Intro to Robotics HW7

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1 Problem 1

A robotic arm has two links with $L_1 = 1.2m$, and $L_2 = 0.8m$. The end effector needs to move from point A [-1.2, 0.9] to B [1.1, 0.8]. The end effector weights $m_e = 1kg$ and each link weights $m_1 = m_2 = 1kg$ with uniform distributed mass. A robotic arm has two links with $L_1 = 1.2m$, and $L_2 = 0.8m$. The end effector needs to move from point A [-1.2, 0.9] to B [1.1, 0.8]. The end effector weights $m_e = 1kg$ and each link weights $m_1 = m_2 = 1kg$ with uniform distributed mass.

Please write down the kinetic energy T and potential energy V of this system. Please derive the equation of motion, i.e., obtain the expression of τ_1 and τ_2. Please write down the mass matrix M, centrifugal and Coriolis terms V, and gravity terms G.

1.1 Kinetic Energy T

System Parameters (Given)

- Length of Link 1: $l_1 = 1.2\,\mathrm{m}$
- Length of Link 2: $l_2 = 0.8\,\mathrm{m}$
- Joint Angles: θ_1, θ_2

Mass Properties

- End-effector Mass: m_e
- Mass of Each Link: m

Center of Link Mass

- $l_{c1} = \frac{l_1}{2}$
- $l_{c2} = \frac{l_2}{2}$

Moment of Inertia

- $I_1 = \frac{1}{12} m l_1^2$
- $I_2 = \frac{1}{12} m l_2^2$

Total kinetic Energy

- $T = T_{link1,trans} + T_{link1,rot} + T_{link2,trans} + T_{link2,rot} + T_{end,effector}$

Position Vector

$$P_{c1} = \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = \begin{bmatrix} \frac{L_1}{2}\cos\theta_1 \\ \frac{L_1}{2}\sin\theta_1 \end{bmatrix}$$

$$P_{c2} = \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + \frac{L_2}{2}\sin(\theta_1+\theta_2) \end{bmatrix}$$

$$P_e = \begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \end{bmatrix}$$

Velovity of link1

$$\mathbf{v}_{c1} = \begin{bmatrix} -\frac{L_1}{2}\sin\theta_1\cdot\dot{\theta}_1 \\ \frac{L_1}{2}\cos\theta_1\cdot\dot{\theta}_1 \end{bmatrix}$$

$$T_1 = \frac{1}{2}m_1 v_{c1}^2 + \frac{1}{2}I_1\omega_1^2$$

Velovity of link2

$$\mathbf{v}_{c2} = \begin{bmatrix} -L_1 sin\theta_1\dot{\theta}_1 - \frac{L_2}{2}sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ L_1 cos\theta_1\dot{\theta}_1 + \frac{L_2}{2}cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$$

Velovity of End-effector

$$\mathbf{v}_e = \begin{bmatrix} -L_1 sin\theta_1\dot{\theta}_1 - L_2 sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ L_1 cos\theta_1\dot{\theta}_1 + L_2 cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$$

Kinetic for link 1

$$T_1 = T_{1t} + T_{1r} = \frac{1}{2}m_1 v_{c1}^2 + \frac{1}{2}I_1\dot{\theta}_1^2$$

where $I_1 = \frac{1}{12}m_1 L_1^2$

$$T_1 = \frac{1}{2}m_1\left(\frac{L_1}{2}\dot{\theta}_1\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_1L_1^2\right)\dot{\theta}_1^2 = \frac{1}{2}m_1L_1^2\left(\frac{1}{4}+\frac{1}{12}\right)\dot{\theta}_1^2 = \frac{1}{6}m_1L_1^2\dot{\theta}_1^2$$

Kinetic for link 2

$$T_2 = T_{2t} + T_{2r} = \frac{1}{2}m_2 v_{c2}^2 + \frac{1}{2}I_2(\dot{\theta}_1+\dot{\theta}_2)^2$$

where $I_2 = \frac{1}{12}m_2 L_2^2$

$$T_2 = \frac{1}{2}m_2\left[L_1^2\dot{\theta}_1^2 + \frac{L_2^2}{4}(\dot{\theta}_1+\dot{\theta}_2)^2 + 2L_1L_2c2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\right] + \frac{1}{2}\left(\frac{1}{12}m_2L_2^2\right)(\dot{\theta}_1+\dot{\theta}_2)^2$$

Kinetic Energy of End Effector (T_e)

$$T_e = \frac{1}{2}m_e v_e^2$$

$$T_e = \frac{1}{2}m_e\left[L_1^2\dot{\theta}_1^2 + L_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2L_1L_2cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\right]$$

Total Kinetic Energy T The total kinetic energy is $T = T_1 + T_2 + T_e$. We substitute the given values $L_1 = 1.2$, $L_2 = 0.8$, $m_1 = m_2 = m_e = 1$. $L_1^2 = 1.44$, $L_2^2 = 0.64$, $L_1L_2 = 0.96$. $I_1 = \frac{1}{12}(1)(1.44) = 0.12$, $I_2 = \frac{1}{12}(1)(0.64) \approx 0.0533$.

$$T_1 = \frac{1}{6}(1)(1.44)\dot{\theta}_1^2 = 0.24\dot{\theta}_1^2$$

$$T_2 = \frac{1}{2}(1)\left[(1.6+0.96c_2)\dot{\theta}_1^2 + 0.16\dot{\theta}_2^2 + (0.32+0.96c_2)\dot{\theta}_1\dot{\theta}_2\right] + 0.0267(\dot{\theta}_1+\dot{\theta}_2)^2$$

$$T_e = \frac{1}{2}(1)\left[(2.08+1.92c_2)\dot{\theta}_1^2 + 0.64\dot{\theta}_2^2 + (1.28+1.92c_2)\dot{\theta}_1\dot{\theta}_2\right]$$

Total Kinetic Energy T

$$T = \frac{1}{2}\left(4.2133 + 2.88c_2\right)\dot{\theta}_1^2 + \frac{1}{2}(0.8533)\dot{\theta}_2^2 + \left(0.8533 + 1.44c_2\right)\dot{\theta}_1\dot{\theta}_2$$

1.2 Potential Energy

$$V = g\left[m_1\left(\frac{L_1}{2}s_1\right) + m_2\left(L_1 s_1 + \frac{L_2}{2}s_{12}\right) + m_e\left(L_1 s_1 + L_2 s_{12}\right)\right]$$

where $m_1 = m_2 = m_e = 1kg,\ L_1 = 0.8m, L_2 = 1m$

$$V = g\left[3.0\sin\theta_1 + 1.2\sin(\theta_1 + \theta_2)\right]$$

Lagrangian and Equation of Motion

The Euler-Lagrange equations for the generalized coordinate q_i (where $q_1 = \theta_1, q_2 = \theta_2$) under generalized force τ_i are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = \tau_i$$

1.3 Mass Matrix M

we take the partial derivative for total kinetic energy

$$\frac{\partial T}{\partial \dot{q}} = \begin{bmatrix}(4.2133 + 2.88c_2)\dot{\theta}_1 + (0.8533 + 1.44c_2)\dot{\theta}_2 \\ 0.8533\dot{\theta}_2 + (0.8533 + 1.44c_2)\dot{\theta}_1\end{bmatrix}$$

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix}M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}$$

M_{11} is coefficient of $\dot{\theta}_1$:

$$M_{11} = 4.2133 + 2.88\cos\theta_2$$

M_{22} is the coefficient of $\frac{1}{2}\dot{\theta}_2^2$:

$$M_{22} = 0.8533$$

$M_{12} = M_{21}$:

$$M_{12} = M_{21} = 0.8533 + 1.44\cos\theta_2$$

so

$$\mathbf{M} = \begin{bmatrix}4.2133 + 2.88c_2 & 0.8533 + 1.44c_2 \\ 0.8533 + 1.44c_2 & 0.8533\end{bmatrix}$$

1.4 Coriolis and Centrifugal Terms (Vector $\mathbf{C\dot{q}}$)

The Euler-Lagrange equations lead to the general form:

$$\mathbf{M\ddot{q}} + \mathbf{C\dot{q}} + \mathbf{G} = \boldsymbol{\tau}$$

It concluding $\dot{M}(q)\dot{q}$ and $-\frac{\partial T}{\partial q_i}$

$$\dot{M} = \begin{bmatrix}-2.88s_2\dot{\theta}_2 & -1.44s_2\dot{\theta}_2 \\ -1.44s_2\dot{\theta}_2 & 0\end{bmatrix}$$

$$-\frac{\partial T}{\partial q} = \begin{bmatrix}0 \\ 1.44s_2\dot{\theta}_1^2 + 1.44s_2\dot{\theta}_1\dot{\theta}_2\end{bmatrix}$$

$$\dot{M}\dot{q} - \frac{\partial T}{\partial q_i} = \mathbf{C\dot{q}} = \begin{bmatrix}-2.88s_2\dot{\theta}_1\dot{\theta}_2 - 1.44s_2\dot{\theta}_2^2 \\ -1.44s_2\dot{\theta}_1^2\end{bmatrix}$$

1.5 Gravity Terms G

The gravity vector $\mathbf{G}$ is obtained by $\mathbf{G} = \frac{\partial V}{\partial \mathbf{q}}$.

$$\mathbf{G} = \frac{\partial V}{\partial q} = \begin{bmatrix} g[3cos\theta_1 + 1.2cos(\theta_1 + \theta_2)] \\ g1.2cos(\theta_1 + \theta_2) \end{bmatrix}$$

1.6 Torques($\boldsymbol{\tau}$)

The required torques $\boldsymbol{\tau} = [\tau_1, \tau_2]^T$ are given by the equation :

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q})$$

$$\tau_1 = (4.2133 + 2.88c2)\ddot{\theta_1} + (0.8533 + 1.44c_2)\ddot{\theta_2} - 2.88s_2\dot{\theta_1}\dot{\theta_2} - 1.44s_2\dot{\theta_2^2} + g[3c_1 + 1.2c_{12}]$$

$$\tau_2 = (0.8533 + 1.44c_2)\ddot{\theta_1} + 0.8533\ddot{\theta_2} - 1.44s_2\dot{\theta_1^2} + g1.2c_{12}$$