Linear Algebra

Assignment 5

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4.1 Vectors in R^n

Problem 24

$u = (1, 2, 3), v = (2, -2, -1), w = (4, 0, -4)$.
$2u + v - w + 3z = \mathbf{0} = (0, 0, 0)$.

$$\begin{aligned}
2u + v - w &\iff \mathbf{0} - 3z \\
&\iff \mathbf{0} + (-3z) \\
&\iff -3z. \\
&\iff \frac{-1}{3}(2u + v - w) = \frac{-1}{3}(-3z) \\
&\iff \frac{-1}{3}(2u + v - w) = z. \\
&\iff z = \frac{-1}{3}(2(1, 2, 3) + (2, -2, 1) + (-1)(4, 0, -4)) \\
&\qquad = \frac{-1}{3}((2, 4, 6) + (2, -2, -1) - (4, 0, -4)) \\
&\qquad = \frac{-1}{3}(2 + 2 + 4, 4 - 2, 6 - 1 - 4) \\
&\qquad = \frac{-1}{3}(10, 2, 1) = (\frac{-10}{3}, \frac{-2}{3}, \frac{-1}{3}).
\end{aligned}$$

Problem 27

a) It is a scalar multiple of z. $v = \frac{3}{2}z = \frac{3}{2}(3, 2, -5) = (\frac{9}{2}, 3, \frac{-15}{2})$.

b) No, it is not a scalar multiple of z. The ratio is different between $\frac{9}{3} = 3, \frac{-6}{2} = -3$.

Problem 34

b) $2w - \frac{1}{2}u = 2(2, -2, 1, 3) - \frac{1}{2}(1, 2, -3, 1) = (4, -4, 2, 6) - (\frac{1}{2}, 1, \frac{-3}{2}, \frac{1}{2}) = (\frac{7}{2}, -5, \frac{7}{2}, \frac{11}{2})$.

4.2 Vector Spaces

Problem 13

Proof. Suppose A, B are matrices of size 4×6. By definition of matrix addition, $A+B = C$ is also of size 4×6, so the addition is closed (1). We also know $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$, clearly commutative (2). Suppose another matrix D of the same size. By associativity of real numbers, $A + B + D = [a_{ij}] + [b_{ij}] + [d_{ij}] = [a_{ij} + b_{ij} + d_{ij}] = [a_{ij} + (b_{ij} + d_{ij})] = [(a_{ij} + b_{ij}) + d_{ij}] = A + (B + D) = (A + B) + D$, we can see addition here is associative (3). We observe that $\mathbf{0}_{4\times 6}$ is the additive identity as $A + \mathbf{0}_{4\times 6} = A$. (4). The fifth axioms holds as $A - A = A + (-A) = [a_{ij} - a_{ij}] = \mathbf{0}_{4\times 6}$. (5). By definition of matrix scalar multiplication, suppose $c, k \in \mathbb{R}$, and A, B are matrices of size 4×6, then (cA) is a matrix of size 4×6 and scalar multiplication is closed (6), it also implies that $c(A + B) = c[a_{ij} + b_{ij}] = [ca_{ij} + cb_{ij}] = [ca_{ij}] + [cb_{ij}] = cA + cB$ (7), and $(c + k)A = (c + k)[a_{ij}] = [(c + k)a_{ij}] = [ca_{ij} + ka_{ij}] = cA + kA$ (8). We also have $cdA = c(dA) = (cd)A$ by scalar multiplication of matrices (9). It is obvious that $1A = A$ (10). Hence, It is a vector space. ∎

Problem 15

It is not a vector space as $x^3 + 5x^2 + (-1)(x^3) = x^3 + 5x^2 - x^3 = 5x^2$, clearly not closed under addition as we can see the result is a polynomial of degree 2.

Problem 17

It is not a vector space, take $p(x) = x$ and take $q(x) = -x$, then $p(x) + q(x) = x - x = 0$ wich is not in the set. Hence, the set is not closed under addition.

Problem 19

As stated in the lecture notes, the set $\mathcal{P}_n$ of all polynomials of degree n or less is a vector space under polynomial addition and scalar multiplication. Therefore, the set of all polynomials of degree four or less is a vector space.

Problem 22

It is not a vector space, take the pair $(1, 1)$. Observe $(-1)(1, 1) = (-1, -1)$, clearly not closed under scalar multiplication.

Problem 24

It is a vector space.
(1),(2) Choose x to be a, b, where $a, b \in \mathbb{R}$. We observe $(a, \frac{1}{2}a)+(b, \frac{1}{2}b) = (a+b, \frac{1}{2}a+\frac{1}{2}b) = (a + b, \frac{1}{2}(a + b)) = (b + a, \frac{1}{2}(b + a))$, clearly closed under addition and it satisfies the commutativity of addition.
(3) Choose a, b, c, where $a, b, c \in \mathbb{R}$. Then $(a, \frac{1}{2}a)+(b, \frac{1}{2}b)+(c, \frac{1}{2}c) = (a, \frac{1}{2}a)+(b+c, \frac{1}{2}(b+c)) = (a + b, \frac{1}{2}(a + b)) + (c, \frac{1}{2}c)$, satisfies the addition associativity.
(4),(5) It is clear that $(0, \frac{1}{2}(0)) = (0, 0)$ is the additive identity, and $(-a, \frac{1}{2}(-a))$ is the additive inverse, where $a \in \mathbb{R}$.
(6),(7) Choose k, d to be real numbers scalars. See that $c(a, \frac{1}{2}) = (ca, \frac{c}{2}a) = (ca, \frac{1}{2}(ca))$, and $c((a, \frac{1}{2}a) + (b, \frac{1}{2}b)) = c(a+b, \frac{1}{2}(a+b)) = (c(a+b), \frac{1}{2}c(a+b)) = (ca+cb, \frac{1}{2}(ca+cb)) = (ca, \frac{c}{2}a) + (cb, \frac{c}{2}b)$.

Problem 26

It is not a vector space.
Suppose A is a matrix of the form $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$, we get a matrix $\begin{bmatrix} a+a & b+b \\ c+c & 2 \end{bmatrix}$ when we add A to itself, clearly it is not closed ander addition.

Problem 31

It is not a vector space.
Choose two singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Adding them, we get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the identity matrix, and we know the identity matrix is invertible, clearly not closed under addition.

Problem 32

It is not a vector space.
Choose two invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Adding them, we get $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, which is a singular matrix, clearly not closed under addition.

Problem 34

Suppose A, B are upper triangular square matrices of size 3, and c, k are real numbers.
(1),(2),(6) From matrix addition and scalar multiplication, we know $A + B$ and cA are upper triangular square matrices of size 3, clearly closed under addition and scalar multiplication. We also know $A + B = B + A$, which satisfies addition commutativity.
(3),(7) Suppose D an upper triangular square matrix of size 3. By matrix addtion, we know $(A + B) + D = A + (B + D)$. We also know that $c(A + B) = cA + cB$.
(4),(5),(10) We clearly see that the zero square matrix of size 3 is an additive identity in this set, and the additive inverse of any matrix A is $-A$ such that $A + (-A) = \mathbf{0}_{3\times3}$. $1A = A$ is obviously satisfied for any matrix.
(8) We observe $(c + k) = A = (c + k)[a_{ij}] = [(c + k)a_{ij}] = [ca_{ij} + ka_{ij}] = c[a_{ij}] + k[a_{ij}] = cA + kA$.
(9) Finally, observe $(ck)A = ck[a_{ij}] = c[ka_{ij}] = c(kA)$.

Problem 36

Problem 37

Problem 38

Problem 41

a)