Linear Algebra

Report 3

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◇ Vector Space

A vector space over the the field of real numbers is a set V, whose members are called *vectors*, together with two operations:

- Vector addition that takes two vectors $\vec{v}$ and $\vec{u}$ from V and produce a third vector denoted by $\vec{u} + \vec{v}$.
- Scalar multiplication that takes a scalar $c \in \mathbb{R}$ and a vector $\vec{u} \in V$, and produces a new vector denoted by $c\vec{v}$.

which satisfy the following axioms, which called *vector space axioms*:

(1) Closure of vector addition.
For every $\vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} \in V$.

(2) Commutativity of vector addition.
For every $\vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

(3) Associativity of vector addition.
For every $\vec{v}, \vec{u}, \vec{w} \in V$, we have $(\vec{v} + \vec{u}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.

(4) Existing of additive identity.
There exists a vector $\vec{\mathbf{0}} \in V$, called the *zero vector*, such that for every $\vec{v} \in V$, we have $\vec{\mathbf{0}} + \vec{v} = \vec{v}$.

(5) Existing of additive inverse.
There exists a vector $-\vec{v} \in V$, called the *additive inverse,* such that for every $\vec{v} \in v$, we have $\vec{v} + (-\vec{v}) = \vec{\mathbf{0}}$.

(6) Closure of scalar multiplication.
For every $\vec{v} \in V$ and scalar $c \in \mathbb{R}$, we have that $c\vec{v} \in V$.

(7) Distributivity of scalar multiplication over vector addtion.
For every $\vec{v}, \vec{u} \in V$ and scalar $c \in \mathbb{R}$, we have $c(\vec{v} + \vec{u}) = c\vec{v} + c\vec{u}$.

(8) Distributivity of scalar multiplication over field addtion.
For every $\vec{v} \in V$ and scalars $c, k \in \mathbb{R}$. we have $(c + k)\vec{v} = c\vec{v} + k\vec{v}$.

(9) Compatibtility of scalar multiplication with field multiplication.
For every $\vec{v} \in V$ and scalars $c, k \in \mathbb{R}$, we have that $(ck)\vec{v} = c(k\vec{v})$.

(10) Unarity. For every $\vec{v} \in V$, we have $1\vec{v} = \vec{v}$.

Example

The set of all 2×2 matrices.

Non-example

The set of polynomials of degree 5 only.

◇ Subspace

Suppose we have a vector space V, we call a set S a *subspace* of V iff $S \subseteq V$ and it satisfies the axioms of the vector space under the operations of vector addition and scalar multiplication inherited from V.

Example

Let V be the $4-$dimensional space, and let S any $3-$dimensional space that passes through the origin. Then S is a subspace of V.

Non-example

Let V be the $4-$dimensional space, and let S any $3-$dimensional space that does not pass through the origin. Then S is not a subspace of V.

◇ A spanning set of a vector space V

A subset $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_k}\}$ is called a spanning set of a vector space V iff we can write every vector in V as a linear combination of the vectors in S.

Example

The set $\{1, x, x^2, x^3\}$ is a spanning set of the space of all polynomials of degree 3 or less.

Non-example

The set $\{(1, 0, 0), (0, 1, 0)\}$ is not a spanning set of $\mathbb{R}^3$.

◇ A linearly independent set

A set $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_k}\}$ of vectors in a vector space is called *linear independent* iff the vector equation $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k} = \vec{\mathbf{0}}$ has only the trivial solution : $c_1 = 0, c_2 = 0, ..., c_k = 0$. In other words, if we cannot obtain a vector in the set by a linear combination of the other vectors in the same set then this set is *linear independent*.

Example

The set $\{(1, -1, 0), (0, 1, 1)\}$ is linear independent.

Non-example

The set $\{(1, 2, 3), (5, 7, 11), (0, -3, -4)\}$ is linear dependent, as $5(1, 2, 3) - (5, 7, 11) + (0, -3, -4) = (0, 0, 0)$.