 & \text{if } x_1 \geq r(s''_{-1}), \end{cases}$$

is increasing, and other cases are similar.

Second, Condition 2 is satisfied, since taking $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, we can see that for every player i, $y_i \in]\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon[$ cannot be a best-reply to $s_{-i} \in]1 - \varepsilon, 1 + \varepsilon[$ for $\varepsilon > 0$ small enough (from the assumptions on l_i and r_i).

This example can be seen as a methodological example, and it could be obviously refined. We will come back to it in Section 5.4, to prove that it does not belong to the class of better-reply secure game (see [14]), or other classes of discontinuous games for which the existence of a Nash equilibrium can be treated using standard theorems in the literature.

5.2 Application of Theorem 4 to Nash equilibrium existence

For two-player games, we can get a simplification of Theorem 5 as follows:

Theorem 6. *Let $G = (u_i, [0,1])_{i=1,2}$ be a w.u.s.c. game. Assume that for every profile of strategy $\bar{x}$ which is a generalized Nash equilibrium and not a Nash equilibrium of G, each u_i has increasing differences* **locally** *at $\bar{x}_{-i}$ (which is true for example if u_i is C^2 with $\frac{\partial^2 u_i(x_i, x_{-i})}{\partial x_i x_{-i}} \geq 0$ for every $x_i \in [0,1]$ and every x_{-i} in some neighborhood of $\bar{x}_{-i}$). Then there exists a Nash equilibrium.*

Proof. We do a contradiciton proof, assuming that there is no Nash equilibrium despite the assumptions in the theorem are true. Since G is w.u.s.c., for every $i \in I$ and every profile of strategies x, we can define $f_i(x_{-i}) \in X_i$ to be the maximum (for the standard order in $\mathbf{R}$) in the set of best-responses of player i against x_{-i}. We note $f(x_1, x_2) = (f_1(x_2), f_2(x_1))$ (in particular, $f : X \to X$ is a selection of the best-response correspondence). We will prove that f has a fixed-point (a contradiction with the assumption that there is no Nash equilibrium) by applying Theorem 4 to f. Let F=co$\overline{f}$ be the convex hull of the closure of f.

To prove that f satisfies the property in Theorem 4, we get, by applying the first assumption in Theorem 6, that for every $i \in N$, u_i has increasing differences **locally at** x_{-i}. By Proposition 5, for every $i \in N$, $f_i(x_{-i})$ is locally increasing with respect to x_{-i} in some neighborhhood of x_{-i}, thus we get the condition in Theorem 4. In particular, there exists a fixed-point of f, thus a Nash equilibrium of G, a contradiction.

□

Example 3. An illustration with Cournot games

Consider a Cournot duopoly model with 2 players $i = 1, 2$, the strategies being the quantities $q_i \in [0, +\infty]$, and the payoff functions

$$u_i(q_1, q_2) = q_i P(q_1 + q_2) - cq_i,$$

where P is a smooth function from $\mathbf{R}$ to $\mathbf{R}$ and $c > 0$. It is well known that if we assume $P'(q_1+q_2) + q_iP''(q_1+q_2) \leq 0$ for every $q = (q_i, q_{-i})$ and $i = 1, 2$, then there exists a Nash equilibrium, assuming that for every $i = 1, 2$, there exists some $M > 0$ large enough such that no best-response to $q_{-i} \in [0, +\infty]$ can belong to $[M, +\infty[$.[6] Indeed, if we consider $s_1 = q_1$ and $s_2 = -q_2$, we get a new payoff function $v_1(s_1, s_2) = s_1P(s_1 - s_2) - cs_2$ and $v_2(s_1, s_2) = -s_2P(s_1 - s_2) + cs_2$, and it's easy to prove under the assumption above that $\frac{\partial^2 v_1(s_1,s_2)}{\partial s_1 \partial s_2} \geq 0$ for every $s_1 \in [0, M]$ and $s_2 \in [-M, 0]$ (and similarly for v_2) which classically implies that the payoff functions v_1 and v_2 have increasing differences (in (s_1, s_2)), thus the game $(v_1, v_2, [0, M], [-M, 0])$ is supermodular and there exists a Nash equilibrium (s_1, s_2) of this game, which provides a Nash equilibrium $(s_1, -s_2)$ of the initial game. Now with Theorem 6, we can relax the assumption $P'(q_1 + q_2) + q_iP''(q_1 + q_2) \leq 0$ for every $q := (q_1, q_2)$ and every $i = 1, 2$:

Corollary 3. *Assume that for every generalized Nash $(\bar{q}_1, \bar{q}_2)$ of $(u_1, u_2, [0, M])$, we have (1) $P'(q_1 + q_2) + q_1P''(q_1 + q_2) \leq 0$ for every $q_1 \in [0, M]$ and for every q_2 on some neighborhood of $\bar{q}_2$, and (2) $P'(q_1 + q_2) + q_2P''(q_1 + q_2) \leq 0$ for every $q_2 \in [0, M]$ and for every q_1 on some neighborhood of $\bar{q}_1$. Then there exists a Nash equilibrium.*

Proof. Indeed, the assumption guarantees that the game $(v_1, v_2, [0, M])$ satisfies the assumption of Theorem 6.

□

The applicability of this theorem rests on the possibility to localize generalized Nash equilibria (or non generalized Nash equilibria) of the game. For example, for the continuous (but potentially non quasi-concave) Cournot Game considered above, if (q_1, q_2) is such that for every $q_1' \leq q_1$, there exists $q_1'' > q_1$ with $u_1(q_1'', q_2) > u_1(q_1', q_2)$, then (q_1, q_2) cannot be a generalized Nash equilibrium (since this requires that q_1 is in the convex hull of two best-responses against q_2, which is impossible from this condition.)

5.3 Some improvement

Remark that in Theorem 5 and Theorem 6 we only require some monotonicity result on the neighborhood of generalized fixed-point. Thus we can weaken w.u.s.c. as follows:

Definition 10. *A Game $G = (u_i)_{i \in N}$ is weakly upper semicontinuous (w.u.s.c.) locally at $\bar{x} \in X$ if for every $i \in N$, there exists $V_{\bar{x}_{-i}}$ an open neighborhood of x_{-i}, such that for every $x_{-i} \in V_{\bar{x}_{-i}}$, $\max \arg\max_{d_i \in [0,1]} u_i(d_i, x_{-i})$ is well defined.*

In Theorem 5 and Theorem 6 we only need to have the existence of a selection of the best-response correspondence, together with local w.u.s.c. at generalized Nash equilibria. For example, we get:

Theorem 7. *Let $G = (u_i, [0, 1])_{i=1,2}$ be a game. Assume for every profile of strategy x which is a generalized Nash equilibrium and not a Nash equilibrium of G, then G is w.u.s.c. at x, and each u_i has increasing differences* **locally** *at x_{-i}. Then there exists a Nash Equilibrium.*

[6] For example, to get this property, we can assume that $q_i \mapsto u_i(q_i, q_{-i})$ is strictly decreasing for $q_i \geq M - \varepsilon$, $\varepsilon > 0$, this independently from q_{-i}. This condition is satisfied for M large enough in usual Cournot games.

5.4 Relation with existing works

For every normal form game $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$ where N is the set of players, where X_i, the strategy space of player $i \in N$, is assumed to be a nonempty convex and compact subset of some Euclidean space (it can be extended), and u_i, the payoff function of player $i \in N$, is assumed to be quasi-concave with respect to x_i, the strategy of player i. Let $\Gamma = \{(x, u(x)) : x \in X\}$ denotes its graph (where $u = (u_i)_{i \in N}$ and $X = \Pi_{i \in N} X_i$) and $\overline{\Gamma}$ be the closure of Γ. We recall that G is *better-reply secure* if for every $(x, v) \in \overline{\Gamma}$ such that x is not a Nash equilibrium, there exist $i \in N$, a deviation $d_i \in X_i$, $\varepsilon > 0$ and $V_{x_{-i}}$ an open neighborhood of x_{-i} such that for every $y_{-i} \in V_{x_{-i}}$, $u_i(d_i, y_{-i}) > v_i + \varepsilon$ (where $v = (v_i)_{i \in N}$). It can be proved that if G is better-reply secure then it admits a Nash equilibrium (see [14]).

Consider the following adaptation of Example 2 to two players, and let us check that it is not better-reply secure (yet, it satisfies the assumptions in Theorem 5 or in Theorem 6). For each player $i = 1, 2$, consider functions $l_i : [0, \frac{1}{2}] \to [0, \frac{1}{2}[$ continuous on $[0, \frac{1}{2}]$ and increasing in a left neighborhood of $\frac{1}{2}$, and $r_i : [\frac{1}{2}, 1] \to]\frac{1}{2}, 1]$ continuous on $[\frac{1}{2}, 1]$, with $\lim_{x \to \frac{1}{2}^-} l_i(x) = \frac{1}{2}$ and $r_i(\frac{1}{2}) > \frac{1}{2}$.

Define

$$u_1(x_1, x_2) = \begin{cases} -|x_1 - l_1(x_2)| & \text{if } x_2 < \frac{1}{2}, \\ -|x_1 - r_1(x_2)| & \text{if } x_2 \geq \frac{1}{2}, \end{cases}$$

for every $x = (x_1, x_2)$ in $[0, 1]^2$, and similarly for u_2 (permuting the variables x_1 and x_2).

We can prove as in Example 2 that the assumptions of Theorem 5 are satisfied, where the unique generalized Nash equilibrium which is not a Nash equilibrium is $(\frac{1}{2}, \frac{1}{2})$.

1. Remark that there is no reason that u_1 has increasing differences, so we cannot use the standard theory of supermodular games on this example.

2. We cannot use Amir and Castro [1], for example because they need one reaction curve to be continuous (which is false here).

3. This game is not better-reply secure, so we cannot use the existence theorem of Reny [14]. Indeed, $((\frac{1}{2}, \frac{1}{2}), (0, 0)) \in \overline{\Gamma}$, since $((\frac{1}{2}, \frac{1}{2}), (0, 0)) = \lim_{x_1 \to \frac{1}{2}^-, x_2 \to \frac{1}{2}^-} (x_1, x_2, u_1(x_1, x_2), u_2(x_1, x_2))$. But for every $i = 1, 2$ there does not exist d_i such that $u_i(d_i, \frac{1}{2}) > 0$ since $u_i \leq 0$.

6 Appendice

6.1 Topological degree of a correspondence

In this subsection, we gather all the properties of topological degree which are used in this paper. Let $N \geq 1$ be an integer, D a nonempty open and bounded subset of $\mathbb{R}^N$. We denote by $\mathcal{C}(cl(D), \mathbf{R}^N)$ the set of correspondences (i.e. multivalued functions) F from $cl(D)$ (the closure of D) to $\mathbf{R}^N$, satisfying the two following properties:

(i) F is upper semicontinuous with nonempty, convex, and compact values, and
(ii) $0 \notin F(x)$ for every $x \in \partial D$.

To every $F \in \mathcal{C}(cl(D), \mathbf{R}^N)$, one can associate an integer $\deg(F) \in \mathbb{Z}$ called the (topological) degree of F, which satisfies the following properties (see, for example, [5]):

1. **(Invariance Under Homotopy Avoiding 0 on the boundary).** Let H be an upper semi-continuous correspondence from $[0,1] \times cl(D)$ to $\mathbb{R}^N$ with nonempty, convex, compact values, and which avoids 0 on the boundary, in the sense that $0 \notin H(\lambda, x)$ for every $(\lambda, x) \in [0,1] \times \partial C$. Then $deg(H_0) = deg(H_1)$ where $H(0,.) = H_0$ and $H(1,.) = H_1$.

2. **(Existence property).** For every $F \in \mathcal{C}(cl(D), \mathbf{R}^N)$, if $\deg(F) \neq 0$, then there exists $x \in C$ such that $0 \in F(x)$.

3. **(Local degree).** If $F \in \mathcal{C}(cl(D), \mathbf{R}^N)$ and if there exists V an open neighborhoods of $\bar{x}$ with $V \cap F^{-1}(0) = \{\bar{x}\}$, then $\deg(F_{|\overline{V}})$ is independent of V in the sense that it is the same for every open neighborhoods $V \subset D$ of $\bar{x}$ satisfying $V \cap F^{-1}(0) = \{\bar{x}\}$. This integer, when it is well defined, is called the local degree of F around $\bar{x}$.

4. **(Additivity).** If $F \in \mathcal{C}(cl(D), \mathbf{R}^N)$, and if $F^{-1}(0) = \{x_1, \ldots, x_k\}$ (for some integer $k > 0$), then

$$\deg(F) = \sum_{i=1}^{k} \deg(F_{|\overline{D_i}})$$

   where for every $i = 1, \ldots, k$, D_i is an open ball of D such that $D_i \cap F^{-1}(0) = \{x_i\}$. That is to say, the degree of F is equal to the sum of local degrees around the x_i.

5. **(Degree Formula).** For every single-valued mapping $F \in \mathcal{C}(cl(D), \mathbf{R}^N)$ such that the restriction of F on D is continuously differentiable, and such that for every $x \in F^{-1}(0)$, the differential $DF(x)$ is invertible, then $\deg(F) = \sum_{x \in F^{-1}(0)} \mathrm{sign}(\det DF(x))$, where $sign(a) = 1$ if $a > 0$ and $sign(a) = -1$ if $a < 0$.

6.2 Proof of the consistency of Definition 2

It is a consequence of the following lemma

Lemma 1. *Let $\bar{x} \in \mathring{C}$ be a generalized fixed-point of f. For every $\varepsilon \in \{0, -1, 1\}$, if $\bar{x}$ has a type ε, then:*

1) There exists some neighborhood V of $\bar{x}$ such that $(\mathrm{id} - F)^{-1}(0) \cap V = \{\bar{x}\}$, where $F := co(\overline{f})$, i.e. $F(x)$ is the convex hull of $\{y : \exists (x_n)_{n \in \mathbb{N}} \in C^{\mathbb{N}}$ converging to x such that $\lim_{n \to +\infty} f(x_n) = y\}$ for every $x \in C$. In particular, $\bar{x}$ is the unique generalized fixed-point of f on V.

2) The local degree of $\mathrm{id} - F$ at $\bar{x}$ is equal to ε.

Proof. Define the correspondence $F = co(\overline{f})$ from C to C. It's straightforward and standard to prove that F has a closed graph, with nonempty, convex and compact values (since $f(x) \in F(x)$) for every $x \in C$, and in particular F is upper semicontinuous. The correspondence F admits nonempty values since $f(x) \in F(x)$ for every $x \in C$, and it has convex values (by definition). Remark also that by assumption on f, the set of fixed-points of F is in the interior of C (since $\bar{x} \in F(\bar{x})$ means that $\bar{x}$ is a generalized fixed-point of f), so we can use topological degree applied to F .

Let $\bar{x} \in \mathring{C}$ be a generalized fixed-point of f, i.e. $\bar{x} \in F(\bar{x})$

1. If $\bar{x}$ is type 0, then there exist $p \neq 0$ in $\mathbf{R}^n$, $\varepsilon > 0$ and $h : B(\bar{x}, \varepsilon) \setminus \{\bar{x}\} \to (0, +\infty)$ such that for every $x \in B(\bar{x}, \varepsilon) \setminus \{\bar{x}\}$, we have
$$\langle x - f(x), p\rangle \geq h(x).$$
Now, let us prove that for every $\lambda \in [0,1]$, the mapping $x \in B(\bar{x}, \frac{\varepsilon}{2}) \mapsto \lambda(x - F(x)) + (1-\lambda)p$ has no zero on the sphere $S(\bar{x}, \frac{\varepsilon}{2})$. Otherwise, there exists $\lambda \in [0,1]$ and there exists $x \in S(\bar{x}, \frac{\varepsilon}{2})$ with $0 \in \lambda(x - F(x)) + (1-\lambda)p$, that is there exists $y \in F(x)$ with
$$\lambda(x - y) + (1-\lambda)p = 0. \tag{16}$$
By definition of F, there exist $k \geq 0$, $(\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k_+$, $\sum_{i=1}^k \lambda_i = 1$, and $y_i \in F(x)$ for every $i = 1, \ldots, k$, there exists a sequence $(x^i_n)_{n\in\mathbb{N}}$ converging to x, such that $f(x^i_n)$ converges to y_i when n tends to $+\infty$, and such that $y = \sum_{i=1}^k \lambda_i y_i$.
In particular, from the choice of p, and since $x^i_n \neq \bar{x}$ for n large enough (since $x \neq \bar{x}$), and for n large enough we have for every $i = 1, \ldots, k$:
$$\langle x^i_n - f(x^i_n), p\rangle \geq h(x^i_n). \tag{17}$$
Multiplying each equation (17) by λ_i and summing, we get, at the limit when n tends to $+\infty$ (from the continuity of h):
$$\langle x - y, p\rangle \geq h(x). \tag{18}$$
or also, since $x - y = -\frac{(1-\lambda)p}{\lambda}$ (λ being different from 0, from $p \neq 0$)
$$-\frac{1-\lambda}{\lambda}\|p\|^2 \geq h(x) > 0 \tag{19}$$
which is impossible. Thus, finally, $\lambda(\mathrm{id} - F) + (1-\lambda)p$ provides some homotopy between $\mathrm{id} - F$ and the constant mapping p on $B(\bar{x}, \frac{\varepsilon}{2})$, with no zero on the sphere $S(\bar{x}, \frac{\varepsilon}{2})$, for all $\lambda \in [0,1]$. In particular, since ε can be taken as small as we want, it proves that F has no fixed-point (taking $\lambda = 1$), except $\bar{x}$, on some neighborhood of $\bar{x}$, which proves assertion 1) in Lemma 1. This also implies that the local degree of $\mathrm{id} - F$ at $\bar{x}$ is well defined and is equal to the local degree of the constant mapping p at $\bar{x}$, which is zero (from Existence property), which assertion 2) in Lemma 1 when $\bar{x}$ has a type 0.

2. If $\bar{x}$ has a type -1, then there exists a linear mapping $\ell : \mathbb{R}^N \to \mathbb{R}^N$ whose sign of jacobian determinant is -1 and such that there exist $\eta > 0$ and $\varepsilon > 0$ such that for every $x \in B(\bar{x}, \varepsilon)$, we have

$$\langle x - f(x), \ell(x - \bar{x})\rangle \geq h(x). \tag{20}$$

Now, let us prove that for every $\lambda \in [0,1]$, the mapping $x \mapsto \lambda(x - F(x)) + (1-\lambda)\ell(x - \bar{x})$ has no zero on the sphere $S(\bar{x}, \frac{\varepsilon}{2})$. Otherwise, there exist $\lambda \in [0,1]$ and $x \in S(\bar{x}, \frac{\varepsilon}{2})$ with $0 \in \lambda(x - F(x)) + (1-\lambda)\ell(x - \bar{x})$, that is there exists $y \in F(x)$ with

$$\lambda(x - y) + (1-\lambda)\ell(x - \bar{x}) = 0. \tag{21}$$

By definition of F, there exist $k \geq 0$, $(\lambda_1, \ldots, \lambda_k) \in \mathbf{R}_+^k$, $\sum_{i=1}^k \lambda_i = 1$, $y_i \in F(x)$ and for every $i = 1, \ldots, k$, there exists a sequence $(x_n^i)_{n\in\mathbb{N}}$ converging to x, such that $f(x_n^i)$ converges to y_i when n tends to $+\infty$, and such that $y = \sum_{i=1}^k \lambda_i y_i$. In particular, from the choice of ℓ, for n large enough, we have for every $i = 1, \ldots, k$:

$$\langle x_n^i - f(x_n^i), \ell(x_n^i - \bar{x})\rangle \geq h(x_n^i). \tag{22}$$

Multiplying each equation (22) by λ_i and summing, we get, at the limit when n tends to $+\infty$,

$$\langle x - y, \ell(x - \bar{x})\rangle \geq h(x). \tag{23}$$

Or, also, since from Equation (21), $x - y = -\frac{(1-\lambda)\ell(x-\bar{x})}{\lambda}$ (λ being different from 0, from $\ell(x - \bar{x}) \neq 0$, a consequence of $x \neq \bar{x}$ and l invertible).

$$-\frac{1-\lambda}{\lambda}\|\ell(x - \bar{x})\|^2 \geq h(x) > 0 \tag{24}$$

which is impossible. Thus, finally, there is an homotopy $(\lambda, x) \in [0,1] \times B(\bar{x}, \frac{\varepsilon}{2}) \mapsto \lambda(x - F(x)) + (1-\lambda)\ell(x - \bar{x})$ between $\mathrm{id} - F$ and the mapping $x \mapsto \ell(x - \bar{x})$ on $B(\bar{x}, \frac{\varepsilon}{2})$ without zero on $S(\bar{x}, \frac{\varepsilon}{2})$ for every $\lambda \in [0,1]$. In particular, taking $\lambda = 1$, and since ε can be taken as small as we want, it proves that F has no fixed-point, except $\bar{x}$, on some neighborhood of $\bar{x}$, which proves assertion 1) in Lemma 1. Moreover, this also implies that the local degree of $\mathrm{id} - F$ at $\bar{x}$ is well defined and is equal to the local degree of $x \mapsto \ell(x - \bar{x})$ at $\bar{x}$, which is 1 (from Degree formula, and since the sign of jacobian determinant of ℓ is -1). This proves assertion 2) in Lemma 1.

3. Last, if $\bar{x}$ is type 1, then the same proof as above gives that F has no fixed-point, except $\bar{x}$, on some neighborhood of $\bar{x}$, and that the local degree of $\mathrm{id} - F$ at $\bar{x}$ is equal to $+1$. □

Now, a byproduct of Lemma 1 is that a generalized fixed-point $\bar{x} \in \mathring{C}$ of f cannot have several types, since a correspondence F cannot have several local degrees at $\bar{x}$ (when this local degree is well defined).

6.3 Proof of Theorem 1

From the previous Lemma 1, we can now prove Theorem 1. Without any loss of generality, since C has a nonempty interior, we can assume that 0 is interior to C (translating C is necessary).

By contradiction, assume that $f : C \to C$ satisfies the assumptions in Theorem 1 but has no fixed-point. Define the correspondence $F = co(\overline{f})$ from C to C, such that for every $x \in C$, $F(x)$ is the convex hull of $\{y : \exists (x_n)_{n\in\mathbb{N}} \in C^{\mathbb{N}}$ converging to x such that $\lim_{n\to+\infty} f(x_n) = y\}$. As already said before, F has a closed graph, with nonempty (since $f(x) \in F(x)$), compact and convex values for every $x \in C$, and in particular F is upper semicontinuous. Remark that from the assumptions on f, the set of fixed-points of F is in the interior of C (since $x \in F(x)$ means that x is a generalized fixed-point of f), and so the topological degree of $F - id$ is well defined.

Consider the homotopy $H : (\lambda, x) \in [0, 1] \times C \mapsto H(\lambda, x) := x - \lambda F(x)$ from $\mathrm{id} - F$ to id on C. For $\lambda \in [0, 1]$ fixed, if $x - \lambda F(x)$ has a zero x on the boundary of C, then $\lambda \neq 0$ (since 0 interior to C) and $\frac{x}{\lambda} \in F(x) \subset C$. This implies $\lambda = 1$ (indeed, for $\lambda < 1$, $\frac{x}{\lambda} \notin C$ from 0 interior to C, x in the boundary of C and C convex), so x is a generalized fixed-point of f. But by assumption, there is no generalized fixed-point of f on the boundary of C. So finally, the homotopy H avoids zero on the boundary. So, by Homotopy property of topological degree, the degree of $\mathrm{id} - F$ should also be equal to the degree of id, i.e. equal to 1 from degree formula.

Now, from Lemma 1 in Section 6.2, the zero of $\mathrm{id} - F$ are isolated, and since C is compact, $(\mathrm{id} - F)^{-1}(0) = \{x_1, \ldots, x_k\}$ is finite. From Additivity property, the degree of $\mathrm{id} - F$ is equal to the sum of local degrees of $\mathrm{id} - F$ at each x_i for $i = 1, \ldots k$, so it should be equal to $\sum_{i=1}^{k} \varepsilon_i$ with $\varepsilon_i = 0$ or $\varepsilon_i = -1$ (from Lemma 1 in Section 6.2 and from the assumption in Theorem 1). Finally, 1=deg($\mathrm{id} - F$) ≤ 0, which is a contradiction, and proves the existence of a fixed-point of f.

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