CHAPTER III

The Fourier transform

III.1 Identification and characterization of a dynamical regime

When, as a result of an experiment or numerical simulation, we have a time-dependent signal $x(t)$ — called a time series — one of the essential tasks is to determine the kind of evolution that produced it. The purpose is to compress information in a way that emphasizes the most significant dynamical characteristics. Are we dealing with an oscillation, more or less complicated in shape, but with a perfectly well-defined period? Are we dealing with a more or less linear superposition of several different oscillations? Or is it something else entirely?

The answer to these questions is not at all trivial, except in very simple situations such as those discussed in the previous chapter. There we were able to ascertain the existence of periodic solutions to the equation of motion of a forced oscillator. A description of these solutions includes a period (or, equivalently, the frequency) and an amplitude: essentially, the amplitude of the limit cycle[26], and the time taken to traverse it. These two characteristics emerge almost automatically as soon as the equations of motion are known. It is important to emphasize that these characteristics remain, and play just as essential a role, for any periodic phenomenon whose underlying mechanism is unknown, or described by equations that are not soluble analytically, by far the most frequent situation in practice.

Certain dynamical regimes are a superposition of oscillations which differ in amplitude, period, ratio of harmonics, etc. We have already seen an example of this in the Mathieu equation and will see others later on. The associated attractor is no longer a limit cycle, but rather a *torus*. This type of regime is called quasiperiodic.

Other regimes are of a nature more difficult to grasp. Given their completely disordered appearance, we call them "chaotic"[27]. When the dynamics are

26. The existence of a limit cycle — or of any kind of attractor — necessarily implies that the system is dissipative, which we will henceforth assume. A conservative system can never have an attractor, since its evolution does not involve contraction of areas in phase space.

27. These regimes are also called turbulent, or aperiodic, or nonperiodic. These adjectives are more or less synonymous, as the terminology has not yet been precisely decided upon. In any case, it is impossible to detect any long term regularity in the corresponding time series $x(t)$.

deterministic (that is, representable by a finite number of nonlinear coupled differential equations or the equivalent), the trajectories in phase space converge onto a *strange attractor*, whose topological properties are radically different from those of a torus.

To answer the questions asked in the beginning of the chapter, we must use "objective" methods of analysis, not merely the observer's judgement of the regularity of the time series. There are several ways to identify and to characterize a dynamical regime. We will explain two frequently used methods: first, the Fourier transform in this chapter[28]; then the Poincaré section, the subject of Chapter IV. In Chapter VI we will see a method developed recently for studying an attractor directly.

III.2 Discrete Fourier transform

III.2.1 SIGNAL DISCRETIZATION

The rapid development of computational methods has meant that a signal $x(t)$ — a continuous function of time — is very often measured by sampling and discretizing. Therefore, an experiment generally provides a discrete sequence[29] of real numbers $x_j (j \in Z)$ regularly spaced at time intervals of Δt (see fig. III.1).

In practice this sequence of numbers is necessarily finite, containing n values for a total length of time $t_{\max} = n \,.\, \Delta t$. The choice of the two quantities n and Δt is determined by practical considerations, such as the acceptable duration of the experiment, and the capacity for storage and processing of the measurements. Fourier transforms can, of course, be applied to continuous functions, as well as to discrete sequences, with integrals replacing summations. We will, however, limit our presentation to discrete sequences, with the understanding that the signal $x(t)$ is, mathematically, an integrable and square integrable function.

III.2.2 DEFINITION OF THE DISCRETE FOURIER TRANSFORM

We define the Fourier transform of a discrete time series x_j to be the operation creating a corresponding discrete series $\hat{x}_k$ such that:

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \exp\left(-i\,\frac{2\pi jk}{n}\right) \tag{8}$$

$$k = 1, \ldots, n$$

$$i = \sqrt{-1} \text{ pure imaginary.}$$

28. The Fourier transform was developed in the nineteenth century by the mathematician Jean Baptiste Fourier during the course of his work on the heat equation. Other kinds of transforms — Rademacher and Hadamard transforms — with similar properties, can also be utilized. Rather than describing them, or the numerous mathematical results on the Fourier transform, we will present here only the practical results most relevant to our subject.

29. The disadvantages of discretization must be compared to the considerable advantage of being able to process the signal entirely numerically. With modern electronic technology, it is possible to sample with a time interval Δt of less than 10^{-6} second and to digitize the signal in real time.

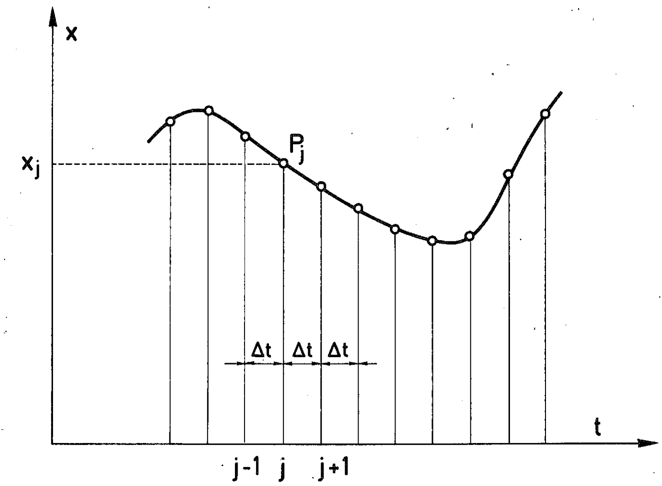

Figure III.1 Discretization of a continuous function.
The continuous function — for example, an experimental signal — is represented by the solid line. The discrete values x_j are taken at the instants ...$j - 1$, j, $j + 1$... equally spaced at time intervals of Δt.
The continuous function is hereafter replaced by the series of points P_j.

For convenience we have taken Δt as the unit of time, so that incrementing j corresponds to evolution over a time interval Δt. We can consider the transformation defined by (8) as a kind of rotation, mapping the vector $(x_1, x_2, ..., x_j, ..., x_n)$ into the vector $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_k, ..., \hat{x}_n)$. It is not a rotation in the usual sense, since the Euclidean length $\Sigma_j x_j^2$ is not conserved. What is conserved by (8) is the Hermitian length[30], as expressed by the Parseval-Plancherel equation:

$$\sum_{j=1}^{n} |x_j|^2 = \sum_{k=1}^{n} |\hat{x}_k|^2.$$

We point out that $\hat{x}$ is a function of the variable conjugate to the time, which is the frequency f. The frequency also varies by the discrete interval Δf :

$$\hat{x}_k = \hat{x}(k \,.\, \Delta f) \quad ; \quad \Delta f = \frac{1}{t_{\max}}.$$

If the series x_j is real-valued, as we have assumed, knowledge of the complex components $\hat{x}_k$ implies that of $2n$ real quantities (real and imaginary parts of each $\hat{x}_k$).

30. The hermitian length is the quantity $\Sigma x x^* = \Sigma |x|^2$. The notation x^* designates the complex conjugate of x.

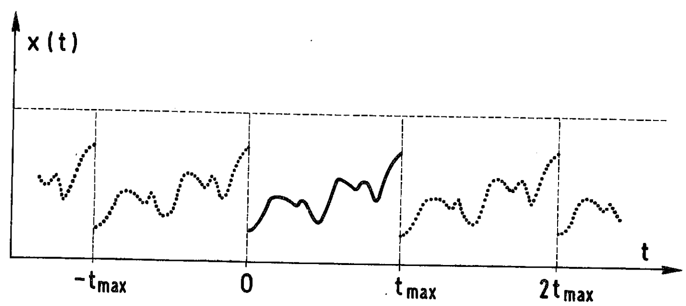

Figure III.2 Sampling of a function $x(t)$ over a finite time.
The measured interval $[0, t_{max}]$ is equivalent to one period of a function defined for all time $t \in]-\infty, \infty[$ and with period t_{max}.

Clearly, there is redundancy in this data, which is expressed by the easily established relation:

$$\hat{x}_k = \hat{x}^*_{n-k}.$$

By the inverse transformation we can go from the vector $\hat{x}$ back to the initial vector:

$$x_j = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \hat{x}_k \exp\left(i \frac{2\pi kj}{n}\right). \tag{8'}$$

Note that this relation can be used to define components x_j not only for $j \in [1, n]$ but for all integer values of j. This function x is periodic in n (in fact $n \,.\, \Delta t$) since:

$$x_{j+n} = x_j.$$

This explains why any signal $x(t)$ can be expressed as a sum of periodic functions as in (8′). The finite interval $[0, t_{max}]$ can simply be considered as representing one period of a function, which, while defined for all t, is periodic in t_{max}, as shown in Figure III.2.

III.2.3 THE WIENER-KHINTCHIN THEOREM

We define the *autocorrelation function* of the signal x_j by:

$$\psi_m = \frac{1}{n} \sum_{j=1}^{n} x_j x_{j+m}.$$

The unit of time is still Δt, so that:

$$\psi_m = \psi(m \,.\, \Delta t).$$

Physically, this function represents the average of the product of the signal values at a given time and at a time $m \,.\, \Delta t$ later. We can therefore deduce from ψ_m whether, and for how long, the instantaneous value of the signal depends on its previous values: hence its name. Or else, we can say that it is a measure of the degree of resemblance of the signal with itself as time passes[31].

Since the series x_j is periodic in n, the autocorrelation function necessarily has the same property:

$$\psi_m = \psi_{m+n}.$$

By applying the inverse Fourier transform, we get:

$$\psi_m = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k,k'=1}^{n} \hat{x}_k \hat{x}_{k'} \exp\left[i \frac{2\pi}{n} (jk + (j+m)k') \right].$$

Using the property $\hat{x}^*_{k'} = \hat{x}_{n-k'}$ and summing over the indices j and k', we establish that:

$$\psi_m = \frac{1}{n} \sum_{k=1}^{n} |\hat{x}_k|^2 \cos\left(\frac{2\pi mk}{n}\right).$$

This signifies that, up to a factor of proportionality, the auto-correlation function is merely the Fourier transform of $|\hat{x}_k|^2$.

Let us now find the inverse relation between $|x_k|^2$ and ψ_m. Let S_k be the function defined by:

$$S_k = \sum_{m=1}^{n} \psi_m \cos\left(\frac{2\pi mk}{n}\right).$$

Substituting into this definition the expression for ψ_m calculated above, we get:

$$S_k = \sum_{\ell=1}^{n} |\hat{x}_\ell|^2 \frac{1}{n} \sum_{m=1}^{n} \cos\left(\frac{2\pi mk}{n}\right) \cos\left(\frac{2\pi m\ell}{n}\right).$$

Using the equality:

$$\cos\left(\frac{2\pi mp}{n}\right) = \frac{1}{2}\left[\exp\left(i\,\frac{2\pi mp}{n}\right) + \exp\left(-\,i\,\frac{2\pi mp}{n}\right)\right]$$

the sum over m becomes a geometric series with n terms. From which:

$$\frac{1}{n} \sum_{m=1}^{n} \cos\left(\frac{2\pi mk}{n}\right) \cos\left(\frac{2\pi m\ell}{n}\right) = \frac{1}{4}\,[\delta^{(n)}_{k+\ell} + \delta^{(n)}_{k-\ell} + \delta^{(n)}_{-k-\ell} + \delta^{(n)}_{-k+\ell}]$$

31. As long as the autocorrelation function is appreciable, the signal remains relatively predictable; knowledge of the signal for a sufficiently long time allows us to calculate with sufficient confidence its value at a later time by extrapolation. On the other hand, when ψ_m approaches zero, the temporal similarity of the signal with itself disappears, and prediction of its evolution becomes impossible.

where $\delta_j^{(n)}$ is 1 if j equals 0 modulo n, and 0 otherwise. Joining this result to the symmetry relation $|\hat{x}_{n-\ell}|^2 = |\hat{x}_\ell|^2$, we find:

$$S_k = |\hat{x}_k|^2 = \sum_{m=1}^{n} \psi_m \cos\left(\frac{2\pi mk}{n}\right)$$

which is the sought-after inversion relation. This result constitutes one form of the Wiener-Khintchin[32] theorem, which says that the function $|\hat{x}_k|^2$ is proportional to the Fourier transform of the autocorrelation function ψ_m of the signal. The graph representing $|\hat{x}_k|^2$ as a function of the frequency $f(f = k \,.\, \Delta f)$ is called the *power spectrum.*

The power spectrum of a real function has the property:

$$|\hat{x}_k|^2 = |\hat{x}_{n-k}|^2$$

which comes from the equality $\hat{x}_k = \hat{x}^*_{n-k}$. This expresses the obvious fact that information about the phase of a component $\hat{x}_k$ is lost[33] when we consider $|\hat{x}_k|^2$.

III.2.4 THE POWER SPECTRUM

Let us examine in closer detail the characteristics of the power spectrum and the kind of information it conveys about the signal $x(t)$. We begin with a time series of n equidistant points, separated by an interval of Δt (fig.III.3 *a*). By applying relation (8) we calculate the $\hat{x}_k$, and then the $|\hat{x}_k|^2$, giving rise to a new function on n discrete points of the abscissa $k/n\,\Delta t$ ($k = 1, ..., n$). The abscissa has the dimension of inverse time — that is, of frequency, from which comes the name of spectrum given to the graph of $|\hat{x}_k|^2$. The interpretation of the ordinate axis depends on the nature of the signal measured.

It is conventional to call this a power spectrum, by analogy to the case when the ordinate represents power, i.e. energy per unit time. Consider a signal $x(t)$ originating in the reception of waves (electromagnetic, sound, etc.) by an antenna. If the detector is linear and does not introduce distortion in the frequency band considered, then the quantity it measures is proportional to the variation of the electric field or of the pressure in the vicinity of the antenna. The theory of waves shows that the power carried by a wave is proportional to the square of its amplitude (just as the energy of a

32. This theorem was proved by Norbert Wiener after work by G. Taylor, who analogically measured a quantity similar to $|\hat{x}_k|^2$ from a turbulent signal (variations in the resistance of a hot-wire anemometer).

33. We note incidentally that loss of phase information is commonly observed in physics in, for example, electron diffusion or in collisions between elementary particles at high energy. The information contained in the phase is associated with the mean reversibility of the signal. In functions such as $|\hat{x}_k|^2$ or ψ_m, which are unchanged when the order of the indices of x_j is reversed, this information cannot be preserved. On the other hand, this would not be true of a correlation function ϕ_m defined by:

$$\phi_m = \frac{1}{n} \sum_{j=1}^{n} (x_j^2 x_{j+m} - x_{j+m}^2 x_j)$$

which changes sign when the time series is treated in the opposite order.

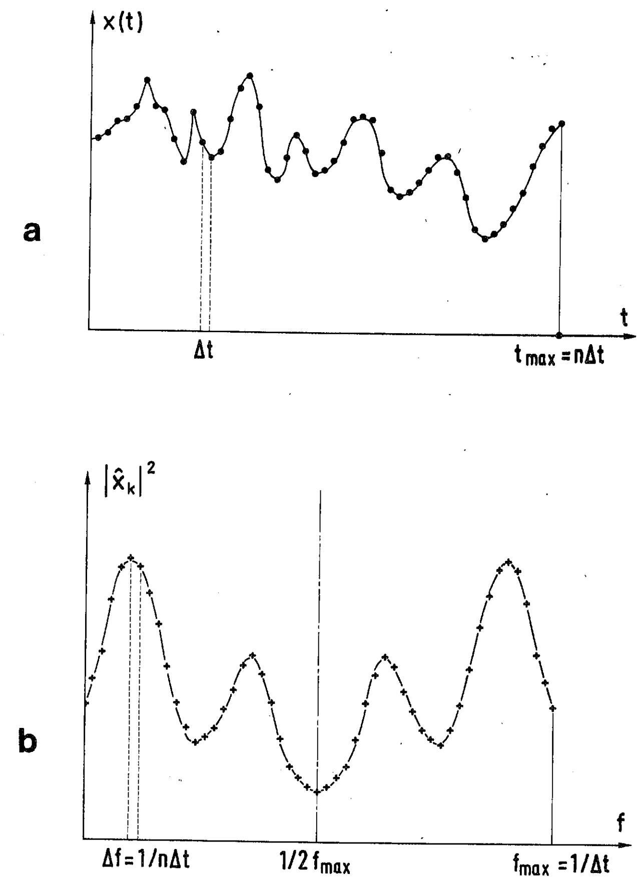

Figure III.3 Time series and power spectrum.
***a)* Graph of the signal sampled at equal time intervals Δt.**
***b)* The (discrete) power spectrum of the signal in *a*).**
Note the symmetry of the spectrum $|\hat{x}_k|^2$ about the vertical axis with abscissa $1/2f_{\max}$.

harmonic oscillator is proportional to the square of its amplitude) averaged over time. By the Parseval-Plancherel formula, we can replace the average over time by an average over frequency. This justifies the appellation "power spectrum", which we keep, for convenience, in all cases[34].

34. In fact, the relationship between power — in the sense of energy received per unit time — and the amplitude of the spectrum of the signal output by amplifiers or, more generally, by a sequence of detectors, can be rather complicated. This is all the more so if the detectors perform a nonlinear transformation of the signal.

The power spectrum in Figure III.3 *b* is that of the signal in Figure III.3 *a*. The step size $f = 1/(n \,.\, \Delta t)$ along the abscissa corresponds to the spectral resolution. To improve the resolution, the product $n \,.\, \Delta t$ must be increased. The highest frequency of the spectrum is $f_{\max} = 1/\Delta t$. To enlarge the frequency domain explored, Δt must be reduced. In any case, because of the relation:

$$|\hat{x}_k|^2 = |\hat{x}_{n-k}|^2$$

the spectrum is symmetric[35] with respect to the vertical line $f = 1/2 f_{\max}$. Consequently, the effective useful frequency domain – that containing non-redundant information – extends only from 0 to $1/(2\,\Delta t)$.

To be completely rigorous, the spectrum is composed of a sequence of "steps", each of width Δf (fig. III.4). In practice, one often merely draws a vertical line segment of height $|\hat{x}_k|^2$ at $f = k \,.\, \Delta f$. Most of the time, one in fact joins the successive points $(k \,.\, \Delta f, |\hat{x}_k|^2)$ by line segments (fig. III.3 *b*). It is not uncommon for the amplitude of $|\hat{x}_k|^2$ to vary over several orders of magnitude, in which case a logarithmic scale is used for the ordinate. The unit of measure along this axis is then the decibel, which is equal to 10 times the decimal logarithm. A difference of α decibels corresponds to an amplitude ratio of $10^{o.1\alpha}$.

III.3 Different kinds of Fourier spectra

III.3.1 PERIODIC SIGNAL

The appearance of the power spectrum clearly depends on the way in which the signal $x(t)$ evolves over time. The interest of the Fourier spectrum is, in fact, that it reveals properties of the evolution which would otherwise remain undetected.

Let us first consider the simple case in which $x(t)$ is a periodic signal of period T, that is:

$$x(t) = x(t + T) = x\left(t + \frac{2\pi}{\omega}\right)$$

such as the signal engendered by a Van der Pol oscillator in the limit cycle regime.

An extreme situation is that in which the period is exactly equal to the duration of measurement:

$$T = t_{\max} = n \,.\, \Delta t.$$

Then, according to (8), the Fourier components are located exactly at the frequencies:

$$\frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \ldots, \frac{n}{T}.$$

35. Recall that this property follows from the fact that the series x_j is assumed to be real-valued.

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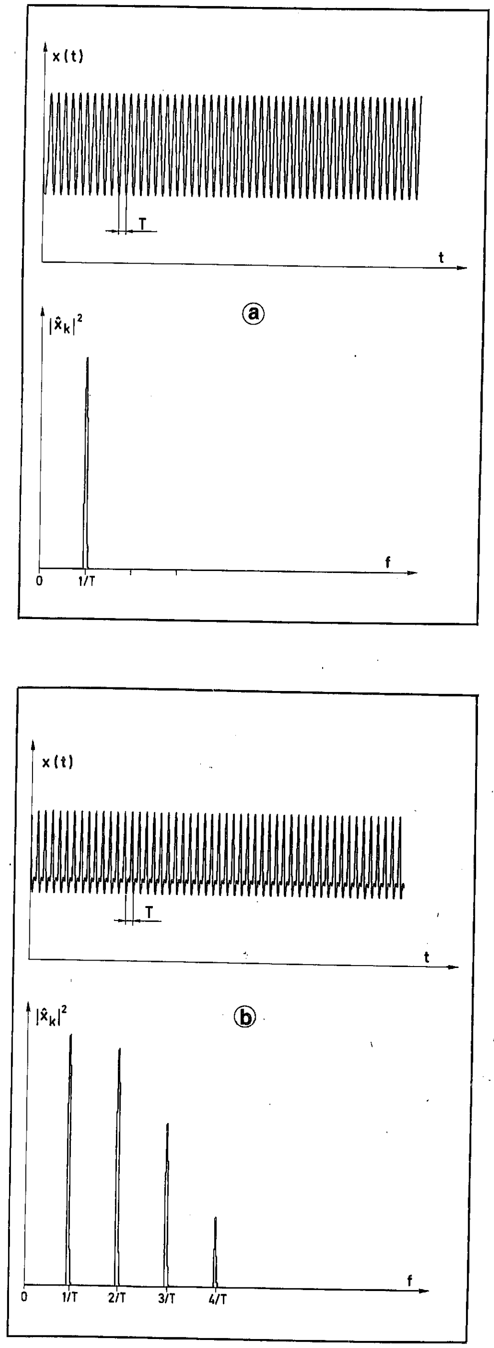

Figure III.4 Periodic functions and their Fourier spectra.
a) Pure sinusoidal function.
b) Periodic function containing harmonics. Only the first four harmonics are shown. (Unless otherwise stated, a logarithmic scale is used for the power in this and subsequent Fourier spectra).

In the simplest case, if $x(t)$ is a circular function — sine or cosine — the spectrum consists of only one nonzero component, with abscissa $1/T$ (or $k = 1$, fig. III.4 *a*). For a signal of a different form, such as a relaxation oscillation, the amplitude of the harmonics of the frequency ($2/T$, $3/T$, ...) is no longer zero. This is why the presence of harmonics in the spectrum is indicative of the non-sinusoïdal nature of the evolution (fig. III.4 *b*).

The preceding results can be easily generalized to the case where the duration of measurement t_{max} is an integer multiple of the signal period T:

$$t_{max} = pT;\ p \text{ a positive integer} > 1.$$

In this case, the nonzero components of the spectrum are still at $1/T$, $2/T$, ..., but the frequency resolution is p times better. It follows that all the components $|\hat{x}_k|^2$ for which k is not an integer multiple of p are zero.

In fact, the situation we have just described is rather academic, since, in practice, the period of the signal is either unknown or known imprecisely. The ratio t_{max}/T is therefore generally not an integer. What are the consequences on the shape of the spectrum? To answer this question, let us calculate the Fourier transform for the simple case of a circular function:

$$x(t) = \exp\left(i\,\frac{2\pi t}{T}\right)$$

$$x_j = \exp\left(i\,\frac{2\pi j\,\Delta t}{T}\right)$$

$$\hat{x}_k = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \exp\left(i\,\frac{2\pi j\,\Delta t}{T}\right)\exp\left(-\,i\,\frac{2\pi jk}{n}\right).$$

Setting: $\phi_k = \Delta t/T - k/n$

$$\hat{x}_k = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \exp\,(i2\pi\phi_k j) = \frac{1}{\sqrt{n}}\exp\,(i2\pi\phi_k)\,\frac{\exp\,(i2n\pi\phi_k) - 1}{\exp\,(i2\pi\phi_k) - 1}$$

$$|\hat{x}_k\hat{x}_k^*| = |\hat{x}_k|^2 = \frac{1}{n}\,\frac{\sin^2\,(n\pi\phi_k)}{\sin^2\,(\pi\phi_k)}.$$

Note that $n\phi_k = (t_{max}/T) - k$ is the difference between the integer k and the (noninteger) ratio of measurement duration to signal period. Therefore $n\phi_k$ is finite for all k.

Under the additional hypothesis of large n — thus small $\pi\phi_k$ — we have, asymptotically:

$$|\hat{x}_k|^2 \simeq n\,\frac{\sin^2\,(n\pi\phi_k)}{(n\pi\phi_k)^2}.$$

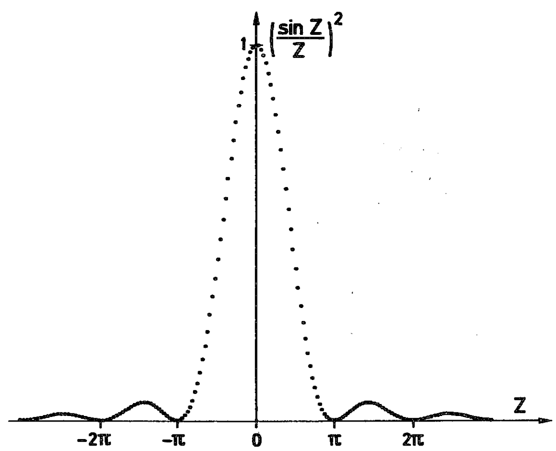

Figure III.5 Graph of the function $\sin^2 z/z^2$.

Therefore the behavior of $|\hat{x}_k|^2$ is like that of the function $\sin^2 z/z^2$, whose graph appears in Figure III.5.

This function has a maximum amplitude of one at $z = 0$, and a series of secondary maxima at $\pm(\ell + 1/2)\pi$ (ℓ a positive integer), whose amplitude decreases like $1/z^2$. It follows that $|\hat{x}_k|^2$ is maximum when k is equal to k_0, the integer closest to $n\,\Delta t/T$; it is there that the discrete variable ϕ_k is close to zero. Moreover, since the integer values of k close to k_0 correspond to the lateral arches of the function $\sin^2 z/z^2$, the amplitude of $|\hat{x}_k|^2$ for $k = k_0 \pm \ell$ is not zero, even though it decreases rapidly with ℓ. The end result is that in addition to the single peak obtained when there is resonance between the signal period and the duration of measurement, there are also secondary peaks, called sidelobes, centered around the frequency $1/T$. The linewidth is of the order of several frequency units Δf, and is widest when the ratio $n\,\Delta t/T$ is a half-integer. The diagram in Figure III.6 illustrates this possibility, contrasting it with the case of resonance.

In summary, the spectrum of a periodic signal of period T is made up of a peak at the frequency $1/T$, its sidelobes, and possibly a certain number of other peaks (and their sidelobes) that are harmonics of the fundamental frequency.

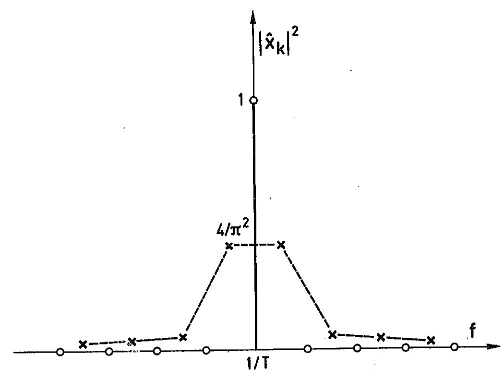

Figure III.6 Form of one spectral component under two different sampling conditions.
– solid line (circles) satisfies the resonance condition $n\ \Delta t/T$ integer,
– dashed line (crosses) satisfies $n\ \Delta t/T$ half-integer.

III.3.2 QUASIPERIODIC SIGNAL

A function y of r independent variables $t_1, t_2, ..., t_r$ is said to be periodic, of period 2π in each of its arguments, when increasing one of these variables by 2π does not change its value:

$$y(t_1, t_2, ..., t_j, ..., t_r) = y(t_1, t_2, ..., t_j + 2\pi, ..., t_r) \qquad j = 1,$$

Such a function is called quasiperiodic in time if its r variables are all proportional to the time t:

$$t_j = \omega_j t \qquad j = 1, ..., r.$$

A quasiperiodic function has r fundamental frequencies:

$$f_j = \frac{\omega_j}{2\pi} \qquad j = 1, ..., r$$

and therefore r periods: $T_j = 1/f_j = 2\pi/\omega_j$.

Before arriving at the form of the Fourier spectrum of such a function, recall that the formal definition above must be related to phenomena with multiple periodicity. We cite one example among many: the astronomical position of a point on the surface of the earth is described by a quasiperiodic law, since it results from the rotation of the earth about its axis ($T_1 = 24$ h), the rotation of the earth around the sun ($T_2 = 365.242$ days), and finally the precession of the earth's axis of rotation (T_3

= 25 800 years, sometimes called a platonic year)[36]. We mention also that the phase trajectory associated with a quasiperiodic function is defined on a *torus*[37] of dimension r, written T^r.

As one might guess, the Fourier spectrum of a function which is quasiperiodic in time generally has a relatively complex appearance. One exception exists, nevertheless: when the quasiperiodic signal $x(\omega_1 t, \ldots, \omega_r t)$ is the sum of periodic functions:

$$x(\omega_1 t, \ldots, \omega_r t) = \sum_{i=1}^{r} x_i(\omega_i t).$$

Then, using the linearity of Equation (8), the power spectrum is the sum of the r spectra of each of the functions $x_i(\omega_i t)$. It is therefore composed of a set of peaks, located at the fundamental frequencies $f_1, f_2, \ldots, f_r$, and of their harmonics:

$$m_1 f_1,\ m_2 f_2,\ \ldots,\ m_r f_r$$

($m_1, m_2, \ldots, m_r$ positive integers). But there is, in general, no reason why the signal should be the sum of r independent periodic terms. If the quasiperiodic function includes a term like the product of circular functions (for example: $\sin(\omega_i t) \sin(\omega_j t)$) then the Fourier spectrum contains fundamental frequencies $|f_i - f_j|$ and $|f_i + f_j|$ and their harmonics, since:

$$\sin(\omega_i t) \sin(\omega_j t) = \frac{1}{2} \cos(|f_i - f_j|\, 2\pi t) - \frac{1}{2} \cos(|f_i + f_j|\, 2\pi t).$$

Generalizing this result, we see that the Fourier spectrum of a quasiperiodic function $x(t)$, which depends nonlinearly on periodic functions of the variables $\omega_i t$, contains components at all frequencies of the form:

$$|m_1 f_1 + m_2 f_2 + \ldots + m_r f_r|$$

where m_i are arbitrary integers.

To further the discussion, we limit ourselves for the moment to the biperiodic case ($r = 2$) and set aside the fact that the resolution Δt of the power spectrum is finite. Each nonzero component of the spectrum of the signal $x(\omega_1 t, \omega_2 t)$ is a peak with abscissa $|m_1 f_1 + m_2 f_2|$ which we may abbreviate by (m_1, m_2). The ratio f_1/f_2 can be either a rational or an irrational number. In the latter case it is shown in number theory that sums of the kind $|m_1 f_1 + m_2 f_2|$ constitute a dense set over the positive reals. In other words, any real positive number is as close as one wishes to such a sum. As a consequence, the spectrum of the signal is itself dense. But this does not mean that it is represented by a continuous function. Indeed, two peaks that are close together on the

36. We neglect the very small perturbations to these fundamental motions. Given the order of magnitude difference in the time scales of the motions, their interaction remains weak.

37. In the simplest case $r = 2$, one can show under fairly general conditions that the trajectory described parametrically by the three functions of time $x(\omega_1 t, \omega_2 t)$, $(\mathrm{d}/\mathrm{d}t)x(\omega_1 t, \omega_2 t)$, and $(\mathrm{d}^2/\mathrm{d}t^2)\, x(\omega_1 t, \omega_2 t)$ is indeed located on a torus in $\mathbb{R}^3$, that is, a torus in the usual sense of the word.

frequency axis have no *a priori* reason to have amplitudes that are close. Most of the time, one therefore observes a very limited number of frequencies for which the lines have a significant amplitude, because the high-order lines – that is, those corresponding to values of m_1 and m_2 greater than ten or so – have amplitudes too low to be detected[38]. In practice, one identifies a quasiperiodic spectrum by looking for the two fundamental frequencies f_1 and f_2 which allow the frequencies of the high amplitude lines to be represented by simple combinations $|m_1 f_1 + m_2 f_2|$ with m_1 and m_2 small: 0, ± 1, ± 2... (see fig. III.7).

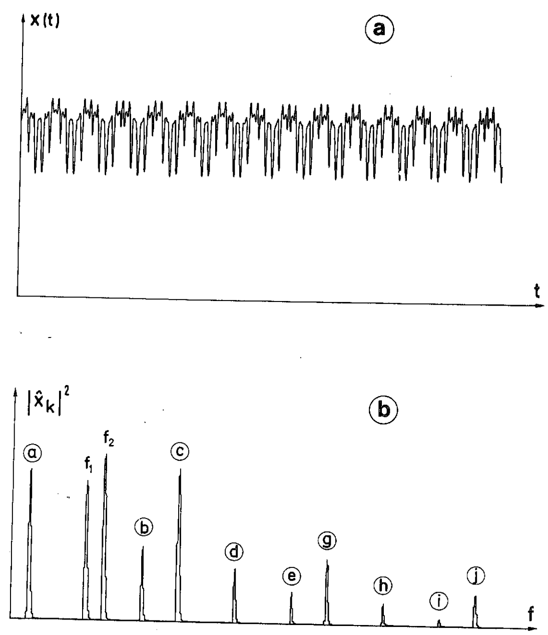

Figure III.7 Quasiperiodic function with two frequencies f_1 and f_2 (the ratio f_1/f_2 is irrational).
***a)* Times series.**
***b)* Fourier spectrum containing, besides the two fundamental frequencies f_1 and f_2, the principal peaks of frequency $f = m_1 f_1 + m_2 f_2$.**

$a \longrightarrow f_2 - f_1$	$e \longrightarrow 5f_1 - f_2$	$j \longrightarrow 5f_1 + f_2$
$b \longrightarrow 3f_1 - f_2$	$g \longrightarrow 3f_1 + f_2$	
$c \longrightarrow f_1 + f_2$	$h \longrightarrow 5f_1$	
$d \longrightarrow 3f_1$	$i \longrightarrow 7f_1 - f_2$	

38. Very generally (but not always, from a strict mathematical point of view), the amplitudes of high-order peaks decrease like exp $(- a_1 |m_1| - a_2 |m_2|)$ where a_1 and a_2 are positive. For large $|m_1|$ and $|m_2|$, these amplitudes therefore become negligible. They are practically undetectable as soon as the indices m_1 and m_2 significantly exceed a_1^{-1} and a_2^{-1} in absolute value.

Under the hypothesis of f_1/f_2 rational, the Fourier spectrum is not dense. It follows that the spectrum is definitely not represented by a continuous function. Since:

$$f_1/f_2 = n_1/n_2 \ (n_1,\ n_2 \text{ integers}).$$

the quasiperiodic signal is in fact periodic with period $T = n_1 T_1 = n_2 T_2$. Indeed, according to the definition, one has:

$$x(\omega_1 t,\ \omega_2 t) = x(\omega_1 t + 2\pi n_1,\ \omega_2 t + 2\pi n_2)$$

$$x(\omega_1 t,\ \omega_2 t) = x\left(\omega_1\left(t + \frac{n_1}{f_1}\right),\ \omega_2\left(t + \frac{n_2}{f_2}\right)\right).$$

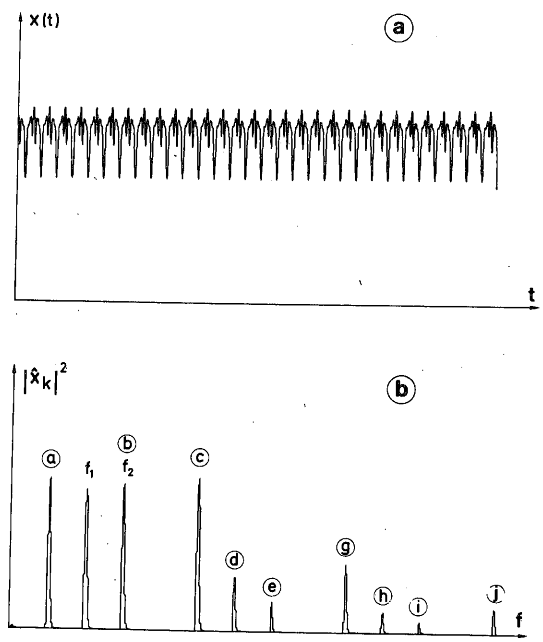

Figure III.8 Quasiperiodic function (f_1/f_2 rational).
a) **Time series.**
b) **Fourier spectrum. The function is almost identical to that shown in Figure III.7, but we have changed f_1 so that $f_1/f_2 = 2/3$. Note that under these conditions, all the peaks are harmonics of the frequency $f = f_2 - f_1 = 1/3 f_2$**

$a \longrightarrow f_2 - f_1$	$d \longrightarrow 6f$	$h \longrightarrow 10f$
$b \longrightarrow 3f (= f_2)$	$e \longrightarrow 7f$	$i \longrightarrow 11f$
$c = 5f$	$g \longrightarrow 9f$	$j \longrightarrow 12f$

One says that there is *frequency locking* of f_1 with f_2. All the lines of the Fourier spectrum are harmonics of the lowest frequency:

$$f_0 = \frac{1}{T} = \frac{f_1}{n_1} = \frac{f_2}{n_2}.$$

Figure III.8 illustrates this possibility. Consecutive lines of the spectrum are always separated by the same distance of $1/T$.

III.3.3 APERIODIC SIGNAL

When the signal $x(t)$ is neither periodic nor quasiperiodic, it is called *aperiodic* (or sometimes nonperiodic). We will meet with several examples of this situation later in the book, and we will see that the Fourier spectrum is then continuous[39]. The real difficulty is that a Fourier spectrum which looks continuous cannot be automatically attributed to an aperiodic signal, because this is also the appearance of the spectrum of a quasiperiodic signal with a very high number of frequencies (in the infinite limit).

If we suppose that the signal is truly aperiodic, we must still decide, for methodological reasons, whether the number of degrees of freedom is small (say, less than ten) or, on the contrary, very large. In the first case, we have the capability of developing a completely deterministic description of the system. By contrast, in the second case, only a probabilistic approach can be achieved with the current state of knowledge. This leads us to introduce the notion of chance, but without challenging the (hidden) determinism of the phenomena. Although we are now venturing beyond the subject of this book, it is appropriate to give an idea of what we mean by the distinction between "deterministic" and "random".

The extreme case of a random signal is what is called "white noise". This name is evocative, alluding to noise – in the sense of sound without any harmonic structure – and to the absence of color in light. One can thus "see" white noise – a white light – or "hear" it – it would be, typically, the sound of a waterfall. White noise is produced by a multitude of independent agents: the atoms of a heated filament emitting white light, or droplets in a waterfall falling on a boulder.

One can consider the signal $x(t)$ resulting from white noise as being "new"[40] at each instant, at least in a first approximation. Let us show that the corresponding

39. The spectrum of an aperiodic signal approaches a continuous function only in the mean. In fact, if we calculate the discrete spectrum of a signal of duration t_{max} by the sampling method described, we find that the amplitudes of neighboring $|\hat{x}_k|^2$ vary a great deal, no matter how large we take t_{max}. The difference between $|\hat{x}_k|^2$ and $|\hat{x}_{k\pm 1}|^2$ is typically comparable to these quantities themselves. To obtain a continuous spectrum (or at least a reasonable approximation to it) starting from the sequence $|\hat{x}_k|^2$, one must therefore either average over many spectra, each taken over a time window of length t_{max}/N (N large), or else, equivalently, average the $|\hat{x}_k|^2$ locally over a certain number of consecutive values of k.

40. To be completely rigorous, the noise is white if, in addition to being independent, the individual emitters each produce a signal which is an impulse of infinitesimal duration. If each of these individual signals has a spectral structure, the light or noise becomes *colored*.

power spectrum is "flat", – in other words, that the amplitude is independent of the frequency – and so possesses the essential characteristic of being devoid of all harmonic structure.

Let $x(t)$ be a signal of zero mean and of variance $\overline{x^2}$. The absence of correlation between signal values at different instants is expressed by the condition[41]:

$$\langle x_j x_{j+m} \rangle = \overline{x^2}\,\delta_m = \langle \psi_m \rangle$$

where δ_m is the Kronecker delta function ($\delta_0 = 1$ and $\delta_m = 0$ for $m \neq 0$). Using the Wiener-Khintchin theorem, we find:

$$\langle |\hat{x}_k|^2 \rangle = \sum_{m=1}^{n} \langle \psi_m \rangle \cos\left(\frac{2\pi mk}{n}\right) = \overline{x^2}$$

showing that the average amplitude of the signal's power spectrum is independent of k. In other words, the average amplitude is independent of the frequency for white noise.

The situation we have just described is quite common in practice; the high frequency structure of numerous natural phenomena is that of white noise. Indeed, noise due to molecular agitation involves the action of an almost infinite number of independent agents: molecules, conductance electrons, elastic vibrations of the atoms in a network, etc. The theory of thermodynamic noise is one of the great success stories of physics in the first half of twentieth century. As a concrete illustration, we consider the classical example of Nyquist noise, named after the person who explained it. Let an RC circuit (R = resistance, C = capacitance) be at thermal equilibrium (in the absence of an externally applied voltage). A sensitive voltmeter measures the voltage fluctuation $V(t)$ across the capacitor or the resistor. The energy stored in the capacitor of capacitance C, as a function of the voltage across it, is equal to $CV^2/2$. This energy is due to thermal fluctuations, and by Ehrenfest's equipartition theorem, its average value is equal to $kT/2$, where k is the Boltzmann constant ($k = R_J/N$, R_J = Joule's constant, N = Avogadro's number) and T is the absolute temperature of the system. Then:

$$\langle V^2 \rangle = \frac{kT}{C}$$

the average being taken either over time, or over the thermodynamic ensemble. But this is an average of instantaneous values, and we would like to know the frequency structure of this noise. We see this as follows. If $V(t)$ is the voltage across the capacitor at an instant t, the voltage measured at a later time $t + \tau$ will come from two factors.

41. In the expressions which follow, the averages are *ensemble averages*. We consider implicitly autocorrelation functions ψ_m and Fourier transforms $|\hat{x}_k|^2$ obtained from series of n values of x, giving an individual realization of these quantities. We then take their arithmetic average, denoted by $\langle\ \rangle$. Note that for a given realization, the mean square distance between $|\hat{x}_k|^2$ and its average value is of the same order of magnitude as the average. This justifies, as we have already said, the fact that spectra of turbulent signals must be either smoothed, or else averaged over an ensemble of realizations, to appear as continuous functions of the frequency.

On the one hand, the capacitor will discharge to the resistor R, giving a voltage $V(t)\ e^{-\tau/RC}$, where (RC) is the time constant of the circuit. On the other hand, new electric charge is deposited on the capacitor plates by thermal fluctuations. These fluctuations are uncorrelated with the initial charge of the capacitor, and thus also uncorrelated with $V(t)$. Since the RC circuit is linear, we have:

$$\langle V(t)V(t+\tau) \rangle = \langle V(t)^2 \rangle \exp^{(-|\tau|/RC)}$$

where we have used the absolute value $|\tau|$ in case τ is negative. The spectrum of the noise is given by the Wiener-Khintchin theorem for continuous variables:

$$S(\omega) = \frac{1}{\pi} \int_0^{\infty} d\tau \ \langle V(t)V(t+\tau) \rangle \cos(\omega\tau)$$

$$S(\omega) = \frac{2RkT}{\pi(1 + \omega^2 R^2 C^2)}.$$

With this formalism, the Parseval-Plancherel relation is written:

$$\int_0^{+\infty} S(\omega)\, d\omega = \langle V^2(t) \rangle = \frac{kT}{C}$$

This formula gives the spectral distribution of the voltage fluctuations across the capacitor of the RC circuit. In the limit of zero capacitance, it suffices to set $C = 0$ in the expression for $S(\omega)$ which gives the classic expression for Nyquist noise:

$$S(\omega) = \frac{2RkT}{\pi}.$$

The thermal noise across a resistor is therefore a white noise, since $S(\omega)$ is independent of ω. The noise amplitude is proportional to the absolute temperature and the resistance. Note that in this limit ($C = 0$), the average V^2 diverges, since $\langle V^2 \rangle = kT/C$. This divergence does not occur in practice, since at very high frequencies, one always finds parasitic capacitance in a circuit, so that C is never exactly zero.

The noise produced by a simple resistor is white noise, whose spectral power is independent of the frequency. By contrast, the thermal fluctuations of the voltage of an RC circuit constitute a *colored noise:* the spectrum is still a continuous function of frequency, but no longer constant.

Having completed this digression on random signals, we return to our main goal, which is to study the signals produced by deterministic dynamics. As we have already indicated, an aperiodic signal has a power spectrum of continuous appearance, like the example of Figure III.9. Therefore, this method of analysis does not distinguish between an aperiodic and a random signal. This limits the application of the Fourier transform, leading us to turn to other methods, notably that of Poincaré sections.

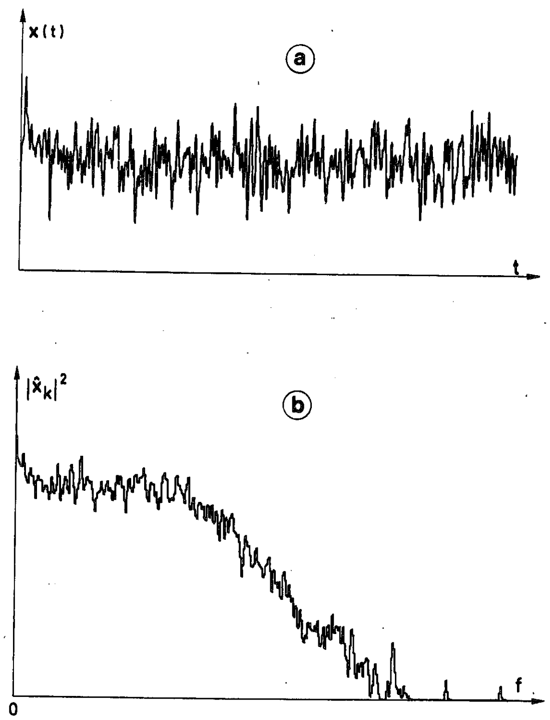

Figure III.9 Aperiodic function.
***a*) Time series.**
***b*) Fourier spectrum. Note the continuous nature of this spectrum in contrast to the line spectra of Figures III.7 and III.8.**

III.4 Fast Fourier Transform (FFT)

We have seen that there exists a close relationship between the periodic, quasiperiodic, or aperiodic nature of a signal $x(t)$, and the form of its power spectrum. To identify the nature of a dynamical regime, it remains to calculate $|\hat{x}_k|^2$ explicitly. In theory, there is no difficulty in using the formula given by Equation (8). But in practice, the extent of the task leaps to the eye as soon as n takes on appreciable values. Yet it is naturally advantageous to choose n large and Δt small to represent the signal $x(t)$ as faithfully as possible. With $n = 10^3$, which is still relatively small, we must already calculate 1 000 sums, each of which contains 1 000 terms! More generally, we see that calculation of the n components of a spectrum requires a number of operations (additions and multiplications) which is on the order of n^2. This rapid increase is a severe limiting factor which was, for a long time, a considerable deterrent to the use of Fourier transforms. Even the advent of the first computers did not greatly improve the situation.

However, when n is a power of two, an algorithm, called the FFT[42], permits calculation of the spectrum with many fewer operations, on the order of $n \log_2 n$. Then, for $n = 2^{10} = 1\,024$, the gain is already a factor of 100; it attains 7 000 for $n = 2^{18}$. The difference in cost which results is considerable. It is in fact thanks to this fantastic economy of scale that analysis by Fourier transform is a routine procedure in research laboratories, enabling great progress to be made in many domains[43], in addition to the study of dynamical systems. We mention, among others, different kinds of spectroscopy: IR, Raman, X, magnetic resonance, photon beating, etc. Digital Fourier analyzers are commercially available, which, combining the FFT algorithm with other tricks, provide the power spectrum of a signal containing several thousand points in a few seconds.

42. FFT abbreviates *Fast Fourier Transform.* This algorithm was developed in 1965 by Cooley and Tukey. They later discovered that the same result had been obtained as early as 1942 by other researchers whose article had received no attention at the time.

43. Signal processing methods (notably filtering and increase of the signal-to-noise ratio) have undergone considerable technical improvement to which implementation of the FFT has greatly contributed.