CS223A - Introduction to Robotics - Homework #1

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Question 1

1. First, rotate about Y_B by θ:

$$ {}^A_BR_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

2. Then rotate about Z_B by ϕ :

$$ {}^A_BR_2 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

3. Then calculate the ${}^A_BR_1 \times {}^A_BR_2$:

$$ {}^A_BR_1 \times {}^A_BR_2 \equiv \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi + \sin\phi & -\cos\theta\sin\phi & \sin\theta \\ \sin\phi & \cos\phi & 0 \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} $$

Question 2

$$ \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\phi & \cos\phi\sin\theta \\ \sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

Question 3

$$ {}^B_AT = \begin{bmatrix} {}^B_AR & {}^BP_{Aorg} \\ 0\ 0\ 0 & 1 \end{bmatrix} $$

$$ {}^B_AT^{-1} = {}^A_BT = \begin{bmatrix} {}^B_AR^T & -{}^B_AR^T \cdot {}^BP_{Aorg} \\ 0\ 0\ 0 & 1 \end{bmatrix} $$

(1)

1. Calculate ${}^B_A R$:

$$ {}^B_A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} $$

2. Calculate ${}^B_A R^T$:

$$ {}^B_A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} $$

$$ -{}^B_A R^T \cdot {}^B P_{Aorg} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta \\ 0 & \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2\cos\theta - 3\sin\theta \\ 2\sin\theta - 3\cos\theta \end{bmatrix} $$

$$ {}^A_B T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \cos\theta & \sin\theta & -2\cos\theta - 3\sin\theta \\ 0 & -\sin\theta & \cos\theta & 2\sin\theta - 3\cos\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(2)

$$ \theta = 45^\circ $$

$$ {}^A_B T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -2 \cdot \frac{\sqrt{2}}{2} - 3 \cdot \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \cdot \frac{\sqrt{2}}{2} - 3 \cdot \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} \\ 0 & -\frac{2}{2} & \frac{2}{2} & -\frac{2}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^A P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{5\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{2}{2} & -\frac{2}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{5\sqrt{2}}{2} + \frac{6\sqrt{2}}{2} - \frac{5\sqrt{2}}{2} \\ -\frac{5\sqrt{2}}{2} + \frac{6\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3\sqrt{2} \\ 0 \\ 1 \end{bmatrix} $$

Question 4

(1)

$$ r_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T $$

$$ r_2 = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T $$

$$ r_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T $$

$$ \|r_1\| = \sqrt{(\tfrac{\sqrt{2}}{2})^2 + (-\tfrac{1}{2})^2 + (-\tfrac{1}{2})^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4}} = 1 $$

$$ \|r_2\| = \sqrt{0^2 + (\tfrac{\sqrt{2}}{2})^2 + (\tfrac{1}{2})^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4}} = 1 $$

$$ \|r_3\| = \sqrt{(\tfrac{\sqrt{2}}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4}} = 1 $$

$$ r_1 \cdot r_2 = \tfrac{\sqrt{2}}{2} \cdot 0 - \tfrac{1}{2} \cdot \tfrac{\sqrt{2}}{2} + \tfrac{1}{2} \cdot \tfrac{\sqrt{2}}{2} = -\tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{4} = 0 $$

$$r_1 \cdot r_3 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 0$$
$$r_2 \cdot r_3 = 0 \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = 0 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} = 0$$

so it is a rotation matrix.

(2)

$$\theta = \arccos(\tfrac{r_{11}+r_{12}+r_{33}-1}{2}) = \arccos(\tfrac{2\sqrt{2}+1}{4})$$
$$K = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

(3)

For homework:

$$\epsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2}} = \frac{1}{2}\sqrt{\sqrt{2} + \frac{3}{2}}$$
$$\epsilon_1 = \frac{r_{32}-r_{23}}{4\epsilon_4} = \frac{-\frac{1}{\sqrt{2}}-\frac{1}{2}}{2\sqrt{\sqrt{2}+\frac{3}{2}}} = \frac{-\sqrt{2}+1}{4\sqrt{\sqrt{2}+\frac{3}{2}}}$$
$$\epsilon_2 = \frac{r_{13}-r_{31}}{4\epsilon_4} = \frac{\frac{1}{\sqrt{2}}+\frac{1}{2}}{4\epsilon_4} = \frac{\sqrt{2}+1}{4\sqrt{\sqrt{2}+\frac{3}{2}}}$$
$$\epsilon_3 = \frac{r_{21}-r_{12}}{4\epsilon_4} = \frac{-\frac{1}{2}-0}{2\sqrt{\sqrt{2}+\frac{3}{2}}} = -\frac{1}{4\sqrt{\sqrt{2}+\frac{3}{2}}}$$