A degree theoretical proof of Singbal's fixed point theorem

Lu YU

August 18, 2024

1 Introduction

Topological degree theory is an important tool to study fixed points. For instance, Brouwer's fixed point theorem can be proved using the Brouwer degree. In an infinite dimensional normed space, the Leray-Schauder degree theory provides a way to prove Schauder's fixed point theorem [Sch30]. The traditional degree-theoretical approach to fixed point theorem (see e.g., [FG95, Sec 7.3], [Cro95, p.139], [CC06, Sec. 2.2], [Dei10, p.60], etc.) is divided into two steps. First, degree theory is used to treat convex subsets with nonempty relative interior. The fixed point property is inherited by a retract. Hence, the second step reducing the general case to the considered one is applying Dugundji's extension theorem [Dug51, Theorem 4.1]. It shows that a closed convex subset of a normed space is a retract.

Tychonoff's fixed point theorem [Tyc35, Satz, p.770] extends Schauder's fixed point theorem to locally convex spaces. Nagumo [Nag51] extends the Leray-Schauder degree theory from normed spaces to locally convex spaces. However, Dugundji's theorem is missing in locally convex spaces, which curbs a proof of Tychonoff's theorem via degree theory.

The purpose of this note is to provide a purely degree-theoretical proof of the Tychonoff's fixed point theorem. First, we prove Singbal's fixed point theorem. Let X be a locally convex TVS.

Theorem 1.1 (Singbal, [Bon62, Appendix])**.** *Let A be a nonempty closed convex subset of X, and let B be a compact subset of A. Let $f : A \to B$ be a continuous map. Then f has a fixed point.*

Remark 1.2. In Theorem 1.1, if X is a Banach space and A is bounded, then the result specializes to [FG95, Theorem 7.30]. The norm structure of X is used in an essential way in the proof in [FG95].

Tychonoff's fixed point theorem follows from Theorem 1.1.

Corollary 1.3 ([Tyc35, §2, Satz])**.** *Let A a nonempty compact convex subset of X. Let $f : A \to A$ be a continuous map. Then f admits a fixed point.*

To give a new proof of Theorem 1.1, we modify Nagumo's degree theory a bit, which allows a proof avoiding Dugundji's extension theorem.

2 Preliminaries

Section 2 contains miscellaneous elementary results that we need. The reader can skip it first and consult it only when necessary.

Lemma 2.1. *Let X be a topological space. Let A and Y be subsets of X. Then $\partial_Y(A \cap Y) \subset \partial_X(A)$.*

Proof. For every $p \in \partial_Y(A \cap Y)$ and every open neighborhood U of p in X, $U \cap Y$ is an open neighborhood of p in Y. Then $U \cap Y \cap A$ and $(U \cap Y) \setminus (A \cap Y)$ are nonempty. Therefore, $U \cap A$ and $U \setminus A$ are nonempty, which shows $p \in \partial_X(A)$. □

Throughout, vector spaces and affine spaces are over $\mathbb{R}$. By TVS, we mean a Hausdorff topological vector space in the sense of [Rud91, 1.6, p.7].

Definition 2.2. Let X be a TVS and S be a subset of X. The *relative interior* $\mathrm{ri}(S)$ is the interior of S inside its affine hull $\mathrm{Aff}(S)$. The *relative boundary* of S is $\mathrm{rb}(S) := S \setminus \mathrm{ri}(S)$.

Lemma 2.3. *In Definition 2.2, let E be an affine subspace of X. Then*

1. $\mathrm{rb}(S \cap E) \subset \partial_X(S)$.
2. $\mathrm{rb}(S) \subset \partial_X(S)$.
3. *If* $E \subset \mathrm{Aff}(S)$*, then* $\mathrm{rb}(S \cap E) \subset \mathrm{rb}(S)$.

Proof.

1. The affine space E contains $S \cap E$, so $\mathrm{Aff}(S \cap E) \subset E$ and hence $S \cap \mathrm{Aff}(S \cap E) \subset S \cap E$. By $S \cap E \subset S$ and $S \cap E \subset \mathrm{Aff}(S \cap E)$, one has $S \cap E \subset S \cap \mathrm{Aff}(S \cap E)$. Thus, one has $S \cap \mathrm{Aff}(S \cap E) = S \cap E$. By Lemma 2.1,
$$\mathrm{rb}(S \cap E) = \partial_{\mathrm{Aff}(S \cap E)}(S \cap E) = \partial_{\mathrm{Aff}(S \cap E)}(S \cap \mathrm{Aff}(S \cap E))$$
is contained in $\partial_X(S)$.
2. Take $E = X$ in Part 1.
3. Take $X = \mathrm{Aff}(S)$ in Part 1.

□

Lemma 2.4. *In Definition 2.2, let E be an affine subspace of X. Then $\mathrm{ri}(S) \cap E \subset \mathrm{ri}(S \cap E)$.*

Proof. Since $\mathrm{Aff}(S) \cap E$ is an affine space containing $S \cap E$, we have $\mathrm{Aff}(S) \cap E \supset \mathrm{Aff}(S \cap E)$. The subset $\mathrm{ri}(S)$ of $\mathrm{Aff}(S)$ is open, so $\mathrm{ri}(S) \cap E$ is open in $\mathrm{Aff}(S) \cap E$, hence also open in $\mathrm{Aff}(S \cap E)$. Therefore $\mathrm{ri}(S) \cap E$ is contained in the interior $\mathrm{ri}(S \cap E)$ of $S \cap E$ inside $\mathrm{Aff}(S \cap E)$. □

For a subset S of a TVS X, write $\mathrm{co}(S)$ for its convex hull. Let $C^0(S)$ for the set of continuous maps $S \to X$. Let $K(S)$ be the set of $f \in C^0(S)$ with $\overline{f(S)}$ compact. Set $K_1(S) := \{\mathrm{Id}_S - f | f \in K(S)\}$. When S is compact, one has $K(S) = C^0(S) = K_1(S)$.

Fact 2.5 ([Nag51, Theorem 1])**.** *Let X be a TVS and B be a closed subset of X. If $\phi \in K_1(B)$, then $\phi(B)$ is closed in X.*

Lemma 2.6. *Let X be a topological space and Y be a TVS. Let $f : X \to Y$ be a continuous map with $\overline{f(X)}$ compact. Then for every* convex *open neighborhood V of 0 in Y, there is a finite subset $\{y_1, \dots, y_n\}$ of $f(X)$ and a continuous map $f_V : X \to \mathrm{co}(y_1, \dots, y_n)$ such that $f(x) - f_V(x) \in V$ for all $x \in X$.*

Proof. The family $\{f(x) + V\}_{x \in X}$ is an open cover of the compact subset $\overline{f(X)}$, so there is a finite subcover $\{f(x_i) + V\}_{i=1}^n$. Take a partition of unity $h_1, \dots, h_n$ given by [Rud87, Thm. 2.13]. Define a map
$$f_V : X \to Y, \quad x \mapsto \sum_{i=1}^n h_i(f(x)) f(x_i).$$

Then f_V is continuous and $f_V(X) \subset \mathrm{co}(f(x_1), \ldots, f(x_n))$, For every $x \in X$ and every $1 \leq i \leq n$ with $h_i(f(x)) > 0$, we have $f(x) \in f(x_i) + V$. By convexity of V, one has

$$f(x) - f_V(x) = \sum_i h_i(f(x))(f(x) - f(x_i)) \in V.$$

□

Lemma 2.7. *Let X be a TVS and $B \subset X$ be a bounded subset, then $\bar{B}$ is bounded. If X is locally convex, then* $\mathrm{co}(B)$ *is also bounded.*

Proof. The first assertion is [Rud91, Theroem 1.13 (f)]. We prove the second. Since X is locally convex, for every open neighborhood U of 0 in X, there is a convex open neighborhood V of 0 in U. By boundedness of B, there is $s > 0$ such that $B \subset tV$ for every $t \geq s$. As tV is convex, we have $\mathrm{co}(B) \subset tV \subset tU$, which shows the boundedness of $\mathrm{co}(B)$. □

3 Review of Brouwer degree theory

We recall the classical Brouwer degree and consider a variation of it.

Let E be a Euclidean space. Let $D \subset E$ be a nonempty bounded open subset. The *Brouwer degree* is a degree function $d(\cdot, D, \cdot, E)$ such that for every continuous map $\phi : \bar{D} \to E$ and every $y \in E \setminus \phi(\partial D)$, the value $d(\phi, D, y, E)$ is an integer, and satisfies the following axioms:

- (d1) if $\phi = \mathrm{Id}_{\bar{D}}$ then $d(\phi, D, y, E) = 1$ for every $y \in D$;
- (d2) $d(\phi, D, y, E) = d(\phi, D_1, y, E) + d(\phi, D_2, y, E)$ whenever D_1, D_2 are disjoint open subsets of D with $y \notin \phi(\bar{D} \setminus (D_1 \cup D_2))$
- (d3) for every continuous map (called a homotopy) $H : [0, 1] \times \bar{D} \to E$ and every continuous map $y : [0, 1] \to E$ with $y(t) \notin H(t, \partial D)$ for all $t \in [0, 1]$, the number $\deg(H(t, \cdot), D, y(t), E)$ is independent of $t \in [0, 1]$.

Moreover, $d(\phi, D, y, E)$ relies only on the affine space underlying E, i.e., it is invariant if we choose another point of E as the new origin. By convention, if $\bar{D}$ is contained in the domain of a map ϕ, then $d(\phi, D, y, E)$ means $d(\phi|_{\bar{D}}, D, y, E)$.

We need a variant of the classical theory. Fix a nonempty bounded closed convex subset $C \subset E$. Let T be a translation of $\mathrm{Aff}(C)$, i.e., there is $x \in E$

with $x + \mathrm{Aff}(C) = T$. (If $x' \in E$ also satisfies $x' + \mathrm{Aff}(C) = T$, then $x - x' \in \mathrm{Span}(C)$.) For a continuous map $\phi : C \to T$, we shall define a degree function $\deg(\phi, C, \cdot) : E \setminus \phi(\mathrm{rb}(C)) \to \mathbb{Z}$.

By [AB06, Lemma 7.33], $\mathrm{ri}(C)$ is a nonempty bounded open subset of $\mathrm{Aff}(C)$ and its closure in $\mathrm{Aff}(C)$ is C. By [AB06, Lemma 5.28], $\partial_{\mathrm{Aff}(C)}(\mathrm{ri}(C)) = \mathrm{rb}(C)$, which allows the following definition.

Definition 3.1. For $y \in E \setminus \phi(\mathrm{rb}(C))$, define

$$\deg(\phi, C, y) = \begin{cases} d(\phi - x, \mathrm{ri}(C), y - x, \mathrm{Aff}(C)) & \text{if } y \in T, \\ 0 & \text{if } y \in E \setminus T. \end{cases}$$

By [FG95, Proposition 2.5], if $y \in T$, then the Brouwer degree $d(\phi - x, D, y - x, \mathrm{Aff}(C))$ is independent of the choice of x, which justifies Definition 3.1. For every $a \in E$, define a map $\phi_a : C - a \to T$, $x \mapsto \phi(x + a)$. By [FG95, Theorem 2.10], the degree $\deg(\phi_a, C - a, y)$ is well-defined and equals $\deg(\phi, C, y)$. In particular, this degree is translation invariant. We omit the ambient space E from the notation, since we may replace E by any other affine subspace containing C and y without changing the degree.

Lemma 3.2. *For a continuous map $f : C \to C$, set $\phi = \mathrm{Id} - f$. If $f(x) \neq x$ for all $x \in \mathrm{rb}(C)$, then $\deg(\phi, C, 0)$ is well-defined and equals 1.*

Proof. By [AB06, Lemma 7.33], $\mathrm{ri}(C)$ contains a point p. The translate $\mathrm{Span}(C) = \mathrm{Aff}(C) - p$ of $\mathrm{Aff}(C)$ contains $\phi(x) = x - f(x)$ for all $x \in C$. By assumption, $0 \notin \phi(\mathrm{rb}(C))$, so $\deg(\phi, C, 0)$ is well-defined. Consider the maps $f' : C - p \to C - p$ defined by $f'(x) = f(x + p) - p$ and $\phi' = \mathrm{Id}_{C-p} - f'$. Then $\phi'(x) = x - f'(x) = x + p - f(x + p) = \phi(x + p) = \phi_p(x)$. Because of $\deg(\phi_p, C - p, 0) = \deg(\phi, C, 0)$ and $0 \in \mathrm{ri}(C - p)$, by translation we can assume $0 \in \mathrm{ri}(C)$. Then by definition $\deg(\phi, C, 0) = d(\phi, \mathrm{ri}(C), 0, \mathrm{Span}(C))$.

Define a map $H : [0, 1] \times C \to E$ by $H(t, x) = t\phi(x) + (1 - t)x$. Then H is continuous and takes value in $\mathrm{Aff}(C)$. One has $H(1, \cdot) = \phi$ and $H(0, \cdot) = \mathrm{Id}_C$.

We claim that $0 \notin H(t, \mathrm{rb}(C))$ for every $t \in [0, 1]$. Otherwise, there exist $t_0 \in [0, 1]$ and $x_0 \in \mathrm{rb}(C)$ with $H(t_0, x_0) = 0$, i.e., $x_0 = t_0 f(x_0)$. By assumption, one has $t_0 < 1$. By [AB06, Lemma 5.28], one has $x_0 \in \mathrm{ri}(C)$, which is a contradiction. The claim is proved.

By (d1) and (d3), one has

$$d(\phi, \mathrm{ri}(C), 0, \mathrm{Span}(C)) = d(\mathrm{Id}_C, \mathrm{ri}(C), 0, \mathrm{Span}(C)) = 1.$$

□

For our presentation of Nagumo's degree, we need a version of the Leray-Schauder reduction theorem.

Lemma 3.3. *Assume* $0 \in C$*. Let* $\{x_1, \ldots, x_n\}$ *be a finite subset of* C *and set* $F = \mathrm{Span}(x_i : 1 \leq i \leq n)$*. For a continuous map* $f : C \to F$*, define* $\phi : C \to E$ *by* $\phi(x) = x - f(x)$*. If* $y \in F \setminus \phi(\mathrm{rb}(C))$*, then* $\deg(\phi, C, y)$ *and* $\deg(\phi, C \cap F, y)$ *are well-defined and coincide.*

Proof. Since $0 \in C$, one has $\mathrm{Span}(C) = \mathrm{Aff}(C)$ which contains $\phi(C)$. From $y \notin \phi(\mathrm{rb}(C))$, the integer $\deg(\phi, C, y) = d(\phi, \mathrm{ri}(C), y, \mathrm{Span}(C))$ is well-defined.

For every $i = 1, \ldots, n$, one has $x_i \in C \cap F$, so $F \subset \mathrm{Span}(C \cap F) = \mathrm{Aff}(C \cap F)$. Conversely, F is an affine space containing $C \cap F$, so $F \supset \mathrm{Aff}(C \cap F)$. Therefore, $F = \mathrm{Aff}(C \cap F)$. Then $\phi(C \cap F) \subset F = \mathrm{Aff}(C \cap F)$. By Lemma 2.3 3, one has $\mathrm{rb}(C \cap F) \subset \mathrm{rb}(C)$, so $y \notin \phi(\mathrm{rb}(C \cap F))$. Then $\deg(\phi, C \cap F, y) = d(\phi, \mathrm{ri}(C \cap F), y, F)$ is well-defined.

By [FG95, Lemma 1.22, p.20], we have

$$d(\phi, \mathrm{ri}(C), y, \mathrm{Span}(C)) = d(\phi, \mathrm{ri}(C) \cap F, y, F).$$

Set $K := F \cap \mathrm{rb}(C)$. It is a bounded closed subset of the Euclidean space F, hence a compact subset. By Lemma 2.4, one has $\mathrm{ri}(C) \cap F \subset \mathrm{ri}(C \cap F)$. Since $\mathrm{ri}(C) \cap \mathrm{rb}(C) = \emptyset$, one has $\mathrm{ri}(C) \cap F \subset \mathrm{ri}(C \cap F) \setminus K$. Since $K = (C \cap F) \setminus (F \cap \mathrm{ri}(C))$, one has $\mathrm{ri}(C \cap F) \setminus K = \mathrm{ri}(C) \cap F$. By $y \notin \phi(K)$, compactness of K and the excision property [FG95, Theorem 2.7(2)], we have

$$d(\phi, \mathrm{ri}(C \cap F), y, F) = d(\phi, \mathrm{ri}(C) \cap F, y, F) = \deg(\phi, C \cap F, y),$$

which completes the proof. $\square$

4 Review of Nagumo degree theory

We rephrase Nagumo's degree theory, taking convex subsets without relative interior into consideration.

Fix a locally convex TVS X. Let A be a nonempty bounded convex closed subset of X. In infinite dimension, Leray's example (see, e.g., [FG95, Sec. 7.1]) shows that there does not exist a well-behaved degree theory on $C^0(A)$. Still, the subclass $K_1(A)$ of compact perturbations of the identity has such a theory. For $\phi \in K_1(A)$ with $\phi(A) \subset \mathrm{Aff}(A)$, we review the Schauder degree $\deg(\phi, A, p)$ defined for all $p \in X \setminus \phi(\partial_X(A))$.

By translation, we may assume $0 \in A$. If $p \notin \mathrm{Span}(A)$, we put $\deg(\phi, A, p) = 0$. Now assume $p \in \mathrm{Span}(A)$. Then there is a finite subset $\{p_1, \dots, p_r\}$ of A with $p \in \mathrm{Span}(p_1, \dots, p_r)$.

Set $f = \mathrm{Id}_A - \phi$. By definition, one has $f \in K(A)$. By Fact 2.5, $\phi(\partial_X(A))$ is closed in X. Since X is locally convex, there is an open convex neighborhood V of 0 in X such that $p - V$ is disjoint from $\phi(\partial_X(A))$.

By Lemma 2.6 and $f(A) \subset \mathrm{Span}(A)$, there is a finite subset $\{y_1, \dots, y_n\}$ of A and a continuous map $f_V : A \to \mathrm{Span}(y_1, \dots, y_n)$ such that $f(x) - f_V(x) \in V$ for all $x \in A$. Define a map

$$\phi_V : A \to X, \quad \phi_V = \mathrm{Id}_A - f_V.$$

Set $E_V := \mathrm{Span}(y_1, \dots, y_n, p_1, \dots, p_r)$, which is a finite dimensional linear subspace of X. Set $A_V := A \cap E_V$, which contains 0. For every finite dimensional affine subspace E of X, the subset$E \cap A$ of E is convex bounded closed.

Lemma 4.1. *The integer* $\deg(\phi_V, A_V, p)$ *is well-defined and independent of the choice of* $\{p_1, \dots, p_r\}$ *and of* V*, whence independent of the choice of* $\{y_1, \dots, y_n\}$ *and* f_V *a fortiori.*

Proof. By construction, one has $\phi_V(A_V) \subset E_V \subset \mathrm{Span}(A_V) = \mathrm{Aff}(A_V)$. Choose another such neighborhood V' and another finite set $\{p'_1, \dots, p'_s\}$ of A with $p \in \mathrm{Span}(p'_1, \dots, p'_s)$. For V', we can similarly choose a finite subset $\{y'_1, \dots, y'_m\}$ of A, a map $f_{V'} : A \to \mathrm{Span}(y'_1, \dots, y'_m)$ and set

$$E_{V'} = \mathrm{Span}(y'_1, \dots, y'_m, p'_1, \dots, p'_s).$$

Replacing V' with $V \cap V'$, we may assume $V \supset V'$.

Set $E = E_V + E_{V'}$, which is a Euclidean subspace of X. Then $C := A \cap E$ a convex bounded closed subset of E containing 0. Since $A_V \subset C$, we have $E_V \subset \mathrm{Span}(C)$. Similarly, we have $E_{V'} \subset \mathrm{Span}(C)$.

Consider a map $H : [0, 1] \times C \to \mathrm{Span}(C)$ defined by

$$H(t, x) = t\phi_V(x) + (1 - t)\phi_{V'}(x).$$

Then H is continuous, $H(0, \cdot) = \phi_{V'}$ and $H(1, \cdot) = \phi_V$. We claim $p \notin H(t, \mathrm{rb}(C))$ for all $t \in [0, 1]$. In particular, $p \notin \phi_V(\mathrm{rb}(C))$.

Otherwise, there exist $t \in [0, 1]$ and $x \in \mathrm{rb}(C)$ with $H(t, x) = p$. Then

$$\begin{aligned} p - \phi(x) =& t(\phi_V(x) - \phi(x)) + (1-t)(\phi_{V'}(x) - \phi(x)) \\ =& t(f(x) - f_V(x)) + (1-t)(f(x) - f_{V'}(x)) \\ \in& tV + (1-t)V' \subset tV + (1-t)V \subset V, \end{aligned}$$

where the last inclusion uses the convexity of V. By Lemma 2.3 1, one has $\mathrm{rb}(C) \subset \partial_X(A)$. Then $\phi(x) \in (p - V) \cap \phi(\partial_X(A))$, which contradicts the choice of V. The claim is proved.

Since $E_V \cap C = A_V$, by Lemma 2.3 3, one has $\mathrm{rb}(A_V) \subset \mathrm{rb}(C)$. From the claim, one has $p \notin \phi_V(\mathrm{rb}(A_V))$.

By Lemma 3.3, both $\deg(\phi_V, C, p) = d(\phi_V, \mathrm{ri}(C), p, \mathrm{Span}(C))$ and $\deg(\phi_V, A_V, p)$ are well-defined and coincide. Similarly, both $\deg(\phi_{V'}, C, p) = d(\phi_{V'}, \mathrm{ri}(C), p, \mathrm{Span}(C))$ and $\deg(\phi_{V'}, A_{V'}, p)$ are well-defined and coincide. From the claim and (d3), we have

$$d(\phi_V, \mathrm{ri}(C), p, \mathrm{Span}(C)) = d(\phi_{V'}, \mathrm{ri}(C), p, \mathrm{Span}(C)),$$

which implies $\deg(\phi_V, A_V, p) = \deg(\phi_{V'}, A_{V'}, p)$. □

Definition 4.2. For every $p \in \mathrm{Span}(A) \setminus \phi(\partial_X(A))$, the common integer in Lemma 4.1 is called the *Schauder degree* of ϕ at p with respect to A, and is denoted by $\deg(\phi, A, p)$.

When X is of finite dimension, we can take $f_V = f$ and $E_V = \mathrm{Span}(A)$, then Definition 4.2 agrees with Definition 3.1.

We show that Definition 4.2 is compatible with the classical counterpart.

Lemma 4.3. *Assume further that X is a linear normed space. Let $D \subset X$ be a nonempty bounded open* convex *subset and $A = \bar{D}$. For every $p \in X \setminus \phi(\partial_X(D))$, let $d(\phi, D, p)$ be the Leray-Schauder degree of ϕ at p with respect to D in the sense of* [FG95, Definition 7.6]. *Then $X = \mathrm{Aff}(D)$ and $d(\phi, D, p) = \deg(\phi, A, p)$.*

Proof. By translation, we assume $0 \in D$. By Fact 2.5, the subset $\phi(\partial D)$ of X is closed. Let $\rho(p, \phi(\partial_X D))$ be the distance of p to $\phi(\partial_X D)$, which is positive. Let V be the open ball in X, centered at 0 with radius $\rho(p, \phi(\partial_X D))$. Then $p - V$ is disjoint from $\phi(\partial_X D)$. Take f_V and E_V correspondingly. Then $D \cap E_V$ is a nonempty open subset of E_V and hence $\mathrm{Aff}(D \cap E_V) = E_V$. Set $K := (\bar{D} \cap E_V) \setminus (D \cap E_V)$. Then $K = (\partial_X D) \cap E_V$ is compact.

We claim that $p \notin \phi_V(K)$. Otherwise, there is $x \in K$ with $p = \phi_V(x)$. Then $\phi(x) = p + f_V(x) - f(x) \in (p - V) \cap \phi(\partial_X D)$, which is a contradiction.

From the claim and the excision property [FG95, Theorem 2.7 (2)], we have

$$d(\phi_V, \mathrm{ri}(A_V), p, E_V) = d(\phi_V, D \cap E_V, p, E_V).$$

By definition, the left (resp. right) hand side is $\deg(\phi, A, p)$ (resp. $d(\phi, D, p)$). $\square$

In the same way, we can show that Definition 4.2 agrees with Nagumo's degree [Nag51, (0), p.502].

Theorem 4.4. *For every $p \in X \setminus \phi(A)$, one has $\deg(\phi, A, p) = 0$.*

Proof. Since A is closed in X, we have $\partial_X(A) \subset A$. Then $p \notin \phi(\partial_X(A))$, so $\deg(\phi, A, p)$ is well defined. Assume to the contrary that it is nonzero. Then $p \in \mathrm{Span}(A)$. By construction, for every convex open neighborhood V of 0 in X, we have

$$d(\phi_V, \mathrm{ri}(A_V), p, \mathrm{Span}(A_V)) = \deg(\phi_V, A_V, p) = \deg(\phi, A, p) \neq 0.$$

By [FG95, Theorem 2.1, p.30], there is $x_V \in A_V$ with $\phi_V(x_V) = p$. Then $\phi(x_V) = p + f_V(x_V) - f(x_V) \in (p - V) \cap \phi(A)$. In particular, the open neighborhood $p - V$ of p in X intersects $\phi(A)$. Hence, one has $p \in \overline{\phi(A)}$. By Fact 2.5, one has $p \in \phi(A)$, which is a contradiction. $\square$

5 Degree approach to Singbal's fixed point theorem

We give a proof of Singbal's fixed point theorem using only degree theory. We intentionally avoid using Brouwer's fixed point theorem, on which Singbal's original proof relies.

Proof of Theorem 1.1. By translation, we may assume $0 \in B$. From Lemma 2.7, by shrinking A to $\overline{\mathrm{co}}(B)$, we may assume that A is bounded. As $\overline{f(A)}$ is a closed subset of the compact space B, it is also compact. Define a map $\phi = \mathrm{Id}_A - f : A \to X$. Then $\phi(A) \subset \mathrm{Span}(A) = \mathrm{Aff}(A)$.

Assume $0 \notin \phi(A)$ to the contrary. By Theorem 4.4, the number $\deg(\phi, A, 0)$ is well-defined and equals 0. Since X is locally convex, there is a convex open neighborhood V of 0 in X with $-V$ disjoint from $\phi(\partial_X(A))$. By Lemma 2.6, there exists a finite subset $\{y_1, \ldots, y_n\}$ of $f(A)$ and a continuous map $f_V : A \to \mathrm{co}(y_1, \ldots, y_n)$ such that $f(a) - f_V(a) \in V$ for all $a \in A$. Set $E_V := \mathrm{Span}(y_1, \ldots, y_n)$, $A_V := A \cap E_V$ and a map $\phi_V : A \to X$ defined as $\mathrm{Id}_A - f_V$. By construction of the Schauder degree in Definition 4.2, one has

$$0 = \deg(\phi, A, 0) = \deg(\mathrm{Id}_{A_V} - f_V, A_V, 0).$$

One has $0 \in A_V$. Since $E_V \subset \mathrm{Aff}(A)$, by Lemma 2.3 2 and 3, for every $x \in \mathrm{rb}(A_V)$, one has $x \in \partial_X(A) \subset A$. By the choice of V, one has $\phi(x) \notin -V$. Then $x - f(x) = \phi(x) \neq f_V(x) - f(x)$, so one obtains $x \neq f_V(x)$. By Lemma 3.2, one has $\deg(\mathrm{Id}_{A_V} - f_V, A_V, 0) = 1$, which is a contradiction. $\square$

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