Due: Tues Sept 13, 11:59 PM

1. Prove that

$$\forall P, Q \in \mathbb{B}.(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

Note that $\mathbb{B}$ is "booleans", so P and Q are propositions. Also note that another way to phrase this questions is to show that the two are equivalent. While you may typically just use the logical equivalences in your logic handout, you **may not** just directly use this equivalence, as we are asking you to prove it.

2. Suppose we have two functions $f : \mathbb{Z} \to \mathbb{B}$ and $g : \mathbb{Z} \to \mathbb{B}$. That is f and g are both functions that take an integer and return a boolean. Prove that

$$(\forall x \in \mathbb{Z}.f(x) \Rightarrow g(x)) \Leftrightarrow \neg(\exists x \in \mathbb{Z}.\neg(f(x) \Rightarrow g(x)))$$

Note that while you may typically just use the logical equivalences in you logic handout, you **may not** just directly use the "quantifier" equivalences, as we are asking you to prove it.

3. Prove that

$$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

Hint: use induction on n.

4. Prove the transitivity of the "subseq-or-equal-to" operator. Namely,

$$\forall A, B, C \in \mathcal{P}(\mathbb{Z}).A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$$

Hint: remember to start with a careful definition of $\subseteq$.
Note: this is true of all sets, and there is nothing special about it being sets of integers. We just didn't want to throw any new notation for "sets of any type" at you on the first homework.

5. On a magical island in a land far far away, there lives a group of 100 theorem-proving dragons. These dragons are *so good* at theorem proving that anytime they learn a set of facts $P_0, P_1, ...P_N$ such that $P_0, P_1, ...P_N \vdash Q$, the dragons instantly formulate the corresponding proof, and add Q to the set of facts they know (possibly then proving other things).

   These magical dragons have either red eyes or green eyes. Unfortunately, if any dragon ever learns its own eye color, it turns into a butterfly and flies away at midnight that night. The island is small enough that every dragon on the island interacts with every other dragon on the island on a daily basis, and each dragon knows and can recognize ever other dragon. So the dragons see each other, know the eye color of the other dragons, and know not to tell the other dragons any information pertaining to their eye color.

   Periodically, an oracle visits the island. This oracle is so truthful that all dragons consider anything she says to be truth, and use it to prove theorems. At the start of day 0, no dragon has any information related to their own eye color, but knows the eye color of every other dragon on the island. However, at noon, the oracle arrives and proclaims "there is at least one dragon on this island with green eyes".

   (a) The dragon's use a predicate ged : Dragon $\rightarrow \mathbb{B}$ (short for Green Eyed Dragon) to consider the eye color of a dragon.

Namely ged(D_i) is true iff dragon D_i has green eyes. The dragons instantly translate the oracle's proposition into a logical statement with the ged predicate. What is the logical statement they come up with?

(b) Suppose there is *exactly* one dragon with green eyes (we will call this dragon D_0). On day 1, this particular dragon proves ged(D_0), learning its own eye color, and turning into a butterfly at midnight. Write down the proof that this dragon performed on day 1. Hint: write down an formal statement that *only* D_0 has green eyes.

(c) Under the scenario above (only D_0 has green eyes), what happens to the red eyed dragons? Note: you do not need to write a proof, just state what happens.

(d) **This last part is extra credit.** It is meant to be harder, and we want a very solid formal proof in the last step for full credit.

   i. Describe in words (not math) what happens in the general case in which N (for any $N > 1$) dragons have green eyes.. Namely, on what day green eyed dragons turn into butterflies and on what day red eyed dragons turn into butterflies.

   ii. Translate your description above into a formal mathematical proposition.
   Hint 1: You will likely want a function bfly : Dragon $\rightarrow \mathbb{N}$ which, given a dragon states on what day that dragon turns into a butterfly. E.g. bfly(D_5) $= 3$ means D_5 turns into a butterfly on day 3.
   Hint 2: The proposition should start

$$\forall n \in \mathbb{N}. |\{d | ged(d)\}| = n \Rightarrow ...$$

That is, for any natural number N, if the size of the set of dragons that have green eyes is n, then...

iii. Prove your statement from the previous step.