Monotone comparative statics of parameterized games

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Abstract

We show that for a family of games with general complementarities, the set of selections of Nash equilibria is a nonempty complete lattice, whose largest and least elements are increasing in the parameter. Moreover, the subset of increasing selections forms a subcomplete sublattice. For the proof, we use comparative statics of fixed points for increasing families of correspondences.

1 Introduction

Comparative statics is a way to analyze the impact of a change in the extraneous parameters of a game model, by comparing the Nash equilibria that result from the change with the original equilibria. Lippman, Mamer and McCardle [LMM87, Sec. 3] establish comparative statics results for certain parameterized games. More precisely, they show that there is an increasing selection from the equilibrium set when the parameters are varied. As a hypothesis, they assume that "the analyst has a well-defined, single valued best response function" ([LMM87, p.292]) which is increasing. However, in general the best responses to a given strategy may not be unique. Thus, limitations of their comparative statics analysis are that it requires an artificial selection of the best response correspondence, and only the subset of Nash equilibria thereby generated are studied.

For parameterized supermodular games, there are two particular increasing selections of the best response correspondence, namely the maximum and minimum best replies. By analyzing the two selections, Milgrom and Roberts [MR90, Theorem 6] prove that the largest and the least Nash equilibrium is increasing in the parameter. Milgrom and Shannon [MS94, Theorem 13] establish an analog for parameterized games with strategic complementarities.

We study the behavior of the *whole* set of Nash equilibria under changing parameters. We say that a parameterized game satisfies the *comparative statics property*, if the set of selections of Nash equilibria is a nonempty complete lattice, whose largest and least elements are increasing, and if the subset of increasing selections forms a subcomplete sublattice. For parameterized supermodular games, the following result is purely order theoretical. It generalizes a topological result due to Topkis (Corollary 3.1.3).

Theorem (Theorem 3.1.2)**.** *Given a parameterized supermodular game, assume that*

1. *at every parameter, the set of feasible strategy profiles is a* subcomplete *sublattice in the lattice of all strategy files;*
2. *for every player, the corresponding payoff function is order upper semicontinuous (in the sense of [MR90, p.1261]) in his/her own action, and has increasing differences relative to his/her own action and the parameters.*

Then the game satisfies the comparative statics property.

For parameterized quasisupermodular games, we give two result: one is purely order theoretical, the other is topological.

Theorem (Theorem 3.3.2)**.** *Given a parameterized quasisupermodular game, assume that for every player, his/her payoff function is order upper semicontinuous in his/her own action. Then the game satisfies the comparative statics property.*

Theorem (Theorem 3.3.5)**.** *Given a parameterized quasisupermodular game, assume that for every player,*

1. *his/her action lattice is equipped with a compact topology finer than its interval topology;*
2. *his/her payoff function is upper semicontinuous in his/her own action.*

Then the game satisfies the comparative statics property.

The text is organized as follows. Section 2 is devoted to the study of selections of fixed points. We inspect increasing families of self-correspondences on complete lattices. We consider the largest and the least selections, as well as the increasing selections. These results are applied in Section 3 to investigate parameterized games. Three classes of games are analyzed separately: families of supermodular games in the sense of Topkis [Top98], families of games with strategic complementarities in the sense of Veinott [Vei92], and families of quasisupermodular games in the sense of Milgrom-Shannon [MS94].

2 Fixed points of parameterized correspondences

In Section 2, we study how the set of fixed points changes with the parameters. Villas-Boas [VB97, Theorem 4] shows that every fixed point of of a lower map is below at least one fixed point of a higher map. For correspondences, Echenique and Sabarwal [ES03, Theorem] give a sufficient condition such that every fixed point of a lower correspondence is smaller than every fixed point of a higher correspondence.

For a correspondence $U : A \to 2^B$, let $\mathrm{Gr}(U) := \{(b, a) \in B \times A : a \in U(b)\}$ be its graph. When $A = B$, let $\mathrm{Fix}(U) = \{a \in A : a \in F(a)\}$ be the set of fixed points of U. Let X be a nonempty lattice. Let T be a nonempty poset. Let $G : T \to 2^X$ be a correspondence such that every value is a nonempty *complete* sublattice of X. Let $F : \mathrm{Gr}(G) \to 2^X$ be a correspondence such that for every $(x, t) \in \mathrm{Gr}(G)$, one has $F(x, t) \subset G(t)$. For every $t \in T$, let

$$F_t : G(t) \to 2^{G(t)}, \quad x \mapsto F(x, t)$$

be the induced correspondence. Intuitively, $\{F_t\}_{t\in T}$ is a family of self-correspondences. Set $\Sigma = \Sigma_F := \{f \in X^T | f(t) \in \mathrm{Fix}(F_t)\}$. Let $\mathcal{P} = \mathcal{P}_F \subset \Sigma$ be the subset of all increasing maps.

2.1 Extreme selections

Definition 2.1.1. [Vei92, Ch. 4, Sec. 3] If for any $t < t'$ in T, every $x \in G(t)$ and $t' \in G(x')$, we have $x \wedge x' \in G(t)$ (resp. $x \vee x' \in G(t')$), then G is called lower (resp. upper) V-ascending. If G is both upper and lower V-ascending, then G is called V-ascending.

Theorem 2.1.2. *Suppose that*

1. *the correspondence G is lower V-ascending;*
2. *for every $(x, t) \in \mathrm{Gr}(G)$, the value $F(x, t)$ is nonempty and chain-bounded below;*
3. *for any $(x, t), (x', t') \in \mathrm{Gr}(G)$ with $t \le t'$ and $x < x'$, every $y \in F(x, t)$ and every $y' \in F(x', t')$, there is $y'' \in F(x, t)$ with $y'' \le y \wedge y'$.*

Then:

1. *For every $t \in T$, the correspondence $F_t : G(t) \to 2^{G(t)}$ has a least fixed point $m(t) \in G(t)$.*
2. *The resulting map $m : T \to X$ is an increasing selection of G, i.e., $m \in \mathcal{P}$.*

Proof.

1. For every $t \in T$, by Assumption 3, the correspondence $F_t : G(t) \to 2^{G(t)}$ is lower C-ascending in the sense of [Yu23c, Definition 2.3]. Since $G(t)$ is a nonempty complete lattice, by Assumption 2 and [Yu23c, Lemma 3.6], F_t has a least fixed point.
2. Assume the contrary. Then there exist $t < t'$ in T with $m(t) \not\le m(t')$. Then one has $m(t) \wedge m(t') < m(t)$. Moreover, one has $m(t) \in F(m(t), t)$ and $m(t') \in F(m(t'), t')$. By Assumption 1, one has $m(t) \wedge m(t') \in G(t)$. By Assumption 2, there is $a \in F(m(t) \wedge m(t'), t)$. By Assumption 3, there is $b \in F(m(t) \wedge m(t'), t)$ with $b \le a \wedge m(t')$. Similarly, $F(m(t) \wedge m(t'), t)$

has an element c with $c \leq b \wedge m(t)$. Then $c \leq m(t) \wedge m(t')$. By [Yu23c, Lemma 3.6], we get $m(t) \leq m(t) \wedge m(t') \leq m(t')$, which is a contradiction.

□

With additional assumptions that every value of F is a subcomplete sublattice of X and H is increasing, Corollary 2.1.3 reduces to Topkis's theorem [Top98, Theorem 2.5.2]. If H is single valued, then we recover [LMM87, Theorem D] and [MR94, Theorem 3 (iii)].

Corollary 2.1.3. *Assume that* X *is* complete. *Let* $H : X \times T \to 2^X$ *be an upper (resp. lower) C-ascending correspondence. Assume that for every* $x \in X$ *and every* $t \in T$*, the value* $H(t, x)$ *is nonempty and chain-bounded above (resp. below). Then:*

1. *For every* $t \in T$*, the element* $M(t) := \max \operatorname{Fix}(H_t)$ *(resp.* $m(t) := \min \operatorname{Fix}(H_t)$*) exists, and the map* $M : T \to X$ *(resp.* $m : T \to X$*) is increasing.*

2. *If* $\sup_X H(x, t) < \inf_X H(x, t')$ *for every* $x \in X$ *and any* $t < t'$ *in* T*, then* M *(resp.* m*) is strictly increasing.*

Proof. By symmetry, it suffices to prove the result in parentheses.

1. Define a correspondence $G : T \to 2^X, \quad t \mapsto X$. Because X is complete and H is lower C-ascending, the result follows from Theorem 2.1.2.

2. Assume the contrary. Then there exist $t < t'$ in T such that $m(t) < m(t')$ is not true. By Point 1, one has $m(t) \leq m(t')$, so $m(t) = m(t') \in H(m(t), t) \cap H(m(t'), t')$. Thus, one has

$$m(t) \leq \sup_X H(m(t), t) < \inf_X H(m(t'), t') \leq m(t'),$$

which is a contradiction.

□

2.2 Increasing selections

Now assume that G is increasing, and that $F : \operatorname{Gr}(G) \to 2^X$ is of nonempty values and V-ascending.

Theorem 2.2.1. *Assume that every value of* F *is chain-complete downwards (resp. upwards) and chain-bounded above (resp. below). Then:*

1. *Under the product order of* X^T*, the subset* Σ *is a nonempty complete lattice;*

2. *As maps* $T \to X$*, the largest element* M *and the least element* m *of* Σ *are increasing, i.e.,* $m, M \in \mathcal{P}$.

Proof. By symmetry, one may prove only the statement without parentheses.

1. Because F is V-ascending, for every $t \in T$, the correspondence $F_t : G(t) \to 2^{G(t)}$ is also V-ascending. By completeness of $G(t)$ and [Yu23c, Theorem 5.2], the poset $\mathrm{Fix}(F_t)$ is a nonempty, complete lattice. Therefore, so is $\Sigma = \prod_{t \in T} \mathrm{Fix}(F_t)$.
2. By Point 1, one has $m(t) = \min \mathrm{Fix}(F_t)$ for every $t \in T$. As F is V-ascending of nonempty chain-bounded values, by Theorem 2.1.2 2, the maps $m, M : T \to X$ are increasing.

$\square$

Definition 2.2.2. [Ern09, p.104] A subset S of a poset P is called join-subcomplete (resp. meet-subcomplete) if for every nonempty subset $A \subset S$, the element $\sup_P A$ (resp. $\inf_P A$) exists in S.

Example 2.2.3. Let $P = 2^{\mathbb{R}}$ be the set of subsets of $\mathbb{R}$. Then $(P, \subset)$ is a complete lattice. The largest (resp. least) element of P is $\mathbb{R}$ (resp. $\emptyset$). Let S be the set of open subsets of $\mathbb{R}$. Then S is a join-subcomplete sublattice of P. One has $\min P, \max P \in S$. We prove that S is not meet-subcomplete in P. For every integer $n \geq 1$, one has $(-1/n, 1/n) \in S$. For the nonempty subset $A = \{(-1/n, 1/n)\}_{n \geq 1}$ of S, the element $\inf_P A = \{0\}$ is not in S.

Lemma 2.2.4. *If every value of F is nonempty and chain-bounded, then $\mathcal{P}$ is a sublattice of Σ.*

Proof. For any $f, g \in \mathcal{P}$, we claim that $f \vee_\Sigma g \in \mathcal{P}$, i.e., the map $f \vee_\Sigma g : T \to X$ is increasing. For every $t \in T$, let $h(t) = f(t) \vee_X g(t)$. Since $f(t), g(t) \in \mathrm{Fix}(F_t) \subset G(t)$ and G is increasing, one has $h(t) \in G(t)$. Because f, g are increasing, so is h. Define a correspondence

$$K : T \to 2^X, \quad t \mapsto \{x \in G(t) : x \geq h(t)\}.$$

Then K is increasing. For every $t \in T$, the subset $K(t) \subset G(t)$ is subcomplete. Define a correspondence

$$H : \mathrm{Gr}(K) \to 2^X, \quad (x, t) \mapsto \{y \in F(x, t) : y \geq h(t)\}.$$

Because F is V-ascending, so is H.

Every value of H is chain-bounded below. Otherwise, there is $(x, t) \in \mathrm{Gr}(K)$ and a nonempty chain $C \subset H(x, t)$ without a lower bound. If $x = f(t)$, then $x = f(t) \in F(f(t), t)$ and hence $x \in H(x, t)$ is a lower bound on C. Thus, one has $x > f(t)$. Similarly, one has $x > g(t)$. Since $F(x, t)$ is chain-bounded below, there is a lower bound $v \in F(x, t)$ on C. Then $v \vee_X f(t) \in F_t(x)$ and $v \vee_X f(t) \vee_X g(t) \in F_t(x)$. Therefore, $v \vee_X h(t) \in H(x, t)$ is a lower bound on C.

For every $t \in T$, one has $f(t) \vee_{\mathrm{Fix}(F_t)} g(t) = \min \mathrm{Fix}(H_t)$. By Theorem 2.1.2, the map $f \vee_\Sigma g$ is increasing. The claim is proved.

Dually, one has $f \wedge_\Sigma g \in \mathcal{P}$. Thus, $\mathcal{P}$ is a sublattice of Σ. $\square$

Theorem 2.2.5 1 generalizes [Vei92, Ch. 4, Theorem 14], which requires a stronger restriction that every value of F is subcomplete in X.

Theorem 2.2.5. *Assume that every value of G is subcomplete in X.*

1. *Suppose that every value of F is chain-complete downwards and chain-bounded above. Then the set $\mathcal{P}$ is a nonempty, join-subcomplete (resp. meet-subcomplete) sublattice of Σ, with $\max \mathcal{P} = \max \Sigma$ and $\min \mathcal{P} = \min \Sigma$. In particular, $\mathcal{P}$ is a complete lattice.*

2. *If every value of F is a chain-complete, then $\mathcal{P}$ is a subcomplete sublattice of Σ.*

Proof. 1. By Lemma 2.2.4, $\mathcal{P} \subset \Sigma$ is a sublattice. By Theorem 2.2.1 2, one has $\max \Sigma, \min \Sigma \in \mathcal{P}$. In particular, $\mathcal{P}$ is nonempty. Consider a nonempty subset U of $\mathcal{P}$. By subcompleteness of the values of G in X, the map

$$b : T \to X, \quad t \mapsto \sup_X \{f(t) : f \in U\}$$

is a selection of G. For every $f \in U$, the map $f : T \to X$ is increasing, so $b : T \to X$ is increasing.

For every $(x,t) \in \mathrm{Gr}(G)$, let $M_t(x)$ be the set of maximal elements of $F_t(x)$. For every $y \in F_t(x)$, by assumption, every nonempty chain in $\{z \in F_t(x) : z \geq y\}$ has an upper bound. By Zorn's lemma, there is $w \in M_t(x)$ with $w \geq y$. For any $x < x'$ in $G(t)$, every $y \in M_t(x)$ and every $y' \in M_t(x')$, as $F : \mathrm{Gr}(G) \to 2^X$ is V-ascending, one has $y \vee y' \in F_t(x')$. Since y' is maximal in $F_t(x')$ and $y' \leq y \vee y'$, one has $y \leq y'$. By [Yu23c, Lemma 3.3] applied to $F_t, M_t : G(t) \to 2^{G(t)}$ and the subset $\{f(t) : f \in U\}$, there is $x_t \in F_t(b(t))$ with $x_t \geq b(t)$.

For every $x \in G(t)$ with $x > b(t)$, since $F(x,t)$ is nonempty, there is $y \in F(x,t)$. As F is upper V-ascending, one has $y \vee x_t \in F(x,t)$ and $y \vee x_t \geq x_t \geq b(t)$. Define a correspondence $K : T \to 2^X, \quad t \mapsto \{x \in G(t) : x \geq b(t)\}$. Then K is increasing and every value of K is subcomplete in X. Define another correspondence $H : \mathrm{Gr}(K) \to 2^X$ by $H(x,t) = \{y \in F(x,t) : y \geq b(t)\}$. Then every value of H is nonempty.

Let $\bar{f} = \sup_\Sigma U$. Since $\bar{f}(t) \in \mathrm{Fix}(F_t) \subset G(t)$ and $\bar{f}(t) \geq f(t)$ for every $f \in U$, one has $\bar{f}(t) \geq b(t)$ and $\bar{f}(t) \in \mathrm{Fix}(H_t) \subset K(t)$. For every $x \in \mathrm{Fix}(H_t)$, one has $x \in H(x,t)$ and $x \in \mathrm{Fix}(F_t)$. Then for every $f \in U$, one has $x \geq b(t) \geq f(t)$. Thus, one has $x \geq \sup_{\mathrm{Fix}(F_t)}\{f(t) : f \in U\} = \bar{f}(t)$. Therefore, $\bar{f}(t) = \min \mathrm{Fix}(H_t)$.

We prove that every value of H is chain-bounded below. Otherwise, there exists $(x,t) \in \mathrm{Gr}(K)$ and a nonempty chain $C \subset H(x,t)$ without lower bound. Since $F(x,t)$ is chain-complete downwards, the element $m := \inf_{F(x,t)} C$ exists. For every $f \in U$, one has $f(t) \in F(f(t),t)$ and $f(t) \leq b(t) \leq x$. If $f(t) = x$, then $b(t)$ is the least element of $H(x,t)$ and hence a lower bound on C, a contradiction. Thus, one has $f(t) < x$. As F is upper V-ascending, one has $f(t) \vee m \in F(x,t)$. For every $c \in C$, one has $f(t) \leq b(t) \leq c$ and $m \leq c$, so $f(t) \vee m \leq c$. Therefore, one gets $f(t) \leq f(t) \vee m \leq m$ for all $f \in U$. It implies $b(t) \leq m$ and hence $m \in H(x,t)$. Then m is a lower bound on C in $H(x,t)$, which is a contradiction.

For any $(x,t),(x',t')$ in $\mathrm{Gr}(K)$ with $x < x'$ and $t \leq t'$, every $y \in H(x,t)$ and every $y' \in H(x',t')$, one has $y \in F(x,t)$ and $y' \in F(x',t')$. As F is lower V-ascending, we have $y \wedge y' \in F(x,t)$. Since $y \geq b(t)$, $y' \geq b(t')$ and b is increasing, we have $y \wedge y' \geq b(t)$, so $y \wedge y' \in H(x,t)$. Because G is increasing, by Theorem 2.1.2, the map $\bar{f} : T \to X$ is increasing, i.e., $\bar{f} \in \mathcal{P}$. Therefore, $\mathcal{P}$ is join-subcomplete in Σ. By [Yu23b, Lemma 2.6], $\mathcal{P}$ is a complete lattice.

2. By 1 and its dual, $\mathcal{P}$ is join-subcomplete and meet-subcomplete in Σ. Thus, $\mathcal{P} \subset \Sigma$ is a subcomplete sublattice.

$\square$

3 Parameterized games

For game models arising from real life, the environment in which the players compete, e.g., costs, technologies, weather, laws, etc., can affect the outcome. Thus, we are led to consider games with parameters, and to study the effects of the parameter on Nash equilibria.

3.1 Parameterized supermodular games

Definition 3.1.1. A parameterized supermodular game $(N, \{X_i\}, G, \{f_i\}, T)$ consists of the following data:

- a nonempty poset T of parameters;
- a nonempty *finite* set of players N;
- for every $i \in N$, a nonempty lattice X_i of the strategies of player i. Let $X = \prod_{i \in N} X_i$;
- an *increasing* correspondence $G : T \to 2^X$ of nonempty values;
- for every $i \in N$, a payoff function $f_i : (\cup_{t \in T} G(t)) \times T \to \mathbb{R}$

such that for every $i \in N$ and every $t \in T$, one has

- for every $x_{-i} \in X_{-i} := \prod_{j \neq i} X_j$, set $S_i^t(x_{-i})$ to be the sublattice $\{x_i \in X_i : (x_i, x_{-i}) \in G(t)\}$ of X_i. Then the function $f_i : (\cdot, x_{-i}, t) : S_i^t(x_{-i}) \to \mathbb{R}$ is supermodular;
- the function $f_i(\cdot, t) : \cup_{t' \in T} G(t') \to \mathbb{R}$ has increasing difference relative to $X_i \times X_{-i}$ in the sense of [Top98, p.42].

Fix such a parameterized supermodular game. Elements of T are interpreted as exogenous factors. For every $t \in T$, the value $G(t)$ should be interpret as the set of feasible joint strategies available at t.

Let S_i^t be the image of $G(t)$ under the projection $X \to X_i$. The quadruple $(N, \{S_i^t\}, G(t), \{f_i(\cdot, t)\}) := \Gamma_t$ is a supermodular game (in the sense of [Yu23b, Definition 5.1]). Let E be the set of maps $h : T \to X$, such that $h(t)$ is a Nash equilibrium of Γ_t (in the sense of [Yu23b, Definition 5.3]) for every $t \in T$. Let $\mathrm{IE} \subset E$ be the subset of increasing maps. The fixed parameterized supermodular game is said to satisfy the comparative statics property, if E is a nonempty complete lattice, if $\max E, \min E \in \mathrm{IE}$, and if IE is a subcomplete sublattice of E.

Theorem 3.1.2. *Assume that*

1. *every value of G is* subcomplete *in X;*
2. *for every $t \in T$ and every $s_{-i} \in S_{-i}^t$, the function $f_i(\cdot, s_{-i}, t) : S_i^t(s_{-i}) \to \mathbb{R}$ is order upper semi-continuous;*
3. *for every $i \in N$ and every $x_{-i} \in X_{-i}$, the function $f_i(x_{-i}, \cdot) : (\cup_{t\in T} S_i^t) \times T \to \mathbb{R}$ has increasing differences.*

Then the game satisfies the comparative statics property.

Proof. For every $(x, t) \in \mathrm{Gr}(G)$, let $S^t(x) = (\prod_{i\in N} S_i^t(x_{-i})) \cap G(t)$. The correspondence

$$\phi : \mathrm{Gr}(G) \to 2^X, \quad (x, t) \mapsto S^t(x)$$

is increasing. Indeed, consider any $(x, t) \le (x', t')$ in $\mathrm{Gr}(G)$, every $y \in S^t(x) \subset G(t)$ and every $y' \in S^{t'}(x') \subset G(t')$. Since G is increasing, we have $y \wedge y' \in G(t)$. For every $i \in N$, we have $(y_i, x_{-i}) \in G(t)$ and $(y_i', x_{-i}') \in G(t')$, then $(y_i \wedge y_i', x_{-i}) \in G(t)$. Therefore, $y \wedge y' \in S^t(x)$. Similarly, $y \vee y' \in S^{t'}(x')$.

Because N is finite, one may define a function

$$g^t(\cdot, x) : S^t(x) \to \mathbb{R}, \quad y \mapsto \sum_{i\in N} f_i(y_i, x_{-i}, t).$$

Define a correspondence $Y : \mathrm{Gr}(G) \to 2^S, \quad (x, t) \mapsto \arg\max_{y\in S^t(x)} g^t(y, x)$. By Assumption 2 and the proof of [Yu23b, Theorem 5.5], for every $(x, t) \in \mathrm{Gr}(G)$, the function $g^t(\cdot, x) : S^t(x) \to \mathbb{R}$ is supermodular, and $Y(x, t)$ is a nonempty complete lattice.

We prove that the function $g : \mathrm{Gr}(\phi) \to \mathbb{R}, \quad (y, x, t) \mapsto g^t(y, x)$ has increasing differences. Consider any $(x, t) \le (x', t')$ in $\mathrm{Gr}(\phi)$, any $y \le y'$ in $\phi(x, t) \cap \phi(x', t')$ and every $i \in N$. One has $(y_i', x_{-i}), (y_i, x_{-i}) \in G(t)$ and $(y_i', x_{-i}'), (y_i, x_{-i}') \in G(t')$. So by Assumption 3, we have

$$f_i(y_i', x_{-i}, t) - f_i(y_i, x_{-i}, t) \le f_i(y_i', x_{-i}, t') - f_i(y_i, x_{-i}, t').$$

By Definition 3.1.1, we have

$$f_i(y_i', x_{-i}, t') - f_i(y_i, x_{-i}, t') \le f_i(y_i', x_{-i}', t') - f_i(y_i, x_{-i}', t').$$

Summing over $i \in N$, we have $g^t(y', x) - g^t(y, x) \leq g^{t'}(y', x') - g^{t'}(y, x')$.

By [Top98, Theorem 2.8.1], the correspondence $Y : \mathrm{Gr}(G) \to 2^S$ is increasing. By [Yu23a, Lemma 3.7], one has $E = \Sigma_Y$ and $\mathrm{IE} = \mathcal{P}_Y$. By Assumption 1 and Theorem 2.2.1, E is a nonempty complete lattice. From Theorem 2.2.5, one has $\max E, \min E \in \mathrm{IE}$, and IE is a subcomplete sublattice of E. □

Corollary 3.1.3 requires topological assumptions. It generalizes Topkis's theorem [Top98, Theorem 4.2.2], which only asserts $\max E, \min E \in \mathrm{IE}$.

Corollary 3.1.3. *Assume the following conditions.*

1. *For every $i \in N$, there is an integer $m_i \geq 0$ such that X_i is a sublattice of $\mathbb{R}^{m_i}$.*
2. *Every value of G is compact in $\mathbb{R}^{\sum_{i=1}^N m_i}$.*
3. *For every $i \in N$, every $t \in T$ and every $s_{-i} \in S^t_{-i}$, the function $f_i(\cdot, s_{-i}, t) : S^t_i(s_{-i}) \to \mathbb{R}$ is upper semicontinuous.*
4. *For every $i \in N$ and every $x_{-i} \in X_{-i}$, the function $f_i(x_{-i}, \cdot, \cdot) : (\cup_{t \in T} S^t_i) \times T \to \mathbb{R}$ has increasing differences.*

Then the game satisfies the comparative statics property.

Proof. For every $t \in T$, from Assumption 2 and [Yu23b, Lemma 5.9 1], the sublattice $G(t) \subset \prod_{i \in N} X_i$ is subcomplete. By Assumption 3 and [Yu23b, Lemma 5.9 2], for every $s_{-i} \in S^t_{-i}$, the function $f_i(\cdot, s_{-i}, t) : S^t_i(s_{-i}) \to \mathbb{R}$ is order upper semicontinuous. The result follows from Theorem 3.1.2. □

Example 3.1.4. Consider the following parameterized supermodular game. Let $N = \{1, 2\}$, and set $T = [0, +\infty)^4$. Let $X_1 = \{(x_{11}, x_{12}) : x_{11} + x_{12} = 1, x_{11} > 0, x_{12} > 0\} \cup \{(0, 0)\} \cup \{(1, 1)\}$. Then X_1 is a complete lattice, which is non-closed subset of $\mathbb{R}^2$. Let $X_2 = \{0, 1\}^2$. Define a correspondence $G : T \to 2^X, \quad t \mapsto X$. Define the payoff function

$$f_1 : X \times T \to \mathbb{R}, \quad (x, t) \mapsto r_{11}x_{11} + r_{12}x_{12} + c_{11}(x) + c_{12}(x),$$

where $x = (x_1, x_2)$, $x_1 = (x_{11}, x_{21}) \in X_1$, $x_2 = (x_{21}, x_{22}) \in X_2$, $t = (r_{11}, r_{12}, r_{21}, r_{22}) \in T$ and

$$c_{11}(x_{11}, x_{21}) = \begin{cases} 0 & x_{11} + x_{21} \leq 1/2, \\ x_{11}(x_{11} + x_{12}) & x_{11} + x_{21} > 1/2, \end{cases}$$

$$c_{12}(x_{12}, x_{22}) = \begin{cases} 0 & x_{12} + x_{22} \leq 1/2, \\ x_{12}(x_{12} + x_{22})) & x_{12} + x_{22} > 1/2. \end{cases}$$

Define the other payoff function

$$f_2 : X \times T \to \mathbb{R}, \quad (x, t) \mapsto r_{21}x_{21} + r_{22}x_{22} + c_{21}(x) + c_{22}(x),$$

where

$$c_{21}(x_{11}, x_{21}) = \begin{cases} 0 & x_{11} + x_{21} \le 1/2, \\ x_{21}(x_{11} + x_{12}) & x_{11} + x_{21} > 1/2, \end{cases}$$

$$c_{22}(x_{12}, x_{22}) = \begin{cases} 0 & x_{12} + x_{22} \le 1/2; \\ x_{22}(x_{12} + x_{22})) & x_{12} + x_{22} > 1/2. \end{cases}$$

Because every chain in X_1 is finite, this model satisfies the conditions of Theorem 3.1.2. Since $c_{11}(\cdot, 0) : [0, 1] \to \mathbb{R}$ is not upper semicontinuous, when $x_2 = (0, 1)$, the function $f_1(\cdot, x_2, 0) : X_1 \to \mathbb{R}, \quad x_1 \mapsto c_{11}(x_{11}, 0) + x_{12}(x_{12} + 1)$ is not upper semicontinuous. In particular, Theorem 3.1.2 is a proper generalization of [Top98, Theorem 4.2.2].

An interpretation is as follows. There are two companies $i = 1, 2$ producing two goods $j = 1, 2$. The number x_{ij} represents the quantity of good j produced by company i. The price of the good j made in company i is r_{ij}. In selling the products indicated by $x_i \in X_i$, company i receives a profit $r_{i1}x_{i1} + r_{i2}x_{i2}$. In addition to companies' respective sales figures, the abundance of good j in the market contributes to common wealth. Let $q_j = x_{1j}+x_{2j}$ be the quantity of good j. To fix ideas, we assume that the thresholds for "abundance" are 1/2. The common wealth corresponding to j is 0 when $q_j \le 1/2$ and q_j^2 when $q_j > 1/2$. When the quantity q_j of good j exceeds the threshold 1/2, its common wealth is shared by the two companies, where company i takes $c_{ij}(x_{1j}, x_{2j}) = x_{ij}q_j$. The flexibility of G can reflex realistic constraints. For example, if the production of good 1 makes use of good 2, which adds a constraint relation $q_1 \le q_2$, then one is led to consider $G(t) \equiv \{x \in X : x_{11} + x_{21} \le x_{12} + x_{22}\}$.

This game model is a close variant of [Top79, Example (e)], which has its roots in [Ros73, Example 2]. A major difference is at the place of common wealth above, Topkis considers the common cost of producing good j, which is allocated to company i as c_{ij} in proportion to its yield.

3.2 Veinott parameterized games

Definition 3.2.1 ([Vei92, Ch. 10, Sec. 2])**.** A Veinott parameterized game consists of the following data:

- a nonempty poset T of parameters;
- a nonempty set of players N;
- for every $i \in N$, a nonempty complete lattice S_i of the strategies of player i. Let $S = \prod_{j \in N} S_j$ and $S_{-i} = \prod_{k \ne i} S_k$;
- for every $i \in N$, a nonempty chain C_i of possible gains of player i;
- for every $i \in N$, a correspondence $F_i : S_{-i} \times T \to 2^{S_i} \setminus \{\emptyset\}$;
- for every $i \in N$, a payoff function $u_i : \mathrm{Gr}(F_i) \to C_i$.

For every $t \in T$, every $i \in N$ and every $s_{-i} \in S_{-i}$, the set $F_i(s_{-i}, t)$ represents the feasible replies of player i to s_{-i} available at t. Fix such a Veinott parameterized game.

Definition 3.2.2. Given $t \in T$, a strategy $s \in S$ is called a Nash equilibrium at t if for every $i \in N$, one has $s_i \in F_i(s_{-i}, t)$ and if for every $s'_i \in F_i(s_{-i}, t)$, one has $u_i(s, t) \geq u_i(s'_i, s_{-i}, t)$. Let E be the set of maps $h : T \to S$, such that for every $t \in T$, $h(t)$ is a Nash equilibrium at t. Let IE $\subset E$ be the subset of increasing maps.

For every $i \in N$, the best reply correspondence of player i is

$$R_i : S_{-i} \times T \to 2^{S_i}, \quad (s_{-i}, t) \mapsto \operatorname*{arg\,max}_{s_i \in F_i(s_{-i}, t)} u_i(s, t).$$

The joint best reply correspondence is

$$R : S \times T \to 2^S, \quad (s, t) \mapsto \prod_{i \in N} R_i(s_{-i}, t).$$

Then for every $t \in T$, the set of Nash equilibria at t coincides with $\mathrm{Fix}(R_t)$.

Definition 3.2.3. Let $f : L \to C$ be a map from a lattice to a chain. If for any $x, y \in L$

- ([LV92, p.4]) either $f(x \wedge y) \vee f(x \vee y) \geq f(x) \vee f(y)$ or $f(x \wedge y) \wedge f(x \vee y) \geq f(x) \wedge f(y)$, then f is called superextremal;
- either $f(x \wedge y) \geq f(x)$ or $f(x \vee y) \geq f(x) \wedge f(y)$, then f is called meet-superextremal;
- either $f(x \wedge y) \geq f(x) \wedge f(y)$ or $f(x \vee y) \geq f(x)$, then f is called join-superextremal.

By the dual of [Yu23d, Proposition 2.8], f is superextremal if and only it is meet-superextremal and join-superextremal.

Theorem 3.2.4 gives a sufficient condition ensuring the existence of increasing Nash equilibria. It reduces to [Vei92, Ch. 10, Theorem 1], when the following additional assumptions are added:

1. for every $i \in N$, the correspondence $F_i : S_{-i} \times T \to 2^{S_i}$ is increasing;
2. for every $i \in N$ and every $(s_{-i}, t) \in S_{-i} \times T$, the set $F_i(s_{-i}, t)$ is a *subcomplete* sublattice of S_i;
3. for every $i \in N$, the chain C_i is complete;
4. for every $i \in N$, every $(s_{-i}, t) \in S_{-i} \times T$ and every $c \in C_i$, the set $\{s_i \in F_i(s_{-i}, t) : u_i(s_i, s_{-i}, t) \geq c\}$ is chain-subcomplete in S_i.

Theorem 3.2.4. *Suppose that for every $i \in N$,*

1. *the correspondence* $F_i : S_{-i} \times T \to 2^{S_i}$ *is V-ascending;*

2. *for every* $(s_{-i}, t) \in S_{-i} \times T$ *and every* $c \in C_i$*, the poset* $\{s_i \in F_i(s_{-i}, t) : u_i(s_i, s_{-i}, t) \geq c\}$ *is chain-complete downwards (resp. upwards) and supported above (resp. below);*

3. *for every nonempty chain* $D \subset S_{-i} \times T$*, the map* $u_i : \mathrm{Gr}(F_i|_D) \to \mathbb{R}$ *is superextremal.*

Then $\mathrm{IE} \neq \emptyset$*.*

Proof. By symmetry, it is enough to prove the statement without parentheses. We claim that for every $i \in N$, the correspondence $R_i : S_{-i} \times T \to 2^{S_i}$ is weakly ascending in the sense of [Kuk13, p.543]. Consider any $(s_{-i}, t) < (s'_{-i}, t')$ in $S_{-i} \times T$, every $s_i \in R_i(s_{-i}, t)$ and every $s'_i \in R_i(s'_{-i}, t')$ with $s_i \wedge s'_i \notin R_i(s_{-i}, t)$. By V-ascendingness of F_i, one has $s_i \wedge s'_i \in F_i(s_{-i}, t)$ and $s_i \vee s'_i \in F_i(s'_{-i}, t')$. Then $u_i(s_i \wedge s'_i, s_{-i}, t) < u_i(s_i, s_{-i}, t)$. Because $D = \{(s_{-i}, t), (s'_{-i}, t')\}$ is a chain in $S_{-i} \times T$, and $(s_i, s_{-i}, t), (s'_i, s'_{-i}, t') \in \mathrm{Gr}(F_i|_D)$, by Condition 3 and Lemma 3.2.6, one has $u_i(s'_i, s'_{-i}, t') \leq u_i(s_i \vee s'_i, s'_{-i}, t')$. The claim is proved.

By Condition 2 and [Yu23d, Corollary 3.4], every value of R_i is nonempty. As per [Kuk13, Theorem 2.2], the correspondence $R_i : S_{-i} \times T \to 2^{S_i}$ has an increasing selection $r_i : S_{-i} \times T \to S_i$. Define a map $r : S \times T \to S$ by $r(s, t) = (r_i(s_{-i}, t))_{i \in N}$. Then r is an increasing selection of the correspondence $R : S \times T \to 2^S$. Because every S_i is a complete lattice, so is S. From Corollary 2.1.3, there is an increasing map $f : T \to S$ such that for every $t \in T$, one has $f(t) \in \mathrm{Fix}(r_t)$. □

Remark 3.2.5. If $C_i = \mathbb{R}$, then Assumption 2 in Theorem 3.2.4 is strictly weaker than that $u_i(\cdot, s_{-i}, t) : F_i(s_{-i}, t) \to \mathbb{R}$ is order upper semicontinuous.

Lemma 3.2.6. *Let* $f : X \to C$ *be a map from a lattice to a chain. Then the following conditions are equivalent.*

1. *The map* f *is superextremal;*

2. *For any* $x, y \in X$*, the condition* $f(x \wedge y) < f(x)$ *implies* $f(x \vee y) \geq f(y)$*.*

Proof. 1. Assume Condition 1. Consider $x, y \in X$ with $f(x \wedge y) < f(x)$ and $f(x \vee y) < f(y)$. One has $f(x \wedge y) \wedge f(x \vee y) \leq f(x \wedge y) < f(x)$ and $f(x \wedge y) \wedge f(x \vee y) \leq f(x \vee y) < f(y)$. Then

$$f(x \wedge y) \wedge f(x \vee y) < f(x) \wedge f(y). \tag{1}$$

One has $f(x) \vee f(y) \geq f(x) > f(x \wedge y)$ and $f(x) \vee f(y) \geq f(y) > f(x \vee y)$. Thus,

$$f(x) \vee f(y) > f(x \wedge y) \vee f(x \vee y). \tag{2}$$

However, (1) and (2) contradict the superextremality of f.

2. Assume Condition 2. Assume the contrary that f is not superextremal. Then there are x, y with $f(x \wedge y) \vee f(x \vee y) < f(x) \vee f(y)$ and $f(x \wedge y) \wedge f(x \vee y) < f(x) \wedge f(y)$.

   If $f(x \wedge y) \leq f(x \vee y)$, then $f(x \wedge y) < f(x)$ and $f(x \wedge y) < f(y)$. By assumption, one has $f(x \vee y) \geq f(y)$ and $f(x \vee y) \geq f(x)$. Thus, one has $f(x \vee y) \geq f(x) \vee f(y) > f(x \wedge y) \vee f(x \vee y) \geq f(x \vee y)$, a contradiction. Therefore, $f(x \wedge y) > f(x \vee y)$.

   Then $f(x \wedge y) < f(x) \vee f(y)$. By Condition 2, one has $f(x \vee y) \geq f(x) \wedge f(y) > f(x \wedge y) \wedge f(x \vee y) = f(x \vee y)$, which is a contradiction.

□

Theorem 3.2.7 gives a condition guaranteeing the increasingness of the largest Nash equilibrium in the parameters.

Theorem 3.2.7. *Assume that for every $i \in N$,*

1. *the correspondence $F_i : S_{-i} \times T \to 2^{S_i}$ is lower Z-ascending: for any $(s_{-i}, t) \leq (s'_{-i}, t')$ in $S_{-i} \times T$, every $s_i \in F_i(s_{-i}, t)$ and every $s'_i \in F_i(s'_{-i}, t')$, one has $s_i \wedge s'_i \in F_i(s_{-i}, t)$;*

2. *for any $(s_{-i}, t) \leq (s'_{-i}, t')$ in $S \times T$, every $s_i \in F_i(s_{-i}, t)$ and every $s'_i \in F_i(s'_{-i}, t')$, the condition $u_i(s_i \wedge s'_i, s_{-i}, t) \leq u_i(s_i, s_{-i}, t)$ implies the existence of $s''_i \in F_i(s'_{-i}, t')$ with $s''_i \geq s_i \vee s'_i$ and $u_i(s''_i, s'_{-i}, t') \geq u_i(s'_i, s'_{-i}, t')$;*

3. *for every $(s_{-i}, t) \in S_{-i} \times T$, the map $u_i(\cdot, s_{-i}, t) : F_i(s_{-i}, t) \to C_i$ is meet-superextremal (resp. join-superextremal);*

4. *for every $(s, t) \in S \times T$, every $c \in C_i$, the subset $\{s_i \in F_i(s, t) : u_i(s_i, s_{-i}) \geq c\}$ is chain-subcomplete downwards (resp. upwards) in $F_i(s, t)$ and chain-bounded above (resp. supported below).*

Then there is an increasing function $f : T \to S$ such that for every $t \in T$, the value $f(t)$ is the largest Nash equilibrium at t.

Proof. By [Yu23d, Corollary 3.4], Assumptions 3 and 4, for every $i \in N$ and every $(s_{-i}, t) \in S_{-i} \times T$, the poset $R_i(s_{-i}, t)$ is *nonempty* and chain-bounded above. Then so is every value of R.

For every $i \in N$, any $(s_{-i}, t) \leq (s'_{-i}, t')$ in $S_{-i} \times T$, every $s_i \in R_i(s_{-i}, t)$ and every $s'_i \in R_i(s'_{-i}, t')$, from Assumption 1, one has $s_i \wedge s'_i \in F_i(s_{-i}, t)$. Then $u_i(s_i \wedge s'_i, s_{-i}, t) \leq u_i(s_i, s_{-i}, t)$. By Assumption 2, there is $s''_i \in F_i(s'_{-i}, t')$ with $s''_i \geq s_i \vee s'_i$ and $u_i(s''_i, s'_{-i}, t') \geq u_i(s'_i, s'_{-i}, t')$. Then $s''_i \in R_i(s'_{-i}, t')$.

From last paragraph, the correspondence $R : S \times T \to 2^S$ is upper C-ascending. Because every S_i is a complete lattice, so is S. The result follows from Corollary 2.1.3 1. □

Definition 3.2.8. Let X be a lattice, and let C, Y be chains. A map $f : X \times Y \to C$ is called almost quasisupermodular, if for any $x, x' \in X$ and any $y \leq y'$ in Y, the condition $f(x \wedge x', y) \leq f(x, y)$ implies $f(x', y') \leq f(x \vee x', y')$, and if the condition $f(x \wedge x', y) < f(x, y)$ implies $f(x', y') < f(x \vee x', y')$.

By definition, every quasisupermodular map $X \times Y \to C$ is almost quasisupermodular.

Lemma 3.2.9. *Fix $i \in N$. Suppose that F_i is increasing, and for every nonempty chain D of $S_{-i} \times T$, the map $u_i : \mathrm{Gr}(F_i|_D) \to C_i$ is almost quasisupermodular. Then the correspondence $R_i : S_{-i} \times T \to 2^{S_i}$ is increasing.*

Proof. Consider any $(s_{-i}, t) \leq (s'_{-i}, t')$ in $S_{-i} \times T$, every $s_i \in R_i(s, t)$ and every $s'_i \in R_i(s', t')$. By assumption, one has $s_i \wedge s'_i \in F_i(s_{-i}, t)$ and $s_i \vee s'_i \in F_i(s'_{-i}, t')$. Then $s_i \wedge s'_i \in R_i(s_{-i}, t)$. (Otherwise, one has $u_i(s_i \wedge s'_i, s_{-i}, t) < u_i(s_i, s_{-i}, t)$. By assumption, one has $u_i(s'_i, s'_{-i}, t') < u_i(s_i \vee s'_i, s'_{-i}, t')$, which is a contradiction.) One has $s_i \vee s'_i \in R_i(s'_{-i}, t')$. (Otherwise, one has $u_i(s'_i, s'_{-i}, t') > u_i(s_i \vee s'_i, s'_{-i}, t')$. By assumption, one has $u_i(s_i \wedge s'_i, s_{-i}, t) > u_i(s_i, s_{-i}, t)$, which is a contradiction.) □

The fixed Veinott parameterized game is said to satisfy the comparative statics property, if E is a nonempty complete lattice, if $\max E, \min E \in \mathrm{IE}$, and if IE is a nonempty subcomplete sublattice of E.

Theorem 3.2.10 generalizes [Vei92, Ch. 10, Theorem 2], which strengthens the almost quasisupermodularity in Assumption 3 to lattice superextremality. This condition is stronger than quasisupermodularity, by [Yu23d, Lemma 2.5].

Theorem 3.2.10. *Assume that for every $i \in N$,*

1. *the correspondence F_i is increasing;*
2. *for every $(s_{-i}, t) \in S_{-i} \times T$, for every $c \in C_i$, the poset $\{s_i \in F_i(s, t) : u_i(s_i, s_{-i}) \geq c\}$ is chain-complete downwards (resp. upwards) and chain-bounded above (resp. below);*
3. *for every nonempty chain D of $S_{-i} \times T$, the map $u_i : \mathrm{Gr}(F_i|_D) \to C_i$ is almost quasisupermodular.*

Then the game satisfies the comparative statics property.

Proof. We prove the statement without parentheses. From Lemma 3.2.9 and Condition 3, the correspondence $R : S \times T \to 2^S$ is increasing. By Condition 2, every value of R is chain-complete downwards in S and chain-bounded above. By Condition 3, for every $i \in N$ and every $(s_{-i}, t) \in S_{-i} \times T$, the map $u_i(\cdot, s_{-i}, t) : F_i(s_{-i}, t) \to C_i$ is quasisupermodular. In virtue of [Yu23d, Corollary 3.4], the correspondence R is of nonempty values. By [Yu23e, Corollary 2.7], every value of R is a complete lattice. Then the result follows from Theorems 2.2.1 and 2.2.5 2. □

3.3 Parameterized quasisupermodular games

Definition 3.3.1. A Veinott parameterized game is called a parameterized quasisupermodular game if for every $i \in N$, the following conditions are satisfied:

- $C_i = \mathbb{R}$ and $F_i \equiv S_i$;

- the function $u_i : S_i \times (S_{-i} \times T) \to \mathbb{R}$ satisfies the single crossing property;
- for every $(s_i, t) \in S_i \times T$, the function $u_i(\cdot, s_i, t) : S_{-i} \to \mathbb{R}$ is quasisupermodular.

Fix a parameterized quasisupermodular game. The assertion $\max E, \min E \in \mathrm{IE}$ in Theorem 3.3.2 reduces to [MR90, Theorem 6], when the quasisupermodularity is strengthened to supermodularity, the single crossing property is strengthened to increasing differences, and the following conditions ([MR90, (A2)]) are added:

- for every $i \in N$ and every $(s_i, t) \in S_i \times T$, the function $u_i : (\cdot, s_i, t) : S_{-i} \to \mathbb{R}$ is order continuous in the sense of [MR90, p.1260];
- for every $i \in N$, the function $u_i : S \times T \to \mathbb{R}$ has a finite upper bound.

Theorem 3.3.2. *Assume that for every $i \in N$ and every $(s_{-i}, t) \in S_{-i} \times T$, the function $u_i(\cdot, s_{-i}, t) : S_i \to \mathbb{R}$ is order upper semicontinuous. Then the game satisfies the comparative statics property.*

Proof. To apply Theorem 3.2.10, we check its conditions. Order upper semicontinuity gives Condition 2. By Lemma 3.3.3, Condition 3 is satisfied. □

Lemma 3.3.3. *Let X be a lattice, and let Y, C be chains. Let $f : X \times Y \to C$ be a map. Assume that f satisfies the single crossing property, and for every $y \in Y$, the function $f(\cdot, y) : X \to C$ is quasisupermodular. Then f is almost quasisupermodular.*

Proof. Consider any $x, x' \in X$ and any $y \le y'$ in Y. Because $f(\cdot, y) : X \to C$ is quasisupermodular, the condition $f(x \wedge x', y) \le f(x, y)$ (resp. $f(x \wedge x', y) < f(x, y)$) implies $f(x', y) \le f(x \vee x', y)$ (resp. $f(x', y) < f(x \vee x', y)$.) Since f satisfies the single crossing property, it implies $f(x', y') \le f(x \vee x', y')$ (resp. $f(x', y') < f(x \vee x', y')$). □

Remark 3.3.4. In Lemma 3.3.3, f may not by quasisupermodular. For example, let $X = \{0, 1, a, b\}$, where a, b are incomparable, $0 = \min X$, $1 = \max X$. Then X is a lattice. Let $Y = \{0, 1\}$. The function $f : X \times Y \to \mathbb{R}$ defined by

$$f(1,0) = f(1,1) = 0, \quad f(a,0) = f(b,0) = 1,$$
$$f(0,0) = f(a,1) = f(b,1) = 2, \quad f(0,1) = 3$$

satisfies the single crossing property. The functions $f(\cdot, 0), f(\cdot, 1) : X \to \mathbb{R}$ are quasisupermodular. However, f is not quasisupermodular, since $f((a,1) \wedge (b,0)) = f(0,0) = 2 = f(a,1)$, but $f(b,0) > f(1,1) = f((a,1) \vee (b,0))$.

By contrast, if assume that $C = \mathbb{R}$, the function f has increasing differences, and if for every $y \in Y$, the function $f(\cdot, y) : X \to \mathbb{R}$ is supermodular, then f is supermodular. Indeed, it suffices to prove that for any $(x, y), (x', y') \in X \times Y$, one has $f(x, y) + f(x', y') \le f(x \wedge x', y \wedge y') + f(x \vee x', y \vee y')$. Since Y is a chain, by symmetry, one may assume that $y \le y'$. Since f has increasing differences, one has $f(x \vee x', y') - f(x', y') \ge f(x \vee x', y) - f(x', y)$. As $f(\cdot, y)$ is supermodular, one has $f(x \vee x', y) - f(x', y) \ge f(x, y) - f(x \wedge x', y)$.

With an additional hypothesis ([MS94, (2)]) that for every $i \in N$ and every $(s_i, t) \in S_i \times T$, the function $u_i(\cdot, s_i, t) : S_{-i} \to \mathbb{R}$ is continuous, the assertion $\max E, \min E \in \mathrm{IE}$ in Theorem 3.3.5 reduces to [MS94, Theorem 13].

Theorem 3.3.5. *Assume that for every $i \in N$,*

1. *the lattice S_i is equipped with a compact topology finer than its interval topology;*[1]
2. *for every $(s_{-i}, t) \in S_{-i} \times T$, the function $u_i(\cdot, s_{-i}, t) : S_i \to \mathbb{R}$ is upper semicontinuous.*

Then the game satisfies the comparative statics property.

Proof. For every $i \in N$, by Lemmas 3.3.3 and 3.2.9, the correspondence R_i is increasing. In particular, every value of R_i is a sublattice of S_i, and R is increasing. For every $(s_{-i}, t) \in S_{-i} \times T$, by compactness of S_i and the upper semicontinuity of $u_i(\cdot, s_{-i}, t)$, the sublattice $R_i(s_{-i}, t) \subset S_i$ is nonempty and compact. By [Yu23a, Theorem 2.11], the sublattice $R_i(s_{-i}, t) \subset S_i$ is subcomplete. Then the result follows from Theorems 2.2.1. and 2.2.5 2. $\square$

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