# Homework 1

## ECE 590

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October 29, 2022

# Question 1: Prove that:

$$\forall P, Q \in \mathbb{B}. P \rightarrow Q \leftrightarrow \neg P \vee Q$$

**Proof is here:**

Definiation: None

Lemmas and Axioms:

1. $A_{LEM}$ $\forall P. \neg P \vee P$
2. $Demorgan's Laws$ :

   2.1 $P \vee Q = \neg (\neg P \wedge \neg Q)$

   2.2 $P \wedge Q = \neg (\neg P \vee \neg Q)$
3. Contrapositive: $P \Rightarrow Q = \neg Q \Rightarrow \neg P$

Proof Goal:

$$\forall P, Q \in \mathbb{B}. P \rightarrow Q \leftrightarrow \neg P \vee Q$$

**Step: 1.** $\forall P, Q \in \mathbb{B}. P \rightarrow Q \leftrightarrow \neg P \vee Q$

Pick any P and Q, and...

**1.1** $\forall P, Q \in \mathbb{B}. (P \rightarrow Q) \rightarrow \neg P \vee Q$

Assume $P \rightarrow Q$ and...

**1.1.1** $\neg P \vee P$

By $U_{spec}$ of $A_{LEM}$.

**1.1.2** $\neg P \vee Q$

By cases on $P \vee \neg P$:

**case 1:** $P$ :

**1.1.2.1** $Q$

By MP on assumption 1.1.

**1.1.2.2** $\neg P \vee Q$

By or-intro on case 1 and 1.2.1.1.

**case 2:** $\neg P$ :

**1.1.2.2** $\neg P \vee Q$

By or-intro on assumption 1.1.

**1.2** $\forall P, Q \in \mathbb{B}. \neg P \vee Q \rightarrow (P \rightarrow Q)$

Assume $(\neg P \vee Q)$ and...

**1.2.1** $P \rightarrow Q$

a. Assume $P$, and...

b. By contradiction, assume $\neg Q$ and...

**1.2.1.1** $P \wedge \neg Q$

By and-intro on assumption 1.2.1.a and 1.2.1.b.

**1.2.1.2** $\neg (\neg P \vee Q)$

By Demorgan's law on 1.2.1.1..

**1.2.1.3** $False\,(\bot)$

Contradiction between 1.2.1.2 and assumption $(\neg P \vee Q)$.

**1.3** $\forall P, Q \in \mathbb{B}. (P \rightarrow Q) \leftrightarrow \neg P \vee Q$

Iff-intro on 1.1 and 1.2.

## Question 2: Prove that:

$$(\forall x \in \mathbb{Z}. f(x) \to g(x)) \leftrightarrow \neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$$

**Proof is here:**

Definiation: None

Lemmas and Axioms:

1. Demorgan's Laws:

   1.1 $P \lor Q = \neg (\neg P \land \neg Q)$

   1.2 $P \land Q = \neg (\neg P \lor \neg Q)$

Proof Goal:

$$(\forall x \in \mathbb{Z}. f(x) \to g(x)) \leftrightarrow \neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$$

**Step: 1.** $(\forall x \in \mathbb{Z}. f(x) \to g(x)) \leftrightarrow \neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$
Pick any $x \in \mathbb{Z}$, and...

**1.1** $(\forall x \in \mathbb{Z}. f(x) \to g(x)) \to \neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$
Assume $(\forall x \in \mathbb{Z}. f(x) \to g(x))$ and...

**1.1.1** $\neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$
By contradition, assume $\exists x \in \mathbb{Z}. \neg (f(x) \to g(x))$

**1.1.1.1** $\neg (f(a) \to g(a))$
By exists-elim on assumption 1.1.1.

**1.1.1.1.1** $(f(a) \to g(a))$
By forall-elim on assumption 1.1.

**1.1.1.1.2** $False\,(\perp)$
Contradiction between 1.1.1.1 and 1.1.1.1.1.

**1.2** $\neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x))) \to (\forall x \in \mathbb{Z}. f(x) \to g(x))$
Assume $\neg (\exists x \in \mathbb{Z}. \neg (f(x) \to g(x)))$ and...
Pick any x, and...

**1.2.1** $(\forall x \in \mathbb{Z}. f(x) \to g(x))$
Assume $f(x)$, and...
By contradicton, assume $\neg g(x)$

**1.2.1.1** $\neg (\exists x \in \mathbb{Z}. \neg (\neg f(x) \lor g(x)))$
Using meaning of implicationon on assumption 1.2.

**1.2.1.2** $\neg(\exists x \in \mathbb{Z}. f(x) \land \neg g(x))$
Using Demorgan's Laws Based on 1.2.1.1

**1.2.1.3** $f(x) \land \neg g(x)$
Using and-intro based on assumption 1.2.1

**1.2.1.4** $\exists x \in \mathbb{Z}. f(x) \land \neg g(x)$
By exists intro on 1.2.1.3 with witness x=x.

**1.2.1.5** $False\,(\bot)$
Contradiction between assumnption 1.2.1.3. and 1.2.1.4.

**1.3** $(\forall x \in \mathbb{Z}. f(x) \to g(x)) \leftrightarrow \neg(\exists x \in \mathbb{Z}. \neg(f(x) \to g(x)))$
Iff-intro on 1.1 and 1.2.

# Question 3: **Prove that:**

$$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i^3 = \frac{1}{4}n^2 \left(n+1\right)^2$$

**Proof is here:**

Definiation: None

Lemmas and Axioms: None

Proof Goal:

$$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i^3 = \frac{1}{4}n^2 \left(n+1\right)^2$$

**Step: 1.** $\forall n \in \mathbb{N}. \sum_{i=0}^{n} i^3 = \frac{1}{4}n^2 \left(n+1\right)^2$

By weak induction on $n$. Cases:

**Base: 0**

Goal: $\sum_{i=0}^{0} i^3 = \frac{1}{4}0^2 \left(0+1\right)^2$

Proof: 0 = 0 (trivial)

**Base: 1.**

Goal: $\sum_{i=0}^{1} i^3 = \frac{1}{4}1^2 \left(1+1\right)^2$

Proof: 1 = 1 (trivial)

**Ind.**

IH: $\sum_{i=0}^{n} i^3 = \frac{1}{4}n^2 \left(n+1\right)^2$

Goal: $\sum_{i=0}^{n+1} i^3 = \frac{1}{4}(n+1)^2 \left((n+1)+1\right)^2$

Proof:

**1.3.1**

$\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + \left(n+1\right)^3$

Definition of summation.

**1.3.2**

$\sum_{i=0}^{n+1} i^3 = \frac{1}{4}n^2 \left(n+1\right)^2 + \left(n+1\right)^3$

Substitution of IH on 1.3.1.

**1.3.3**

$\frac{1}{4}n^2 \left(n+1\right)^2 + \left(n+1\right)^3 = \frac{1}{4}(n+1)^2 \left((n+1)+1\right)^2$

Because $\frac{1}{4}n^2 \left(n+1\right)^2 + \left(n+1\right)^3$

= $\left(n+1\right)^2 \left[\frac{1}{4}n^2 + (n+1)\right]$

= $\frac{1}{4}\left(n+1\right)^2 \left(n^2 + 4n + 4\right)$

$$= \frac{1}{4}(n+1)^2((n+1)+1)^2$$

## Question 4: Prove that:

$$\forall A, B, C \in P(\mathbb{Z}). A \subseteq B \land B \subseteq C \to A \subseteq C$$

**Proof is here:**

Definiation:

1. $\forall A, B \in P(\mathbb{Z}). A \subseteq B$ is equivalent to $\forall x. (x \in A \to x \in B)$

Lemmas and Axioms: None

Proof Goal:

$$\forall A, B, C \in P(\mathbb{Z}). A \subseteq B \land B \subseteq C \to A \subseteq C$$

**Step: 1.** $\forall A, B, C \in P(\mathbb{Z}). A \subseteq B \land B \subseteq C \to A \subseteq C$
Pick any A, B and C, and...

**1.1** $A \subseteq C$
Assume $A \subseteq B \land B \subseteq C$ and...

**1.1.1** $\forall x. (x \in A \to x \in C)$
Pick any $x$ and...

**1.1.1.1** $x \in A \to x \in C$
Using Definition 1 and universal-elim on 1.1.1.
Assume $x \in A$.

**1.1.1.1.1** $A \subseteq B$
Using and-elim on assumption 1.1.

**1.1.1.1.2** $x \in A \to x \in B$
Using Definition 1 and universal-elim on 1.1.1.1.1.

**1.1.1.1.3** $x \in B$
By implication-elim with assumption 1.1.1.1 on 1.1.1.1.2.

**1.1.1.1.4** $B \subseteq C$
Using and-elim on assumption 1.1.

**1.1.1.1.5** $x \in B \to x \in C$
Using Definition 1 and universal-elim on 1.1.1.1.4.

**1.1.1.1.6** $x \in C$
By implication-elim with 1.1.1.1.3 on 1.1.1.1.5.

# Question 5:

Definiation:

1. $D_i$: Dragons on the island.
2. $ged(D_i)$: If dragon $D_i$ is green eyed, $ged(D_i)$ is true, otherwise is false.

**(a)** The oracle's proposition: 'There is at least one dragon on thisisland with green eyes ' means that:

$$\exists D_i.ged\,(D_i)$$

**(b)** This particular dragon($D_0$)'s proof is here:

Axioms:

1. at least one dragon is green-eyed:

$$\exists D_i.ged\,(D_i)$$

2. all dragons except $D_0$ are not green-eyed:

$$\forall D_i.\,(\neg\,(D_i = D_0) \rightarrow \neg ged(D_i))$$

3. $A \vee (B \wedge \neg B) \leftrightarrow A$

Proof Goal:

$$ged(D_0)$$

**Step: 1.** $ged(D_0)$

**1.1** $ged(D_x)$
By exists-elim on assumption axiom 1.

**1.2** $\forall D_i.\,(D_i = D_0 \vee \neg ged(D_i))$
By meaning of implication on axiom 2.

**1.3** $D_x = D_0 \vee \neg ged(D_x)$
By universal-elim on 1.2.

**1.4** $D_x = D_0 \vee \neg ged(D_x) \wedge ged(D_x)$
By and-intro on 1.1 and 1.3

**1.5** $D_x = D_0$
Using axiom 3 on 1.4.

**1.6** $ged(D_0)$
By MP on 1.1 and 1.5.

**(c)** Under the scenario above (only $D_0$ has green eyes), what happens to the red eyed dragons?:

At 1st midnight, only one dragon turn into butterfly, so all red eyed dragons know that that dragon is the only green-eyed one, so they all turn into butterflies at next midnight.

**(d)**

i All green-eyed dragons will turn into butterflies on the Nth day, and all red-eyed dragons will turn into butterflies on the (N+1)thday.

ii $$\forall n \in \mathbb{N}. \left|\{d|ged(d)\}\right| = n \to (\forall D_i. (i < n \to bfly(D_i) = n) \wedge (i \geq n \to bfly(D_i) = n + 1))$$

iii Proof:

Axioms:

1. at least one dragon is green-eyed:
$$\exists D_i.ged\,(D_i)$$
2. all dragons except n dragons (n>1) are not green-eyed:
$$\forall D_i. (\neg\,((D_i = D_0) \vee (D_i = D_1) \vee .... \vee (D_i = D_{n-1})) \to \neg ged(D_i))$$
and it is equivalent to
$$\forall D_i. ((D_i = D_n) \vee (D_i = D_{n+1}) \vee .... \vee (D_i = D_{99}) \to \neg ged(D_i))$$
3. $\neg\,(A \vee B) \to C \leftrightarrow (\neg A \to C) \vee (\neg B \to C)$
4. If there are n green-eyed dragon, then there won't be any dragon turning into butterfly before nth midnight:
$\forall n \in \mathbb{N}. \left|\{d|ged(d)\}\right| = n \to \forall D_i.\neg\,(bfly(D_i) = n - 1)$

**Step: 1.** $\forall n \in \mathbb{N}. |\{d|ged(d)\}| = n \rightarrow$
$(\forall D_i. (i < n \rightarrow bfly(D_i) = n) \wedge (i \geq n \rightarrow bfly(D_i) = n + 1))$
Assume $\forall n \in \mathbb{N}. |\{d|ged(d)\}| = n$.
By weak induction on $n$. Cases:

**Base: 2**

Goal: $(\forall D_i. (i < 2 \rightarrow bfly(D_i) = 2) \wedge (i \geq 2 \rightarrow bfly(D_i) = 3))$
Proof:

**1.2.1** $\forall D_i. (i < 2 \rightarrow bfly(D_i) = 2)$
Assume $i < 2$, and...

**1.2.1.1** $\forall D_i. (\neg ((D_i = D_0) \vee (D_i = D_1)) \rightarrow \neg ged(D_i))$
By axiom 2, and...

**1.2.1.2** $\forall D_i. ((D_i = D_0) \vee (D_i = D_1) \vee \neg ged(D_i))$
By meaning of implication on 1.2.1.1.
Assume $\forall D_i. (D_i = D_0) \rightarrow ged(D_i)$.

**1.2.1.3** $\neg (bfly(D_0) = 1)$
By axiom 4.

**1.2.1.4** $ged(D_1)$
Based on 1.2.1.3

**1.2.1.5** $bfly(D_0) = 2 \wedge bfly(D_1) = 2$
Based on assumption 1.2.1.2 and 1.2.1.4.

**1.2.2** $\forall D_i. (i \geq 2 \rightarrow bfly(D_i) = 3)$
Assume $i \geq 2$, and...

**1.2.2.1** $\forall D_i. ((D_i = D_n) \vee (D_i = D_{n+1}) \vee .... \vee (D_i = D_{99}) \rightarrow \neg ged(D_i))$
By axiom 2, and...

**1.2.2.2** $\forall D_i. ((D_i = D_n) \rightarrow \neg ged(D_i)) \wedge \forall D_i. ((D_i = D_{n+1}) \rightarrow \neg ged(D_i))$
$\wedge .. \wedge \forall D_i. ((D_i = D_{99}) \rightarrow \neg ged(D_i))$
By axiom3.

**1.2.2.3** $\neg ged(D_n) \wedge \neg ged(D_{n+1}) \wedge ... \wedge \neg ged(D_{99})$
Based on the proof in (b).

**1.2.2.4** $bfly(D_n) = 3 \wedge bfly(D_{n+1}) = 3 \wedge ... \wedge bfly(D_{99})$=3
Based on 1.2.1.5 and 1.2.2.3.

**Ind.**

IH: $\forall n \in \mathbb{N}. |\{d|ged(d)\}| = n \rightarrow$
$(\forall D_i. (i < n \rightarrow bfly(D_i) = n) \wedge (i \geq n \rightarrow bfly(D_i) = n + 1))$
Proof: $\forall n \in \mathbb{N}. |\{d|ged(d)\}| = n + 1 \rightarrow$

$(\forall D_i.\,(i < n+1 \rightarrow bfly(D_i) = n+1) \wedge (i \geq n+1 \rightarrow bfly(D_i) = n+1+1))$

**1.3.1** $\forall D_i.\,(i < n+1 \rightarrow bfly(D_i) = n+1)$
Assume $i < n+1$, and...

**1.3.1.1** $\forall D_i.\,(\neg\,((D_i = D_0) \vee (D_i = D_1) \vee ... \vee (D_i = D_n)) \rightarrow \neg ged(D_i))$
By axiom 2, and...

**1.3.1.2** $\forall D_i.\,((D_i = D_0) \vee (D_i = D_1) \vee ... \vee (D_i = D_n) \vee \neg ged(D_i))$
By meaning of implication on 1.3.1.1.
Assume $\forall D_i.\,(D_i = D_0) \vee (D_i = D_1) \vee ... \vee (D_i = D_{n-1}) \rightarrow ged(D_i)$.

**1.3.1.3** $\neg\,(bfly(D_0) = n)$
By axiom 4.

**1.3.1.4** $ged(D_n)$ Based on 1.2.1.3

**1.3.1.5** $bfly(D_0) = n+1 \wedge bfly(D_1) = n+1 \wedge ... \wedge bfly(D_n) = n+1$

**1.3.2** $\forall D_i.\,(i \geq n \rightarrow bfly(D_i) = n+1)$
Assume $i \geq n$, and...

**1.3.2.1** $\forall D_i.\,((D_i = D_n) \vee (D_i = D_{n+1}) \vee .... \vee (D_i = D_{99}) \rightarrow \neg ged(D_i))$
By axiom 2, and...

**1.3.2.2** $\forall D_i.\,((D_i = D_n) \rightarrow \neg ged(D_i)) \wedge \forall D_i.\,((D_i = D_{n+1}) \rightarrow \neg ged(D_i)) \wedge .. \wedge \forall D_i.\,((D_i = D_{99}) \rightarrow \neg ged(D_i))$
By axiom3.

**1.3.2.3** $\neg ged(D_n) \wedge \neg ged(D_{n+1}) \wedge ... \wedge \neg ged(D_{99})$
Based on the proof in (b).

**1.3.2.4** $bfly(D_n) = n+1 \wedge bfly(D_{n+1}) = n+1 \wedge ... \wedge bfly(D_{99}) = n+1$
Based on 1.3.1.5 and 1.3.2.3.