Assignment 3

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June 29, 2025

Section 2.3

Exercise 5:

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Then, B is the inverse of A by definition.

Exercise 12:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{bmatrix} \xRightarrow{-R_1 \to R_1} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 3 & -3 & 0 & 1 \end{bmatrix}$$
$$\xRightarrow{(R_2 - 3R_1) \to R_1} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Notice that the left matrix is in **RREF** but is not I_2. Hence, the initial matrix in **not** invertible.

Alternative Approach: Let $A = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$. Then, $det(A) = -1 \times -3 - 1 \times 3 = 0$. Then, A is **not** invertible.

Exercise 23:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{(R_2 - 3R_1) \to R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(R_3-2R_1)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2\to R_2} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/4 & 1/4 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_3-5R_2)\to R_3} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/4 & 1/4 & 0 \\ 0 & 0 & 5 & 7/4 & -5/4 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_3\to R_3} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/4 & 1/4 & 0 \\ 0 & 0 & 1 & 7/20 & -1/4 & 1/5 \end{array}\right]$$

Since the left matrix is I_3, the inverse of the initial matrix is $\begin{bmatrix} 1 & 0 & 0 \\ -3/4 & 1/4 & 0 \\ 7/20 & -1/4 & 1/5 \end{bmatrix}$.

Exercise 47:

b) We can represent the system as $AX = B$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

We need to get the inverse first.

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(R_2-R_1)\to R_2} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(R_3-R_1)\to R_3} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & -4 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{-1}{4}R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 0 & -1/4 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{(R_1-2R_2)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 & 0 & -1/4 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{2}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

$$\xrightarrow{(R_1-R_3)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/4 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

Since the left matrix is I_3,

$$A^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & -1/4 \\ 1/2 & -1/2 & 0 \end{bmatrix} \Rightarrow X = A^{-1}B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence, the solution set of the system is $\{(0,-1,1)\}$.

Exercise 55:

$A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$ is **singular** if and only if $det(A) = -12 + 2x = 0 \Rightarrow x = 6$.

Exercise 68:

Theorem. If A, B, and C are square matrices and $ABC = I$, then B is invertible and $B^{-1} = CA$.

Proof. Let A, B, and C be square matrices where $ABC = I$. Then,

$$\begin{aligned}
&(AB)C = I && \text{by associativity of matrix multiplication} \\
&\Rightarrow (AB)^{-1} = C && \text{by the definition of inverse} \\
&\Rightarrow B^{-1}A^{-1} = C && \text{by a previously proven theorem} \\
&\Rightarrow (B^{-1}A^{-1})A = CA && \text{multiplying both sides by } A \text{ from the right} \\
&\Rightarrow B^{-1}(A^{-1}A) = CA && \text{by associativity of matrix multiplication} \\
&\Rightarrow B^{-1}I = CA && \text{by the definition of inverse} \\
&\Rightarrow B^{-1} = CA && \text{since } I \text{ is the identity matrix.}
\end{aligned}$$

Hence, B is invertible and $B^{-1} = CA$ as desired. ∎

Exercise 76:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

a) $A^2 - 2A + 5I = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{0}_{2\times 2}.$

b) $A\frac{1}{5}(2I - A) = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \right) = I_2$ as desired.

c) Now, let's generalize! Let A be any square matrix that satisfies $A^2-2A+5I = \mathbf{0}$. We know that

$AA - 2A + 5I = \mathbf{0}$ by the definition of matrix exponentiation

$\Rightarrow A^{-1}(AA) - 2A^{-1}A + 5A^{-1}I = A\mathbf{0}$

multiplying both sides by A from the left

$\Rightarrow (A^{-1}A)A - 2A^{-1}A + 5A^{-1}I = A\mathbf{0}$

by the associativity of matrix multiplication

$\Rightarrow IA - 2I + 5A^{-1}I = A\mathbf{0}$ by the definition of inverse

$\Rightarrow A - 2I + 5A^{-1} = A\mathbf{0}$ because I is the multiplicative identity

$\Rightarrow A - 2I + 5A^{-1} = \mathbf{0}$ because $A\mathbf{0} = \mathbf{0}$ for any A

$\Rightarrow (A - 2I + 5A^{-1}) + (2I - A) = \mathbf{0} + 2I - A$

adding $2I - A$ to both sides

$\Rightarrow (A - A) + (2I - 2I) + 5A^{-1} = \mathbf{0} + 2I - A$

by associativity of matrix addition

$\Rightarrow \mathbf{0} + \mathbf{0} + 5A^{-1} = \mathbf{0} + 2I - A$ by definition of additive inverse

$\Rightarrow 5A^{-1} = 2I - A$ because $\mathbf{0}$ is the additive identity

$\Rightarrow \dfrac{1}{5}(5A^{-1}) = \dfrac{1}{5}(2I - A)$ multiplying both sides by $\frac{1}{5}$

$\Rightarrow \left(\dfrac{1}{5} \cdot 5\right) A^{-1} = \dfrac{1}{5}(2I - A)$

by associativity of scalar multiplication

$\Rightarrow 1A^{-1} = \dfrac{1}{5}(2I - A)$ by the definition of multiplicative inverse

$\Rightarrow A^{-1} = \dfrac{1}{5}(2I - A)$ as desired.

Very much unnecessary low-level details? Sorry :(

Section 2.4

Exercise 9:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

Note that $A \xrightarrow{R_1 \leftrightarrow R_3} B$. Applying the same ERO to I_3 we get $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Exercise 31:

$$\begin{aligned} A = \begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix} &\xrightarrow{\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & -1/4 \\ 3 & -1 \end{bmatrix} \xrightarrow{(R_2 - 3R_1) \to R_2} \begin{bmatrix} 1 & -1/4 \\ 0 & -1/4 \end{bmatrix} \\ &\xrightarrow{-4R_2 \to R_2} \begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix} \xrightarrow{(R_1 + \frac{1}{4}R_2) \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

We first construct the elementary matrices corresponding to these four EROs and their inverses as follows:

$$E_1 = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}, \text{ and } E_4 = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}.$$

$$E_1^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix}, \text{ and } E_4^{-1} = \begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix}.$$

Since applying an ERO is equivalent to multiplying from the left by the matrix corresponding to it, applying the four EROs above is equivalent to $E_4E_3E_2E_1A$. Hence, $E_4E_3E_2E_1A = I_2$. Multiplying both sides from the left by $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ and using associativity of matrix multiplication multiple times and definition of identity, we get $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. Therefore,

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix}.$$

Exercise 39:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix} \xrightarrow{(R_3 - aR_1) \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & c \end{bmatrix}$$

$$\xRightarrow{(R_3 - bR_2) \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\xRightarrow{\frac{1}{c}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

We first construct the elementary matrices corresponding to these three EROs and their inverses as follows:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b & 1 \end{bmatrix}, \text{ and } E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{bmatrix}.$$

Similar to the previous question, $A = E_1^{-1} E_2^{-1} E_3^{-1}$. Then,

$$A^{-1} = \left(E_1^{-1} E_2^{-1} E_3^{-1}\right)^{-1} = E_3 E_2 E_1 I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a/c & -b/c & 1/c \end{bmatrix}.$$

Here we used the theorem that states that the inverse of a product of matrices is the product of their inverses in reverse order.

Exercise 41:

a) **True** because $I_n \xRightarrow{1R_1 \to R_1} I_n$.

b) **False** because $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is elementary while $2E = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ is not.

c) **True.**

Proof. Let E be an elementary matrix. By the definition of inverse, $EE^{-1} = I$. Since, E is elementary, multiplying E^{-1} by E from the left is equivalent to applying a single ERO to E^{-1} which transforms it into I. We can apply the *inverse* of this ERO to I to get E^{-1} back. We can see that this inverse always exists by studying the following three forms of EROs:

(i) $R_i \leftrightarrow R_j$: the inverse is $R_i \leftrightarrow R_j$.

(ii) $cR_i \to R_i$: the inverse is $\frac{1}{c}R_i \to R_i$.

(iii) $(R_i + cR_j) \to R_i$: the inverse is $(R_i - cR_j) \to R_i$.

Therefore, E^{-1} is elementary by definition. ■