Introduction to Bayesian Inference

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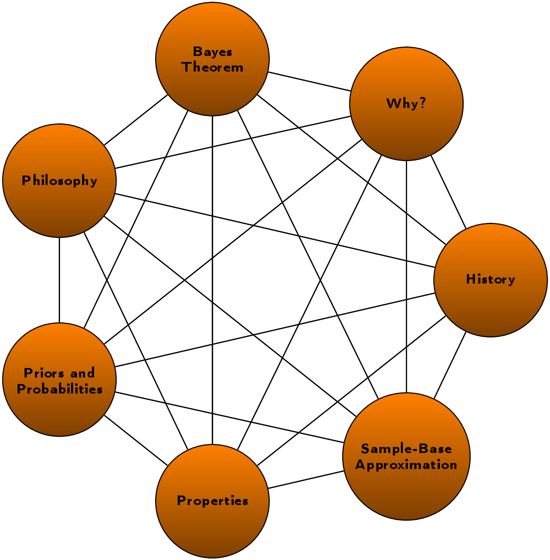
outline


Bayes Theorem
Why?
Philosophy
History
Priors and Probabilities
Sample-Base Approximation
Properties

Why?

Why are you interested in Bayesianism?

Why? – some advantages

Things Ecologist say. . .

- small sample sizes: exact inference
- missing data: easy to impute
- integrate other information
- derived quantities
- complex, hierarchical process models

multiple sources of variation (space,time)
“honest" epistemology

Theoretical

- conditional on the observed data
- probabilistic statements
- evidential
- coherence, decision making
- good frequentist properties

– shrinkage, decision-theory

- model selection

Why? – disadvantages

- what is probability (basing inference on something that doesn't exist!!!)?
- objective basis for science?
- misalignment: probability theory and human psychology
- biased (towards the prior)[1]
- language dependence

[1] A Bayesian would claim that given a prior, it would be irrational to believe in anything other than the posterior.

History

Neo Bayesian Revival (>1992)

Gelfand and Smith 1992 - Sampled based approximations of Bayesian posteriors [2]

Revival (~1920s -)

Subjective Bayesian, Decision Theory

- Ramsey (1926)
- De Finetti (1937)
- Savage (1954)
- (Wald, 1939, 1954)

Hypothesis Testing, Logical/Objective Bayesism

- Jeffreys (1939)
- Jaynes (2003)

Hierarchical Bayesian

- Good (1953,65)

Relationship to Compact Coding Theory

- Rissasen (1978), Wallace (1968)

Prediction

Frequentist "lethal blow"[3]

Rallied against use of prior probabilities in statistical inference

- Sir Ronald A. Fisher (1925,1935,...)

Maximum likelihood, significance tesing, ANOVA, sufficiency, randomized experiments
Inductive inference

- Jerzy Neyman & Egon Pearson (1933)

Hypothesis testing, confidence intervals, Type-I/II error rates
Inductive Behaviour

Philosophical developments

- Karl Popper (1959,1963) and Falsificationism

Anti-Induction: scientific progress is by falsifying theories, only

[3] S. Zabell 1989

History - Frequentism's Ascendency

frequentists:

- reject probabilistic confirmation of models [4]
- reject Bayesian notions of probability
- frequentists care about *good frequency properties*

Estimation: unbiased, efficiency, obtain minimum variance
Hypothesis testing: Type-I error rates, most powerful tests[5]

Frequencies as probabilities

- probabilities only meaningful as long-run frequencies of events

$$\mathbf{Y}: \{H, T, H, H, H, T, H, T, T, H, T, H, T, \ldots\} \tag{1}$$

- The probability of flipping a coin and getting a head is...

$$p(y = \text{Head}) \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{i}^{n} \mathbb{I}[y_i = \text{Head}] \tag{2}$$

the mean counts of heads in the long-run

[4] Some use the AICc for model confirmation
[5] Neyman-Pearson-type testing, not Fisherians

Frequentism

Fisher's p-value

- continuous index of evidence against a hypothesis H
- NEVER prove a hypothesis H, only disprove

Neyman-Pearson α and β

- long-run error rates of Type-I and Type-II
- bound Type-I at $\alpha \leq 0.05$ and hopefully maximize power $(1-\beta)$ with high n and most powerful tests
- never confirm a hypothesis: only act so as to "not be wrong too often"

Bayesians

Model probabilities

probabilistic confirmation of hypotheses

- $p(H_k|Y)$ what is the probability of Hypothesis H_k given the data?

Bayes Factors

evidence in favour of one hypothesis over another

- $BF = \frac{p(Y|H_1)}{p(Y|H_2)}$.

find hypothesis that is more likely to be true

History - Inverse Probability

from late 1700's to ~1920's: <u>Method of Inverse Probability</u>,the bread and butter of applied analyses

- Rev. Thomas Bayes (1778)
- Simone-Pierre Laplace (1774)

Bayes

"PROBLEM: Given the number of times in which an unknown event ($y \in [0,1]$) has happened and failed: Required the chance that the probability of its happening in a single trial lies between any two degrees of probability that can be named... By *chance* I mean the same as *probability*."

- *probability* of a *probability*: $p(\theta|y)$
- two types of probabilities

Laplace

developed Bayes Theorem close to its modern form:

$$\overbrace{p(\theta|Y)}^{\text{posterior}} = \frac{\overbrace{p(\theta)}^{\text{prior}}\overbrace{\mathcal{L}(Y|\theta)}^{\text{likelihood}}}{\underbrace{\int_\theta p(\theta)\mathcal{L}(Y|\theta)d\theta}_{\text{marginal likelihood}}} \tag{3}$$

Bayes great innovation: two types of probability

Probability of an *observable* event Y:

$p(y = \text{Head}) \equiv \theta$

- θ is *like a parameter in a model*
- $p(y = \text{Head}|\theta) = \text{Bern}(y; \theta) \rightarrow$ What we now call a likelihood
- how the data was generated: $y \sim \text{Bern}(\theta)$

Probability distribution for the *parameter* $p(\theta)$

for inference...

- Before data: $p(\theta)$ (the *prior* probability distribution)
- After data: $p(\theta|Y)$ (the *posterior* probability distribution)

Bayes Theorem

- conditional probability: want a probability distribution $p(\theta|y)$ conditional on observed data y -> need a likelihood $f(Y|\theta)$ and prior probability distribution $p(\theta)$.

$$\begin{aligned} p(\theta|Y) &= \frac{p(\theta)\mathcal{L}(Y|\theta)}{\int_\theta p(\theta)\mathcal{L}(Y|\theta)d\theta} \text{where} \quad \ldots \\ p(\theta) &\equiv \text{prior information (before the data)} \\ \mathcal{L}(Y|\theta) &\equiv \text{ likelihood (information from the data)} \quad ^6 \\ p(\theta|Y) &\equiv \text{distibuion of } \theta \text{ after the data} \\ \text{denominator} &: \cancel{\text{marginal likelihood}}\text{(often ignore)} \end{aligned} \tag{4}$$

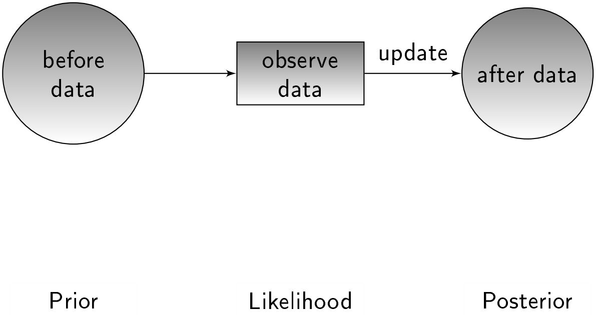
more common form...

$$p(\theta|Y) \propto \mathcal{L}(Y|\theta)p(\theta)$$

- posterior is a mixture of information in prior and likelihood

[6] marginal likelihood ignored when using sample-based approximation

- posterior is a mixture of information in prior and likelihood



What's in a Posterior

Mixture of information

- $\mathcal{L}(Y|\theta)$: Likelihood, specified by model. Similar between Bayesian and non-Bayesian analyses[7]
- $p(\theta)$ ... where do they come from?

How to specify priors (HUGE topic)

- a previous posterior distribution
- elicitation from experts, previous studies
- Priors as degrees-of-beliefs: **Subjectivist/personalist** Bayesians
- Default prior and reference priors: **Objective/logical** Bayesians
- adhoc

[7] Frequentists reserve the term likelihood for a function of θ for fixed y, whereas Bayesians consider "joint probability density of the data" given θ.

Posterior Inference (for estimation θ)

- Probabilistic statements about abstract quantity (θ) (*only* Bayesians can do)
- Posterior probability necesarily depends on a *prior*

“to make an Omelette, you must crack a few eggs" (Savage)

The joy of Posterior Inference

can make statements like...

- what is the probability that $\theta > 0$?
- what is the most probable value of θ? (MAP)
- what is the expected value of θ? (posterior mean)
- what is a *high probability region* of θ (Q% credibility interval)

Posterior Inference (estimation example)

Example 1:

- men's height, n=20 observations.
- $y_i \sim \mathcal{N}(175, 10^2)$

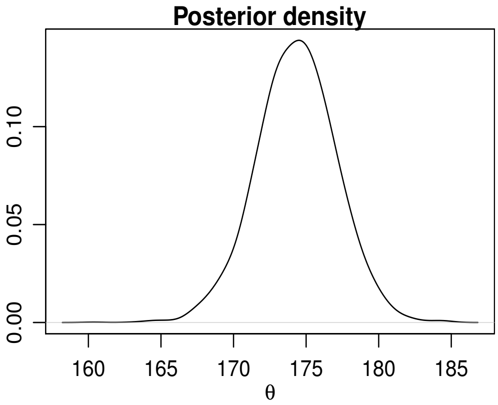
```
y <- c(183.46, 182.32, 178.31, 181.36, 165.12, 185.68, 170.47,
178.11, 174.86, 182.03, 180.09, 172.88, 177.94, 177.26, 182.58,
171, 173.74, 177.78, 180.02, 163.05)
```

- estimate $\theta = [\mu, \sigma^2]$: mean population height and variance
- priors: $p(\mu) = \mathcal{N}(0, 90^2)$, $p(\sigma^2) = \mathcal{IG}(0.1, 0.1)$
- specify a likelihood: $\mathcal{L}(\mathbf{y}|\mu, \sigma^2) = \prod_i^n \mathcal{N}(y_i; \mu, \sigma^2)$

now run a Gibbs sampler to approximate the posterior $p(\mu, \sigma^2|\mathbf{y})$ ...

Why are Posteriors so Great!

- IS a probability distribution



- easy to interpret

Why are Posteriors so Great!

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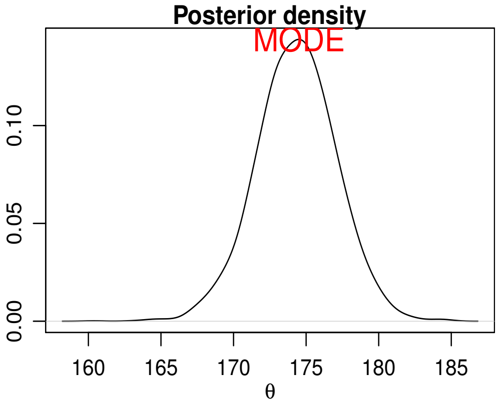

- Posterior mode: most probable value
- Posterior mean $\mathbb{E}[\theta] = \int p(\theta|Y)\theta d\theta$: expected value

Why are Posteriors so Great!

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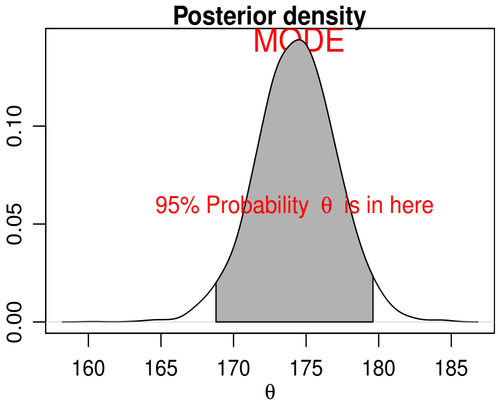
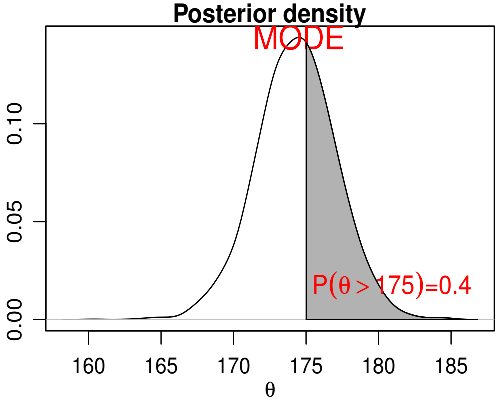


- Posterior mode: most probable value
- Posterior mean $\mathbb{E}[\theta] = \int p(\theta|Y)\theta d\theta$: expected value
- 95%CI of θ

Why are Posteriors so Great!

- IS a probability distribution



- Posterior mode: most probable value
- Posterior mean $\mathbb{E}[\theta] = \int p(\theta|Y)\theta d\theta$: expected value
- What is the probability that $\theta > X$? Area of $p(\theta|Y) > X$

Posterior Estimation vs Maximum Likelihood Estimation

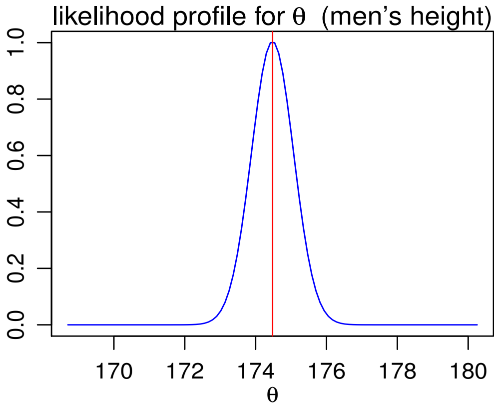
Bayesian vs. frequentist estimates: compare posteriors to maximum-likelihood method

method of maximum likelihood

- Choose θ such that we *maximize* the likelihood ($\mathcal{L}$) of seeing **y**
- interpretation "It would be very (un)likely to see the data that I saw, if the value of θ were X"
- Most common method among Frequentists (single model estimation)
- $\hat{\theta}_{\mathrm{MLE}}$ is NOT the "most probabilty value of θ
- optimality: unbaised, efficient [8]

[8] but see shrinkage estimators for high-dimensional problems

Posterior Estimation vs Maximum Likelihood Estimation



- frequentist point estimates: `glm(y~1)`

MLE	se	95CI-low	95CI-hi
172.3	1.48	169.4	175.17

- compare to (approximate)[9] Posterior descriptive statistics

$E[\theta]$	SD	95CI-low	95CI-hi
172.21	1.51	169.21	175.16

nearly the same

Posterior Inference (estimation example 2)

Example 2:

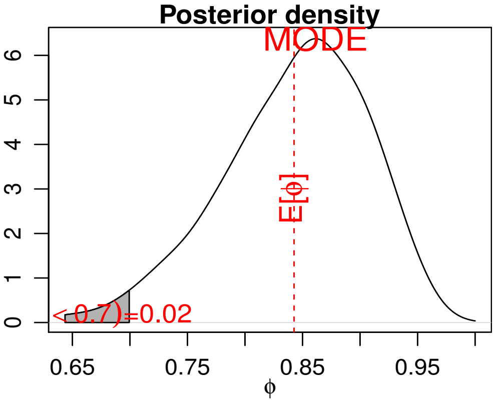
- survival $[0 = \text{died}, 1 = \text{survived}]$, $n = 30$ observations.
- $s_i \sim \text{Bern}(0.9)$

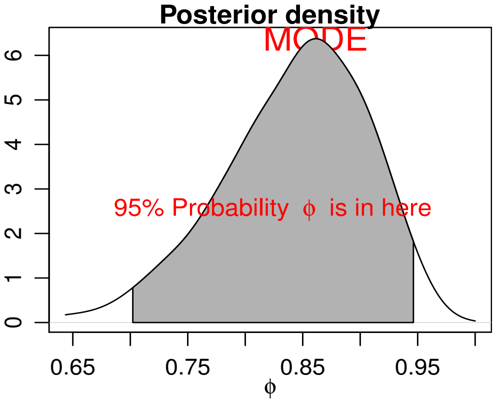
```
s <-
c(1,1,1,1,0,1,0,1,1,1,1,1,1,1,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1)
```

- estimate $\theta = [\phi]$: mean population survival
- priors: $p(\phi) = \text{Beta}(1, 1)$
- specify a likelihood: $\mathcal{L}(\mathbf{s}|\phi, n_s) = \prod_s^n \text{Bern}(s_i; \phi, n_s)$

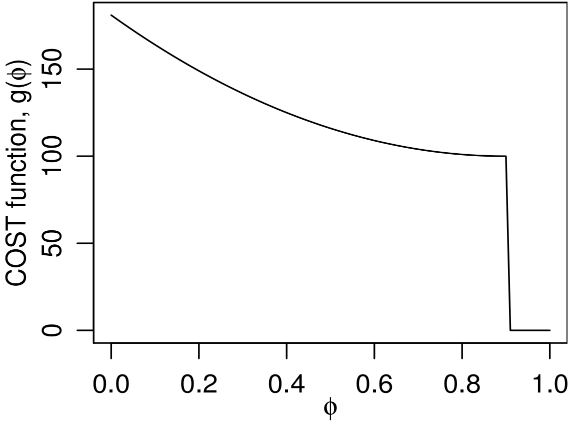
now run a Gibbs sampler to approximate the posterior $p(\phi|\mathbf{s})$

Posterior density

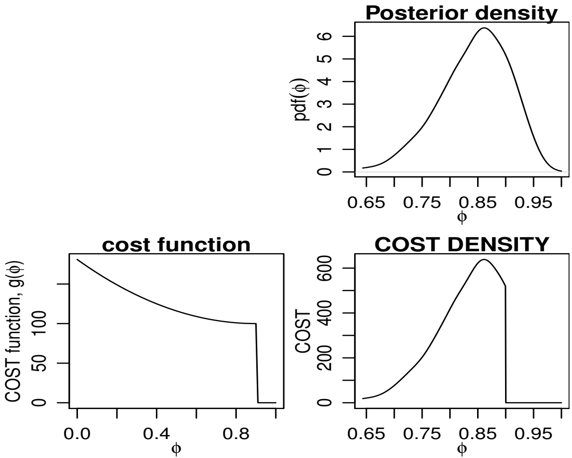



Posterior density
MODE
95% Probability ϕ is in here
0
1
2
3
4
5
6
0.65
0.75
0.85
0.95
ϕ

What if you have a "cost function" $g(phi)$? e.g., cost of conservation action conditional on the estimated values of ϕ?



Posterior inference: cost functions



$\mathbb{E}[\text{COST}] = \int_0^1 g(\phi)p(\phi|s)d\phi$

- full cost including all uncertainty in ϕ

Posterior Estimation vs Maximum Likelihood Estimation

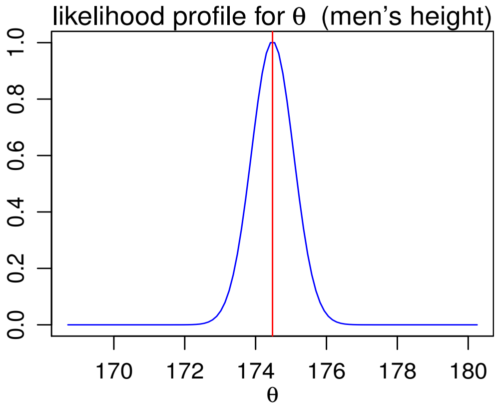
How do Bayesian posterior estimates compared to more-familiar (frequentist) point-estimates based on maximum likelihood?

method of maximum likelihood

- Most common method among Frequentists (single model estimation)
- "It would be very (un)likely to have seen the data that I saw, if the value of θ were X"
- Choose θ: that which *maximize's* the likelihood of seeing $\mathbf{y}$
- $\hat{\theta}_{\text{MLE}}$ is NOT the "most probabilty value of θ
- optimality properties: unbaised, efficient [10]

[10] but see shrinkage estimators for high-dimensional problems

Posterior Estimation vs Maximum Likelihood Estimation



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- compare to (approximate)[11] Posterior descriptive statistics

$E[\theta]$	SD	95CI-low	95CI-hi
172.21	1.51	169.21	175.16

nearly the same

Posterior Estimation vs Maximum Likelihood Estimation

for n getting LARGE, and for WEAK priors

- Posterior Mode $\theta_{\mathsf{MAP}} \to \hat{\theta}_{\mathsf{MLE}}$
- Posterior Confidence Intervals $\to$ Confidence Intervals

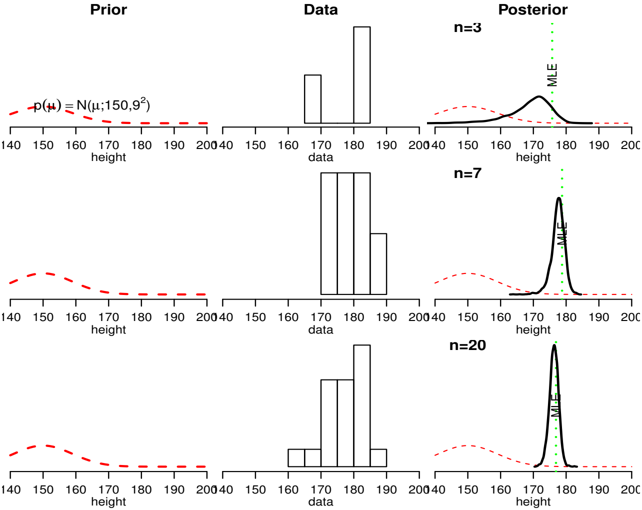
for low n and/or for STRONG priors

- shrinkage: $\theta \to$ Prior expectation.
- Posterior mean $\bar{\theta}$ is “biased" towards the priors

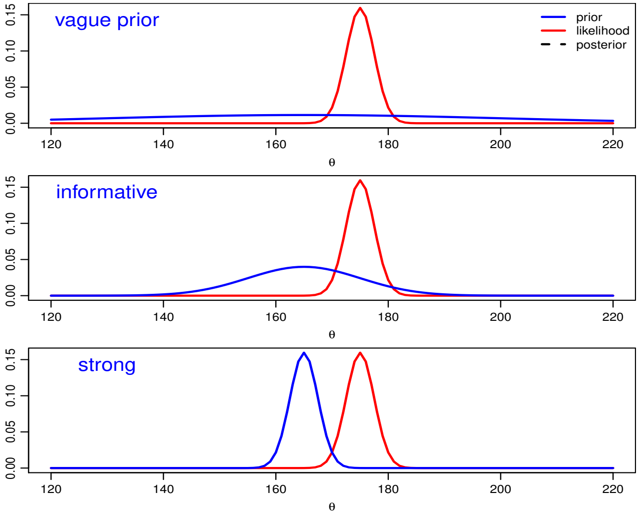
role of priors (from an estimation perspective)

- Priors retard/accerlate rate of convergence of $\bar{\theta} \to$ truth
- At low samples-sizes, “sensible" priors induce *shrinkage* and have better estimation properties than MLEs
- Key POINTS: you must be a master of prior distributions.

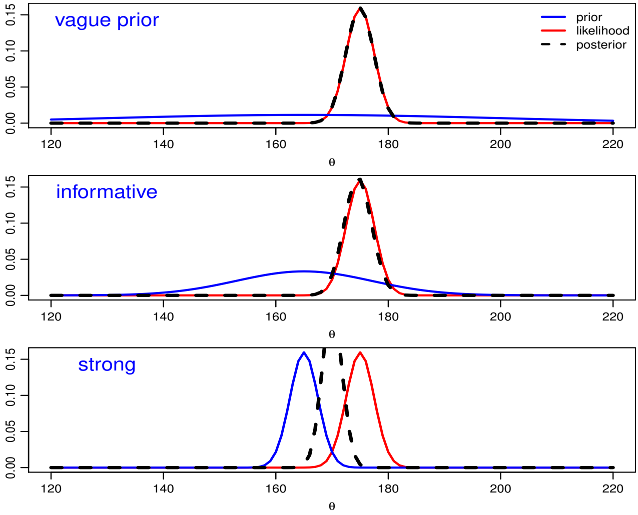
Posteriors and Sample Size



Posteriors and Prior information



Posteriors and Prior information



Priors

Most Biologists are reluctant Bayesians

- **Frequentist** vs. **Bayesian**: often desire that point estimates are identical between posteriors and MLEs
- but, only for: i) weak priors, and ii) large-samples sizes
- key point: **Be a Master of Priors!**

Priors and Philosophy of Probabilities

Subjective Personalist Bayesians

Probabilities are your “degree of belief"

- priors: prior beliefs
- posteriors: bring your beliefs into alignment with posterior
- decision making

Objective Logical Bayesians

Probabilities are continuous extension of Aristolean logic, deductive

- Probabilities capture “degree of truth"
- Priors: non-informative, set by default (Jeffrey's Priors, reference priors, language-invariant priors)

e.g., $p(\phi) = \text{Beta}(0.5, 0.5)$ (Jeffrey's Prior)

- Elicit priors from previous studies (posterior becomes new prior)

Priors and Philosophy of Probabilities

Instrumentalist

priors useful for good estimation properties

- shrinkage, efficiency

Frequentist

Principal principle: “probabilities (p(Event)) should align with long-run frequencies of Event"

- probabilities do not exist in reality

Other

- Quantum mechanics
- Propensities (Karl Popper)

Probability Distributions in JAGS/BUGS

- you must express your prior information probabilitistically

Know the distributions and their parameters (JAGS Manual)

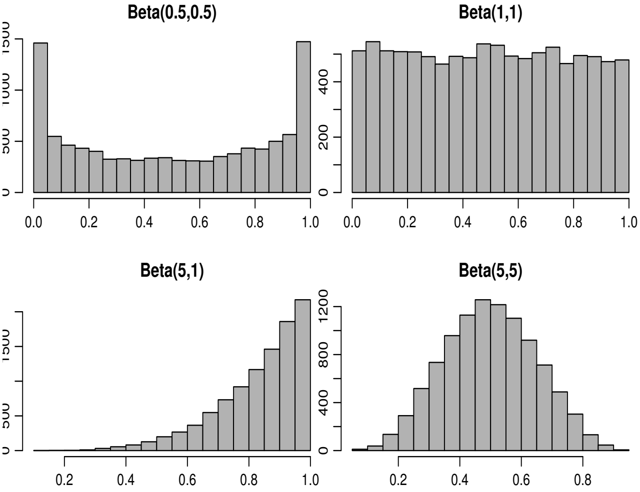
Name	Usage	Density	Lower	Upper
Beta	`dbeta(a,b)` $a > 0,\ b > 0$	$\frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}$	0	1
Chi-square	`dchisqr(k)` $k > 0$	$\frac{x^{\frac{k}{2}-1}\exp(-x/2)}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}$	0	
Double exponential	`ddexp(mu,tau)` $\tau > 0$	$\tau\exp(-\tau\lvert x-\mu\rvert)/2$		
Exponential	`dexp(lambda)` $\lambda > 0$	$\lambda\exp(-\lambda x)$	0	
F	`df(n,m)` $n > 0,\ m > 0$	$\frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\left(\frac{n}{m}\right)^{\frac{n}{2}} x^{\frac{n}{2}-1}\left\{1+\frac{nx}{m}\right\}^{-\frac{(n+m)}{2}}$	0	
Gamma	`dgamma(r, lambda)` $\lambda > 0,\ r > 0$	$\frac{\lambda^r x^{r-1}\exp(-\lambda x)}{\Gamma(r)}$	0	
Generalized gamma	`dgen.gamma(r,lambda,b)` $\lambda > 0,\ b > 0,\ r > 0$	$\frac{b\lambda^{br}x^{br-1}\exp\{-(\lambda x)^b\}}{\Gamma(r)}$	0	
Logistic	`dlogis(mu, tau)` $\tau > 0$	$\frac{\tau\exp\{(x-\mu)\tau\}}{[1+\exp\{(x-\mu)\tau\}]^2}$		
Log-normal	`dlnorm(mu,tau)` $\tau > 0$	$\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} x^{-1}\exp\left\{-\tau(\log(x)-\mu)^2/2\right\}$	0	
Noncentral Chi-squre	`dnchisqr(k, delta)` $k > 0, \delta \geq 0$	$\sum_{r=0}^{\infty}\frac{\exp(-\frac{\delta}{2})(\frac{\delta}{2})^r}{r!}\frac{x^{(k/2+r-1)}\exp(-\frac{x}{2})}{2^{(k/2+r)}\Gamma(\frac{k}{2}+r)}$	0	
Normal	`dnorm(mu,tau)` $\tau > 0$	$\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}}\exp\{-\tau(x-\mu)^2/2\}$		
Pareto	`dpar(alpha, c)` $\alpha > 0,\ c > 0$	$\alpha c^{\alpha}x^{-(\alpha+1)}$	c	

Intuiting Probability Distributions

- easy to learn in R

e.g., $r \sim \text{Beta}(a, b)$

```
r <- rbeta(10000, 0.5, 0.5)
```



Sample-based inference

Posteriors

often no 'analytical' solution to $(\theta|Y)$

Solution: Sampling

- it is a Probability Distribution!!!
- find a way to sample from posterior
- with enough samples: mean(samples) = Posterior Expectation

assuming $\theta_j \sim p(\theta|y)$ for $j = 1, \ldots, J$

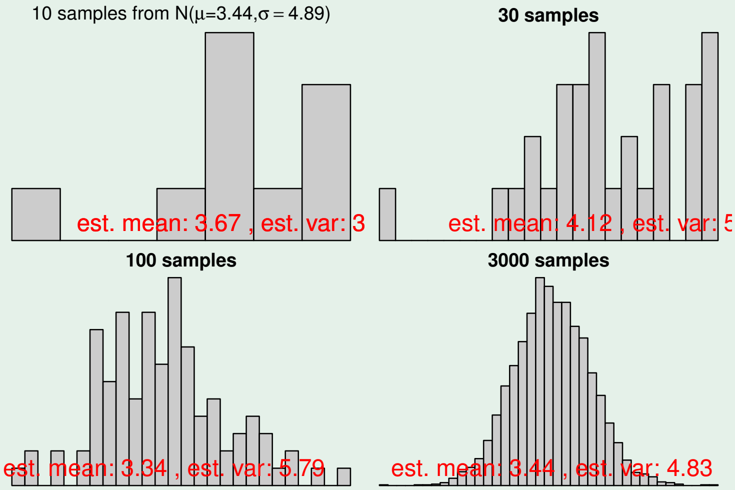
$$\begin{array}{lll}
\text{Expected Value} & = \int \theta p(\theta|y) d\theta & \approx \frac{1}{J} \sum_j^J \theta_j \\
\text{Standard Error}(\theta) & = SE(\theta) & \approx SD(\theta_j) \\
\text{Probability } \theta > 0 & = \int \mathbb{I}[\theta > 0] p(\theta|y) d\theta & \approx \frac{1}{J} \sum_j^J \mathbb{I}[\theta_j > 0]
\end{array}$$

Sampling Algorithms

MCMC; Gibbs Sampling; Metropolis-Hastings; Slice-Sampling; Importance Sampling; "Conjugate Priors";

example: estimate mean and variance of θ

$\theta_{\mathtt{true}} = 3.44$; $\mathrm{Var}(\theta)_{\mathtt{true}} = 4.89$



break-down joint posterior into (simpler) conditional distributions

- difficult: sampling $P(\beta_0, \beta_1, \beta_2, \sigma^2 | Y)$
- easy: sampling $P(\beta_0, \beta_1, \beta_2, | \sigma^2, Y)$ then $P(\sigma^2 | \beta_0, \beta_1, \beta_2, Y)$ then repeat

approximates the joint posterior

algorithm

- initialize: $\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}, \sigma^{2(0)}$

$$\begin{aligned}
\{\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}\} &\sim P(\beta | \sigma^{2(0)}, Y) \\
\sigma^{2(1)} &\sim P(\sigma^2 | \beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, Y) \\
\{\beta_0^{(2)}, \beta_1^{(2)}, \beta_2^{(2)}\} &\sim P(\beta | \sigma^{2(1)}, Y) \\
\sigma^{2(2)} &\sim P(\sigma^2 | \beta_0^{(2)}, \beta_1^{(2)}, \beta_2^{(2)}, Y)
\end{aligned} \tag{5}$$

- repeat 1000's or 1000000 's times

BUGS to the rescue

Previously, Bayesian analysis demanded custom-coding MCMC algorithms

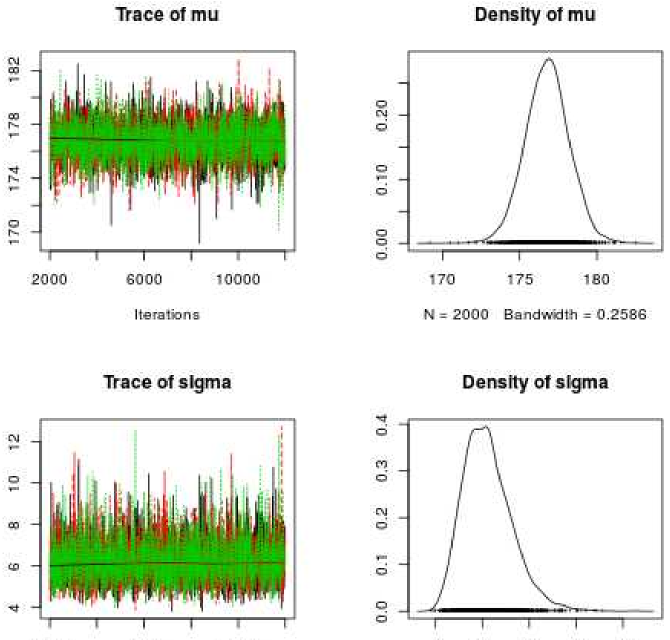
WinBUGS & OpenBUGS & JAGS

automatically use appropriate sampling techniques; so we don't have to worry

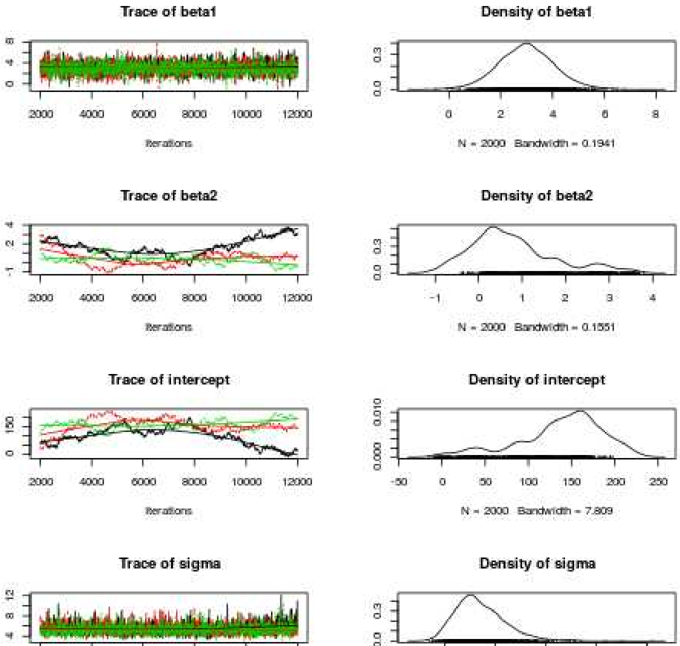
BUT you must: Monitor the MCMC!

- give reasonable **initial values**
- ensure **convergence**: no trend; independent chains give same answer
- ensure adequate **mixing**: independent samples

MCMC: Good mixing



MCMC: Poor convergence



MCMC parameters in JAGS

- `n.chains`: num. of MCMC chains; more is better
- `n.adapt`: discard first samples; let algorithm 'adapt'
- `n.burn`: discard extra samples; allow algorithm to reach stationary distribution
- `n.iter`: total number of sample; more is better
- `thin`: take every k^{th} iteration for a sample; decorrelates one sample from the next; higher is better
- total samples: number of samples to approximate your Posterior; target at least 2000 to 5000

MCMC: what to do with bad mixing

- run longer chains
- ensure long enough adaption phase
- misspecified priors
- bad initial values?

Time to open up R and JAGS

'JAGS: Just Another Gibbs Sampler'

Uses BUGS-like syntax (similar to OpenBUGS, WinBUGS)

- `rjags` Package: R friendly JAGS interface
- easy easy easy Bayesian *estimation*
- not so easy for *model selection*

Don't worry about 'samplers': JAGS does the hard work

- specify **likelihood** (how the data arose) and the **priors**

Bayesian Analysis Example

example model: counts of survival of 30 animals

```
s <-
c(1,1,1,1,0,1,0,1,1,1,1,1,1,1,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1)
```

Same data: three models

- Exercise 1: Bernoulli model: estimate mean survival with `dbeta` priors
- Exercise 2: Bernoulli model with `logit-normal` priors
- Exercise 3: logistic regression model

Bayesian Analysis Example 1

- open up R and `rjags`
- go to the file `BayesCMR_workshop/PART3_introJAGS/R_jags_intro.R`

JAGs models in general

In every analysis, we will precede as follows:

1. Data pre-processing in R
2. Write the JAGS syntax -> save to a JAGS file
3. Assemble the JAGS data
4. Initialize random variables (aka stochastic nodes)
5. Compile the jags model
6. Burn-in phase
7. Sample from the posteriors
8. Do what you want with posteriors!

JAGS steps 1,2

1) Data pre-processing in R

duh

2) Write the JAGS syntax -> save to a JAGS file

all syntax looks like this basic structure...

```
model{
  # SPECIFY PRIORS
  phi ~ dbeta(pr.phi[1],pr.phi[2])
  # SPECIFY LIKELIHOOD | parameter
  for(i in 1:n){
     y[i] ~ dbern(phi)
  }
}
```

Always need section for:

- priors
- likelihood | parameter

JAGS steps 3

3) Assemble the JAGS data

- Jags wants a NAMED LIST of data.
- The names in the JAGS syntax must match the names in the list

```
jags.data<-list(
  y = c(1,0,0,0,0,1,0,0,2),  # response variable
  n = length(n), # length of data (sample size) for loop
  pr.phi = c(4,1) # prior parameters
)
```

JAGS steps 4

4) Initialize random variables (aka stochastic nodes)

- make a function that will *seed* random values for all RANDOM VARIABLES.
- the function must return a NAMED LIST with names equal to the names of random variables in the jags syntax

```
jags.inits.f <- function(){ # no arguments
  ret<-list(
       phi=runif(1,0,1)
      )
  return(ret)}
```

What is a random variable?

Every with a tilde $\sim$ after it (except the data y)

JAGS stap 5,6,7,8

5) Compile the jags model

```
m <- jags.model(file ="my.first.model", data=jags.data,
   inits = jags.inits.f, n.chains=3, n.adapt=1000)
```

6) Burn-in phase

```
update(m,1000)
```

7) Sample from the posteriors

```
post <- coda.samples(model=m, variable.names=c("phi"),
   n.iter=30000, thin=300)
```

8) Do what you want with posteriors!

```
summary(post)
```

Exercises

There are three exercises

- Exercise 1: A simple coin-flip survival model with Beta priors
- Exercise 2: A simple coin-flip survival model with logit-Normal priors
- Exercise 3: A logistic regression model