Modern Algebra

Assignment 3

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3 Subgroups

Problem 2

$\langle \frac{1}{2} \rangle$ in $\mathbf{Q} = \{\cdots, \frac{-3}{2}, -1, \frac{-1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$
$\langle \frac{1}{2} \rangle$ in $\mathbf{Q}^* = \{\cdots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \cdots\}$

Problem 4

Proof. Suppose we have a group G. We pick $g \in G$ and suppose its order is $n \in \mathbb{Z}$. To prove that an element t has order n we need to prove that $t^n = e$ and there is no integer $s < n$ which satisfies $t^s = e$. We know that $g^n = e$. Observe that $(g^{-1})^n = (g^n)^{-1} = e^{-1}$.But the inverse of the identity element is the identity element. Thus, $(g^{-1})^n = e$. For the second part, assume that we have $s \in \mathbb{Z}$, and $(g^{-1})^s = e$. We know that $(g^{-1})^s = (g^s)^{-1} = e$. Now, multiply both sides by g^s to get

$$\begin{aligned} (g^s)^{-1} g^s &= e g^s \\ e &= g^s. \end{aligned}$$

We know that n is the least integer that satisfies $g^n = e$. So, $s >= n$, which contradicts our assumption that $s < n$. So, s Therefore, we proved that for any group, any element and its inverse have the same order. ■

Problem 6

b) $|a| = 4, |b| = 3, |a+b| = 12.$

Problem 7

$$\begin{aligned}
(a^4c^{-2}b^4)^{-1} = (a^{6-2}c^{-2}b^{7-3})^{-1} &= ((a^6a^{-2})c^{-2}(b^7b^{-3}))^{-1} \\
&= ((ea^{-2})c^{-2}(eb^{-3}))^{-1} \\
&= (a^{-2}c^{-2}b^{-3})^{-1} \\
&= (a^{-2}(c^{-2}b^{-3}))^{-1} \\
&= ((c^{-2}b^{-3})^{-1}(a^{-2})^{-1}) \text{ Using } \textit{Socks-Shoes Property} \\
&= (b^3c^2a^2) \text{ Using } \textit{Socks-Shoes Property}
\end{aligned}$$

Problem 10

We observe the cayley table of D_4 and get that:
$\{R_0, R_{90}, R_{180}, R_{270}\}, \{R_{180}, R_0, H, V\}, \{R_{180}, R_0, D, L\}$ are the possible subgroups from D_4.

Problem 19

Proof. Let a is a group element which has an infinite order. That implies that there is no $s \in \mathbb{Z}^+$ that satisfies $a^n = e$. We pick $n, m \in \mathbb{Z}$. We want to proof that if $m \neq n$ then $a^m \neq a^n$. Assume, $m \neq n$, we need to show $a^m \neq a^n$. For the sake of contradiction, assume that $a^m = a^n$. Without loss of generality, assume $m > n$. Then, we multiply both sides by $(a^n)^{-1}$:

$$\begin{aligned}
a^m(a^n)^{-1} &= a^n(a^n)^{-1} \\
a^ma^{-n} &= e \\
a^{m-n} &= e.
\end{aligned}$$

We see that we found an integer $m - n$ such that $g^{m-n} = e$. But we know that if an element g has infinite order then there is no integer s such that $g^s = e$. So we see that we clearly reached a contradiction. So, $a^m \neq a^n$ must be true. Therefore, we proved that $a^m \neq a^n$ when $m \neq n$ for every group element with infinite order. ■

Problem 30

H must be the group of even integers.

Proof. We see that the group is proper, so it cannot be the group of integers. We know that the group has the elements $18, 30, 40$, and so it must has their multiples. Since these elements are in the group, we can get any linear combination of them using addition. That is

$$(18x + 30y + 40z), \text{ where } x, y, z \in \mathbb{Z}.$$

By *Bezout's identity*, any linear combination of numbers is a multiple of their *gcd.* Thus, the group these elements create is the group that contains multiples of their gcd, which is

$$\langle gcd(18, 30, 40)\rangle = \langle 2\rangle.$$

Therefore, we can see H is clearly the group of all even integers. ■

Problem 34

Proof. Since H, K are subgroups of G, then every element in both H, K is an element in G. By definition, we know that the intersection between two sets is the shared elements among these sets. We pick $s, t \in K \cap H$. We know that $s, t \in H$, then $st \in H$. We also know that $s, t \in K$, then $st \in K$. Thus $st \in H \cap K$ and the operation is closed. Since H, K are groups, they both have the identity element. So, it is clear that $e \in H \cap K$. Since, $a \in H$ and $a \in K$ then $a^{-1} \in H, a^{-1} \in K$. Therefore, we showed that $H \cap K$ is a subgroup of G. ■

For the second part, the same proof extend to any number of subgroups.