CS-101 PROJECT

Project Name	Distribution of Primes and Rational Approximations of π		
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INTRODUCTION

As a Mathematics and Computing student, the author inevitably ends up spending a lot of time on sites such as Stack Overflow and Math Stack Exchange. This paper stems from a question on math stack exchange asked by dwymark and answered by Greg Martin [1] . The aim of this paper was not well defined at the point of starting it and the author just started working on the problem, coming up with results on the way. So across this paper we will answer some smaller questions slowly leading up to the last and most elegant of the results the author was able to arrive at.

The author won't go into the details of the question here but he does suggest reading it entirely to get the full idea. The user starts by plotting (p, p) in polar coordinates [2] and notes that they form galactic like spirals with some missing arms. Upon zooming out further the pattern gives away to what looks like emerging rays. The user merely asks the source of these patterns.

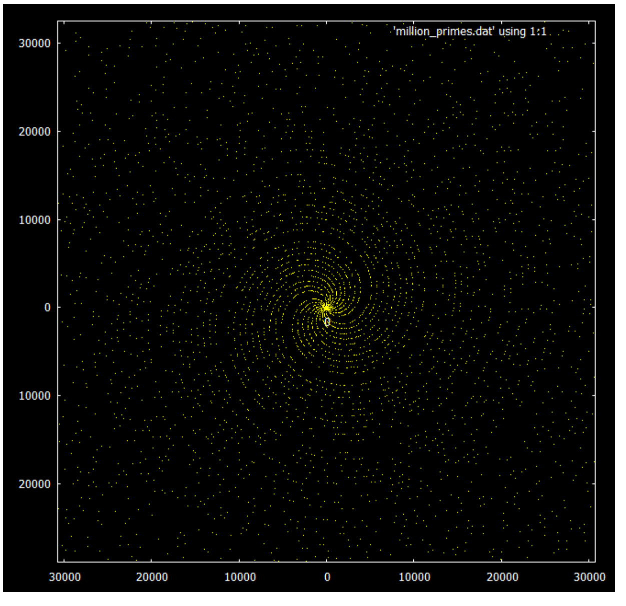

Fig 1: spiral pattern formed by plotting primes < 30'000, having 20 arms.

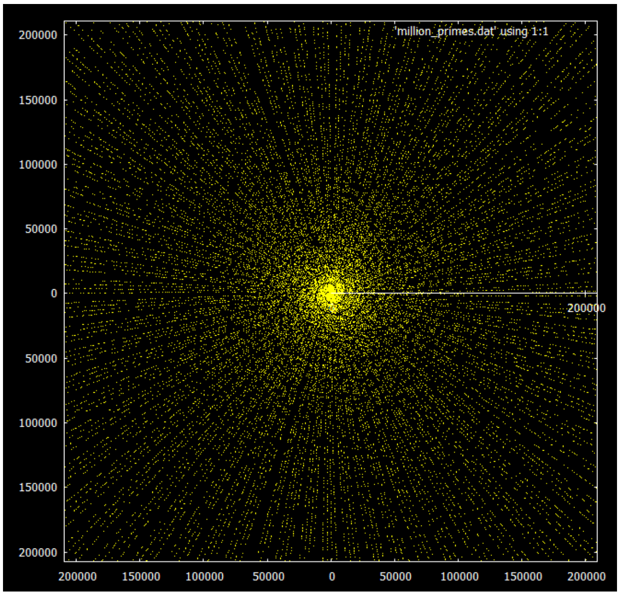

Fig 2: emerging rays pattern formed when plotting primes < 200'000, having 280 arms.

POLAR COORDINATES

Understanding what is being plotted is fundamental to understanding the results. The polar coordinates are represented by 2 numbers, (r, θ). r is the distance of the point from the origin and θ the angle it makes with the horizontal (in radians). One must realize that the points given by (r, θ) and $(r, \theta+2\pi)$ are the same, as an angle shift of 2π refers to a complete rotation about the origin.

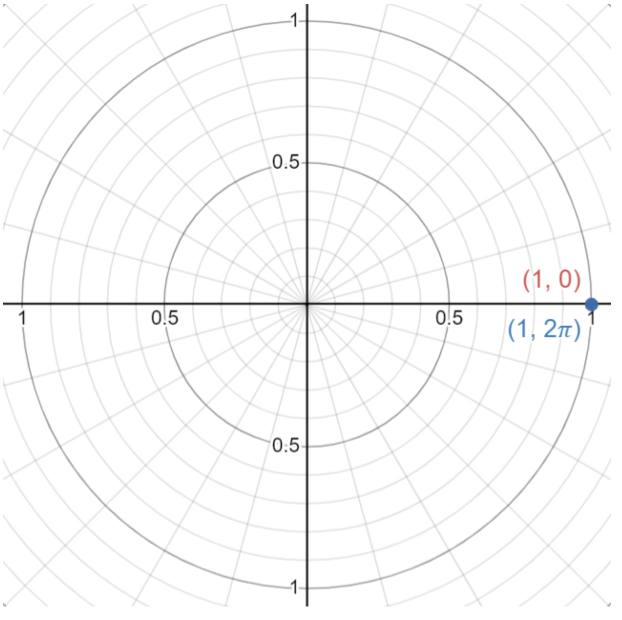

Fig 3: Plot of coordinates (1, 0) and (1, 2π) in polar coordinates.

WHAT ARE WE PLOTTING?

One might be inclined to think that the patterns formed suggest some divine hidden symmetry within the primes. However, the fact that the person asking the question jumped straight into prime numbers makes the puzzle a bit misleading.

If we consider all the whole numbers, instead of just primes and zoom out. We obtain very symmetrical spirals.

But this means that the question of where the spirals come from is completely separate to what happens when we limit our view to primes. However, both of these puzzles are independently very interesting and we will try exploring them to our best efforts.

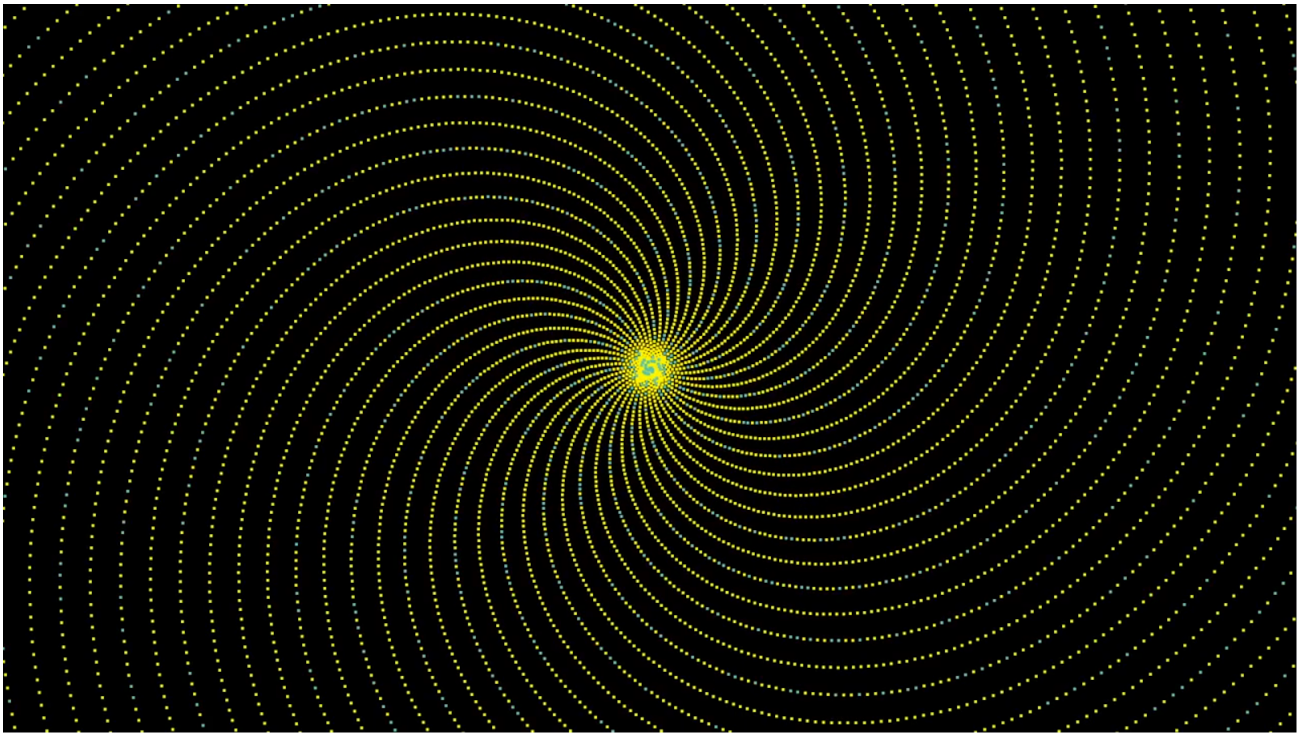

Fig 4: Plot of (x, x) in polar coordinates where x is a whole number. Green dots represent primes. We obtain similar spirals, this time with 44 arms instead of 20.

NON PRIME SPIRALS

To kick things off, let's zoom into the plot. We notice a smaller spiral pattern, this time with only 6 arms. This offers a good starting point to explain what was happening in the larger patterns

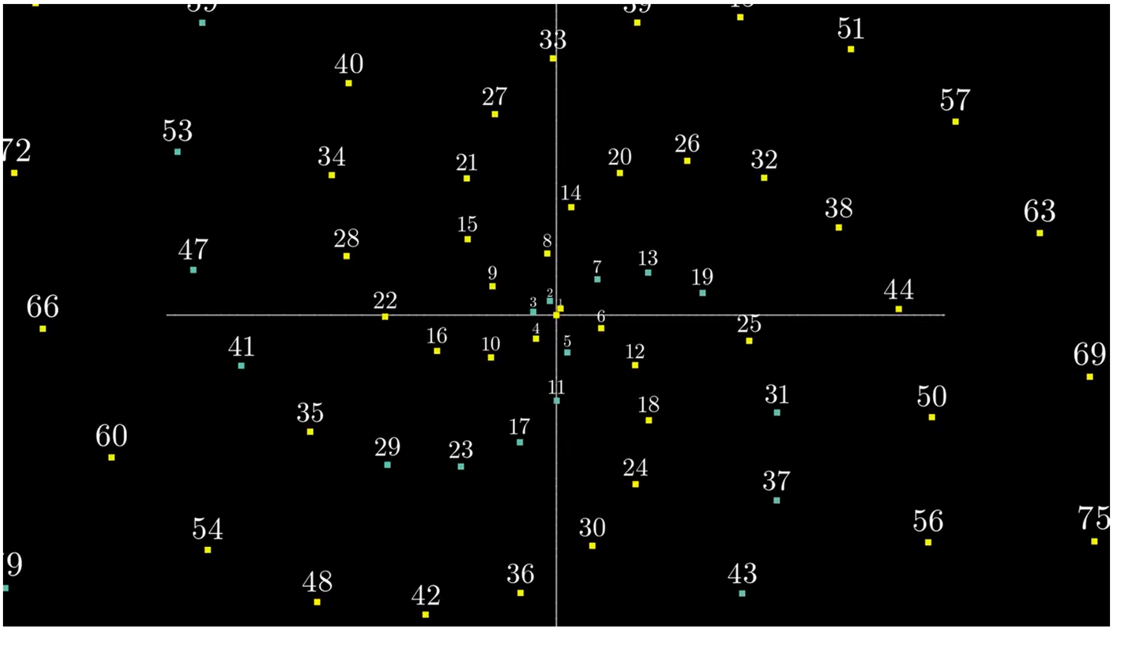

Fig 5: Zoomed in version of Fig 4, depicting a spiral pattern with 6 arms.

Notice how all multiples of 6 form one arm of the spiral. The next one is every integer 1 above a multiple of 6 and so on.

The explanation of this lies in the method of plotting the graph. Remember that each step forward in the sequence involves a turn of 1 radian. So when we count up by 6, we've turned 6 radians from the origin which is just a little shy of 2π radians - a full turn. Hence, counting up by 6 almost makes a full turn. Hence between every consecutive multiple of 6, the angle is small enough to give the illusion of a single curving line.

When we limit our view to the prime numbers, all but 2 of these arms go away.

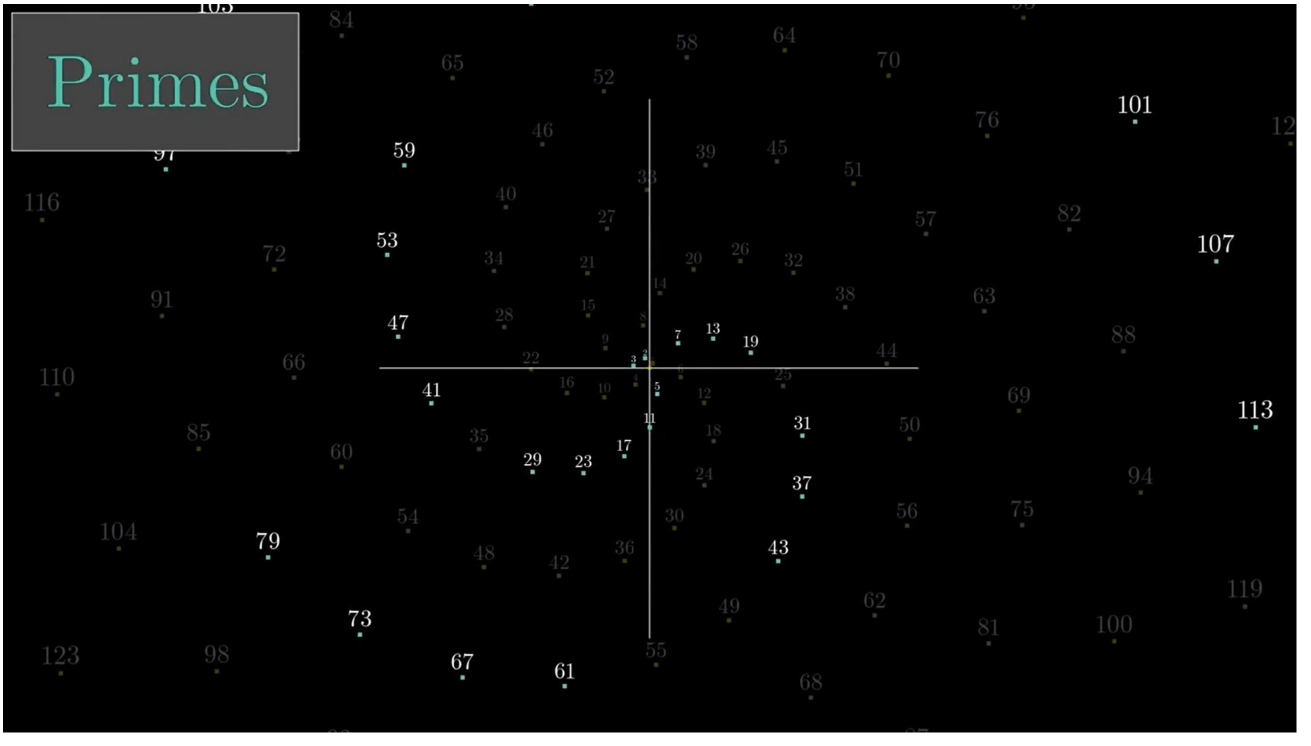

Fig 6: Highlighting prime numbers in Fig 5.

This is because every prime number can be written in the form 6*k*+1 or 6*k*-1 for some integer *k* [3] . Each one of the sequence where we are counting up by 6 is called a *residue class mod 6* [4] . Looking at our diagram, each of our arm corresponds to a residue class mod 6.

LARGER SCALE

In the same way 6 steps is close to a full turn, 44 steps is very close to a full turn.

To prove this we can calculate the number of turns after 44 steps :

number of turns = number of steps / number of steps per turn

number of turns = $44/2\pi$

number of turns = 7.0028...

So taking 44 steps is the same as taking very slightly more than 7 turns. We recognize this pattern with the common approximation $\pi \approx 22/7$.

What this means is that when we count up by multiples of 44 each point has almost the same angle as the previous, just a little but bigger. So as we continue on and on we get this very gentle spiral as the angle increases very slowly.

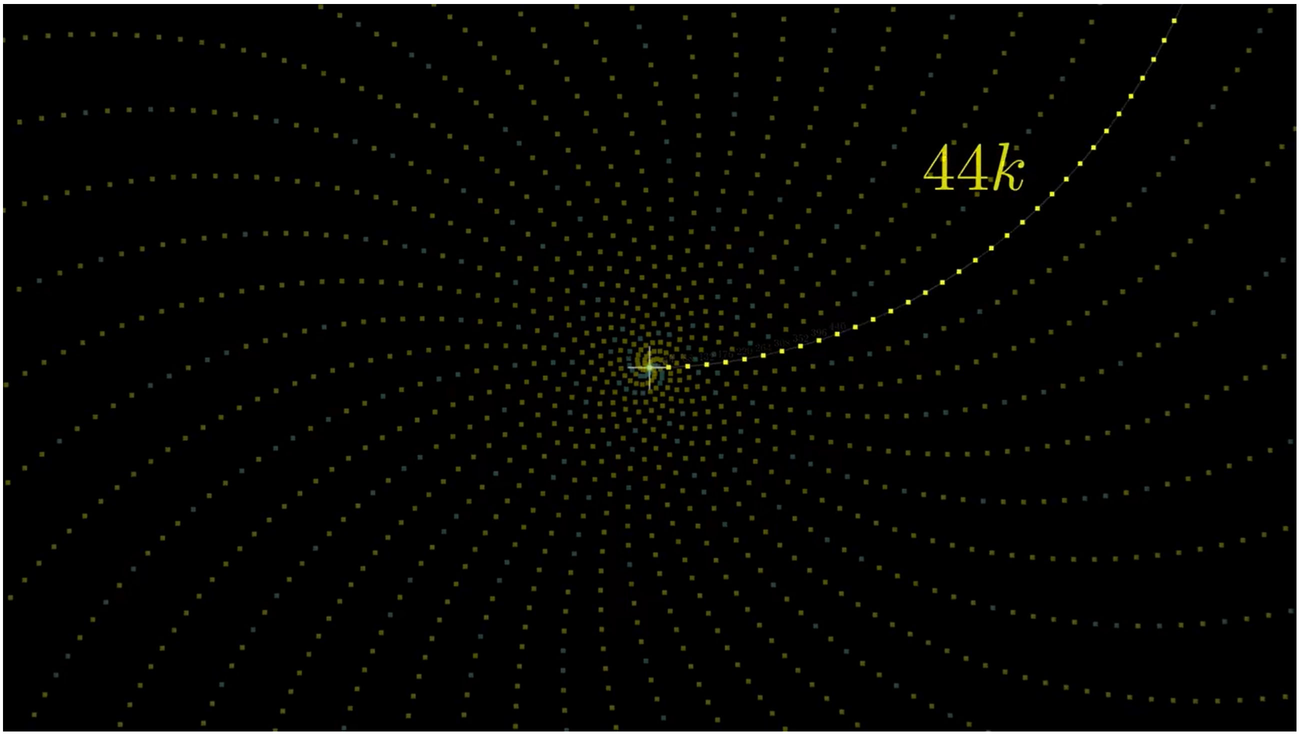

Fig 7: Highlight of one residue class mod 44.

When we limit our view to only primes, each residue class that remains is one which has no common factor with 44. And among all those arms, the prime numbers seem to be randomly distributed.

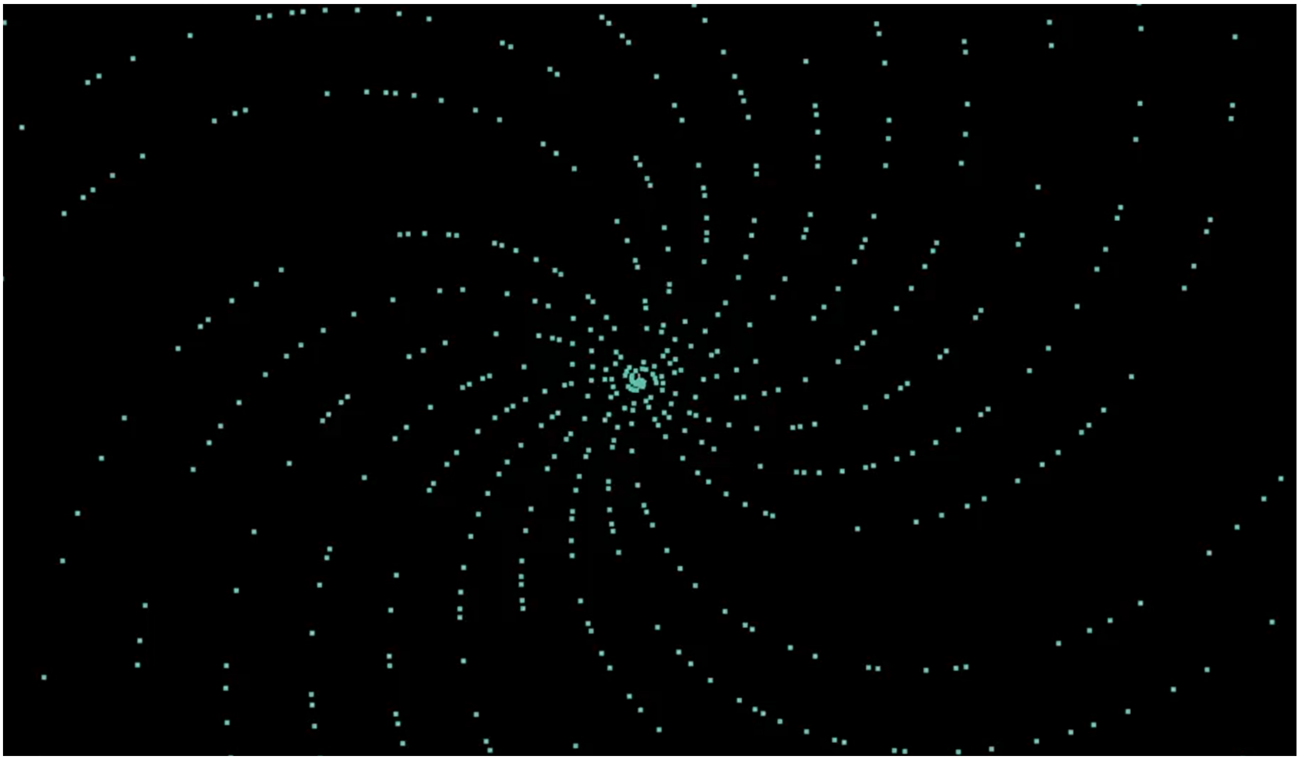

Fig 8: Focusing on the primes in Fig 7.

SOME NOTATION

We are interested in numbers smaller than 44, which are co-prime with 44. We can manually count that there are 20 such numbers. This is a fact that we can write as $\varphi(44) = 20$ [5] . Here φ, as we have learnt in class, represent Euler's totient function. Which is defined to be the number of integers from 1 to n which are co-prime to n.

CONCLUSION

In short, what the user was seeing are two unrelated pieces of number theory but illustrated in one drawing.

The first is that 44/7 is a very close approximation of 2π, which results in the residue classes mod 44 to be cleanly separated out.

The second is that many residue classes contain zero prime numbers or sometimes one so they won't show up. But on the other hand primes do show up plentifully enough in the other residue classes that it makes the spiral arms visible.

At this point it is easy to predict what is going on in the larger scale. Just as 6 radians is close to a full turn and 44 radians is even closer to 7 full turns, it just so happens that 710 radians is extremely close to a whole number of full turns.

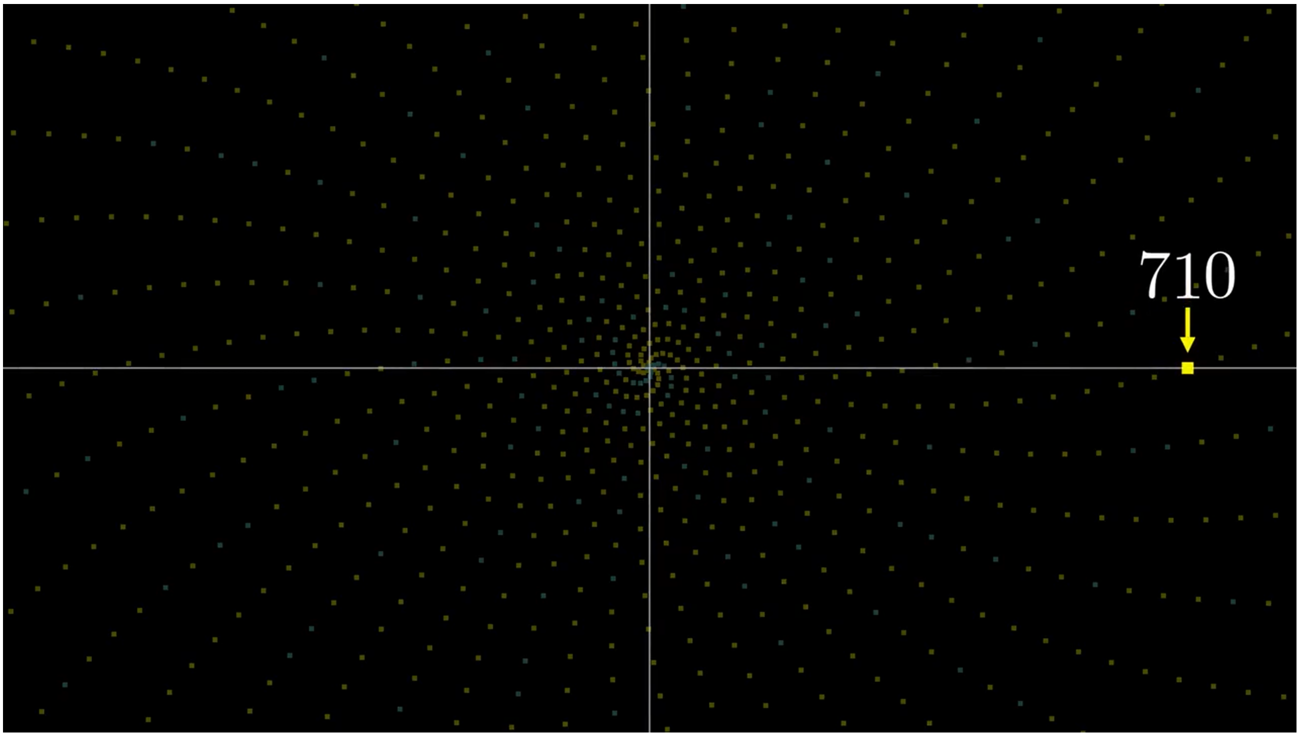

Fig 9: Visual representation showing how close 710 radians is to a whole number of turns.

Visually, we can see how close 710 is to a whole number of turns as it ends up lying on exactly the x axis. Analytically, we can use the same process we did for 44.

number of turns = number of steps / number of steps per turn number of turns = 710/2π

number of turns = 113.00000959...

Some of us have seen this in a different form, saying $\pi \approx 355/113$.

For us, what that means is that even very far out, the angle change is so small that the sequence of a residue class mod 710 looks like a straight line.

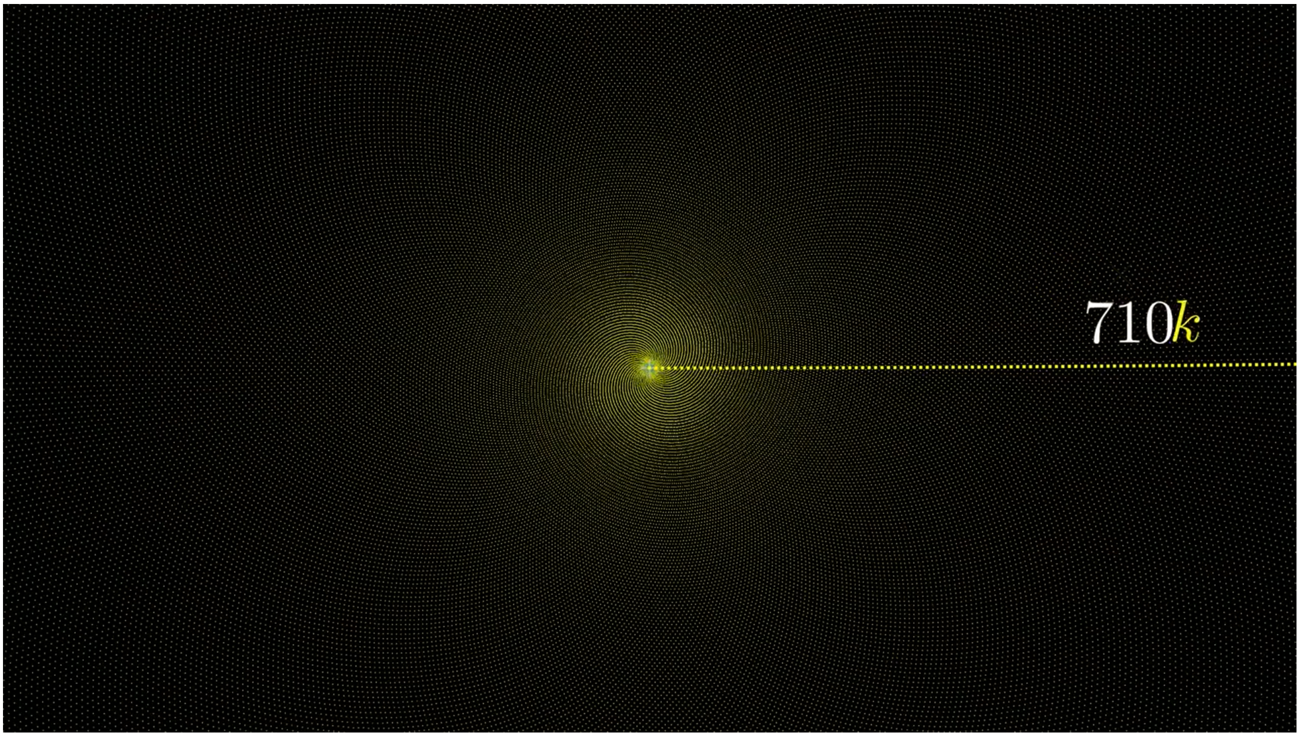

Fig 9: zooming out the graph to note the pattern.

Further, when we limit our view to the primes, the result is analogous to what we have observed before. $\varphi(710) = 280$ - which explains the 280 rays observed by the user in Fig 2. This does not prove that either of the remaining residue classes should contain primes, however we can empirically it seems as though the primes are evenly distributed amongst the remaining residue classes.

This observation of the even distribution of primes is probably one of the most interesting in this whole deal. It relates to a deep fact in number theory known as Dirichlet's theorem [6] . However, it is beyond the scope of this paper.

REFERENCES

- [1] Initial Question asked on Math Stack Exchange.
- [2] Polar Co-ordinates.
- [3] Proof that every prime (>3) is of the form 6*k*+1 or 6*k*-1 for some integer *k*.
- [4] Modular Arithmetic
- [5] Euler's totient function
- [6] Dirichlet's theorem on arithmetic progressions