Problem 11

ⓐ $A\vec{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 24 \end{bmatrix} = 4\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

$x$ is an eigenvector of $A$ as $A\vec{x} = 4x$

ⓑ $A\vec{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -16 \\ 12 \end{bmatrix}$

Clearly $x$ is not an eigenvector as there is no $\lambda \in \mathbb{R}$ such that $A\vec{x} = \lambda\vec{x}$

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Problem 23

ⓐ $\chi_A(\lambda) = \det(\lambda I - A) = \left|\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}\right|$

$= \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 & 2 \\ 2 & \lambda-5 & 2 \\ 6 & -6 & \lambda+3 \end{vmatrix}$

$= (\lambda-1)((\lambda-5)(\lambda+3)+12) + 2(2(\lambda+3)-12)$
$+ 2(-16 - 6(\lambda-5))$

$= (\lambda-1)(\lambda-5)(\lambda+3) + 12(\lambda-1) + 4(\lambda+3) - 24$
$- 24 - 12(\lambda-5)$

Problem 23

$= (\lambda^2 - 5\lambda - \lambda + 5)(\lambda + 3)$

$= \lambda^3 - 3\lambda^2 - 13\lambda + 15 + \cancel{12\lambda - 12} + 4\lambda + 12 - \cancel{48}$
$\cancel{-12\lambda + 60} = \lambda^3 - 3\lambda^2 - 9\lambda + 27$

Thus, the characteristic equation is

$$\boxed{\lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0}$$

(b) The roots of the characteristic Polynomial are $\lambda_1 = -3$, $\lambda_2 = 3$. These are the eigenvalues of A

To find the corresponding eigenvectors, we find the corresponding eigenspaces to $\lambda_1 = -3$ and to $\lambda_2 = 3$

$(\lambda_1 = -3)$ we solve the homogeneous system $(-3I - A)\vec{x} = \vec{0}$ by Gassian elimination.

$$(-3I - A) = \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix}$$

we apply the EROs:

$$\begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{(R_2 - 2R_1) \to R_2} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{9}R_2 \to R_2} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \\ 6 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{(R_3 - 6R_1) \to R_3} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \\ 0 & -9 & 3 \end{bmatrix}$$

$$\xrightarrow{(R_3 + 9R_2) \to R_3} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The homogeneous system corresponding to the REF is

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the general solution is $z = t$, $y = \frac{1}{3}t$,
$x = \frac{1}{3}t$

Thus a basis for the eigenspace of A corresponding to $-3$ is

$$\text{span}\left(\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right).$$

$(\lambda = 3)$

$$(3I - A) = \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 6 & -6 & 6 \end{bmatrix}$$

$$\xrightarrow{(R_3 - 3R_1) \to R_3} \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The homogeneous system corresponding to the REF is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, a general solution is $z = t$, $y = s$, where $t, s \in \mathbb{R}$, and $x = s - t$.

Thus the eigenspace of $A$ corresponding to 3 is

$$\left\{ \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} ; s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Problem 49

$$\chi_A(\lambda) = \det(\lambda I - A) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ -7 & 3 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda - 5 & 0 \\ 7 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 5)(\lambda - 3)$$
$$= \lambda^2 - 8\lambda + 15$$

We cheek that A satisfies the characteristic equation:

$$A^2 - 8A + 15 = 0$$

$$\begin{bmatrix} 5 & 0 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -7 & 3 \end{bmatrix} - 8\begin{bmatrix} 5 & 0 \\ -7 & 3 \end{bmatrix} + 15\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 25 & 0 \\ -56 & 9 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ -56 & 24 \end{bmatrix} + \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 0 \\ 0 & -15 \end{bmatrix} + \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 72

~~Since~~ we solve the homogeneous system

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\left[\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, a general solution is $z = 0, y = 0, x = t$ where $t \in \mathbb{R}$.

so $E_A(3) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

and the basis of this eigenspace is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

so $\dim(E_A(3)) = 1$

## Problem 7g.

By definition, $Ax = \lambda x$, where $\lambda$ is the eigenvalue of $A$.

we multiply both sides by $A$ from the left

$$AAx = A\lambda x$$
$$A^2x = \lambda Ax$$
$$Ax = \lambda(\lambda x)$$
$$\boxed{Ax = \lambda^2 x}$$

Hence, we have that the square of each of the eigenvalues of $A$ is itself:

$$\lambda^2 = \lambda$$
$$\lambda^2 - \lambda = 0$$
$$\lambda = 1, \quad \lambda = 0$$

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## Problem 8o

By definition of the eigenvalues, we have

$$Ax = \lambda x$$

we multiply both sides by $A$ from the left,

$$A^2x = \lambda Ax$$

we keep multiplying till we reach $k$:

$$A^k x = \lambda A^{k-1}x = \lambda^k x$$

we know that $A^k$ is the zero matrix.

Hence, $\lambda^k = 0$.

Problem 81

Proof.

To show that $r$ is an eigenvalue we have to show there exists $\vec{v}$ such that $A\vec{v} = r\vec{v}$.

Pick $\vec{v}$ to be the vector whose entries are 1s

$\vec{v} = (1, 1, \dots, 1, 1)$ of size $n$.

we have:

$$AV = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \vec{u}$$

By definition of matrix multiplication, we have the each entry of the vector $\vec{u}$ is the sum of the ~~corr~~ entries of the corresponding row in A. Thus, we have that

$$\vec{u} = \begin{bmatrix} r \\ r \\ r \\ \vdots \\ r \end{bmatrix}_{n \times 1} = r \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1} = r\vec{v}$$

Therefore, $\vec{v}$ is an eigenvector of A ~~corre~~ corresponding with the eigenvalue $r$.

Example

let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

we see the sum of entries of each row is 3.

let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and observe $A\vec{v}$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 3\vec{v}$$

Hence, 3 is an eigenvalue of A.