Nash equilibria in discontinuous and non-quasiconcave games

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Abstract

We give several results on the existence of Nash equilibria for games with possibly discontinuous and non-quasiconcave payoff functions. The central hypotheses of our results are various variants of better-reply security introduced by Reny. We also study qualitative games satisfying multi-player security.

Keywords — Nash equilibrium, fixed point, discontinuous game, qualitative game, better-reply security

1 Introduction

Nash [Jr.50] and subsequently Debreu, Glicksberg and Fan (see, e.g., [DM86, Thm. 1]) establish the existence of equilibria for games with continuous quasiconcave payoff functions. Still, in many classical settings in economics, such as the Hotelling location game, Cournot oligopoly, Bertrand oligopoly and auctions, the well-being of individuals depends on the strategies in a discontinuous way.

A well-known result due to Reny [Ren99, Thm. 3.1] gives a Nash equilibrium existence result for quasiconcave, better-reply secure (possibly discontinuous) games. Roughly, a game is *better-reply secure* if for every nonequilibrium strategy x and every limit u of payoff vectors from strategies approaching x, some player i has a securing strategy that guarantees a payoff better than u_i, even if the other players deviate slightly from x. This hypothesis is sufficiently general and, in numerous

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cases, easy to verify. For instance, Reny [Ren99, Example 5.2] applies this theorem to show that multi-unit pay-your-bid auctions have Nash equilibria.

Bich [Bic09, Def. 3.4] introduces *strong better-reply security*, which is a reinforcement of Reny's better-reply security condition. Then [Bic09, Thm. 3.2] provides the existence of Nash equilibria for strongly better-reply secure games possibly *without* quasiconcavity assumption.

McLennan, Monteiro and Tourky [MMT11, Thm. 3.4] prove that their weakening *multiply security* of Reny's better reply security ensures existence of Nash equilibria in a larger class of games. A novel feature is that they allow players to use multiple securing strategies. By [MMT11, p.1647], the McLennan-Monteiro-Tourky theorem contains the Nishimura-Friedman theorem [NF81, Thm. 1] concerning continuous games without quasiconvexity. Theorem 2.8 is a generalization of the McLennan-Monteiro-Tourky theorem, by allowing less rigid securing strategies. Theorems 2.2 and 2.8 are both generalizations of the Reny's theorem.

Barelli and Meneghel [BM13, Def. 2.1] generalize multiply security to *continuous security.* Multiply security requires that there exist finitely many constant securing deviations. By contrast, continuous security allow semicontinuous securing deviations, which may have an infinite number of choices. For continuously secure games, [BM13, Thm. 2.2] show that there exist Nash equilibria. In Theorem 2.2, we generalize the Barelli-Meneghel theorem to normal form games with infinitely many players.

In another direction, Yu [Yu99, Theorems 3.1, 3.2] studies the existence of Nash equilibria for games with finitely many players and semicontinuous payoff functions. Yang and Song [YS22] propose a systematic approach to Nash equilibrium existence, by reducing games with infinitely many players to the case with finitely many players. Thereby, they [YS22, Thm. 3.1] obtain a version of Yu's theorem for games with arbitrarily many players. Corollary 2.11 is a simultaneous extension of the two versions.

Yang and Song's technique can be applied to a setting more general than normal form games, namely qualitative games. For qualitative games whose preference correspondence are of open preimages, [YS22, Thm. 3.2] shows that Nash equilibria exist. Adopting a method of Borglin and Keiding [BK76, Cor. 3], Toussaint [Tou84, Thm. 2.4] also gives an existence result for qualitative games. Toussaint's result extends the fixed point theorem of Gale and Mas-Colell [GMC75, p.10] to infinite number of correspondences on subsets of infinite dimensional spaces. Theorem 3.4 unifies the two theorems. We also give a variant of the Barelli-Meneghel theorem for qualitative games in Theorem 3.3.

Notation and conventions

For correspondences $F, G : X \to 2^Y$ between two sets X, Y and a subset S of X, by $F \subset_S G$ we mean $F(s) \subset G(s)$ for every $s \in S$. For a family of sets $\{E_i\}_{i \in I}$ and $i_0 \in I$, let $p_{i_0} : \prod_{i \in I} E_i \to E_{i_0}$ be the projection. For a correspondence $H : X \to 2^{\prod_{i \in I} E_i}$, the correspondence $X \to 2^{E_{i_0}}, \quad x \mapsto p_{i_0}(H(x))$ is denoted by H_{i_0}. Let X be a topological space. Let E be a topological vector space (TVS) in the sense of [Rud91, 1.6]. A correspondence $X \to 2^E$ is called *Kakutani*, if it has closed graph and nonempty convex values.

2 Normal form games

We establish existence of Nash equilibria for certain classes of normal form games. We allow the payoff functions to be discontinuous.

Recall that a *normal form game* $G = (X_i, u_i)_{i \in N}$ refers to the following data: N is a nonempty set of players. Every $i \in N$ has a nonempty set X_i of strategies. Let $X = \prod_{i \in N} X_i$ and $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$. Every $i \in N$ has a payoff function $u_i : X \to \mathbb{R}$. A profile $x \in X$ is called a *Nash equilibrium* of G, if for every $i \in N$ and every $y_i \in X_i$, one has $u_i(x) \geq u_i(y_i, x_{-i})$. For every $i \in N$, the corresponding best reply correspondence is defined as

$$R_i : X \to 2^{X_i}, \quad x \mapsto \arg\max_{X_i} u_i(\cdot, x_{-i}).$$

The joint best reply correspondence $\prod_{i \in N} R_i : X \to 2^X$ is denoted by R. By definition, the set of Nash equilibria of G coincides with the set of fixed points of R.

Fix a normal form game $G = (X_i, u_i)_{i \in N}$, and assume that X_i is a compact convex subset of a TVS E_i for every $i \in N$. Let $E = \prod_{i \in N} E_i$.

Theorem 2.1 generalizes [NF81, Thm. 1] to discontinuous games. The Nishimura-Friedman theorem assumes further that N is finite, the E_i are finite dimensional, and the u_i are continuous.

Theorem 2.1. *Assume that R has no maximal element. Assume that for every $x \in X$ which is not a Nash equilibrium, there exists an open neighborhood U of x in X and $p \in E^*$, such that for any $x^1, x^2 \in U$ which are not Nash equilibria, every $y^1 \in R(x^1)$ and every $y^2 \in R(x^2)$, the condition $(p, y^1 - x^1) \cdot (p, y^2 - x^2) > 0$ holds. Then G admits a Nash equilibrium.*

Proof. For every $x \in X$ that is not Nash equilibrium, take p and U as in the assumption. Then either $p(y' - x') > 0$ for all $x' \in U$ and $y' \in R(x')$ or $-p(y' - x') >$

0 for all $x' \in U$ and $y' \in R(x')$. The correspondence $R : X \to 2^X$ satisfies the condition [Ura00, (K1), p.90]. By Urai's fixed point theorem [Ura00, Thm. 1], R admits a fixed point, which is a Nash equilibrium. $\square$

For every $i \in N$, define a correspondence

$$B_i : X \times \mathbb{R} \to 2^{X_i}, \quad (x, a) \mapsto \{y_i \in X_i | u_i(y_i, x_{-i}) \geq a\}.$$

Theorem 2.2 is a generalization of [BM13, Theorem 2.2], which assumes further that N is finite, the E_i are locally convex, and the payoff functions $u_i : X \to \mathbb{R}$ are bounded.

Theorem 2.2. *Assume that each E_i^* separates E_i. Assume that for every $x \in X$ that is not a Nash equilibrium, there exists $\alpha_x \in \mathbb{R}^N$, an open neighborhood U_x of x in X, and a Kakutani correspondence $\varphi_x : U_x \to 2^X$ such that*

1. *$\varphi_{x,i} \subset_{U_x} B_i(\cdot, \alpha_{x,i})$ for all $i \in N$, and*
2. *for every $y \in U_x$, there is $i \in N$ with $y_i \notin \mathrm{co}(B_i(y, \alpha_{x,i}))$.*

Then G has a Nash equilibrium.

Proof. Let C be the set of Nash equilibria of G. We check the conditions of Lemma 2.4 a). The assumption implies Condition (i) for φ_x. For every $x \in X \setminus C$, define a correspondence $B_x : X \to 2^X, \quad y \mapsto \prod_{i \in N} \mathrm{co}(B_i(y, \alpha_{x,i}))$. Then Condition (ii) is verified. Assumption 2 becomes Condition (iii).

We check Condition (iv). For any finitely many points $x(1), \ldots, x(m) \in X \setminus C$ and every $i \in N$, there is $1 \leq j_i \leq m$ such that $\alpha_{x(j),i} \leq \alpha_{x(j_i),i}$ for every $1 \leq j \leq m$. For every $z \in U_{x(j_i)}$, we have $B_{x(j),i}(z) \supset B_i(z, \alpha_{x(j),i}) \supset B_i(z, \alpha_{x(j_i),i}) \supset \varphi_{x(j_i),i}(z)$, where the last inclusion is from Assumption 1.

Now by Lemma 2.4 a), the set C is nonempty. $\square$

Remark 2.3. The authors of [BM13, p.816] emphasize that they do not require the φ_x in the notion of continuously secure game [BM13, Def. 2.1] to be convex valued. However, their proof does not seem to be sufficient for that purpose. In fact, values of the correspondence Φ in the proof [BM13, p.823] may not compact nor closed, as shown by [AB06, Example 5.34]. A remedy for this is assuming convexity of the values of φ_x as in Theorem 2.2, and another without convexity is in Theorem 2.7.

Lemma 2.4. *Let C be a subset of X. If for every $x \in X \setminus C$, there exists an open neighborhood U_x of x in X and two correspondences $F_x, B_x : U_x \to 2^X$ such that:*

(i) the correspondence F_x is Kakutani;

(ii) the values of B_x are convex;

(iii) for every $y \in U_x$, there is $i \in N$ with $y_i \notin B_{x,i}(y)$;

(iv) for every $i \in N$ and any finitely many points $x(1), \dots, x(m) \in X \setminus C$, there is $1 \leq j_i \leq m$ satisfying $F_{x(j_i),i}(z) \subset B_{x(j),i}(z)$ for all $1 \leq j \leq m$ and $z \in \cap_{k=1}^m U_{x(k)}$.

Suppose that

a) either for every $i \in N$, the dual E_i^* separates E_i, or

b) for every $x \in X \setminus C$ and every $i \in N$, there is a finite dimensional vector subspace $E_{x,i}$ of E_i containing $F_{x,i}(U_x)$.

Then C is nonempty.

Proof. Assume the contrary $C = \emptyset$. For every $x \in X$, define a correspondence $B'_x : U_x \to 2^X$ by $B'_x = \prod_{i \in N} B_{x,i}$. One has $B_x \subset_{U_x} B'_x$. By Condition (ii), every value of B'_x is convex. By Condition (iii), the correspondence $B'_x : U_x \to 2^X$ has no fixed points.

By [SW99, Ch. I, 1.3], for every $x \in X$, there is a open neighborhood U'_x of x in X, such that the closure Z_x of U'_x in X is contained in U_x. By Tychonoff's theorem, X is compact. Thus, there exist finitely many points $x(1), \dots, x(n) \in X$ with $X \subset \cup_{j=1}^n U'_{x(j)}$. For every $x \in X$, the index set $\Lambda(x) := \{1 \leq j \leq n | x \in Z_{x(j)}\}$ is nonempty. Denote $\{1 \leq j \leq n | x \in U_{x(j)}\}$ by $\Lambda'(x)$. The sets $V_x := \cap_{j \in \Lambda'(x)} U_{x(j)}$ and $W_x := V_x \cap \cap_{j \notin \Lambda(x)} (X \setminus Z_{x(j)})$ are open neighborhoods of x in X.

For every $x' \in W_x$ and every $j \in \Lambda(x')$, one has $x' \in W_x \cap Z_{x(j)}$, so $j \in \Lambda(x)$. We get

$$\Lambda(x') \subset \Lambda(x), \quad \forall x' \in W_x. \tag{1}$$

By Condition (iv), for every $x \in X$ and every $i \in N$, there is $j_{i,x} \in \Lambda'(x)$ with $F_{x(j_{i,x}),i} \subset_{V_x} B_{x(j),i}$ for every $j \in \Lambda'(x)$. For every $x \in X$, define a correspondence $H_x : V_x \to 2^X, \quad y \mapsto \prod_{i \in N} F_{x(j_{i,x}),i}(y)$. By construction, one has

$$H_x \subset_{V_x} B'_{x(j)}, \quad \forall j \in \Lambda'(x). \tag{2}$$

By Condition (i), the correspondence $H_x : V_x \to 2^X$ is Kakutani.

By compactness of X, the open cover $\{W_x\}_{x \in X}$ of X admits a finite subcover $\{W_{w(k)}\}_{k=1}^s$. By [Rud87, Thm. 2.13], there is a partition of unity $\{h_k : X \to [0,1]\}_{k=1}^s$ subordinate to the cover $\{W_{w(k)}\}_{k=1}^s$. Define a correspondence

$$H : X \to 2^X, \quad x \mapsto \sum_{k=1}^s h_k(x) H_{w(k)}(x).$$

By Lemma 2.6 and [AB06, Theorem 17.32 3], the correspondence H is Kakutani.

For every $x \in X$, every $j \in \Lambda(x)$ and every $1 \leq k \leq s$ with $h_k(x) > 0$, we have $x \in W_{w(k)} \subset V_{w(k)}$. By (1), we have $j \in \Lambda(w(k)) \subset \Lambda'(w(k))$, then $w(k) \in Z_{x(j)} \subset U_{x(j)}$. By (2), we have $H_{w(k)}(x) \subset B'_{x(j)}(x)$. As $B'_{x(j)}(x)$ is convex, one has

$$H(x) \subset B'_{x(j)}(x), \quad \forall x \in X, \forall j \in \Lambda(x). \tag{3}$$

We prove that $H : X \to 2^X$ has a fixed point, of which the proof diverges in the two cases.

a) Under the product topology, E is a TVS. We prove that E^* separates E. Indeed, for every nonzero $v \in E$, there is $i \in N$ with $v_i \neq 0$. By assumption, there is $u_i \in E_i^*$ with $u_i(v_i) \neq 0$. Let $p_i : E \to E_i$ be the projection. Then $u := u_i \circ p_i \in E^*$ satisfies $u(v) \neq 0$. By Lemma 2.5, the correspondence H has a fixed point $x_0 \in X$.

b) For every $i \in N$, there is a finite dimensional vector subspace E'_i of E_i containing $\cup_{j=1}^n E_{x(j),i}$. For every $1 \leq j \leq n$, one has $\emptyset \neq F_{x(j),i}(x(j)) \subset E_{x(j),i} \subset E'_i \cap X_i$, so $E'_i \cap X_i$ is nonempty. By [Rud91, Thm. 1.21 (b)], E'_i is closed in E_i. Set $E' := \prod_{i \in N} E'_i$. Then it is a closed, locally convex vector subspace of E. Therefore, $X \cap E' = \prod_{i \in N} E'_i \cap X_i$ is nonempty, convex and compact.

   For every $x \in X$ and every $1 \leq k \leq s$ with $h_k(x) > 0$, we have $x \in V_{w(k)}$. For every $i \in N$, we have $x \in U_{x(j_{i,w(k)})}$, so $F_{x(j_{i,w(k)}),i}(x) \subset E_{x(j_{i,w(k)}),i} \subset E'_i$. It implies $H_{w(k)}(x) \subset E'$ and hence $H(x) \subset E'$. From Kakutani's fixed point theorem, the restriction $E' \cap X \to 2^{E' \cap X}, \quad x \mapsto H(x)$ of H has a fixed point x_0.

Take $j \in \Lambda(x_0)$. Then by (3), one has $x_0 \in H(x_0) \subset B'_{x(j)}(x_0)$, which contradicts that $B'_{x(j)} : U_{x(j)} \to 2^X$ has no fixed points. □

Lemma 2.5 is a direct corollary of Kakutani's fixed point theorem. Every locally convex TVS V satisfies that V^* separates V, but the converse fails.

Lemma 2.5. *Let V be a TVS such that V^* separates V. Let C be a nonempty compact convex subset of V. Let $F : C \to 2^C$ be a Kakutani correspondence. Then F admits a fixed point.*

Proof. Let τ, τ' be the original topology and the weak topology of V respectively. By [Rud91, p.65], (V, τ') is a locally convex TVS. By [Rud91, (a), p.62], since C is compact in τ, one has $\tau|_C = \tau'|_C$. Kakutani's fixed point theorem applied to $(C, \tau'|_C)$ shows that F has a fixed point. □

Lemma 2.6. *Let C be a compact subset of a TVS V. Let S be a topological space. Let $f : S \to \mathbb{R}$ be a continuous function, and let $F : S \to 2^C$ be a closed graph correspondence. Then the correspondence $S \to 2^V, \quad x \mapsto f(x)F(x)$ has a closed graph.*

Proof. Take a net $\{(x_\alpha, y_\alpha)\}_\alpha$ in $S \times V$ converging to (x, y) with $y_\alpha \in f(x_\alpha)F(x_\alpha)$ for all α. For every α, there is $z_\alpha \in F(x_\alpha)$ with $f(x_\alpha)z_\alpha = y_\alpha$. Since f is continuous, we have $f(x_\alpha) \to f(x)$. Since C is compact, there is a subnet (still denoted by the original one) $z_\alpha \to z$ for some $z \in C$. Because F has a closed graph, we have $z \in F(x)$ and $y = f(x)z \in f(x)F(x)$. □

As an alternative to bypass the problem mentioned in Remark 2.3, we require hypothesis 2 in Theorem 2.7. It is stronger than [BM13, Def. 2.1 (b)], as it involves closed convex hull instead of convex hull.

Theorem 2.7. *Assume that every E_i is locally convex. Assume that for every $x \in X$ that is not a Nash equilibrium, there exists $\alpha_x \in \mathbb{R}^N$, an open neighborhood U_x of x in X and a correspondence $\varphi_x : U_x \to 2^X$ with closed graph and nonempty values, such that*

1. *$\varphi_{x,i} \subset_{U_x} B_i(\cdot, \alpha_{x,i})$ for all $i \in N$, and*
2. *for every $y \in U_x$, there is $i \in N$ with $y_i \notin \overline{\mathrm{co}}(B_i(y, \alpha_{x,i}))$.*

Then G has a Nash equilibrium.

Proof. Let C be the set of Nash equilibria of G. For every $x \in X \setminus C$, define a correspondence $F_x : U_x \to 2^X, \quad y \mapsto \overline{\mathrm{co}}(\varphi_x(y))$. By local convexity of the E_i, compactness of X and [AB06, Thm. 17.35 1], F_x has closed graph and is Kakutani. For every $x \in X \setminus C$, define a correspondence $B_x : X \to 2^X, \quad y \mapsto \prod_{i \in N} \overline{\mathrm{co}}(B_i(y, \alpha_{x,i}))$.

We check Condition (iv) of Lemma 2.4 a). For any finitely many points $x(1), \ldots, x(m) \in X \setminus C$ and every $i \in N$, there is $1 \le j_i \le m$ such that $\alpha_{x(j),i} \le \alpha_{x(j_i),i}$ for all $1 \le j \le m$. For every $z \in U_{x(j_i)}$, we have

$$F_{x(j_i),i}(z) \subset \overline{\mathrm{co}}(\varphi_{x(j_i),i}(z)) \overset{\text{(a)}}{\subset} \overline{\mathrm{co}}(B_i(z, \alpha_{x(j_i),i})) \subset \overline{\mathrm{co}}(B_i(z, \alpha_{x(j),i})) = B_{x(j),i}(z),$$

where (a) uses Assumption 1.

The remaining of the proof is similar to that of Theorem 2.2. □

To state our result of McLennan-Monteiro-Tourky's type, we review some notation from [MMT11, §3]. For every $i \in N$, fix a correspondence $\mathcal{X}_i : X \to 2^{X_i}$.

Let $\mathcal{X} = \prod_i \mathcal{X}_i : X \to 2^X$ and call it a *restriction operator*. For every $i \in N$, define correspondences $B_{\mathcal{X},i}, C_{\mathcal{X},i} : X \times \mathbb{R} \to 2^{X_i}$ by

$$B_{\mathcal{X},i}(x,a) = \{y_i \in \mathcal{X}_i(x) | u_i(y_i, x_{-i}) \geq a\}, \quad C_{\mathcal{X},i}(x,a) = \mathrm{co}(B_{\mathcal{X},i}(x,a)).$$

(If $\mathcal{X} = \mathrm{Id}$, then $B_{\mathcal{X},i} = B_i$.)

Theorem 2.8 generalizes [MMT11, Theorem 3.4], as Remark 2.9 explains. Loosely speaking, we allow more flexible securing strategies.

Theorem 2.8. *Assume that for every $x \in X$ which is not an equilibrium, there is open neighborhood U_x of x in X, a finite dimensional vector subspace E_x of E and $\alpha_x \in \mathbb{R}^N$ such that*

I) there is a Kakutani correspondence $F_x : U_x \to 2^{X \cap E_x}$;

II) $F_{x,i}(z) \subset C_{\mathcal{X},i}(z, \alpha_{x,i})$ for all $z \in U_x$ and $i \in N$;

III) for every $z \in U_x$, there is $i \in N$ with $z_i \notin C_{\mathcal{X},i}(z, \alpha_{x,i})$.

Then G has a Nash equilibrium.

Proof. For every $x \in X$ which is not an equilibrium, define a correspondence $B_x : U_x \to 2^X, \quad z \mapsto \prod_{i \in N} C_{\mathcal{X},i}(z, \alpha_{x,i})$. Then B_x satisfies Condition (ii). The remaining proof is parallel to that of Theorem 2.2, except that we apply Lemma 2.4 b) instead of Lemma 2.4 a). □

Remark 2.9.

- Instead of Assumptions I) and II) in Theorem 2.8, the original assumption in [MMT11, Definitions 3.1-3.3] is as follows.

  4. there is a finite closed cover $U_x = \cup_{j=1}^{J(x)} C_j$, such that for every $1 \leq j \leq J(x)$, there is $y(x,j) \in X$ with $y(x,j)_i \in B_{\mathcal{X},i}(z, \alpha_{x,i})$ for all $z \in C_j$ and $i \in N$.

  Given Assumption 4, we define a correspondence

  $$F_x : U_x \to 2^X, \quad z \mapsto \mathrm{co}(y(x,j) | z \in C_j).$$

  Then $F_x(U_x)$ is contained in a finite dimensional vector subspace $E_x := \mathrm{Span}(y(x,1), \ldots, y(x, J(x)))$ of E. By Lemma 2.10, the correspondence $F_x : U_x \to 2^X$ has closed graph, so it is Kakutani. For every $z \in U_x$ and every $i \in N$, we have $F_{x,i}(z) \subset C_{\mathcal{X},i}(z, \alpha_{x,i})$. Thus, Assumption 4 implies Assumptions I) and II).

- Strictly speaking, the authors do not assume that the E_i are Hausdorff while we need this hypothesis. Nevertheless, [MMT11, p.1648] writes that the generalization to non-Hausdorff TVS "is a mathematical refinement without any known economic applications."

Lemma 2.10. *Let S be a topological space with a finite closed cover $C_1, \dots, C_n$. Let V be a TVS. For every $1 \le j \le n$, take $v_j \in V$. Define a correspondence $F : S \to 2^V, \quad x \mapsto \mathrm{co}(v_j | x \in C_j)$. Then F has closed graph.*

Proof. Take any net $\{(x_\alpha, z_\alpha)\}_{\alpha \in I}$ in the graph of F converging to some (x, z) in $S \times V$. It remains to show $z \in F(x)$.

For every $\alpha \in I$, we can write $z_\alpha = \sum_{i=1}^n a_\alpha^i v_i$, where $\sum_{i=1}^n a_\alpha^i = 1$, every $a_\alpha^i \ge 0$, and $a_\alpha^i = 0$ whenever $x_\alpha \notin C_i$. Since $[0, 1]$ is compact, we may find a subnet (still denoted by the original one) such that $a_\alpha^i \to a^i$ in $[0, 1]$ for all $1 \le i \le n$. Then $\sum_{i=1}^n a^i = 1$ and $z = \sum_{i=1}^n a^i v_i$. If j is an index with $x \notin C_j$, then there is $\alpha_0 \in I$ such that for every $\alpha \ge \alpha_0$ we have $x_\alpha \notin C_j$ and hence $a_\alpha^j = 0$. Taking limit, we get $a^j = 0$. Therefore, $z \in \mathrm{co}(v_i | x \in C_i) = F(x)$. □

Corollary 2.11 generalizes [Yu99, Thm. 3.2] from games with finitely many players to infinite setting. When the functions u_i are furthermore continuous, Corollary 2.11 specializes to [YS22, Thm. 3.1]. A function $f : S \to \mathbb{R}$ defined on a convex subset S of a real vector space is *quasiconcave*, if $f(\lambda x + (1 - \lambda) y) \ge \max\{f(x), f(y)\}$ for all $x, y \in S$ and $\lambda \in [0, 1]$.

Corollary 2.11. *Assume that for every $i \in N$,*

1. *the payoff function $u_i : X \to \mathbb{R}$ is upper semicontinuous;*
2. *For every $x_i \in X_i$, the function $u_i(x_i, \cdot) : X_{-i} \to \mathbb{R}$ is lower semicontinuous;*
3. *For every $x_{-i} \in X_{-i}$, the function $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasiconcave.*

Then there exists a Nash equilibrium.

Proof. We check the conditions of Theorem 2.8. Take the restriction operator $\mathcal{X} = \mathrm{Id}$. For every $x \in X$ that is not equilibrium, there exist $i_x \in N$ and $y_{i_x} \in X_{i_x}$ with $u_{i_x}(y_{i_x}, x_{-i_x}) > u_{i_x}(x)$. Set $\alpha_{x,i_x} = (u_{i_x}(y_{i_x}, x_{-i_x}) + u_{i_x}(x))/2$. By Assumption 2, for every $i \ne i_x$, the minimum

$$\alpha_{x,i} := \min_{z \in X_{-i}} u_i(x_i, z) \tag{4}$$

exists. By Assumption 1, there is an open neighborhood V_x of x in X satisfying

$$u_{i_x}(z) < \alpha_{x,i_x}, \quad \forall z \in V_x. \tag{5}$$

By Assumption 2, there is an open neighborhood W_x of x_{-i_x} in X_{-i_x} satisfying

$$u_{i_x}(y_{i_x}, z_{-i_x}) > \alpha_{x,i_x}, \quad \forall z_{-i_x} \in W_x. \tag{6}$$

Let $U_x := V_x \cap (X_i \times W_x)$, which is an open neighborhood of x in X. Define a correspondence $F_x : U_x \to X$ be of constant value (y_{i_x}, x_{-i_x}). Then Condition I) holds.

From (6), for every $z \in U_x$, one has $F_{x,i_x}(z) = y_{i_x} \in B_{i_x}(z, \alpha_{x,i_x}) \subset C_{\mathcal{X},i}(z, \alpha_{x,i_x})$. By (4), for every $i \in N \setminus \{i_x\}$, one has $F_{x,i}(z) = x_i \in B_i(z, \alpha_{x,i}) \subset C_{\mathcal{X},i}(z, \alpha_{x,i})$. Thus, Condition II) is verified.

By Assumption 3, for every $i \in N$, the values of $B_{\mathcal{X},i}$ are convex, so $B_{\mathcal{X},i} = C_{\mathcal{X},i}$. By (5), we have $z_{i_x} \notin B_{\mathcal{X},i_x}(z, \alpha_{x,i_x})$. Therefore, Condition III) is satisfied. The result follows from Theorem 2.8. $\square$

3 Qualitative games

Qualitative games are natural extensions of normal form games, as Example 3.2 shows.

Definition 3.1. Let N be a nonempty set of players. For every $i \in N$, let X_i be a nonempty set of strategies. Let $X = \prod_{i\in N} X_i$. For every $i \in N$, let $P_i : X \to 2^{X_i}$ be a correspondence. Then $G := (X_i, P_i)_{i\in N}$ is called a *qualitative game* with preference correspondences $\{P_i\}_{i\in N}$. A profile $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for all $i \in N$ is called a *Nash equilibrium* of G.

Example 3.2. Given a normal form game $(X_i, u_i)_{i\in N}$, for every $i \in N$ define a correspondence

$$P_i : X \to 2^{X_i}, \quad x \mapsto \{y_i \in X_i | u_i(y_i, x_{-i}) > u_i(x)\}.$$

Then $(X_i, P_i)_{i\in N}$ is a qualitative game satisfying $x_i \notin P_i(x)$ for all $x \in X$ and $i \in N$. The sets of Nash equilibria for the two games coincide.

Fix a qualitative game $G = (X_i, P_i)_{i\in N}$, and assume that X_i is a compact convex subset of a TVS E_i for every $i \in N$. Set $E := \prod_{i\in N} E_i$.

Theorem 3.3. *Assume that for every $x \in X$ that is not an equilibrium, there exist an open neighborhood U_x of x in X, a Kakutani correspondence $F_x : U_x \to 2^X$ and a player $i_x \in N$, such that*

1. *for every $z \in X$ with $x \in U_z$, one has $F_{z,i_x}(x) \subset \operatorname{co}(P_{i_x}(x))$,*
2. *and $x_{i_x} \notin \operatorname{co}(P_{i_x}(x))$.*

Suppose that

(A) either every E_i^* separates E_i

(B) or for every $x \in X$ that is not an equilibrium, $F_x(U_x)$ is contained in a finite dimensional vector subspace E_x of E.

Then there exists a Nash equilibrium.

Proof. Assume the contrary that there is no Nash equilibrium. By compactness of X, the open cover $\{U_x\}_{x\in X}$ admits a finite subcover $\{U_j\}_{j=1}^n$ with corresponding correspondences $F_j : U_j \to 2^X$. By [Rud87, Thm. 2.13], there is a partition of unity $\{h_j : X \to [0,1]\}_{j=1}^n$ subordinate to the cover $\{U_j\}_{j=1}^n$. Define a correspondence

$$F : X \to 2^X, \quad x \mapsto \sum_{j=1}^n h_j(x)F_j(x).$$

By Lemma 2.6 and [AB06, Theorem 17.32 3], the correspondence F is Kakutani.

For every $x \in X$ and every $1 \le j \le n$ with $h_j(x) > 0$, one has $x \in U_j$. By Assumption 1, one has $F_{j,i_x}(x) \subset \mathrm{co}(P_{i_x}(x))$. By convexity of $\mathrm{co}(P_{i_x}(x))$, one has

$$F_{i_x}(x) = \sum_{j=1}^n h_j(x)F_{j,i_x}(x) \subset \mathrm{co}(P_{i_x}(x)).$$

Together with Assumption 2, it implies that F has no fixed points.

(A) Since E^* separates E, this case contradicts Lemma 2.5.

(B) For every $1 \le j \le n$, there is a finite dimensional vector subspace $E_{x(j)}$ of E containing $F_j(U_j)$. Then $E' := \sum_{j=1}^n E_{x(j)}$ has finite dimension and contains $F(X)$. By Kakutani's fixed point theorem, the restriction $F : E' \cap X \to 2^{E'\cap X}$ has a fixed point, which is a contradiction.

□

A correspondence $F : X \to 2^Y$ from a topological space X to a set Y is called of *open preimages*, if for every $y \in Y$, the subset $\{x \in X | y \in F(x)\}$ of X is open.

Theorem 3.4. *Assume that for every $i \in N$,*

1) the set $\{x \in X | P_i(x) \neq \emptyset\}$ is open in X, and

2) for every $x \in X$ with $P_i(x) \neq \emptyset$, there exists an open neighborhood $U_{i,x}$ of x in X and a correspondence of open preimages $Q_{i,x} : U_{i,x} \to 2^{X_i}$, such that $P_i \subset_{U_{i,x}} Q_{i,x}$ and $x_i' \notin \mathrm{co}(Q_{i,x}(x'))$ for all $x' \in U$.

Then there exists an equilibrium.

Proof. Assume the contrary. For every $x \in X$, let $I(x) = \{i \in N | P_i(x) \neq \emptyset\}$. As x is not a Nash equilibrium, $I(x)$ is nonempty. Define a correspondence

$$F : X \to 2^X, \quad x \mapsto \prod_{i \in I(x)} P_i(x) \prod_{j \notin I(x)} X_j.$$

Then the values of F are nonempty. For every $x \in X$, fix $i_x \in I(x)$ and $y_{i_x} \in P_{i_x}(x)$. Define a map

$$f : X \to X, \quad x \mapsto (y_{i_x}, x_{-i_x}).$$

By Assumption 1), the set $V_x := \{y \in X | P_{i_x}(y) \neq \emptyset\}$ is an open neighborhood of x in X. Take another open neighborhood $U_{i_x,x}$ of x in X and a correspondence $Q_{i_x,x} : U_{i_x,x} \to 2^{X_{i_x}}$ given by Assumption 2). Set $W_x := U_{i_x,x} \cap V_x$, which is an open neighborhood of x in X. Define a correspondence

$$G_x : X \to 2^X, \quad z \mapsto \begin{cases} Q_{i_x,x}(z) \times \prod_{j \neq i_x} X_j & \text{if } z \in W_x, \\ \emptyset & \text{if } z \notin W_x. \end{cases}$$

In the terminology of [YP83, Def. 5.1], G_x is of class $\mathscr{L}$, and $F : X \to 2^X$ is $\mathscr{L}$-majorized since $F \subset_{W_x} G_x$ for all $x \in X$. However, this contradicts the Yannelis-Prabhakar theorem [YP83, Cor. 5.1]. □

Remark 3.5. Theorem 3.4 unifies Toussaint's theorem and Yong-Song's theorem.

- The condition (ii) of [Tou84, Theorem 2.4] assumes that the P_i are KF-majorized in the sense of [Tou84, p.100]. Then for every $i \in N$ and every $x \in X$ with $P_i(x) \neq \emptyset$, there is a KF-majorant $Q_{i,x} : X \to 2^{X_i}$ of P_i at x. By definition, $Q_{i,x}$ is KF, and there is an open neighborhood $U_{i,x}$ of x in X with $P_i \subset_{U_{i,x}} Q_{i,x}$. It implies that $Q_{i,x}$ is of open preimages, the values of $Q_{i,x}$ are convex, and $x'_i \notin Q_{i,x}(x')$ for all $x' \in U$. Therefore, Toussaint's condition (ii) is stronger than Assumption 2). Thus, we recover [Tou84, Theorem 2.4] from Theorem 3.4.

- Assume that for every $i \in N$, the correspondence P_i is of open preimages and for every $x \in X$, the set $P_i(x)$ is convex and $x_i \notin P_i(x)$. Then $\{x \in X | P_i(x) \neq \emptyset\} = \cup_{y_i \in X_i} \{x \in X | y_i \in P_i(x)\}$ is open in X, so Assumption 1) holds. For every $i \in N$ and every $x \in X$, the pair $U_{i,x} = X$ and $Q_{i,x} = P_i$ satisfy Assumption 2). Thus, Theorem 3.4 reduces to [YS22, Theorem 3.2].

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