Modern Algebra

Assignment 2

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2 Groups

Problem 1

b) Division of non-zero integers is not closed, as $\frac{1}{2} \notin \mathbb{Z}$.

d) Multiplying 2×2 matrices with integer entries is closed. Pick $A, B, C 2 \times 2$ marices, let $AB = C$ we know from the Linear Algebra course that C has the same number of rows as A and the same number of columns in B. Thus, C is a 2×2 matrix.

Problem 2

a) Subraction of integers is not associative, as $(5-3)-2 = 0 \neq 5-(3-2) = 4$.

b) Division by non-zero rationals is not associative, as $(16/8)/4 = \frac{1}{2} \neq 16/(8/4) = 8$.

e) Exponentiation of integers is not associative, as $(2^2)^3 = 64 \neq 2^{(2^3)} = 2^8 = 256$.

Problem 3

c) Let $g(x) = x^2, f(x) = x+1$, we observe that

$$f(g(x)) = x^2 + 1 \neq g(f(x)) = x^2 + 2x + 1.$$

So, $f(g(x)) \neq g(f(x))$ and function composition of polynomials with real coefficients is not commutative.

d) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, we observe that

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}, BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

So, the multiplication of 2×2 matrices with real entries is not commutative.

Problem 5

a) The inverse of 13 is 7, as $(13 + 7) \mod 20 = 0$

b) The inverse of 13 is 13, as $(13 * 13) \mod 14 = 169 \mod 14 = 1$.

Problem 7

First reason: The set is not closed under addition, as $3 + 5 = 8$ wich is an even number.
Second reason: It does not have the identity element.

Problem 14

$(ab)^3 = (ab)(ab)(ab)$. Since multiplication is associative, we can remove the paranthases. So, $(ab)^3 = (ab)(ab)(ab) = ababab$.
$(ab^{-2}c)^{-2} = (ab^{-2}c)^{-1}(ab^{-2}c)^{-1}$. By the *Socks-Shoes Property* and the associativity of multiplication, $((ab^{-2})c)^{-1} = c^{-1}(ab^{-2})^{-1} = c^{-1}(b^{-2})^{-1}a^{-1} = c^{-1}b^2a^{-1}$. Thus,

$$(ab^{-2}c)^{-1}(ab^{-2}c)^{-1} = c^{-1}b^2a^{-1}c^{-1}b^2a^{-1}.$$

Problem 16

Proof. We construct the cayley table of the operation:

	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

From the table, we observe that the set is closed under multiplication modulo 40, 25 is the identity element, and every element has an inverse which is the number itself. The modular multiplication is associative by definition. Therefore, the set $\{5, 15, 25, 35\}$ with the operation multiplication modulo 40 is a group. ■

We know that $U(8) = \{1, 3, 5, 7\}$. We construct the cayley table of $U(8)$:

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

We see that the group is similar to $U(8)$, they both have four elements and each element is the inverse of itself.

Problem 18

From the cayley table of D_4, we can conclude that:

$$K = \{R_0, R_{180}\}, L = \{R_0, R_{180}, H, V, D, L\}$$

Problem 33

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Problem 34

First part:

Proof. We need to proof that $(ab)^2 = a^2b^2$ iff $ab = ba$. We start by the forward direction. Assume $(ab)^2 = a^2b^2$ is true, so we have that $(ab)(ab) = abab = a^2b^2$. We multiply each side by b^{-1} from the right to get

$$\begin{aligned} aba(bb^{-1}) &= a^2b^2b^{-1} \\ aba(e) &= a^2b(bb^{-1}) \\ aba &= a^2be = a^2b \end{aligned}$$

Then, we multiply each side by a^{-1} from the left:

$$\begin{aligned}
a^{-1}aba &= a^{-1}a^2b \\
(a^{-1}a)ba &= a^{-1}aab \\
eba &= (a^{-1}a)ab \\
ba &= eab \\
ba &= ab.
\end{aligned}$$

Hence, we proved that $ab = ba$ if $(ab)^2 = a^2b^2$.

For the backward direction, we assume that $ab = ba$. Then, $(ab)^2 = (ab)(ab) = abab = (ab)ab = ((ab)a)b = (a(ba))b = (a(ab))b = ((aa)b)b = (a^2b)b = a^2bb = a^2b^2$. Thus, we proved that if $ab = ba$ then $(ab)^2 = a^2b^2$. Therefore, we proved that $(ab)^2 = a^2b^2$ iff $ab = ba$. ∎

Second part:

Proof. We need to proof that $(ab)^{-2} = b^{-2}a^{-2}$ iff $ab = ba$. We start by the forward direction. Assume $(ab)^{-2} = b^{-2}a^{-2}$ is true, so we have that $(ab)^{-1}(ab)^{-1} = (b^{-1}a^{-1})(b^{-1}a^{-1}) = b^{-1}a^{-1}b^{-1}a^{-1} = b^{-2}a^{-2}$. We multiply each side by a from right to get

$$\begin{aligned}
b^{-1}a^{-1}b^{-1}a^{-1}a &= b^{-2}a^{-2}a \\
b^{-1}a^{-1}b^{-1}a^{-1}a &= b^{-2}a^{-1}a^{-1}a \\
b^{-1}a^{-1}b^{-1}(a^{-1}a) &= b^{-2}a^{-1}(a^{-1}a) \\
b^{-1}a^{-1}b^{-1}e &= b^{-2}a^{-1}e \\
b^{-1}a^{-1}b^{-1} &= b^{-2}a^{-1}
\end{aligned}$$

Then, we multiply each side by b from the left:

$$\begin{aligned}
bb^{-1}a^{-1}b^{-1} &= bb^{-2}a^{-1} \\
(bb^{-1})a^{-1}b^{-1} &= bb^{-1}b^{-1}a^{-1} \\
(bb^{-1})a^{-1}b^{-1} &= (bb^{-1})b^{-1}a^{-1} \\
ea^{-1}b^{-1} &= eb^{-1}a^{-1} \\
a^{-1}b^{-1} &= b^{-1}a^{-1}
\end{aligned}$$

We take the inverse of both sides:

$$\begin{aligned}
(a^{-1}b^{-1})^{-1} &= (b^{-1}a^{-1})^{-1} \\
ba &= ab.
\end{aligned}$$

Hence, we proved that $ab = ba$ if $(ab)^{-2} = b^{-2}a^{-2}$.

For the backward direction, we assume that $ab = ba$. Then, $(ab)^{-2} = (ab)^{-1}(ab){-1} = b^{-1}a^{-1}b^{-1}a^{-1} = (b^{-1}a^{-1})b^{-1}a^{-1} = ((b^{-1}a^{-1})b^{-1})a^{-1} = (b^{-1}(a^{-1}b^{-1}))a^{-1} = (b^{-1}(b^{-1}a^{-1}))a^{-1} = ((b^{-1}b^{-1})a^{-1})a^{-1} = (b^{-2}a^{-1})a^{-1} = b^{-2}a^{-1}a^{-1} = b^{-2}a^{-2}$. Thus, we proved that if $ab = ba$ then $(ab)^{-2} = b^{-2}a^{-2}$. Therefore, we proved that $(ab)^{-2} = b^{-2}a^{-2}$ iff $ab = ba$. ■

Problem 47

Proof. Suppose G is a group with the property that the square of every element is the identity, then every element is the inverse of itself. We want to prove that the group is *Abelian.* Choose $a, b \in G$, we observe that:

$$\begin{aligned}
(ab)^2 &= e \\
(ab)(ab) &= e \\
abab &= e \\
ababb^{-1} &= eb^{-1} \\
aba(bb^{-1}) &= b^{-1} \\
abae &= b^{-1} \\
aba &= b^{-1} \\
a^{-1}aba &= a^{-1}b^{-1} \\
(a^{-1}a)ba &= a^{-1}b^{-1} \\
eba &= a^{-1}b^{-1} \\
ba &= a^{-1}b^{-1}
\end{aligned}$$

By our hypothesis, we know that $b^{-1} = b, a^{-1} = a$. Thus, $ba = ab$. Therefore, we proved that if the group with the property that the square of every element is the identity, then the group is Abelian. ■