# Linear Algebra

Assignment 10

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## 6.1 Introduction to Linear Transformations

### Problem 29

a) To find the kernel we find the set of the $3 \times 1$ matrices which are mapped to the zero vector $\vec{\mathbf{0}}_{2\times1}$ such that $Ax = \vec{\mathbf{0}}_{2\times1}$, so the kernel is the nullspace of $A$. We solve the homogenous system $Ax = \vec{0}$ to get $N(A)$. We start by applying EROs to $A$.

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 4 & 0 & 11 \\ 0 & -2 & 3 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 0 & \frac{11}{4} \\ 0 & -2 & 3 \end{bmatrix}$$
$$\xrightarrow{\frac{-1}{2}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 0 & \frac{11}{4} \\ 0 & 1 & \frac{-3}{2} \end{bmatrix}$$

We see we get the system

$$x + \frac{11}{4}z = 0$$
$$y - \frac{3}{2}z = 0$$

, the non-pivot column of the ERO matrix is the $3^{rd}$ column and $z$ is the free variable. So we let $z = t, t \in \mathbb{R}$, and we get $x = -\frac{11}{4}t, y = \frac{3}{2}t$.

$$\text{So } N(A) = \left\{ \begin{bmatrix} -\frac{11}{4}t \\ \frac{3}{2}t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -\frac{11}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = Span \left( \begin{bmatrix} -\frac{11}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right)$$

b) $nullity(T) = nullity(A) = 1$, the number of the non-pivot columns.

c) We use the fact that $range(T) = C(A)$. To find the basis of the column space, we find the columns in $A$ that correspond to the pivot column in the ERO matrix:

$$\text{basis of } C(A) = \left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}, C(A) = span \left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$$

d) We use the fact that $rank(T) + nullity(T) = rank(A) + nullity(A) = n$, where $n$ is the number of columns of $A$. So $rank(T) = 3 - nullity(T) = 3 - 1 = 2$.

**Problem 40**

Observe that $T(x, y, z) = (x, y, 0)$ can be written in the form $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

$A$ is clearly in REF, and has two pivot columns which are the $1^{st}, 2^{nd}$ columns, so $rank(A) = 2$. Using the fact that $rank(T) + nullity(T) = rank(A) + nullity(A) = n$ we deduce that $nullity(T) = 3 - 2 = 1$.
To find the kernel, we solve the system $A\vec{x} = \vec{b}$ when $\vec{b} = \vec{\mathbf{0}}_{3\times 3}$. We see that $A$ is already in REF, so we don't need to apply the Gaussian elimination method. We get the system : $x = 0, y = 0$ and $z$ is a free varible since it corresponds to the third column which is a non-pivot column. So

$$ker(T) = N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

Clearly, the geometric representation of the kernel is the set of all points in the $z - axis$ in $\mathbb{R}^3$.

To find the range, we use that $range(T) = C(A)$. Since $A$ is in REF, we pick the columns in $A$ that are pivot columns to get the basis of the column space.

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = range(T)$$

Which is interpretted geometrically as the set of every point in $xy - plane$ in $\mathbb{R}^3$.

## 6.3 Matrices for Linear Transformations

### Problem 9

We can see that this linear transformation is from $\mathbb{R}^2 \to \mathbb{R}^3$, so firstly, we find the images of the standard basis of $\mathbb{R}^2$, namely, $T(e_1), T(e_2)$.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the standard matrix is

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

and the image of $\vec{v} = T(\vec{v}) = A\vec{v} =$

$$\begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 4 \end{bmatrix}$$

### Probme 40

a) Since the domain is $R^4$, we find the images of the standard basis, namely, $T(e_1), T(e_2), T(e_3), T(e_4)$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, the standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

and the image of $\vec{v} =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

b) We see

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \text{ So, } [T(1,0,0,1)]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \text{ So, } [T(0,1,0,1)]_B = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \text{ So, } [T(0,1,0,1)]_B = \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix}.$$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \text{ So, } [T(0,1,0,1)]_B = \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix}.$$

Thus, the matrix of $T$ relative to the bases $B$ and $B'$ is

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & \frac{3}{4} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

### Problem 44

We construct vectors of the coffecients of $B$ and $B$ and find the matrix of $T$ relative to the bases $B$ and $B'$:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = (1)\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \text{ so} T[(1,0,0)]_B = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} = (1)\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \text{ so} T[(1,0,0)]_B = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = (1)\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}, \text{ so} T[(1,0,0)]_B = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}.$$

Thus, the matrix of $T$ relative to the bases $B$ and $B'$ is

$$A = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$