Linear Algebra

Assignment 3

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2.3 The inverse of a matrix

Problem 5

We need to show that $AB = I_3$, we can see that clearly as follows:

$$\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{-4}{3} & \frac{-5}{3} & 1 \\ \frac{-4}{3} & \frac{-8}{3} & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 12

We keep applying the same EROs to booth the matrix and the identity matrix of the same size to arrive to the inverse as follows:

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{(R_2 - 3R_1) \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right].$$

We can see we got a matrix with row of zeros. Therefore, this inverse is not invertible.

Problem 23

We keep applying the same EROs to booth the matrix and the identity matrix of the same size to arrive to the inverse as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(R_2-3R_1)\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_3-2R_1)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_3-5R_2)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 5 & \frac{7}{4} & \frac{-5}{4} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{-1}{4} & \frac{1}{5} \end{array}\right].$$

So the inverse of the matrix $= \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & \frac{-1}{4} & \frac{1}{5} \end{bmatrix}$.

Problem 47

b) Suppose the matrix of the coefficients A, the matrix of varibles X, and the matrix of constants B, so $AX = B$. To solve the equation, we multiply each side by the

A^{-1} from the left. We get A^{-1} as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(R_2-R_1)\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_3-R_1)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & -4 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{4}R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{-1}{4} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{(R_1-2R_2)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{-1}{4} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{2}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{(R_1-R_3)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & 0 \end{array}\right]$$

Problem 55

A is singular iff its determinant is equal to zero. Hence, $det(A) = -12 + 2x = 0$, when it is singular, so $x = 6$.

Problem 68

Proof. Suppose A, B, C are square matrices and $ABC = I$. By properties of matrix multiplication, we use the associativity to get $(AB)C = I$, and we know that the result of the matrix multiplication is a matrix. Since we multiply square matrices we will not care about size as it will not change through multiplication by other square matrices of the same size. Hence, $(AB)C = DC = I$. By definition of the inverse matrix, we know that C is clearly invertible and $D = C^{-1}$. By a theorem proved in class, if (AB) is invertible, then A is invertible and B is invertible. For the second part, start with $ABC = I$, we know each of these matrices is invertible, so we multiply from the left by A^{-1} and from right by C^{-1} to get $B = A^{-1}IC^{-1} = A^{-1}C^{-1}$. Hence, $B^{-1} = (A^{-1}C^{-1})^{-1} = CA$. ■

Problem 76

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

a) $A^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $2A \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$.

So $A^2 - 2A + 5I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$.

b) $\frac{1}{5}(2I - A) = \frac{1}{5}(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}) = \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

We can get A^{-1} quickly by interchanging the main daigonal entries and multiply

entries of other diagonal by -1 and multiply the new matrix by the determinant of the old one. So we get $\frac{1}{1-(-4)} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}(2I - A)$.

Due to time constraints, I'll complete using scanned pen and paper solution.

c)