Modern Algebra

Assignment 3

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3 Subgroups

Problem 2

$\langle \frac{1}{2} \rangle$ in $\mathbf{Q} = \{\cdots, \frac{-3}{2}, -1, \frac{-1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$
$\langle \frac{1}{2} \rangle$ in $\mathbf{Q}^* = \{\cdots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \cdots\}$

Problem 4

Proof. Suppose we have a group G. We pick $g \in G$ and suppose its order is $n \in \mathbb{Z}$. To prove that an element t has order n we need to prove that $t^n = e$ and there is no integer $s < n$ which satisfies $t^s = e$. We know that $g^n = e$. Observe that $(g^{-1})^n = (g^n)^{-1} = e^{-1}$.But the inverse of the identity element is the identity element. Thus, $(g^{-1})^n = e$. For the second part, assume that we have $s \in \mathbb{Z}$, and $(g^{-1})^s = e$. We know that $(g^{-1})^s = (g^s)^{-1} = e$. Now, multiply both sides by g^s to get

$$\begin{aligned} (g^s)^{-1}g^s &= eg^s \\ e &= g^s. \end{aligned}$$

We know that n is the least integer that satisfies $g^n = e$. So, $s >= n$, which contradicts our assumption that $s < n$. So, s Therefore, we proved that for any group, any element and its inverse have the same order. ■

Problem 6

b) $|a| = 4, |b| = 3, |a + b| = 12.$

Problem 7

$$
\begin{aligned}
(a^4c^{-2}b^4)^{-1} = (a^{6-2}c^{-2}b^{7-3})^{-1} &= ((a^6a^{-2})c^{-2}(b^7b^{-3}))^{-1} \\
&= ((ea^{-2})c^{-2}(eb^{-3}))^{-1} \\
&= (a^{-2}c^{-2}b^{-3})^{-1} \\
&= (a^{-2}(c^{-2}b^{-3}))^{-1} \\
&= ((c^{-2}b^{-3})^{-1}(a^{-2})^{-1}) \text{ Using } \textit{Socks-Shoes Property} \\
&= (b^3c^2a^2) \text{ Using } \textit{Socks-Shoes Property}
\end{aligned}
$$

Problem 10

We observe the cayley table of D_4 and get that:
$\{R_0, R_{90}, R_{180}, R_{270}\}, \{R_{180}, R_0, H, V\}, \{R_{180}, R_0, D, L\}$ are the possible subgroups from D_4.

Problem 19

Proof. Let a is a group element which has an infinite order. That implies that there is no $s \in \mathbb{Z}^+$ that satisfies $a^n = e$. We pick $n, m \in \mathbb{Z}$. We want to proof that if $m \neq n$ then $a^m \neq a^n$. Assume, $m \neq n$, we need to show $a^m \neq a^n$. For the sake of contradiction, assume that $a^m = a^n$. Without loss of generality, assume $m > n$. Then, we multiply both sides by $(a^n)^{-1}$:

$$
\begin{aligned}
a^m(a^n)^{-1} &= a^n(a^n)^{-1} \\
a^m a^{-n} &= e \\
a^{m-n} &= e.
\end{aligned}
$$

We see that we found an integer $m - n$ such that $g^{m-n} = e$. But we know that if an element g has infinite order then there is no integer s such that $g^s = e$. So we see that we clearly reached a contradiction. So, $a^m \neq a^n$ must be true. Therefore, we proved that $a^m \neq a^n$ when $m \neq n$ for every group element with infinite order. ■

Problem 30

H is clearly the group of even integers.

Problem 34

Proof. Since H, K are subgroups of G, then every element in both H, K is an element in G. By definition, we know that the intersection between two sets is the shared elements among these sets. We pick $s, t \in K \cap H$. We know that $s, t \in H$, then $st \in H$. We also know that $s, t \in K$, then $st \in K$. Thus $st \in H \cap K$ and the operation is closed. Since H, K are groups, they both have the identity element. So, it is clear that $e \in H \cap K$. Since, $a \in H$ and $a \in K$ then $a^{-1} \in H, a^{-1} \in K$. Therefore, we showed that $H \cap K$ is a subgroup of G. ■

For the second part, the same proof extend to any number of subgroups.