Existence of pure stationary equilibria for stochastic games

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Abstract

We study stochastic games with complementarities that are similar to Curtat's supermodular stochastic games. In Curtat's case, the state set is an interval, and the game satisfies the strict diagonal dominance assumption. Instead, we assume the state set to be countable and weaken the strict diagonal dominance assumption to concavity. Parallel to Curtat's theorem, we give the existence of a discounted, pure stationary equilibrium that is *increasing* in the state.

Keywords— stochastic game, equilibrium, fixed point, supermodularity

1 Introduction

In a stochastic game, if the state space, the player set and the action sets are finite, then for every discount factor, there is at least one discounted equilibrium in stationary strategies (see, e.g., [Sol22, Sec. 8.3]). In [Sob73], the existence of Nash equilibrium is claimed when the state set and the action sets are compact metric spaces. However, as [Fed78, p.457] and [Lev13, p.1974] point out, its proof is flawed. For uncountable state sets and under absolute continuity condition, Levy [Lev13, Footnote 9] mentions that incorrect proofs of the existence of equilibria have also appeared in [Cha99]. In fact, a counterexample (with state set $[0, 1]$) is raised up in [LM15, Sec. 3.4]. By contrast, if the state set is at most countable and if every action set is a compact metric space, Federgruen [Fed78, Theorem 1] proves that then under continuity condition, there is a stationary Nash equilibrium for every discount factor.

Curtat considers stochastic games with complementarities, which are called *supermodular stochastic games* in [Cur96, p.185]. The state spaces are compact *intervals* of Euclidean spaces, and the games are assumed to satisfy the *strict diagonal dominance*. Curtat [Cur96, Theorem 4.6] proves that for such games, there is a stationary Nash equilibrium which is increasing in the state. Curtat's theorem is also explained in [Ami03, Theorem 1 (a)].

We investigate a class of stochastic games similar to Curtat's supermodular stochastic games. As [Cur96, p.184] mentions, strict diagonal dominance together with supermodularity implies the strict concavity. We keep the supermodularity assumption, but to a certain degree we weaken strict diagonal dominance to concavity. Nevertheless, the state set is not assumed to be a compact interval, but a countable set. In Theorem 5.3, we prove that such stochastic game has a pure stationary Nash equilibrium for every discount factor, which is increasing with the state variable.

Notation and conventions

The product of topological spaces are endowed with the product topology. A topological space X is endowed with the Borel σ-algebra $\mathcal{B}_X$. Let $P(X)$ be the set of probability Borel measures. Let $C(X)$ be the set of continuous real-valued functions on X. Let T (resp. L) be a poset (resp. lattice). Then a correspondence $F : T \to 2^L$ is called *increasing* if for any $t \leq t'$ in T, any $x \in F(t)$, $x' \in F(t')$, one has $x \wedge x' \in F(t)$ and $x \vee x' \in F(t')$.

2 Model

We present a model of stochastic games to be considered. Let S be a nonempty, at most countable poset of states. Endow S with the discrete topology. Let I be a finite nonempty set of players. For every $i \in I$, let X^i be a topological space of actions for player i. Set $X := \prod_{i\in I} X^i$. For every $s \in S$, let $A^i(s) \subset X^i$ be a nonempty subset of actions feasible for player i at state s. Set $A(s) := \prod_{i\in I} A^i(s)$. Then $\mathrm{SA} := \sqcup_{s\in S} A(s)$ (resp. $\mathrm{SA}^i := \sqcup_{s\in S} A^i(s)$) is the set of all feasible action profiles (resp. action for player i) at all states. Endow them with the disjoint union topology. For every $i \in I$, let $r^i : \mathrm{SA} \to \mathbb{R}$ be the payoff function for player i. The transition rule is a map $q : \mathrm{SA} \to P(S)$. Fix a discount factor $\lambda_i \in [0, 1)$ for each player $i \in I$.

Definition 2.1. A *stochastic game* lasts for countably infinitely many stages. An initial state $s_0 \in S$ is given. At each stage $t \geq 0$, the following takes place:

- The current state s_t is announced to all players.
- Each player $i \in I$ chooses an action $a^i_t \in A^i(s_t)$. The players' choices are made simultaneously and independently.
- The action profile $(a^i_t)_{i\in I}$ is publicly announced to all players.
- Each player $i \in I$ receives a stage payoff $r^i(s_t, a_t)$.
- A state $s_{t+1} \in S$ is drawn according to the probability $q(\cdot|s_t, a_t)$ on S, and the game proceeds to stage $t + 1$.

We introduce some notation before defining mixed strategies. For a compact *metric* space X, let $F(X)$ be the vector space of Radon measures on X in the sense of [Cho69b, p.185]. Every $f \in C(X)$ defines a seminorm

$$p_f : F(X) \to [0, +\infty), \quad \mu \mapsto |\mu(f)|.$$

From [Rud91, p.26], the family $\{p_f\}_{f\in C(X)}$ of seminorms induces a topology on $F(X)$, called the weak*-topology, making it a locally convex topological vector space. We use only the weak*-topology.[1] By [Rud87, Thm. 2.18], every finite Borel measure on X is

[1] There are other useful topologies on $F(X)$. For example, by [Rud87, Thm. 6.19], the vector space $F(X)$ is the dual of the Banach space $(C(X), \|\cdot\|_\infty)$. The dual norm induces a topology on $F(X)$ finer than the weak*-topology.

regular in the sense of [Rud87, p.47], hence a Radon measure. Then by compactness of X and [Cho69b, p.40 and Cor. 12.7], $P(X)$ is a convex compact subset of $F(X)$. From [Cho69a, 25.1, p.108], the map $X \to P(X)$ taking the Dirac measures (in the sense of [Cho69b, Prop. 12.9]) is a topological embedding which identifies X with the set of extreme points of $P(X)$. From [Dud66, Thm. 12], the weak*-topology on $P(X)$ is metrizable since X is metrizable.[2]

Definition 2.2. For every integer $t \geq 0$, the set of histories till stage t is $H_t := \mathrm{SA}^t \times S$. Let $H := \sqcup_{t \geq 0} H_t$ denote the set of all histories, and let $H_\infty = \mathrm{SA}^{\mathbb{N}}$ denote the space of plays. Let $\mathcal{H}$ be the product of the Borel σ-algebras on H_∞ of each factor SA. Then $(H_\infty, \mathcal{H})$ is a measurable space.

In particular, $H_0 = S$ is the set of initial states.

Definition 2.3. A mixed strategy of player $i \in I$ is a map π^i assigns to each history

$$h_t = (s_0, a_0, \dots, s_{t-1}, a_{t-1}, s_t) \in H$$

an element of $P(A^i(s_t))$, such that for every $B \in \mathcal{B}_{A^i(s_t)}$, the function $\pi^i(B|\cdot) : \mathrm{SA}^t \times \{s_t\} \to [0,1]$ is measurable. The set of all strategies of player i is denoted by Π^i, and the set of strategy profiles is denoted by $\Pi := \prod_{i \in I} \Pi^i$.

Let $\mathrm{SP} = \sqcup_{s \in S} \prod_{i \in I} P(A^i(s)) = \{(s, \mu^i)_{i \in I} : s \in S, \mu^i \in P(A^i(s))\}$. Then SA is a subset of SP. Because I is finite, one may extend r^i to a function $\mathrm{SP} \to \mathbb{R}$ by

$$r^i(s,\mu) := \int_{A^N(s)} \cdots \int_{A^1(s)} r^i(s,a) d\mu^1(a^1) \dots d\mu^N(a^N).$$

Similarly, the extension $q : \mathrm{SP} \to P(S)$ is such that for every $s' \in S$

$$q(s'|s,\mu) = \int_{A^N(s)} \cdots \int_{A^1(s)} q(s'|s,a) d\mu^1(a^1) \dots d\mu^N(a^N).$$

3 Assumptions

1. For every $i \in I$, assume that X^i is a sublattice of an Euclidean space $\mathbb{R}^{n_i}$. For every $s \in S$, assume that $A^i(s)$ is a convex compact subset of X^i.
2. The correspondence $A^i(\cdot) : S \to 2^{X_i}$ is increasing.
3. For every $i \in I$, the function $r^i : \mathrm{SA} \to \mathbb{R}$ is bounded.
4. For every $s \in S$, the function $r^i(s, \cdot) : A(s) \to \mathbb{R}$ is continuous.
5. For any $s, s' \in S$, the map $q(s'|s,\cdot) : A(s) \to [0,1]$ is continuous.
6. For every $s \in S$, every $i \in I$ and every $a^{-i} \in A^{-i}(s)$, the function $r^i(s, \cdot, a^{-i}) : A^i(s) \to \mathbb{R}$ is concave.
7. For every $s \in S$, every $i \in I$ and every $a^{-i} \in A^{-i}(s)$, the function $q(s, \cdot, a^{-i}) : A^i(s) \to P(S)$ is stochastically concave, i.e., for every increasing function $f : S \to [0, +\infty)$, the function $\int_S f(s') dq(s'|s, \cdot, a^{-i}) : A^i(s) \to [0, +\infty]$ is concave.
8. For every $s \in S$, every $i \in I$ and every $a^{-i} \in A^{-i}(s)$, the function $r^i(s, \cdot, a^{-i}) : A^i(s) \to \mathbb{R}$ is supermodular.

[2]By contrast, the weak*-topology on $F([0,1])$ is not metrizable.

9. For every $s \in S$, every $i \in I$ and every $a^{-i} \in A^{-i}(s)$, the function $q(s, \cdot, a^{-i}) : A^i(s) \to P(S)$ is stochastically supermodular in the sense of [Cur96, p.180].
10. The function $r^i : \mathrm{SA} \to \mathbb{R}$ is increasing.
11. The function $q : \mathrm{SA} \to P(S)$ is stochastically increasing in the sense of [Cur96, p.180].
12. For every $i \in I$ and every $a^{-i} \in X^{-i}$, the function

$$r^i(\cdot, a^{-i}) : \sqcup_{s \in S : a^{-i} \in A^{-i}(s)} A^i(s) \to \mathbb{R}$$

has *increasing difference*, i.e., for any $s \leq s'$ in S, any $a^i \leq a'^i$ in X^i with $a^i \in A^i(s')$ and $a'^i \in A^i(s)$, one has

$$r^i(s', a'^i, a^{-i}) + r^i(s, a^i, a^{-i}) \geq r^i(s', a^i, a^{-i}) + r^i(s, a'^i, a^{-i}). \tag{1}$$

13. For every $i \in I$ and every $a^{-i} \in X^{-i}$, the function $q(\cdot, a^{-i}) : \sqcup_{s \in S : a^{-i} \in A^{-i}(s)} A^i(s) \to P(S)$ has stochastic increasing difference.
14. For every $s \in S$ and every $i \in I$, the function $r^i(s, \cdot) : A^i(s) \times A^{-i}(s) \to \mathbb{R}$ has increasing difference.

Remark 3.1. By Assumption 2, for every $s \in S$, the value $A^i(s)$ is a sublattice of X^i. Moreover, in Assumption 12, one has $a^i \in A^i(s)$ and $a'^i \in A^i(s')$, so (1) makes sense. For any $(s, a), (s', a') \in \mathrm{SA}$, if $s \leq s'$ in S and $a^j \leq a'^j$ in X^j for every $j \in I$, then we define $(s, a) \leq (s', a')$. The relation $\leq$ is a partial order on SA, which is used in Assumptions 10 and 11.

Remark 3.2. To get the existence of stationary Nash equilibria in the case of infinitely many states and actions, one has to assume assumptions other than mere continuity. In fact, Levy and McLennan [LM15, Theorem 3.1] construct a stochastic game without stationary equilibria, where I is finite, $S = [0, 1]$ is compact, every $A^i(s)$ is finite, the payoff functions r^i and the transition q are continuous.

We compare our assumptions with those in related literature.

Remark 3.3. Curtat's assumption [Cur96, Cardinal Complementarity, p.185] contains simultaneously Assumptions 8, 9, 12, 13 and 14. Curtat [Cur96, Smoothness] and Amir [Ami03, Sec. 2] assume that for every $i \in I$, the set $A^i(s) = A^i$ is a compact Euclidean interval independent of s. In this case, Assumptions 1 and 2 are satisfied. Assumptions 3, 4 and 5 are weaker than the hypothesis [Cur96, Smoothness] (see also [Ami03, (A1), p.444]), where the payoff functions and the transition function are assumed to be twice continuously differentiable. We don't need [Ami03, (A3)], which by [Ami03, Lemma 1] is an assumption stronger than the uniqueness of Nash equilibrium. In return, we adopt Assumptions 6, 7 as well as the increasingness of r^i in a^i. The combination of Assumptions 8, 9, 12, 13 and 14 is weaker than [Ami03, (A2)], as we do not require that the transition has increasing differences in (a^i, a^{-i}). [Ami03, (A4)] is similar to Assumptions 10 and 11.

4 Discounted return

For every $\pi \in \Pi$, every integer $t \geq 1$ and every $s \in S$, an element

$$(s_0, a_0, \ldots, s_{t-1}, a_{t-1}) \in \mathrm{SA}^t$$

defines a product probability $\prod_{i\in I}\pi^i(s_0,a_0,\dots,s_{t-1},a_{t-1},s)$ on $A(s)$. Taking sum over $s\in S$ of this probability weighted by $q(s|s_{t-1},a_{t-1})$, one gets a probability measure on SA. For every subset $B\subset S$, the function $q(B|\cdot):\mathrm{SA}\to[0,1]$ is the supremum of a countable family $\{q(B'|\cdot):\mathrm{SA}\to[0,1]\}_{B'}$ of functions, where B' runs through finite subsets of B. By Assumption 5, for every B', the function $q(B'|\cdot):\mathrm{SA}\to[0,1]$ is continuous and hence measurable. By [Kal21, I, Lem. 1.10], the function $q(B|\cdot)$ is also measurable. From the measurability assumption in Definition 2.3, one gets a stochastic kernel in the sense of [HLL12, Def. C.1] on SA given SA^t. Every $s_0\in H_0$ induces a probability $\prod_{i\in I}\pi^i(s_0)$ on $A(s_0)$. Extended by zero to a probability on SA, it induces a probability $P_{s_0,\pi}$ on the measurable space $(H_\infty,\mathcal{H})$ via the Ionescu-Tulcea extension theorem (see, e.g., [HLL12, Prop. C.10]). This probability is supported on the the preimage of s_0 under the map $H_\infty\to H_0$ taking the initial state. Let $\mathbf{E}_{\pi,s_0}:L^1(H_\infty,\mathcal{H},P_{s_0,\pi})\to\mathbb{R}$ be the corresponding expectation operator.

Since I is finite, by Assumption 3, there is $R>0$ with $\|r^i\|_\infty\le R$ for every $i\in I$. By Assumption 4, for every integer $t\ge 0$, the function

$$H_\infty\to\mathbb{R},\quad (s_0,a_0,s_1,a_1,\dots)\mapsto r^i(s_t,a_t)$$

is $\mathcal{H}$-measurable and bounded by R, hence a random variable with expectation.

Definition 4.1. For every $i\in I$, the function

$$V^i:H_0\times\Pi\to\mathbb{R},\quad (s_0,\pi)\mapsto(1-\lambda_i)\mathbf{E}_{s_0,\pi}[\sum_{t\ge 0}\lambda_i^t r^i(s_t,a_t)]$$

is well-defined. It is called the *total expected λ_i-discounted return* to player i.

One has

$$\|V^i\|_\infty\le\|r^i\|_\infty. \tag{2}$$

Definition 4.2. For every $i\in I$, let $\Delta^i:=\prod_{s\in S}P(A^i(s))\subset\Pi^i$ be the subset of stationary strategies for player i. Then $\Delta:=\prod_{i\in I}\Delta^i$ is the subset of Π consisting of stationary strategy profiles. Let $\Delta_p^i:=\prod_{s\in S}A^i(s)\subset\Delta^i$ be the subset of pure stationary strategies for player i. Set $\Delta_p:=\prod_{i\in I}\Delta_p^i$ to be the set of pure stationary strategy profile. For every $i\in I$, let $J^i\subset\Delta_p^i$ be the subset of increasing strategies. Set $J:=\prod_{i\in I}J^i$.

For every $\delta\in\Delta$, the function $V^i(\cdot,\delta):S\to\mathbb{R}$ is bounded measurable. One has

$$V^i(s,\delta)=(1-\lambda_i)r^i(s,\delta(s))+\lambda_i\int_S V^i(s',\delta)dq(s'|s,\delta(s)). \tag{3}$$

Let $B(S)$ be the space of bounded measurable functions on S. Then $B(S)$ is a vector space with the sup norm $\|f\|_\infty:=\sup_{s\in S}|f(s)|$. Let $B_R(S)=\{f\in B(S):\|f\|_\infty\le R\}$ be the closed ball of radius R.

Lemma 4.3. *The metric space $B_R(S)$ is complete.*

Proof. For every Cauchy sequence $\{f_n\}_{n\ge1}$ in $B_R(S)$, we prove that it converges. For every $\epsilon>0$, there is an integer $N\ge1$ such that $\|f_n-f_m\|_\infty<\epsilon$ for all integers $m,n\ge N$. In particular, for every $s\in S$,

$$|f_n(s)-f_m(s)|<\epsilon, \tag{4}$$

so $\{f_n(s)\}$ is a Cauchy sequence in $\mathbb{R}$. This sequence converges to a real number $f(s)$. Then f_n converges to f pointwisely, so the function f is measurable and bounded by R. Thus, $f \in B_R(S)$. Letting $m \to +\infty$ in (4), one gets $|f_n(s) - f(s)| \leq \epsilon$ for every $s \in S$. Thus, $\|f_n - f\|_\infty \leq \epsilon$ for every integer $n \geq N$. Therefore, the sequence $\{f_n\}_n$ converges to f in $B_S(R)$. $\square$

Lemma 4.4. *Let $I(S)$ be the subset of $B(S)$ of increasing functions. Then $I(S)$ is closed in $B(S)$.*

Proof. Let $f_n \to f$ be a converging sequence in $B(S)$ with $f_n \in I(S)$ for all $n \geq 1$. For any $s \leq s'$ in S, one has $f_n(s) \leq f_n(s')$. Taking limit in n, one gets $f(s) \leq f(s')$, so $f \in I(S)$. $\square$

Lemma 4.5. *For every pure, stationary, increasing strategy profile $\gamma \in J$, the function $V^i(\cdot, \gamma) : S \to \mathbb{R}$ is increasing and takes values in $[-R, R]$.*

Proof. Define an operator $T : B(S) \to B(S)$ by

$$T(v)(s) := (1 - \lambda_i) r^i(s, \gamma(s)) + \lambda_i \int_S v(s') dq(s'|s, \gamma(s))$$

for every $v \in B(S)$. (Because v is bounded measurable, the integral makes sense.) Moreover,

$$\|Tv\|_\infty \leq (1 - \lambda_i)\|r^i\|_\infty + \lambda_i \|v\|_\infty, \tag{5}$$

so the function Tv is bounded. It is measurable. Thus, T is well-defined.

The map T is a contraction. Indeed, for any $v, v' \in B(S)$ and every $s \in S$, one has

$$\begin{aligned}
&|Tv(s) - Tv'(s)| = |\lambda_i \int_S [v(s') - v'(s')] dq(s'|s, \gamma(s))| \\
\leq &\lambda_i \int_S |v(s') - v'(s')| dq(s'|s, \gamma(s)) \\
\leq &\lambda_i \|v - v'\|_\infty.
\end{aligned}$$

Therefore, $\|Tv - Tv'\|_\infty \leq \lambda_i \|v - v'\|_\infty$. In particular, T has at most one fixed point in $B(S)$. By (3), the function $V^i(\cdot, \gamma) : S \to \mathbb{R}$ is the unique fixed point.

The map T preserves $I(S)$. In fact, for every $v \in I(S)$ and any $s \leq \bar{s}$ in S, as γ is increasing, one has $\gamma(s) \leq \gamma(\bar{s})$ in X. By Assumption 10, one has $r^i(s, \gamma(s)) \leq r^i(\bar{s}, \gamma(\bar{s}))$. Because $v : S \to \mathbb{R}$ is increasing, from Assumption 11, one has

$$\int_S v(s') dq(s'|s, \gamma(s)) \leq \int_S v(s') dq(s'|\bar{s}, \gamma(\bar{s})).$$

Hence, one obtains $Tv(s) \leq Tv(\bar{s})$ and $Tv \in I(S)$.

From (5), the map T preserves $B_R(S)$. By Lemmas 4.3 and 4.4, the metric space $I_R(S) := I(S) \cap B_R(S)$ is complete. It is nonempty as it contains the constant 0 function. From Banach's fixed point theorem (see, e.g., [Pat+19, Thm. 1.1]) and $T(I_R(S)) \subset I_R(S)$, the map T has a fixed point in $I_R(S)$. Therefore, $V^i(\cdot, \gamma) \in I_R(S)$. $\square$

5 Equilibria

Definition 5.1. [Fed78, p.456] An element $\pi^* \in \Pi$ is called a λ-*discounted equilibrium point of policies* (λ-DEP) if for every $s \in S$, every $i \in I$ and every $\pi^i \in \Pi^i$, one has

$$V^i(s, \pi^*) \geq V^i(s, \pi^i, (\pi^*)^{-i}).$$

A λ-DEP lying in Δ (resp. Δ_p) is called stationary (resp. pure stationary). Lemma 5.2 is a variant of [Fed78, Lem. 2.3], where it is attributed to Blackwell [Bla65, Thm. 6 (f)] and a coefficient α is missing. For the convenience of readers, a proof is included.

Lemma 5.2. *A stationary strategy profile $\delta \in \Delta$ is a λ-DEP if for every $s \in S$ and every $i \in I$, one has $V^i(s, \delta) = \max_{P(A^i(s))} T^i_{\delta,s}$, where $T^i_{\delta,s} : F(A^i(s)) \to \mathbb{R}$ is a map defined by*

$$\mu \mapsto (1 - \lambda_i) r^i(s, \mu, \delta^{-i}(s)) + \lambda_i \int_S V^i(s', \delta) dq(s'|s, \mu, \delta^{-i}(s)). \tag{6}$$

Proof. For every $i \in I$, when the players other than i use strategy δ^{-i}, the stochastic game induces a Markov decision problem (MDP) for player i. For every $s \in H_0$ and every $\pi^i \in \Pi^i$, let $\mathbf{E}_{s,\pi^i}$ be the expectation operator on the measurable space of plays for this MDP defined in [Sol22, p.11].

For every history $h_t \in H$ till stage t, one has

$$\begin{aligned}
&\mathbf{E}_{s,\pi^i}[(1 - \lambda_i) r^i(s_t, \cdot, \delta^{-i}(s_t)) + \lambda_i V^i(s_{t+1}, \delta)|h_t] \\
=&(1 - \lambda_i) r^i(s_t, \pi^i(h_t), \delta^{-i}(s_t)) + \lambda_i \sum_{s' \in S} q(s'|s_t, \pi^i(h_t), \delta^{-i}(s_t)) V^i(s', \delta) \\
=&T^i_{\delta,s_t}(\pi^i(h_t)) \leq V^i(s_t, \delta).
\end{aligned}$$

By [Sol22, Lem. 1.28], one has $V^i(s, \pi^i, \delta^{-i}) \leq V^i(s, \delta)$. Therefore, δ is an λ-DEP. $\square$

Theorem 5.3. *There is a pure stationary λ-DEP that is* increasing *in the state.*

The idea of the proof is applying Kakutani's fixed point theorem (see, e.g. [Pat+19, Thm. 10.1]). We consider the best reply correspondence restricted to increasing strategies. For this correspondence, we verify the hypothesis of Kakutani's theorem.

Proof. For every $i \in I$, we check that J^i is nonempty and closed in Δ^i_p. From [Yu24, Thm. 2.11] , Assumptions 1 and 2, for every $s \in S$, the nonempty set $A^i(s)$ is subcomplete sublattice of $\mathbb{R}^{n_i}$. Therefore, the map

$$S \to X^i, \quad s \mapsto \min A^i(s)$$

is increasing, hence belongs to J^i. In particular, J^i is nonempty. The convergence in the product topology of $\Delta^i_p = \prod_{s \in S} A^i(s)$ is the pointwise convergence. Let $f_\alpha \to f$ be a converging net in Δ^i_p with $f_\alpha \in J^i$ for every α. Then for any $s \leq \bar{s}$ in S, one has $f_\alpha(s) \leq f_\alpha(\bar{s})$. Taking limit, one has $f(s) \leq f(\bar{s})$, so $f \in J^i$ and J^i is closed in Δ^i_p.

By Assumption 1 and Tychonoff's theorem (see, e.g., [Mun14, Thm. 37.3]), Δ^i_p is compact. Moreover, both Δ^i_p and J^i are convex subsets of the locally convex topological vector space $\prod_{s \in S} \mathbb{R}^{n_i}$.

By Corollary 5.8 and Assumption 4, for every $\delta \in \Delta_p$ and every $s \in S$, the function

$$F(A^i(s)) \to \mathbb{R}, \quad \mu \mapsto r^i(s, \mu, \delta^{-i}(s))$$

is linear and continuous. Similarly, from Assumption 5, for every $s' \in S$, so is the function

$$F(A^i(s)) \to \mathbb{R}, \quad \mu \mapsto q(s'|s, \mu, \delta^{-i}(s)).$$

From (2), the countable family $\{V^i(s', \delta)\}_{s' \in S}$ of reals is bounded. By Lemma 5.5, because $\sum_{s' \in S} q(s'|s, \mu, \delta^{-i}(s)) = 1$ for every $\mu \in F(A^i(s))$, the series

$$\sum_{s' \in S} q(s'|s, \cdot, \delta^{-i}(s)) V^i(s', \delta)$$

converges pointwise to a continuous function on $F(A^i(s))$. Therefore, the function (6) is also linear and continuous on $F(A^i(s))$.

For every $i \in I$ and every $s \in S$, define a correspondence

$$\Phi^{i,\delta} : S \to 2^{X^i}, \quad s \mapsto A^i(s) \cap \operatorname{argmax}_{P(A^i(s))} T^i_{\delta,s}.$$

By Bauer's maximum principle (see, e.g., [NP06, Cor. A.3.3]), the subspace $\Phi^{i,\delta}(s)$ is nonempty and closed in $A^i(s)$.

For every $i \in I$, define a correspondence

$$B^i : J \to 2^{J^i}, \quad \delta \mapsto \{\gamma^i \in J^i : \forall s \in S, \gamma^i(s) \in \Phi^{i,\delta}(s)\}.$$

By Lemma 4.5 and Assumption 7, for every $\delta \in J$, the function

$$\int_S V^i(s', \delta) dq(s'|s, \cdot, \delta^{-i}(s)) : A^i(s) \to \mathbb{R}$$

is concave. Together with Assumption 6, it implies that $T^i_{\delta,s} : A^i(s) \to \mathbb{R}$ is a concave function. In particular, $\Phi^{i,\delta}(s) = \operatorname{argmax}_{A^i(s)} T^i_{\delta,s}$ is a convex set.

By Lemma 4.5 and Assumption 13, the function

$$\int_S V^i(s', \delta) dq(s'|\cdot, \delta^{-i}(s)) : S \times A^i \to \mathbb{R}$$

has increasing difference. For any $s \leq s'$ in S, every $a^i \in A^i(s')$ and every $a'^i \in A^i(s)$, one has

$$\begin{aligned} &r^i(s', a'^i, \delta^{-i}(s')) - r^i(s', a^i, \delta^{-i}(s')) \overset{\text{(a)}}{\geq} r^i(s, a'^i, \delta^{-i}(s')) - r^i(s, a^i, \delta^{-i}(s')) \\ \overset{\text{(b)}}{\geq} &r^i(s, a'^i, \delta^{-i}(s)) - r^i(s, a^i, \delta^{-i}(s)), \end{aligned}$$

where (a) (resp. (b)) uses Assumption 12 (resp. 14).

Thus, the function $T^i_{\delta,\cdot}(\cdot) : \sqcup_{s \in S} A^i(s) \to \mathbb{R}$ has increasing difference. By Assumptions 8 and 9, the function $T^i_{\delta,s} : A^i(s) \to \mathbb{R}$ is supermodular. Then from Assumption 2 and [Top98, Thm. 2.8.1], the correspondence $\Phi^{i,\delta}$ is increasing. In particular, for every $s \in S$, the value $\Phi^{i,\delta}(s)$ is a sublattice of $A^i(s)$. By [Yu24, Thm. 2.11], this sublattice is subcomplete. In particular, $\max \Phi^{i,\delta}(s)$ exists. The map $\max \Phi^{i,\delta} : S \to X^i$ is increasing, so it belongs to $B^i(\delta)$. In particular, every value of B^i is nonempty.

A convex combination of two increasing maps $S \to A^i$ is still increasing, so every value of B^i is convex. Since $B^i(\delta) = J^i \cap \prod_{s\in S} \Phi^{i,\delta}(s)$, it is closed and hence compact in Δ_p^i.

Define a correspondence

$$B : J \to 2^J, \quad \delta \mapsto \prod_{i\in I} B^i(\delta).$$

Then every value of B is nonempty, convex and compact. Moreover, $J = \prod_{i\in I} J^i$ is a nonempty convex compact subset of the locally convex topological vector space $\prod_{i\in I, s\in S} R^{n_i}$. From Lemma 5.4, the graph of B is closed. By Kakutani's fixed point theorem, B has a fixed point. By Lemma 5.2, every fixed point is a pure stationary *increasing* λ-DEP. $\square$

Lemma 5.4. *The correspondence $B^i : J \to 2^{J^i}$ has closed graph.*

Proof. Let $\{(\delta_\alpha, \gamma_\alpha^i)\}_\alpha$ be a net on the graph of B^i converging to $(\delta, \gamma^i) \in J \times J^i$. For every $i \in I$, every $s \in S$ and every α, one has $\gamma_\alpha^i(s) \in \Phi^{i,\delta}(s)$. For every $a^i \in A^i(s)$, one has

$$\begin{aligned}
&(1-\lambda_i) r^i(s, a^i, \delta_\alpha^{-i}(s)) + \lambda_i \int_S V^i(s', \delta_\alpha) dq(s'|s, a^i, \delta_\alpha^{-i}(s)) \\
\leq &(1-\lambda_i) r^i(s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) + \lambda_i \int_S V^i(s', \delta_\alpha) dq(s'|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)).
\end{aligned} \tag{7}$$

By Assumption 5, for every subset T of S and every $s' \in T$, one has $q(s'|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) \to q(s'|s, \gamma^i(s), \delta^{-i}(s))$. By Fatou's lemma (see, e.g. [Roy68, Thm. 11, p.226]), one gets

$$q(T|s, \gamma^i(s), \delta^{-i}(s)) \leq \liminf_\alpha q(T|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)).$$

Similarly, one has

$$q(S \setminus T|s, \gamma^i(s), \delta^{-i}(s)) \leq \liminf_\alpha q(S \setminus T|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)),$$

or equivalently

$$\limsup_\alpha q(T|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) \leq q(T|s, \gamma^i(s), \delta^{-i}(s)).$$

Therefore, one obtains $\lim_\alpha q(T|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) = q(T|s, \gamma^i(s), \delta^{-i}(s))$. From [Fed78, Lem. 2.1], the net of functions $\{V^i(\cdot, \delta_\alpha)\}$ on S converges pointwise to $V^i(\cdot, \delta)$. From [Roy68, Prop. 18, p.232], one has

$$\int_S V^i(s', \delta_\alpha) dq(s'|s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) \to \int_S V^i(s', \delta) dq(s'|s, \gamma^i(s), \delta^{-i}(s)).$$

Similarly, one has $\int_S V^i(s', \delta_\alpha) dq(s'|s, a^i, \delta_\alpha^{-i}(s)) \to \int_S V^i(s', \delta) dq(s'|s, a^i, \delta^{-i}(s))$. From Assumption 4, one has

$$\begin{gathered}
r^i(s, \gamma_\alpha^i(s), \delta_\alpha^{-i}(s)) \to r^i(s, \gamma^i(s), \delta^{-i}(s)), \\
r^i(s, a^i, \delta_\alpha^{-i}(s)) \to r^i(s, a^i, \delta^{-i}(s)).
\end{gathered}$$

Taking limits in (7), one has

$$\begin{aligned}&(1-\lambda_i)r^i(s,a^i,\delta^{-i}(s))+\lambda_i\int_S V^i(s',\delta)dq(s'|s,a^i,\delta^{-i}(s))\\ \leq&(1-\lambda_i)r^i(s,\gamma^i(s),\delta^{-i}(s))+\lambda_i\int_S V^i(s',\delta)dq(s'|s,\gamma^i(s),\delta^{-i}(s)).\end{aligned}$$

Hence, one has $\gamma^i(s)\in\Phi^{i,\delta}(s)$. Then $\gamma^i\in B^i(\delta)$. □

Lemma 5.5. *Let X be a topological space. Let $\{f_n : X\to[0,+\infty)\}_{n\geq 1}$ be a sequence of continuous functions. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence in $\mathbb{R}$. If $\sum_{n\geq 1} f_n$ converges pointwise to a continuous function on X, then so does $\sum_{n\geq 1} a_n f_n$.*

Proof. By assumption, there is $B>0$ such that $|a_n|\leq B$ for every $n\geq 1$. For every $x\in X$, one has $\sum_{n\geq 1}|a_nf_n(x)|\leq B\sum_{n\geq 1}f_n(x)$, so $\sum_{n\geq 1}a_nf_n$ converges pointwise. For every integer $N>0$, set $S_N:=\sum_{n=1}^N(B+a_n)f_n$, which is a continuous function on X. Then $B\sum_{n\geq 1}f_n+\sum_{n\geq 1}a_nf_n=\sum_{n\geq 1}(B+a_n)f_n$ is the supremum of $\{S_N\}_{N\geq 1}$, so it is lower semicontinuous on X. Similarly, the function $-B\sum_{n\geq 1}f_n+\sum_{n\geq 1}a_nf_n$ is upper semicontinuous on X. As $\sum_{n\geq 1}f_n$ is continuous on X, the function $\sum_{n\geq 1}a_nf_n$ is simultaneously upper semicontinuous and lower semicontinuous. Therefore, $\sum_{n\geq 1}a_nf_n$ is continuous. □

Remark 5.6. Lemma 5.5 is used implicitly in [Fed78, (2.8)] and [Idz05, p.227].

For a compact space K, the set $C(K)$ of real continuous functions on K is a Banach space with the sup norm.

Lemma 5.7. *Let X,Y be topological spaces. Let $f: X\times Y\to\mathbb{R}$ be a continuous function. If Y is compact, then the map*

$$X\to C(Y),\quad x\mapsto f(x,\cdot)$$

is continuous.

Proof. Fix $x_0\in X$ and $\epsilon>0$. For every $y\in Y$, since f is continuous at (x_0,y), there is an open neighborhood U_y (resp. V_y) of $x_0\in X$ (resp. $y\in Y$) such that for every $(u,v)\in U_y\times V_y$, one has $|f(x_0,y)-f(u,v)|<\epsilon$. Then $\{V_y\}_{y\in Y}$ forms an open covering of Y. By the compactness of Y, there is a finite subcover $\{V_{y_i}\}_{i=1}^n$. Hence $U:=\cap_{i=1}^n U_{y_i}$ is an open neighborhood of $x_0\in X$. For every $x\in U$ and every $y\in Y$, there is $1\leq i\leq n$ with $y\in V_{y_i}$. Since $(x,y)\in U_{y_i}\times V_{y_i}$, one has $|f(x_0,y)-f(x,y)|<\epsilon$. Therefore, $\|f(x_0,\cdot)-f(x,\cdot)\|_\infty\leq\epsilon$. □

Corollary 5.8. *Let X,Y be compact metric spaces. Let $f: X\times Y\to\mathbb{R}$ be a continuous function. Fix $\nu\in F(Y)$. Then the function*

$$F(X)\to\mathbb{R},\quad \mu\mapsto\int_X\int_Y f(x,y)d\nu(y)d\mu(x)$$

is linear continuous.

Proof. The linearity follow from that of integration. By Lemma 5.7, since the map $C(Y)\to\mathbb{R},\quad g\mapsto\int_Y g d\nu$ is continuous, the function $X\to\mathbb{R},\quad x\mapsto\int_Y f(x,y)d\nu(y)$ is continuous. By the definition of weak*-topology on $F(X)$, the proof is completed. □

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