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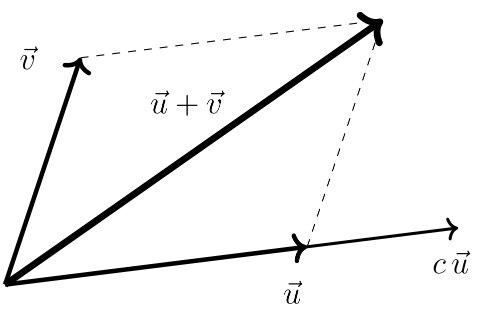
LINEAR ALGEBRA



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Introduction

Algebra is one of the main fields of contemporary Mathematics. Other fields include analysis, geometry, number theory, topology, logic, combinatorics, and graph theory. Algebra originated by the need of developing tools to solve equations. The Babylonians (1894 BC – 539 BC) in Mesopotamia (present-day Iraq and Syria) developed methods for solving linear and quadratic equations and they studied the Pythagorean theorem. They used to inscribe their mathematics on clay tablets. The word "algebra" stems from the Arabic word "Al-jabr" which means "restoring". This word was used by the Persian mathematician and philosopher Muhammad ibn Musa Al-Khwarizmi (780 – 850) in his influential textbook "*The Compendious Book on Calculations by Completion and Balancing*". Al-Khwarizmi used to work in the House of Wisdom in Baghdad, Iraq. Al-Khawarizmi in this Arabic mathematical treatise presented a systematic solution to linear and quadratic equations, and he established algebra as an independent field of mathematics. In the 12^{th} century his work was translated to Latin and European universities have used it as a main textbook on mathematics until the 16^{th} century.

In modern algebra mathematicians study abstract objects such as vector spaces, groups, rings, and fields. Vector spaces are studied in linear algebra. The field of linear algebra arose from the study of systems of linear equations. Linear algebra develops matrix theory and its application in solving systems of linear equations, and it studies the theory of vector spaces and linear transformations. Linear algebra has tremendous applications in science, engineering, industry, economics, and business.

This set of notes is based on the textbook "*Elementary Linear Algebra*" by Ron Larson and David Falvo.

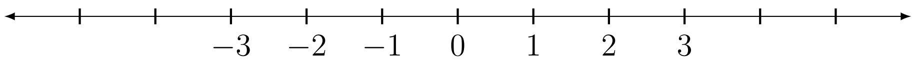
Chapter 1

Matrices and Linear Equations

In this chapter we will introduce matrices and define arithmetic operations on them such as matrix addition, scalar multiplication, and matrix multiplication. We will then use matrices to develop algorithms capable of solving systems of liner equations.

The set of all integers is $\mathbb{Z} = \{\ldots, -3, -2, -1,\ 0,\ 1,\ 2,\ 3, \ldots\}$. The set of all positive integers is $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$. The set of all real numbers is denoted by $\mathbb{R}$. We can think of the real numbers as the points on the real number line.

Real Number Line $\mathbb{R}$



The set $\mathbb{R}$ of real numbers is equipped with addition and multiplication of real numbers. We call $\mathbb{R}$ together with these two operations the *field of real numbers*. This means that both operations, addition and multiplication, are commutative and associative, the number 0 is the additive identity, and the number 1 is the multiplicative identity, moreover, every real number has an additive inverse, and every nonzero real number has a multiplicative inverse, and finally, we have that multiplication distributes over addition. We will use the field of real numbers to construct objects called matrices, and subsequently define operations on matrices.

1.1 Matrices

Definition 1.1.1. (Matrix)

Let m and n be positive integers. An $m \times n$ *matrix* is a rectangular array of real numbers with m rows and n columns. In such case, we say that the *size* of the matrix is "m by n", written as $m \times n$.

We write $A_{m\times n}$ to say that A is a matrix of size $m \times n$. The plural of "matrix" is "matrices". We represent a matrix A with m rows and n columns as shown below where each entry $a_{ij} \in \mathbb{R}$.

$$A = A_{m\times n} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{matrix} a \\ a \\ \vdots \\ a \end{matrix}}\right\} m \text{ rows}$$

For a shortcut notation of the above representation we simply write

$$A = [a_{ij}].$$

The entry a_{ij} is a real number and it is located in the i^{th} row and j^{th} column of the matrix A where i, j are integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$. We call the entry a_{ij} the (i, j)-entry in matrix A. For example, a_{23} is the entry in the 2^{nd} row and 3^{rd} column of the matrix.

In general, we can consider matrices where their entries come from a certain field such as the field of the complex numbers $\mathbb{C}$ or the finite field $\mathbb{F}_2$ of two elements. Nevertheless, in this exposition we will only work with the field of real numbers $\mathbb{R}$.

When the number of rows of a matrix is equal to the number of columns, that is, when $m = n$, we say that the matrix is a *square matrix*. An $n \times n$ matrix is called a square matrix of order n. In a square matrix of order n, the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called the *main diagonal entries*. A matrix of size $1 \times n$ is called a *row matrix*, and a matrix of size $m \times 1$ is called a *column matrix*.

Example 1.1. Here are examples of matrices of different sizes.

- A 2×2 matrix
$$\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$$
- A 3×2 matrix
$$\begin{bmatrix} 1 & -4 \\ 6\sqrt{5} & 2 \\ 7 & 8.1 \end{bmatrix}$$
- A 2×3 matrix
$$\begin{bmatrix} 1 & 4 & \pi \\ 6 & -3.7 & \sqrt{2} \end{bmatrix}$$
- A 1×4 matrix
$$\begin{bmatrix} 1 & 9 & 4 & 8 \end{bmatrix}$$
- A 3×1 matrix
$$\begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix}$$
- A 1×1 matrix
$$\begin{bmatrix} 3 \end{bmatrix}$$

Definition 1.1.2. (Equality of Matrices)

We say that matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are *equal* if and only if they have the same size $m \times n$, and $a_{ij} = b_{ij}$ for every i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Thus, two matrices are *equal* if and only if they have the same size and the corresponding entries in every position are equal.

Example 1.2.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

If we know that $\begin{bmatrix} 1 & 2 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} x & 2 \\ 7 & y \end{bmatrix}$, then it must be that $x = 1$ and $y = 9$.

♣ Special Matrices

• The *zero matrix* $\mathbf{0}_{m\times n}$ is the $m \times n$ matrix whose all of its entries are zeros.

$$\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{0}_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

• The *identity matrix* I_n of order n is the $n \times n$ matrix $I_n = [\delta_{ij}]$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that the identity matrix of order n is an $n \times n$ matrix whose main diagonal entries are ones and all the remaining entries are zeros.

$$I_1 = [1], \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

• A *diagonal matrix* is a square matrix where every entry not on the main diagonal is zero. Note that this means that entries on the main diagonal can be zero or nonzero. The zero matrix $\mathbf{0}_{n\times n}$ and the identity matrix I_n are diagonal matrices. Here are more examples of diagonal matrices.

$$\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

• A *lower triangular matrix* is a square matrix where every entry above the main diagonal is zero. So a matrix $A = [a_{ij}]$ is lower triangular if $a_{ij} = 0$ whenever $i < j$.

The zero matrix $\mathbf{0}_{n\times n}$, the identity matrix I_n, and diagonal matrices are lower triangular matrices. Here are more examples.

$$\begin{bmatrix} 6 & 0 \\ 2 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 1 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 4 & 0 & 7 & 0 \\ 5 & 6 & 0 & 3 \end{bmatrix}.$$

• An *upper triangular matrix* is a square matrix where every entry below the main diagonal is zero. So a matrix $A = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. The zero matrix $\mathbf{0}_{n\times n}$, the identity matrix I_n, and diagonal matrices are upper triangular matrices. Here are more examples.

$$\begin{bmatrix} 5 & 2 \\ 0 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 9 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 9 & 4 & 8 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

We next intend to define arithmetic operations in the world of matrices such as matrix addition, scalar multiplication, and matrix multiplication.

♣ Matrix Addition

Definition 1.1.3. (Matrix Addition)

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The *sum* $A + B$ of A and B is the $m \times n$ matrix $C = [c_{ij}]$ whose (i, j)-entry c_{ij} is obtained by adding the corresponding entries a_{ij} and b_{ij} in A and B, respectively. That is,

$$A + B = C = [c_{ij}] \text{ where } c_{ij} = a_{ij} + b_{ij}$$

for every $1 \le i \le m$ and $1 \le j \le n$.

Two matrices can be added only when they have the same size. Otherwise, if they have different sizes, then their addition is undefined.

Example 1.3. Addition of two 2×2 matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 1+0 & 2-2 \\ 3-3 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 1.4. Addition of two 3×3 matrices.

$$\begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 6 \\ -1 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 7 \\ -1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 0+5 & 2+7 \\ 4-1 & 5-3 & 6+2 \\ -1+0 & -2+1 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 9 \\ 3 & 2 & 8 \\ -1 & -1 & 2 \end{bmatrix}.$$

♣ Scalar Multiplication

Definition 1.1.4. (Scalar Multiplication)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let c be a real number. The scalar multiple of A by c is the $m \times n$ matrix cA obtained by multiplying every entry of A by the number c. In other words, $cA = [c \cdot a_{ij}]$.

We define the matrix $-A$ to be the matrix obtained when we scalar multiply A by the real number -1, that is, $-A = (-1)A$. Moreover, for matrices A and B of the same size, we define $A - B = A + (-B)$.

Example 1.5. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & -4 \\ -1 & 3 \end{bmatrix}$. Compute $3A$, $-B$, and $3A - B$.

(i) $3A = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 0 \\ 3 \cdot 5 & 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 0 \\ 15 & 3 \end{bmatrix}$.

(ii) $-B = (-1)B = \begin{bmatrix} -2 & 0 \\ -1 & 4 \\ 1 & -3 \end{bmatrix}$.

(iii) $3A - B = 3A + (-B) = \begin{bmatrix} 3 & 6 \\ 12 & 0 \\ 15 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 11 & 4 \\ 16 & 0 \end{bmatrix}$.

The next two theorems present properties of the matrix operations of addition and scalar multiplication.

Theorem 1.1.5.

Let A, B, C be matrices of the same size and c, d be real numbers. Then the following properties are true.

(i) $A + B = B + A$. *(Commutativity of addition)*

(ii) $(A + B) + C = A + (B + C)$. *(Associativity of addition)*

(iii) $A + \mathbf{0} = A$. *(Zero matrix is the additive identity)*

(iv) $A + (-A) = \mathbf{0}$. *(Existence of additive inverses)*

Proof. We prove the first one and we leave the others for the reader. Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size. Then using the definition of matrix addition and the fact that addition of real numbers is commutative we have the following.

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A.$$

Thus $A + B = B + A$. So we have shown that matrix addition is commutative. ■

Theorem 1.1.6.

Let A, B, C be matrices of the same size and c, d be real numbers. Then the following properties are true.

(i) $(cd)A = c(dA)$.

(ii) $1A = A$.

(iii) $c(A+B) = cA + cB$.

(iv) $(c+d)A = cA + dA$.

(v) *If* $cA = \mathbf{0}$, *then* $c = 0$ *or* $A = \mathbf{0}$.

Proof. We prove the third one and we leave the others for the reader. Let $S = A+B$ where $S = [s_{ij}]$. Choose any real number c. We have that

$$\begin{aligned} c(A+B) = c\,S = c[s_{ij}] &= c[a_{ij} + b_{ij}] = [c(a_{ij} + b_{ij})] = [ca_{ij} + cb_{ij}] \\ &= [ca_{ij}] + [cb_{ij}] = c[a_{ij}] + c[b_{ij}] = c\,A + c\,B. \end{aligned}$$

■

Example 1.6. For the matrices A and B below, find a 2×2 matrix X such that $3X + A = B$.

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Using the properties in the theorem above we have that

$$\begin{aligned} 3X + A = B &\iff (3X+A) + (-A) = B + (-A) \iff 3X + (A + (-A)) = B - A \\ &\iff 3X + \mathbf{0} = B - A \iff 3X = B - A \\ &\iff \frac{1}{3}(3X) = \frac{1}{3}(B-A) \iff \left(\frac{1}{3} \cdot 3\right) X = \frac{1}{3}(B-A) \\ &\iff 1X = \frac{1}{3}(B-A) \iff X = \frac{1}{3}(B-A). \end{aligned}$$

Therefore, $X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$.

♣ Matrix Multiplication

Definition 1.1.7. (Matrix Multiplication)

Let $A = [a_{ij}]$ be an $m \times k$ matrix and $B = [b_{ij}]$ be an $k \times n$ matrix. Their *product* AB is the $m \times n$ matrix whose (i, j)-entry is equal to the sum of the products of the corresponding entries from the i^{th} row of A and j^{th} column of B.

More precisely, for each $1 \le i \le m$ and $1 \le j \le n$ let c_{ij} denote the (i,j)-entry in the product matrix, so $AB = [c_{ij}]$, then

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ik} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{l=1}^{k} a_{il}b_{lj}.$$

So to compute the (i,j)-entry of the product AB we sum the products of the entries of the i^{th} row of A by the corresponding entries of the j^{th} column of B.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a_{i1}} & \mathbf{a_{i2}} & \cdots & \mathbf{a_{ik}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{11} & b_{12} & \cdots & \mathbf{b_{1j}} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \mathbf{b_{2j}} & \cdots & b_{2n} \\ b_{31} & b_{32} & \cdots & \mathbf{b_{3j}} & \cdots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & \mathbf{b_{kj}} & \cdots & b_{kn} \end{bmatrix}} = \underset{AB}{\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & \mathbf{c_{ij}} & \cdots & c_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}}$$

The slogan for matrix multiplication is "*row of first matrix by column of second matrix*" since the (i,j)-entry of the product AB is calculated using the i^{th} row of A and j^{th} column of B. The product AB is defined only when the number of columns of A is equal to the number of rows of B. Otherwise, we say that the product is undefined. Moreover, the product AB has the same number of rows as A and the same number of columns as B.

Example 1.7. Compute the product of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Matrix A has size 4×3 and B has size 3×2. Since the number of columns of A is equal to the number of rows of B, the product AB is defined and has size 4×2. Let $AB = [c_{ij}]$. To compute c_{11} we sum the products of the corresponding elements from the 1^{st} row of A and the 1^{st} column of B. So

$$c_{11} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = (1 \cdot 2) + (0 \cdot 1) + (4 \cdot 3) = 2 + 0 + 12 = 14.$$

To compute c_{12} we sum the products of the corresponding elements from the 1^{st} row of A and the 2^{nd} column of B. So

$$c_{12} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = (1 \cdot 4) + (0 \cdot 1) + (4 \cdot 0) = 4 + 0 + 0 = 4.$$

Similarly we compute c_{32} using the 3^{rd} row of A and the 2^{nd} column of B.

$$c_{32} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = (3 \cdot 4) + (1 \cdot 1) + (0 \cdot 0) = 12 + 1 + 0 = 13.$$

We continue in this manner until computing all the entries of AB.

$$AB = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}_{4\times 3} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}_{4\times 2}.$$

Note that BA is undefined as the number of columns of B (2 columns in B) is not equal to the number of rows of A (4 rows in A).

Example 1.8. Compute the following multiplications of matrices.

(i) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$ (ii) $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$

(iii) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}.$

Example 1.9. Consider the matrices $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$.

(i) $CD = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$

(ii) $DC = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$

So $CD = DC$. This example shows that it is possible for two matrices to commute. But is it always the case?

Example 1.10. Consider the matrices $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

(i) $AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}.$

(ii) $BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$.

Observe that $AB \neq BA$. In general, matrix multiplication is not commutative.

For real numbers a, b, c with c is nonzero, if $ac = bc$, then $a = b$. This is called the *cancellation property*. Such phenomenon does not hold in general for matrices as the next example demonstrates.

Example 1.11. Consider the matrices $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$.

(i) $AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$.

(ii) $BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$.

Observe that C is not a zero matrix and $AC = BC$, however, $A \neq B$.

For real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$. Again, such property does not hold in general for matrices as shown the next example.

Example 1.12. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Here we have that $AB = \mathbf{0}$, however, neither A nor B is a zero matrix.

The next theorem presents properties of matrix multiplication.

Theorem 1.1.8.

Given matrices A, B, C of suitable sizes and $c \in \mathbb{R}$, the following hold.

(i) $(AB)C = A(BC)$. *(Associativity of multiplication)*

(ii) $A(B + C) = AB + AC$. *(Left distributivity)*

(iii) $(B + C)A = BA + CA$. *(Right distributivity)*

(iv) $c(AB) = (cA)B = A(cB)$.

Proof. We prove the second property. Let's denote the entries of the matrices involved as follows: $A_{m\times k} = [a_{ij}]$, $B_{k\times n} = [b_{ij}]$, $C_{k\times n} = [c_{ij}]$, $A(B + C) = [d_{ij}]$, $B + C = [e_{ij}]$ $AB = [r_{ij}]$, $AC = [s_{ij}]$, and $AB + AC = [t_{ij}]$. It is sufficient to show that $d_{ij} = t_{ij}$.

$$\begin{aligned} d_{ij} &= \sum_{l=1}^{k} a_{il}e_{lj} = \sum_{l=1}^{k} a_{il}(b_{lj} + c_{lj}) = \sum_{l=1}^{k}(a_{il}b_{lj} + a_{il}c_{lj}) \\ &= \sum_{l=1}^{k} a_{il}b_{lj} + \sum_{l=1}^{k} a_{il}c_{lj} = r_{ij} + s_{ij} = t_{ij}. \end{aligned}$$

Therefore, $A(B + C) = AB + AC$ as desired. ■

In the world of real numbers, the number 1 plays the role of multiplicative identity in the sense that if we multiply 1 by any real number we get that number itself. The identity matrix plays the role of the multiplicative identity in the world of matrices as the next lemma states. The proof is left for the reader.

Lemma 1.1.9.

Let A be an $m \times n$ matrix. Then

$$AI_n = A \quad \text{and} \quad I_m A = A.$$

Definition 1.1.10. (Power of a Matrix)

Let A be an $n \times n$ matrix. We define

- $A^0 = I_n$,
- $A^{k+1} = A^k A$ for $k \geq 0$.

It follows that $A^1 = A^0 A = I_n A = A$, and $A^2 = A^1 A = AA$, and $A^3 = A^2 A = AAA$, and $A^4 = A^3 A = AAAA$, and so on. In general, for positive k we have that

$$A^k = \underbrace{AAA \cdots A}_{k \text{ times}}.$$

Theorem 1.1.11.

Let A be a square matrix of order n. Then the following hold for any integers $m \geq 0$ and $k \geq 0$.

(i) $A^m A^k = A^{m+k}$.

(ii) $(A^m)^k = A^{mk}$.

Proof. We prove the first equality and leave the second for the reader. Fix an arbitrary natural number m. We aim to prove that $A^m A^k = A^{m+k}$ by induction on k. For the base case, by definition $A^0 = I_n$ and the fact that I_n is a multiplicative identity, we have that,

$$A^m A^0 = A^m I_n = A^m = A^{m+0}.$$

For the induction step, suppose that $A^m A^k = A^{m+k}$ is true, our goal is to prove that $A^m A^{k+1} = A^{m+(k+1)}$. Using the definition of the power of a matrix, and the associativity of matrix multiplication, and the induction hypothesis we have that,

$$A^m A^{k+1} = A^m(A^k A) = (A^m A^k)A \overset{IH}{=} A^{m+k}A = A^{(m+k)+1} = A^{m+(k+1)}.$$

Thus, the induction step holds and the proof is complete. ■

♣ Matrix Transpose

Definition 1.1.12. (Matrix Transpose)

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of A, denoted by A^T, is the $n \times m$ matrix $A^T = [\hat{a}_{ij}]$ where $\hat{a}_{ij} = a_{ji}$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. In other words, A^T is the matrix whose columns are the rows of A.

Example 1.13. Consider the 2×3 matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Then its transpose is the following 3×2 matrix,

$$A^T = [\hat{a}_{ij}] = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \\ \hat{a}_{31} & \hat{a}_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Theorem 1.1.13.

Let A, B be matrices of suitable sizes.

(i) $(A^T)^T = A$.

(ii) $(A + B)^T = A^T + B^T$.

(iii) $(AB)^T = B^T A^T$.

Definition 1.1.14. (Symmetric Matrix)

A square matrix A is called *symmetric* if and only if $A = A^T$. A square matrix A is called *skew-symmetric* if and only if $A = -A^T$.

Observe that if $A = [a_{ij}]$ is a symmetric matrix of order n, then $a_{ij} = a_{ji}$ for every $1 \leq i, j \leq n$. This means that the main diagonal acts as a mirror where every entry on the left hand side of the main diagonal is equal to its mirror image on the right hand side of the main diagonal.

Example 1.14. The following matrices are symmetric. Notice the symmetry across the main diagonal.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{bmatrix}.$$

Observe that if $A = [a_{ij}]$ is a skew-symmetric matrix of order n, then $a_{ij} = -a_{ji}$ for every $1 \leq i, j \leq n$. This forces all main diagonal entries to be 0s.

Example 1.15. The following matrices are skew-symmetric.

$$A = \begin{bmatrix} 0 & -4 & 5 \\ 4 & 0 & 7 \\ -5 & -7 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}.$$

Theorem 1.1.15.

Let A be any matrix and B be any square matrix.

(i) The matrix AA^T is symmetric.

(ii) The matrix $B + B^T$ is symmetric.

Proof. (i) Let A be any matrix. We will show that AA^T is symmetric by showing that the transpose of AA^T is itself.

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

(ii) Let B be a square matrix. To show that $B + B^T$ is symmetric we proceed in a similar way.

$$(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T.$$

■

1.2 Systems of Linear Equations

Definition 1.2.1. (Linear Equation)

A *linear equation* in n variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The *variables* $x_1, x_2, \ldots, x_n$ range over the real numbers $\mathbb{R}$, and $a_1, a_2, \ldots, a_n, b$ are fixed real numbers. We call the number a_i the *coefficient* of the variable x_i and we call b the *constant term*.

We also use the letters x, y, z, s, t for variables in linear equations as well.

Example 1.16. The following are linear equations.

(i) $3x + 2y = 6$.

(ii) The equation of a straight line: $y = mx + b$, where $m, b \in \mathbb{R}$.

(iii) $2x + y - \pi z = \sqrt{2}$.

(iv) $x_1 + (\sin \frac{\pi}{4})x_2 + 12x_3 + 7x_4 = e^2$.

Example 1.17. The following equations are not linear.

(i) $xy + z = 2$.

(ii) $e^x - 2y = 4$.

(iii) $\sin x + 3y + z = 4$.

(iv) $\frac{1}{x} + \frac{3}{y} = 7$.

Definition 1.2.2. (Solution of a Linear Equation)

A *solution* of a linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

is a sequence $(s_1, s_2, \ldots, s_n)$ of real numbers which satisfy the equation when the number s_i is substituted for the variable x_i for every $1 \leq i \leq n$. We call the set of all solutions of an equation the *solution set* of the equation.

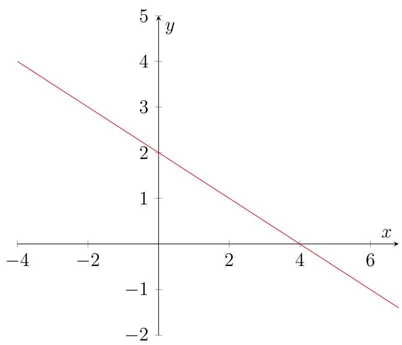
Example 1.18. Find the solution set of the linear equation $x + 2y = 4$.
Observe that $(2, 1)$, $(4, 0)$, and $(0, 2)$ are all solutions of this equation. To find all solutions of an equation involving two variables, we solve for one variable in terms of the other. Let us solve for x in terms of y to get

$$x = 4 - 2y.$$

In this form, we call the variable y a *free variable* meaning that y can take on *any* real number value. On the other hand, the variable x here is not free as its value is determined once a value is assigned for y. For instance, assigning $y = 3$ results to $x = -2$. To represent all solutions, we introduce a new variable t called a *parameter* representing the values of the free variable. Thus, a solution will have the form $y = t$ and $x = 4 - 2t$ where t is *any* real number. Thus, we have an infinite solution set written as

$$\big\{(4-2t, t) \mid t \in \mathbb{R}\big\} = \big\{(4,0),\ (-4,4),\ (0,2),\ (3,0.5),\ \ldots\big\}.$$

Observe that when we sketch all the points which belong to this solution set in the cartesian plane we will get a straight line, namely, the line determined by the equation $y = \frac{-1}{2}x + 2$ as shown below.

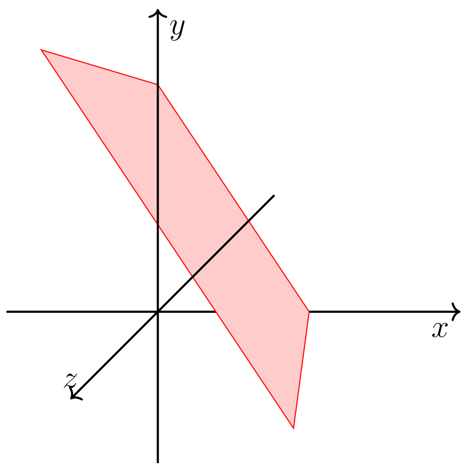


Example 1.19. Find the solution set of the equation $3x + 2y - z = 3$.

We solve for z to get that $z = 3x + 2y - 3$. In this form, both x and y are free variables. When we set $x = s$ and $y = t$ for *any* real numbers s, t of our choice, we get that $z = 3s + 2t - 3$. Therefore, the solution set of this linear equation is

$$\{(s,\ t,\ 3s + 2t - 3) \mid s, t \in \mathbb{R}\}.$$

For instance, $(1, 0, 0)$, $(1, 1, 2)$, and $(-1, 6, 6)$ are solutions. When we sketch all points of the solution set in the 3D space we will get a plane.



The plane above represents the solution set of the system.

Definition 1.2.3. (System of Linear Equations)

A *system* of linear equations is a finite list of linear equations. We write a system of m linear equations in the same n variables $x_1, \ldots, x_n$ as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Note that a_{ij} is the coefficient of x_j in the i^{th} equation.

Definition 1.2.4. (Solution of a System of Linear Equations)

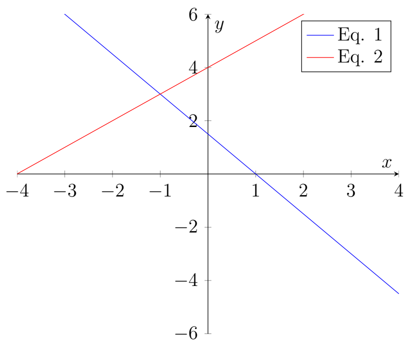
A *solution* of a system of linear equations is a sequence $(s_1, s_2, \ldots, s_n)$ of real numbers which is a solution to *all* linear equations of the system simultaneously.

When we say "solve the system of linear equations" we mean find *all* solutions of the system, that is, find the solution set of the system.

Example 1.20. Solve the following system of linear equations.

$$3x + 2y = 3 \tag{1}$$
$$-x + y = 4 \tag{2}$$

We graph the straight lines representing the solution sets of the two equations and see that there is only one point, namely $(-1, 3)$, which lies on both lines.



We can see that $(-1, 3)$ is a solution to the above system since it is a solution to both equations, however, $(1, 0)$ is not a solution to the system because it is not a solution to the the second equation. To find all solutions of the system we proceed as follows. From Equation (1), we see that $y = -1.5x + 1.5$. From Equation (2), we get that $y = x + 4$. Therefore, $-1.5x + 1.5 = x + 4$, and so $-2.5x = 2.5$, which implies that $x = -1$. Using any of the two equation, when $x = -1$, it forces $y = 3$. Thus, there is only one pair of numbers which satisfies both equations simultaneously, namely the pair $(1, 3)$. This system has exactly one solution and its solution set is $\{(-1, 3)\}$.

Definition 1.2.5. (Consistent and Inconsistent Systems)

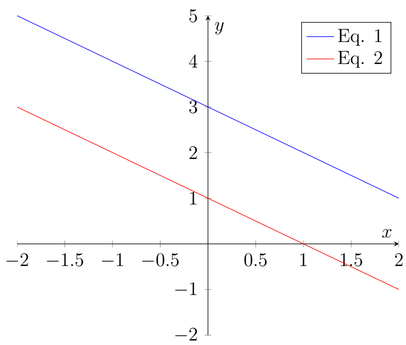
A system that has no solutions is said to be *inconsistent*. A system with at least one solution is called *consistent*.

Example 1.21. Show that the system below is inconsistent.

We need to show that no pair of real numbers (s, t) satisfies both equations simultaneously.

$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

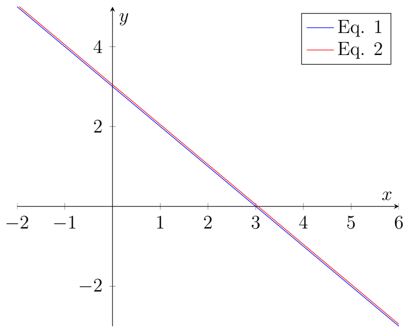
To see this, suppose the system above has a solution. So there is a pair (s, t) of real numbers which satisfies both equations when s substitutes x and t substitutes y. So $s + t = 3$ and $s + t = 1$. Thus, $s = 3 - t$ and also $s = 1 - t$. It follows that $3 - t = 1 - t$, and thus $2 = 0$, which is a contradiction! Geometrically, this system is represented by two parallel distinct straight lines (one line for each linear equation). The graph below shows that no point lies on both lines and so no pair of numbers satisfies both equations simultaneously. The solution set here is the empty set $\emptyset$.



Example 1.22. Solve the following system of linear equations.

$$x + y = 3 \tag{1}$$
$$2x + 2y = 6 \tag{2}$$

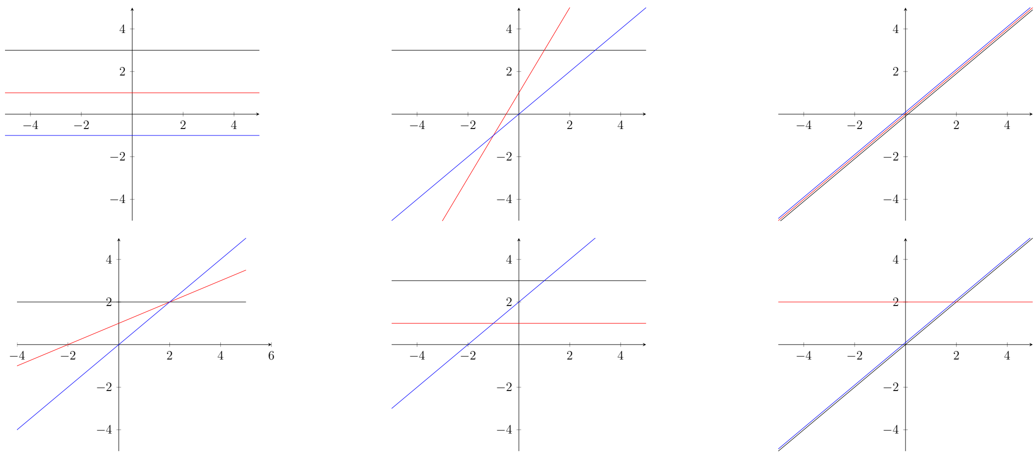
Solving for y in the first equation, we get that $y = 3 - x$. Thus, $\{(t, 3 - t) \mid t \in \mathbb{R}\}$ is the solution set of the first equation. Observe that the second equation has the same solution set as well. This shows that this system of linear equations has infinitely many solutions such as $(0, 3)$, $(1, 2)$, $(2, 1)$, and $(8, -5)$. Notice that the second equation is obtained from the first one by multiplying both sides of the first equation by 2. Geometrically, the both lines representing the solution sets of these two equations coincide as seen below.



Example 1.23. Consider a system of three linear equations in 2 variables x and y. The general form of such system is as follows.

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \\ a_{31}x + a_{32}y &= b_3 \end{aligned}$$

The solution set of each of these equation can be represented by a straight line. The solutions of the whole system corresponds to the points of intersection which lie on the three lines at the same time. Below are possible scenarios of plotting the three straight lines representing a given system of three linear equations in 2 variables.



In all of the systems of linear equations studied above there were only three cases: the system is inconsistent, the system has exactly one solution, or the system has infinitely many solutions. Is it possible to have other options for the number of solutions of a system of linear equations? For example, is there a system of linear equations in 7 variables which has exactly 4 solutions?

We will utilize the theory we developed earlier of matrices to show that these are actually the only three options. Towards this goal, we will start by representing a system of linear equations by a matrix equation. Consider the following matrices.

$$A = A_{3\times 4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \qquad X = X_{4\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad B = B_{3\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Now, consider the matrix equation $AX = B$. The left hand side of this equation is the 3×1 column matrix AX where

$$AX = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix}$$

Now the equation $AX = B$ says that every entry of the matrix AX is equal to its corresponding entry in the matrix B, and so the first entry of AX is b_1, the second entry is b_2, and the third entry is b_3, meaning that we have the following system of three equation in 4 variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \end{aligned}$$

In general, given a system of m linear equations in n variables $x_1, \dots, x_n$, say,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we can represent this system by the matrix equation

$$AX = B$$

where the matrices A, X, and B are given below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix A is called the *coefficient matrix*, the matrix X is called the *column matrix of variables*, and the matrix B is called the *column matrix of constant terms.*

Observe that the first column of A has the coefficients of the variable x_1, the second column of A has the coefficients of the variable x_2, and so on. Moreover, we can tell using this matrix equation when a sequence of real numbers of length n is a solution to the system. A sequence $(s_1, s_2, \ldots, s_n)$ of real numbers is a solution of the system $AX = B$ if and only if the matrix equation $AS = B$ is satisfied where $S = [s_1 \; s_2 \; \cdots \; s_n]^T$. In other words, a column matrix $S_{n\times 1}$ is a solution to the system $AX = B$ if and only if when we multiply S from the left by the coefficient matrix $A_{m\times n}$ we get the matrix of constant terms $B_{m\times 1}$.

We now have the sufficient tools to prove the following result on the number of solutions of a system of linear equations. There are only three options for the solution set of a system of linear equations.

Theorem 1.2.6.

For any system of linear equations precisely one of the following holds:

(i) The system has no solutions (inconsistent system).

(ii) The system has exactly one solution (consistent system).

(iii) The system has infinitely many solutions (consistent system).

Proof. Suppose we have a system of m linear equations in the variables $x_1, x_2, \ldots, x_n$. If the system has no solutions, then it is the first case. If the system has exactly one solution, then it is the second case. Otherwise, assume the system has at least two solutions, say $(r_1, r_2, \ldots, r_n)$ and $(s_1, s_2, \ldots, s_n)$. As discussed above, we represent this system as a matrix equation $AX = B$, where A is the coefficient matrix, X is the column of the variables, and B is the column of the constant terms. Consider the column matrices $X_1 = [r_1 \; r_2 \; \ldots \; r_n]^T$ and $X_2 = [s_1 \; s_2 \; \ldots \; s_n]^T$. Since R and S are solutions to the system, we know that $AX_1 = B$ and $AX_2 = B$. Next, choose any $c \in \mathbb{R}$. We claim that the column matrix $X_1 + c\,(X_1 - X_2)$ is a solution of the system. To see this, we need to show that $A(X_1 + c(X_1 - X_2)) = B$. We show this using the properties of matrix addition, scalar multiplication, and matrix multiplication.

$$\begin{aligned} A(X_1 + c(X_1 - X_2)) &= AX_1 + A(c\,(X_1 - X_2)) \\ &= B + c\,(A(X_1 - X_2)) \\ &= B + c\,(AX_1 - AX_2)) \\ &= B + c\,(B - B) \\ &= B + c\,\mathbf{0}_{m\times 1} \\ &= B + \mathbf{0}_{m\times 1} = B. \end{aligned}$$

Therefore, the column matrix $X_1 + c\,(X_1 - X_2)$ satisfies the equation $AX = B$ meaning that $X_1 + c\,(X_1 - X_2)$ is a solution to the given system. Moreover, since X_1

and X_2 are distinct solutions, one can show that different choices of c give different matrices of the form $X_1 + c\,(X_1 - X_2)$. As we have infinitely many real numbers to choose from for the scalar c we can generate infinitely many solutions of the system. Therefore, we proved that if a system has at least two distinct solutions, then it must have infinitely many solutions, and this is the third case of the theorem. $\blacksquare$

1.3 Elementary Row Operations

We aim to develop an algorithm to solve systems of linear equations. Let us start with a system which is easy to solve.

Example 1.24. Solve the following system of linear equations.

$$\begin{aligned} x - 2y + 3z &= 9 \qquad (1)\\ y + 3z &= 5 \qquad (2)\\ z &= 2 \qquad (3) \end{aligned}$$

We are looking for triples (x, y, z) of real numbers which satisfy the above three equations simultaneously. Clearly, from the Equation (3) we must have that $z = 2$. When we substitute 2 for z in Equation (2) we get that $y = -1$. Finally, by substituting 2 for z and -1 for y in Equation (1) we get that $x = 1$. Thus, this system has exactly one solution namely $(1, -1, 2)$.

We used "back-substitution" to solve the system above. We started from the last equation and obtained a value for the variable z. Then we used this information to know a value for y using the second equation. We went all the way up until we obtained a value for x using the first equation.

Our goal is to start with any system of linear equations and change it to a system of linear equations which has exactly the same solutions as the original system and it is easy to solve like the one in the example above. This motivates the following notions.

Definition 1.3.1. (Equivalent Systems)

Two systems of linear equations in the same variables are called *equivalent* if and only if they have the same set of solutions.

A strategy to solving systems of linear equations would be as follows:

- First: Reduce the system to an equivalent system that is easy to solve.
- Second: Solve the easy system, possibly using back substitution.

It remains to make precise what we mean by "reduce the system". In other words, how can we change a system of linear equations without changing its set of solutions! The answer is in the following theorem which provides operations we can perform on systems of linear equations in order to obtain simpler systems without changing the set of solutions of the original system.

Theorem 1.3.2.

Any of the following operations produces an equivalent system of linear equations.

(I) Interchange two equations of the system.

(II) Multiply one equation by a nonzero real number.

(III) Replace an equation by the result of adding it to a multiple of another equation.

When we apply these operations it will be easier to omit the variables and just work with the coefficients and constant terms of the system of linear equations in hand. Consequently, given a system of linear equations as below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we define its *augmented matrix* to be the $m \times (n+1)$ matrix which is obtained by appending the column of constant terms B on the right of the coefficient matrix A, thus, the augmented matrix is:

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}.$$

Observe that we can extract the original system of linear equations from its augmented matrix.

Example 1.25. Consider the following system of linear equations.

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad - 4z &= 6 \end{aligned}$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}.$$

Extract the original system from this augmented matrix. For instance, the second row $[-1 \quad 3 \quad -1 \quad -3]$ of the augmented matrix above tells us that the second equation of the original system is $-x + 3y - z = -3$.

It is clear that the augmented matrix has all the information about the original system: it includes the information about the number of equations, the number of variables, the coefficients of the variables, and all the constant terms of the linear equations in the system. The idea now is to apply the operations above to the augmented matrix instead of applying them to the system itself. Clearly, the i^{th} row of the augmented matrix of a system of linear equations represents the i^{th} equation of the system. Accordingly, we now translate the operations we introduced above to corresponding operations on the rows of the augmented matrix of a system.

♣ Elementary Row Operations (EROs) on Matrices

An *elementary row operation* is one of the following operations performed on a matrix. We denote the i^{th} row of a matrix by R_i.

1. (Type I) Interchange two rows.

   We write $R_i \leftrightarrow R_j$ when we interchange the i^{th} and j^{th} rows.

2. (Type II) Multiply a row by a nonzero real number.

   We write $cR_i \to R_i$ when the new R_i is obtained by multiplying the old R_i by a real number $c \neq 0$.

3. (Type III) Add a multiple of a row to another row. More precisely, replace a row by the row obtained by adding it to a multiple of another row.

   We write $(R_i + cR_j) \to R_i$ when the new R_i is obtained by adding the old R_i and cR_j where $i \neq j$.

To be more precise we should write $cR_i^{\text{old}} \to R_i^{\text{new}}$ and $(R_i^{\text{old}} + cR_j) \to R_i^{\text{new}}$ for Type-II and Type-III operations, respectively, however, we will omit the superscripts "old" and "new".

Applying an elementary row operation on an augmented matrix of a system produces a new augmented matrix which corresponds to a new system of linear equations which is equivalent to the original system. Thus, elementary row operations do not change the set of solutions of the systems they are representing.

Definition 1.3.3. (Row-Equivalence)

A matrix A is *row-equivalent* to a matrix B if and only if B is obtained from A by applying finitely many elementary row operations.

Lemma 1.3.4.

Let $G_1 = [A_1 \mid B_1]$ and $G_2 = [A_2 \mid B_2]$ be augmented matrices of systems of linear equations. If G_1 is row-equivalent to G_2, then the system represented by G_1 is equivalent to the system represented by G_2.

Theorem 1.3.5.

The row-equivalence relation is an equivalence relation on the set of all matrices. That is, row-equivalence is reflexive, symmetric, and transitive.

Here are examples of applying elementary row operations to a given matrix.

Original matrix	ERO	New matrix
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$

Original matrix	ERO	New matrix
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\frac{1}{2}R_1 \to R_1$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$

Original matrix	ERO	New matrix
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$(R_3 - 2R_1) \to R_3$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$

It turns out that Type-I operations can be performed by applying Type-II and Type-III operations as the next lemma shows, and so we do not really need Type-I operations.

Lemma 1.3.6.

In any matrix, we can interchange row R_i and row R_j using only Type-II and Type-III operations.

Proof. Let A be a matrix. To swap row R_i and row R_j apply the following sequence of Type-II and Type-III elementary row operations.

(1) $(R_i + R_j) \to R_i$

(2) $(R_j - R_i) \to R_j$

(3) $(R_i + R_j) \to R_i$

(4) $-R_j \to R_j$

Let us illustrate the application of these EROs on a 3×3 matrix to interchange the first and second rows. Take $i = 1$ and $j = 2$.

$$\begin{aligned}
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{bmatrix} &\xRightarrow{(R_1+R_2)\to R_1} \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{bmatrix} \\
&\xRightarrow{(R_2-R_1)\to R_2} \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ -a_1 & -a_2 & -a_3 \\ c_1 & c_2 & c_2 \end{bmatrix} \\
&\xRightarrow{(R_1+R_2)\to R_1} \begin{bmatrix} b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \\ c_1 & c_2 & c_2 \end{bmatrix} \\
&\xRightarrow{-R_2\to R_2} \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_2 \end{bmatrix}.
\end{aligned}$$

■

We next introduce the form of the augmented matrix of a system of linear equations which is easy to solve by back-substitution. We call it *row-echelon form*.

Definition 1.3.7. (Row-Echelon Form)

A matrix is said to be in *row-echelon form* (REF) if and only if it satisfies the following three conditions:

(i) All rows consisting entirely of zeros occur at the bottom of the matrix.

(ii) The first nonzero entry of every row is 1 (called the *leading* 1 or the *pivot*).

(iii) Any leading 1 is farther to the right than the leading 1 in the row above.

In an REF matrix, a column that has a leading 1 is called a *pivot column*.

A matrix in row-echelon form with the additional property that all entries directly above and below leading 1s are zeros is said to be in reduced row-echelon form.

Definition 1.3.8. (Reduced Row-Echelon Form)

A matrix A is said to be in *reduced row-echelon form* (RREF) iff

- A is in row-echelon form.
- Every pivot column has zeros above and below its leading 1.

Example 1.26. The following matrices are in row-echelon form. The leading 1s are circled. Matrices A and B are not in reduced row-echelon form, however, matrices C and D are in reduced row-echelon form.

(i) $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 & -5 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (iv) $D = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example 1.27. The following matrices are not in row-echelon form. The first nonzero entry of the second row in matrix A is not 1. In matrix B, there is a row of zeros which is not at the bottom of the matrix. In matrix C, the leading 1 of the fourth row is not on the right of the leading one in the third row.

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 & -9 & 2 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 8 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

Lemma 1.3.9.

Suppose that $A_{m\times n}$ is a matrix in row-echelon form. Then the following hold.

- *The number of leading 1s in A is at most* $\min(m, n)$.
- *All the entries directly below a leading 1 must be zeros. (Otherwise, the third property will be violated.)*
- *Every column of A has at most one leading 1.*

In a matrix, a row whose all of its entries are zeros is called a *zero row*. A *nonzero row* is a row which has at least one nonzero entry.

Lemma 1.3.10.

Let $A = [a_{ij}]$ be a matrix in row-echelon form. If the entry a_{ij} is a leading 1, then $i \leq j$. Consequently, when A is a square matrix in REF, then leading 1s must occur on or above the main diagonal.

Proof. Suppose that $A = [a_{ij}]$ is a matrix in REF and assume that the entry a_{ij} is a leading 1. So this leading 1 lies in the i^{th} row and j^{th} column. Since all zero rows of A occur at the bottom, all rows above the i^{th} row must be nonzero rows and so each one of them has exactly one leading 1. Thus, there are exactly $i-1$ leading 1s in the rows before the i^{th} row. Moreover, we know that a_{ij} must be to the right of all these $i-1$ leading 1s, meaning that these $i-1$ leading 1s live in the first $j-1$ columns. Moreover, each column of A has at most one leading 1, and thus there are at most $j-1$ leading 1s in the first $j-1$ columns. In other words, we must have that $i-1 \leq j-1$, and so $i \leq j$ as desired. ■

Corollary 1.3.11.

Any square matrix in REF is an upper triangular matrix.

Proof. Let $A = [a_{ij}]$ be a square matrix in REF. Suppose that a_{ik} be a nonzero entry in A. Then the i^{th} row is a nonzero row. Let a_{ij} be the leading 1 of the i^{th} row. Thus, $j \leq k$. By the above lemma, we know that $i \leq j$. Since $i \leq j$ and $j \leq k$, we get $i \leq k$. This implies that the entry a_{ik} is on or above the main diagonal. We have shown that every nonzero entry is on or above the main diagonal, so A is an upper triangular matrix. ■

Lemma 1.3.12.

Suppose that $A_{n\times n}$ is a square matrix in RREF. Then either $A = I_n$ or A has a row of zeros.

Lemma 1.3.13.

Suppose that the identity matrix I_n is row-equivalent to a matrix A. Then A has no row of zeros.

Proof. Suppose that I_n is row-equivalent to A. For the sake of contradiction, assume that A has a row of zeros. Without loss of generality, assume that the first row of A consists entirely of zeros. Since I_n is row-equivalent to A, we also have that A is row-equivalent to I_n. It follows that we can apply EROs to transform A to I_n. Now consider the system of linear equations $AX = B$ where $B = [1\ 0\ 0\ \cdots\ 0]^T$. It follows that the first equation of this system is $0\,x_1 + 0\,x_2 + \ldots + 0\,x_n = 1$ which has no solution, and consequently, the system $AX = B$ is inconsistent.

Next, we apply the EROs used to transform A to I_n to the augmented matrix $[A \mid B]$ to get the augmented matrix $[I_n \mid C]$. By Lemma 1.3.4, since the system $AX = B$ is inconsistent, the system $I_nX = C$ is also inconsistent. However, $I_nX = C$ is indeed consistent since the column matrix C is a solution. We got a contradiction! Therefore, A has no row of zeros. ■

1.4 Gaussian Elimination

We now have all the ingredients and tools to present an algorithm to solve systems of linear equations.

Algorithm 1.4.1. (Gaussian Elimination)

1. Start with any system of linear equations.
2. Write down the augmented matrix of the given system.
3. Apply elementary row operations to transform the augmented matrix into a matrix in row-echelon form.
4. Extract the system corresponding to the matrix in row-echelon form and solve it by back-substitution.

One should doubt the third step of the algorithm. Can we always reduce a matrix to one in row-echelon form by applying elementary row operations? In other words, is every matrix row-equivalent to a matrix in REF? Luckily, the answer is yes!

Theorem 1.4.2.

Any matrix is row-equivalent to a matrix in row-echelon form.

We will give a strategy of transforming a matrix to a matrix in row-echelon form. Recall that every column in a matrix in REF has at most one leading 1.

Roughly speaking, we move across the columns from the first column all the way to the last column (from left to right), placing one leading 1 (if possible) in each column using Type-I and Type-II EROs, and then use this leading 1 together with Type-III ERO to change all entries directly below this leading 1 to zeros (i.e. killing all the entries below the pivot). When it is not possible to place a leading 1 in a column we skip it and move to the next column.

Here is a more precise algorithm.

Algorithm 1.4.3. (Row-Echelon Form)

1. Start with any matrix and label all of its entries by the label "uncovered".
2. When all entries of the original matrix are covered, then STOP. Otherwise, jump to Step (3).
3. If there is at least one nonzero entry in the first column of the uncovered matrix, then do the following:
   (a) Choose a row of the uncovered matrix whose first entry is nonzero and move it to the very top of the uncovered matrix using a Type–I ERO.
   (b) Use a Type–II ERO to make the first entry of the first row of the uncovered matrix 1 (it will be a leading 1 or a pivot).
   (c) Use this pivot and a Type-III ERO to make all the entries directly below this pivot zeros.
   (d) Cover the first row and the first column of the uncovered matrix and jump to Step (2).
4. Otherwise, all the entries of the first column of the uncovered matrix are zeros, then cover this first column and jump to Step (2).

By adjusting the algorithm above by killing all the entries directly above leading 1s as well we get the following theorem.

Theorem 1.4.4.

Any matrix is row-equivalent to a matrix in reduced row-echelon form. Moreover, such RREF matrix is unique.

Example 1.28. Solve the following system of linear equations by Gaussian elimination.

$$\begin{aligned} y + z - 2w &= -3 \\ x + 2y - z &= 2 \\ 2x + 4y + z - 3w &= -2 \\ x - 4y - 7z - w &= -19 \end{aligned}$$

We obtain the augmented matrix and then apply EROs to get a matrix in REF.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix} \xRightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

$$\xRightarrow{(R_3-2R_1)\to R_3} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

$$\xRightarrow{(R_4-R_1)\to R_4} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

$$\xRightarrow{(R_4+6R_2)\to R_4} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

$$\xRightarrow{\frac{1}{3}R_3\to R_3} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & -2 & -3 \\ 0 & 0 & \mathbf{1} & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

$$\xRightarrow{\frac{-1}{13}R_4\to R_4} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & -2 & -3 \\ 0 & 0 & \mathbf{1} & -1 & -2 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{bmatrix}.$$

Next, we obtain the system represented by the very last matrix which is in row-echelon form.

$$\begin{aligned} x + 2y - z \quad &= 2 && (1) \\ y + z - 2w &= -3 && (2) \\ z - w &= -2 && (3) \\ w &= 3 && (4) \end{aligned}$$

We now proceed by back-substitution. From Equation (4) we see that $w = 3$. Using this information and Equation (3) we must have that $z = -2 + w = -2 + 3 = 1$. Next, using Equation (2) we get that $y = -3 + 2w - z = -3 + 2(3) - 1 = 2$. Finally, using Equation (1), we get that $x = 2 + z - 2y = 2 + 1 - 2(2) = -1$. Therefore, this system has exatly one solution, namely the sequence $(-1, 2, 1, 3)$. **Check that**

this sequence of real numbers satisfies all of the four equations of the original system!

Example 1.29. Solve the following system of linear equations by Gaussian elimination.

$$\begin{aligned} x - y + 2z &= 4 \\ x \qquad + z &= 6 \\ 2x - 3y + 5z &= 4 \\ 3x + 2y - z &= 1 \end{aligned}$$

We start by obtaining the the augmented matrix and then apply EROs to reduce it to a matrix in REF.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xRightarrow{(R_2-R_1)\to R_2} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xRightarrow{(R_3-2R_1)\to R_3} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix}$$

$$\xRightarrow{(R_4-3R_1)\to R_4} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \xRightarrow{(R_3+R_2)\to R_3} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

$$\xRightarrow{(R_4-5R_2)\to R_4} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \xRightarrow{R_3\leftrightarrow R_4} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\xRightarrow[\frac{-1}{2}R_4\to R_4]{\frac{-1}{2}R_3\to R_3} \begin{bmatrix} \mathbf{1} & -1 & 2 & 4 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & \mathbf{1} & \frac{1}{2} \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}.$$

The fourth row R_4 of the REF matrix represents the equation $0x + 0y + 0z = 1$ which has no solutions at all. So the system associated with this REF matrix is inconsistent, and consequently the original system is inconsistent.

A variation of the Gaussian elimination algorithm proceeds by continuing the reduction process until we obtain a matrix in a reduced row-echelon form.

Algorithm 1.4.5. (Gauss-Jordan Elimination)

1. Start with any system of linear equations.
2. Write down the augmented matrix of the given system.
3. Apply EROs to the augmented matrix until a matrix in RREF is obtained.
4. Read off the solution of the associated system.

Example 1.30. Use Gauss-Jordan elimination to solve the following system of linear equations.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

As before, we start with the augmented matrix and proceed by applying EROs.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xRightarrow[(R_3-2R_1)\to R_2]{(R_2+R_1)\to R_2} \begin{bmatrix} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xRightarrow{(R_3+R_2)\to R_3} \begin{bmatrix} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xRightarrow{\frac{1}{2}R_3\to R_3} \begin{bmatrix} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix} \xRightarrow{(R_1+2R_2)\to R_1} \begin{bmatrix} \mathbf{1} & 0 & 9 & 19 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$$

$$\xRightarrow[(R_1-9R_3)\to R_1]{(R_2-3R_3)\to R_2} \begin{bmatrix} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}.$$

Therefore, $x = 1$, $y = -1$, and $z = 2$. This shows that the original system has a unique solution, namely the triple $(1, -1, 2)$. **Check that this triple satisfies all three equations of the original system.**

Example 1.31. Solve the following system of linear equations.

$$\begin{aligned} 2x + 4y - 2z &= 0 \\ 3x + 5y &= 1 \end{aligned}$$

We proceed by Gaussian elimination.

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xRightarrow{\frac{1}{2}R_1\to R_1} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xRightarrow{(R_2-3R_1)\to R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

$$\xRightarrow{-1R_2\to R_2} \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 \\ 0 & \mathbf{1} & -3 & -1 \end{bmatrix} \xRightarrow{(R_1-2R_2)\to R_1} \begin{bmatrix} \mathbf{1} & 0 & 5 & 2 \\ 0 & \mathbf{1} & -3 & -1 \end{bmatrix}.$$

The REF matrix corresponds to the following system of linear equations.

$$\begin{aligned} x + 5z &= 2 \\ y - 3z &= -1 \end{aligned}$$

In the last REF matrix, observe that the first column (the column of the variable x) and the second column (the column of the variable y) are both pivot columns (contain leading 1s), however, the third column (the column of the variable z) is a non-pivot column. This observation guides us to treat z as the free variable when solving the system. Thus, choose any real number $t \in \mathbb{R}$ to get the following solution:

$$\begin{aligned} z &= t, \\ y &= -1 + 3t \\ x &= 2 - 5t \end{aligned}$$

So this system has infinitely many solutions of the form $(2 - 5t,\ -1 + 3t,\ t)$ where t is a real number. For example, $(2, -1, 0)$, $(-3, 2, 1)$, and $(7, -4, -1)$ are all solutions of the system.

Definition 1.4.6. (Homogeneous System of Linear Equations)

A system of linear equations in which all of its constant terms are zeros is called a *homogeneous* system.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Clearly, every homogeneous system is consistent since it has at least the *trivial solution*, namely, $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Thus, any homogeneous system either has exactly one solution (the trivial solution) or infinitely many solutions.

Example 1.32. Solve the following homogeneous system of linear equations.

$$\begin{aligned} x - y + 3z &= 0 \\ 2x + y + 3z &= 0 \end{aligned}$$

We will proceed by Gauss-Jordan elimination. We start by the augmented matrix and then apply elementary row operations to get a matrix in RREF.

$$\begin{bmatrix} \mathbf{1} & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{(R_2-2R_1)\to R_2} \begin{bmatrix} \mathbf{1} & 0 & 2 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2\to R_2} \begin{bmatrix} \mathbf{1} & 0 & 2 & 0 \\ 0 & \mathbf{1} & -1 & 0 \end{bmatrix}.$$

The final RREF matrix corresponds to the following system of linear equations.

$$\begin{aligned} x + 2z &= 0 \\ y - z &= 0 \end{aligned}$$

The first two columns of the RREF matrix are ptivot columns The third column has no leading 1 and consequently we take the variable z to be a free variable. So let $t \in \mathbb{R}$, a solution of the system has the form $x = -2t$, $y = t$, $z = t$. The solution set of the system is

$$\{(-2t,\, t,\, t) \mid t \in \mathbb{R}\}.$$

Therefore, the system has infinitely many solutions.

Remark. We note that when we use Gaussian elimination to solve homogeneous systems we may just apply EROs to the coefficient matrix (instead of the augmented matrix), this is because the last column of the augmented matrix (column of constant terms) will stay a column of zeros in all steps of the Gaussian elimination process.

The system in the example above has 2 equations and 3 variables. This has led to columns without leading 1s in the REF matrix and so free variables were present giving infinitely many solutions.

This will always be the case for homogeneous systems. To see this, consider the coefficient matrix $A_{m\times n}$ of any homogeneous system. Assume further that the number of equations is strictly less than the numbers of variables, so $m < n$. Towards solving the system, we start with the augment matrix $[A \mid 0]$ and then apply EROs to obtain a matrix $[A' \mid 0]$ in row echelon form. We know that the number of leading 1s in $[A' \mid 0]$ is at most $\min(m, n) = m$. Since $m < n$, there will be columns of A' in $[A' \mid 0]$ without leading ones, these columns correspond to free variables. Once we have a free variable, the homogeneous system has infinitely many solutions. Consequently, we have the theorem below.

Theorem 1.4.7.

Every homogeneous system of linear equations with fewer equations than variables has infinitely many solutions.

A more general result would be as follows.

Theorem 1.4.8.

Every consistent system of linear equations with fewer equations than variables has infinitely many solutions.

We finish this section by presenting a general analysis for any system of linear equations.

Algorithm 1.4.9. (Solving a System of Linear Equations)

Given any system $AX = B$ of linear equations with coefficient matrix A and column of constant terms B, we do the following:

- Form its augmented matrix $G = [\,A \mid B\,]$.
- Apply EROs on $G = [\,A \mid B\,]$ to obtain a matrix $\bar{G} = [\,\bar{A} \mid \bar{B}\,]$ in row-echelon form (REF).
- If the last nonzero row of $\bar{G}$ is $[0\ 0\ 0\ \cdots\ 0\ 1]$, then the system is inconsistent. In this case, the last column $\bar{B}$ of $\bar{G}$ is a pivot column.
- Otherwise, if $\bar{B}$ is not a pivot column, then the system is consistent. In this case, the number of leading 1s is at most the number of variables (number of columns in A). We now check the following.
  - ($\star$) If the number of leading 1s is equal to the number of variables, then the system has a unique solution. Here, all columns of $\bar{A}$ are pivot columns.
  - ($\star$) Otherwise, the number of leading 1s will be strictly less than the number of variables giving rise to non-pivot columns in $\bar{A}$. The corresponding variables to these non-pivot columns are free variables, and so the system has infinitely many solutions.

Example 1.33. Consider the system of linear equations below. For which values of a the system has no solution? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} ax + y + z &= 1 \\ x + ay + z &= a \\ x + y + az &= a^2 \end{aligned}$$

We use Gaussian elimination as follows.

$$\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1 & a \\ 1 & 1 & a & a^2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & a & a^2 \\ 1 & a & 1 & a \\ a & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[(R_3 - aR_1) \to R_3]{(R_2 - R_1) \to R_2} \begin{bmatrix} 1 & 1 & a & a^2 \\ 0 & a-1 & 1-a & a-a^2 \\ 0 & 1-a & 1-a^2 & 1-a^3 \end{bmatrix} \qquad \text{(Assume } a \neq 1\text{)}$$

$$\xrightarrow[\frac{1}{1-a} R_3 \to R_3]{\frac{1}{a-1} R_2 \to R_2} \begin{bmatrix} 1 & 1 & a & a^2 \\ 0 & 1 & -1 & -a \\ 0 & 1 & 1+a & 1+a+a^2 \end{bmatrix}$$

$$\xrightarrow{(R_3 - R_2) \to R_3} \begin{bmatrix} 1 & 1 & a & a^2 \\ 0 & 1 & -1 & -a \\ 0 & 0 & 2+a & 1+2a+a^2 \end{bmatrix} \qquad \text{(Assume } a \neq -2\text{)}$$

$$\xrightarrow{\frac{1}{2+a} R_3 \to R_3} \begin{bmatrix} 1 & 1 & a & a^2 \\ 0 & 1 & -1 & -a \\ 0 & 0 & 1 & \frac{1+2a+a^2}{2+a} \end{bmatrix}.$$

We can see that:

- The system has infinitely many solutions when $a = 1$.
- The system has no solutions when $a \neq 1$ and $a = -2$.
- The system has one solution when $a \neq 1$ and $a \neq -2$. The solution is:

$$x = a^2 - \frac{a(a+1)^2}{a+2} + a - \frac{(a+1)^2}{a+2}, \quad y = -a + \frac{(a+1)^2}{a+2}, \quad z = \frac{(a+1)^2}{a+2}.$$

For instance, when $a = 0$, the unique solution is $(-0.5,\ 0.5,\ 0.5)$. When $a = -1$, the unique solution is $(0, 1, 0)$.

1.5 LU-Decomposition

We will give another strategy to solve systems of linear equations. Recall that a square matrix L is lower triangular if all entries above the main diagonal are zeros. A square matrix U is upper triangular if all entries below the main diagonal are zeros. For example, take

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 7 & 6 & 5 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Definition 1.5.1. (LU-Decomposition)

An *LU-decomposition* of a matrix A is a pair of matrices L and U where L is lower triangular, U is upper triangular, and such that $A = LU$.

Example 1.34.

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU.$

(b) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU.$

Once we have obtained an LU-decomposition of a matrix A, say $A = LU$, then we can solve the system of linear equations $AX = B$ easily. So, $AX = (LU)X = L(UX) = B$. First, we let $Y = UX$ and solve the system $LY = B$. Second, we solve the system $UX = Y$. Note that as L is lower triangular, and U is upper triangular, it is easy to find Y and X by forward and backward substitution.

Example 1.35. Solve the linear system

$$\begin{aligned} x - 3y &= -5 \\ y + 3z &= -1 \\ 2x - 10y + 2z &= -20 \end{aligned}$$

This system is expressed as a matrix equation $AX = B$ where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix}, \text{ and } \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

From a previous example we know an LU-decomposition of A where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

.

The idea is to decompose the original system $AX = B$ to two simpler systems. Observe that as $AX = B$ we get that $(LU)X = B$ and so $L(UX) = B$. First, let $Y = UX$, and we proceed to solve the system $LY = B$ which is easy to solve as L is a lower triangular matrix. So we solve the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix}$$

We get that $y_1 = -5$, $y_2 = -1$, and $y_3 = -14$.

Second, we solve the system $UX = Y$. So we solve the system

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

Using back substitution, we get $z = -1$, $y = 2$, and $x = 1$. Therefore, the solution of the original system $AX = B$ is

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Chapter 2

Invertible Matrices

2.1 Inverse of a Matrix

Which square matrices have multiplicative inverses and which don't?

Definition 2.1.1. (Invertible Matrix)

A square matrix A of size $n \times n$ is *invertible* if and only if there exists a matrix B such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

We call B the *inverse* of A and denote it by A^{-1}. A matrix which has no inverse is called *noninvertible* or *singular*.

Lemma 2.1.2.

If A is invertible, then its inverse is unique.

Proof. Suppose that matrices B and C are both inverses of A. This means that $AB = I$, $BA = I$, $AC = I$, and $CA = I$. It follows that,

$$B = BI = B(AC) = (BA)C = IC = C.$$

Therefore, $B = C$. This shows that the inverse of A is unique. ∎

Example 2.1. Show that $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$. To establish this, we compute

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Also,

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Therefore,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Example 2.2. Since $I_n I_n = I_n$, it follows that the identity matrix I_n is always invertible, and it is its own inverse, that is, $I_n^{-1} = I_n$.

Lemma 2.1.3.

Let $A = [a_{ij}]$ be a square matrix of size $n \times n$.

- *If A has a row of zeros, then A is not invertible.*
- *If A has a column of zeros, then A is not invertible.*

Proof. We will prove the first statement. Suppose that the k^{th} row of A is a zero row. It follows that $a_{k1} = 0$, $a_{k2} = 0$, $a_{k3} = 0$, $\dots$, $a_{kn} = 0$. Now, choose any matrix $B = [b_{ij}]$ of size $n \times n$. Let $AB = C = [c_{ij}]$. We aim to compute the k^{th} row of C. So fix any $1 \leq j \leq n$, then by definition of matrix multiplication, we get that

$$c_{kj} = a_{k1}b_{1j} + a_{k2}b_{2j} + \dots + a_{kn}b_{nj} = 0 \cdot b_{1j} + 0 \cdot b_{2j} + \dots + 0 \cdot b_{nj} = 0.$$

Therefore, the k^{th} row of $C = AB$ is a row consisting entirely of zeros, which means that AB is not the identity matrix. It follows that no matrix B can be found such that $AB = I$ because AB always contains a row of zeros. Therefore, A is not invertible. ■

Theorem 2.1.4.

Let A and B be $n \times n$ matrices. If $AB = I_n$, then $BA = I_n$.

Example 2.3. Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

We need to find a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = I$. Thus, if B exists, we would have that

$$AB = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 4c & b + 4d \\ -a - 3c & -b - 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, by equality of matrices, we get

$$\begin{aligned} a + 4c &= 1 \\ -a - 3c &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} b + 4d &= 0 \\ -b - 3d &= 1 \end{aligned}$$

We obtain two systems of linear equations, the first in the variables a and c, and the second system in the variables b and d. Their augmented matrices are:

$$\begin{bmatrix} 1 & 4 & 1 \\ -1 & -3 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 4 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

Observe that both augmented matrices have the same coefficient matrix, namely the matrix A we started with. We will solve the systems by Gauss-Jordan elimination. Observe that we will apply the same sequence of elementary row operations to both augmented matrices. Accordingly, we will solve the two systems simultaneously and apply the EROs to the following new matrix which has A as its left half and the columns of the constant terms of each system as its right half. Notice that the right half is identity matrix.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \xRightarrow{EROs} \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

Thus, the two augmented matrices are reduced to the following matrices in RREF.

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

From the first matrix we get that $a = -3$ and $c = 1$, and from the second matrix we get that $b = -4$ and $d = 1$. Thus, we found the inverse of A, it is the matrix

$$B = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Observe that here we did reduce $[A \mid I]$ to $[I \mid A^{-1}]$ using EROs.

In general, when we want to find the inverse of an $n \times n$ matrix $A = [a_{ij}]$ we need to find an $n \times n$ matrix $B = [x_{ij}]$ such that $AB = I_n$. For simplicity let us take $n = 3$.

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By equating the first columns of AB and the identity matrix we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By equating the second columns of AB and the identity matrix we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

By equating the third columns of AB and the identity matrix we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To find the matrix $B = [x_{ij}]$ we need to find all the entries x_{ij} and so we need to solve the three systems above. Observe that all of these systems have the same coefficient matrix A. It follows that we will apply the same sequence of EROs to solve the three systems. To do this simultaneously we will apply EROs to the matrix below aiming to reduce A to a matrix in RREF.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right].$$

One can show that the systems above will all be consistent (and so the inverse exists) if and only if the matrix A (left side) can be reduced to the identity matrix. In this case, the matrix $[A \mid I_n]$ will be transformed $[I_n \mid A^{-1}]$. Based on the above discussion we state the following important observation.

Theorem 2.1.5.

A square matrix A is invertible if and only if A is row-equivalent to the identity matrix I_n.

This theorem is the justification for the correctness of the algorithm below which we will use to find the inverse of a matrix.

Algorithm 2.1.6. (Matrix Inverse)

Given a square matrix A of size $n \times n$.

- Adjoin I_n to the right of A to get the matrix $[A \mid I]$.
- Apply EROs to reduce $[A \mid I]$ to $[\bar{A} \mid B]$ where $\bar{A}$ is in RREF.
  - ($\star$) If $\bar{A} = I_n$, then $A^{-1} = B$. (Thus, $[A \mid I]$ is reduced to $[I \mid A^{-1}]$.)
  - ($\star$) If $\bar{A} \neq I_n$, then A is not invertible.

Example 2.4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}.$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xRightarrow[(R_3+6R_1)\to R_3]{(R_2-R_1)\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xRightarrow[(R_3+4R_2)\to R_3]{(R_1+R_2)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\xRightarrow{-1R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xRightarrow[(R_2+R_3)\to R_2]{(R_1+R_3)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right].$$

Therefore, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Example 2.5. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}.$$

We start with $[A \mid I]$ and aim to reduce A to a matrix in RREF.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \xRightarrow{EROs} \left[\begin{array}{ccc|ccc} 1 & 0 & 4/7 & 1/7 & 2/7 & 0 \\ 0 & 1 & -2/7 & 3/7 & -1/7 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right].$$

As the left part of the matrix has a row of zeros, we cannot reduce A to the identity matrix I. It follows that A has no inverse, so A is noninvertible (or singular).

Theorem 2.1.7.

Let A be a 2×2 matrix, say,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is invertible if and only if $ad - bc \neq 0$. Moreover, if A is invertible, then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. We proceed towards reducing $[A \mid I]$ to $[\bar{A} \mid B]$ where $\bar{A}$ is a matrix in RREF. We assume that $a \neq 0$ and leave the case $a = 0$ to the reader.

$$[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{a}R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_2 - cR_1) \to R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right] \quad (\text{if } ad - bc \neq 0)$$

$$\xrightarrow{\frac{a}{ad-bc}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{(R_1 - \frac{b}{a}R_2) \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = [I \mid A^{-1}].$$

Therefore, $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided that $ad - bc \neq 0$. ■

The number $ad - bc$ from the previous theorem is given a special name.

Definition 2.1.8. (Determinant of 2×2 Matrix)

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The *determinant* of A is the number

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Example 2.6. Find the inverse of the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$.

(i) $\det(A) = 3(2) - (-1)(-2) = 6 - 2 = 4$.Thus, the inverse A^{-1} exists and

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

(ii) $\det(B) = 3(2) - (-6)(-1) = 6 - 6 = 0$. Thus, B^{-1} does not exist; and we say that B is singular.

Definition 2.1.9.

For an invertible matrix A and a positive integer k, we define

$$A^{-k} = (A^{-1})^k = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ times}}.$$

Theorem 2.1.10.

Let A be an invertible matrix, let k be a positive integer, and let $c \in \mathbb{R}$ be a nonzero number. Then the matrices A^{-1}, A^k, cA, A^T are all invertible matrices, and moreover their inverses are:

(i) $(A^{-1})^{-1} = A$.

(ii) $(A^k)^{-1} = (A^{-1})^k$.

(iii) $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(iv) $(A^T)^{-1} = (A^{-1})^T$.

Proof. To show that matrix D is the inverse of matrix C we need to show that $CD = I$.

(i) Be definition of A^{-1} we know that $A^{-1}A = I$. Thus, $(A^{-1})^{-1} = A$.

(ii) $A^k(A^{-1})^k = (AAA\cdots A)(A^{-1}A^{-1}\cdots A^{-1}) = I$. Thus, $(A^k)^{-1} = (A^{-1})^k$.

(iii) $(cA)(\frac{1}{c}A^{-1}) = c(A(\frac{1}{c}A^{-1})) = c(\frac{1}{c}(AA^{-1})) = (c\frac{1}{c})(AA^{-1}) = 1I = I$. Thus,
$$(cA)^{-1} = \frac{1}{c}A^{-1}.$$

(iv) $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$. Thus, $(A^T)^{-1} = (A^{-1})^T$.

This completes the proof. ■

Theorem 2.1.11.

Let A and B be invertible matrices of size $n \times n$. Then their product AB is invertible, and moreover,
$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. We proceed using that fact that matrix multiplication is associative.
$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(B(B^{-1}A^{-1})) = A((BB^{-1})A^{-1}) \\ &= A(IA^{-1}) = AA^{-1} = I.\end{aligned}$$

Therefore, the inverse of AB is $B^{-1}A^{-1}$. ■

By mathematical induction we can show the following generalization.

Corollary 2.1.12.

Let $A_1, A_2, \ldots, A_k$ be invertible matrices of size $n \times n$. Then the inverse of their product is the product of their inverses in the reverse order, in symbols,
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}.$$

Example 2.7. Consider the matrices A and B below.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}.$$

We can find $(AB)^{-1}$ as follows.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & \frac{-7}{3} \end{bmatrix}.$$

Theorem 2.1.13. (Cancellation Property)

Suppose that A is an invertible matrix, and B, C are any matrices of suitable sizes.

(i) If $AB = AC$, then $B = C$.

(ii) If $BA = CA$, then $B = C$.

Proof. Since A is invertible, the inverse A^{-1} exists. Now assume that $AB = AC$. We proceed as follows.

$$\begin{aligned} AB = AC &\implies A^{-1}(AB) = A^{-1}(AC) \\ &\implies (A^{-1}A)B = (A^{-1}A)C \\ &\implies IB = IC \implies B = C. \end{aligned}$$

The proof of the second statement is similar. ■

Lemma 2.1.14.

Let A and B be square matrices of same size. If A is noninvertible, then AB is also noninvertible.

Proof. Suppose that A is noninvertible, but AB is invertible for the contrary. This means that there is a matrix C such that $(AB)C = I_n$. Since multiplication is associative, it follows that $A(BC) = I$ showing that BC is the inverse of A and so matrix A is invertible, a contradiction! So AB must be noninvertible. ■

Corollary 2.1.15.

Let A and B be square matrices of same size. If AB is invertible, then A and B are both invertible as well.

Theorem 2.1.16.

Suppose that $AX = B$ is a system of n linear equations in n variables. If the coefficient matrix A is an invertible matrix, then the system has a unique solution given by the $n \times 1$ column matrix $A^{-1}B$.

Proof. Suppose that A is invertible. It follows that A^{-1} exists. Let $X = [x_1\ x_2\ \cdots\ x_n]^T$ be a column matrix of real numbers. Then

$$\begin{aligned} X \text{ is a solution of the system} &\iff AX = B \\ &\iff A^{-1}(AX) = A^{-1}B \\ &\iff (A^{-1}A)X = A^{-1}B \\ &\iff IX = A^{-1}B \\ &\iff X = A^{-1}B. \end{aligned}$$

This shows that X satisfies the system if and only if $X = A^{-1}B$. Moreover, as A^{-1} is unique, the sequence $A^{-1}B$ is the only solution of the system. ■

2.2 Elementary Matrices

Definition 2.2.1. (Elementary Matrix)

An $n \times n$ matrix is called an *elementary matrix* if and only if it can be obtained from the identity matrix I_n by a single application of an elementary row operation.

Example 2.8. The following matrices are elementary.

(i) Apply $R_2 \leftrightarrow R_3$ to the identity matrix I_4 to get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ii) Apply $3R_2 \to R_2$ to the identity matrix I_4 to get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iii) Apply $(R_1 + 7R_3) \to R_1$ to the identity matrix I_4 to get:

$$\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 2.9. The following matrices are not elementary.

(i) A non-square matrix is not elementary.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(ii) No elementary row operation can produce a row of zeros starting from the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) At least two EROs are required to obtain the matrix below from I_3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem 2.2.2.

Let E be an elementary matrix of size $m \times m$, and let A be any $m \times n$ matrix. Then the product EA is the matrix obtained by performing on A the same elementary row operation that was performed on I_m to obtain E.

The theorem above says that when we want to apply an ERO on a matrix A we can achieve this by multiplying the matrix A from the *left* by a suitable elementary matrix E. The next example demonstrates this fact.

Example 2.10. Below we multiply an elementary matrix E with a matrix A to obtain the matrix EA. Observe that the matrix EA on the right hand side can also be obtained by applying on A the same ERO that was used to obtain E from I.

- Here E was obtained from I_3 by $R_1 \leftrightarrow R_2$.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix}} = \overset{EA}{\begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}}.$$

- Here E was obtained from I_3 by $2R_2 \to R_2$.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix}} = \overset{EA}{\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 4 & 12 & -8 \\ 0 & 1 & 3 & 1 \end{bmatrix}}.$$

- Here E was obtained from I_3 by $(R_2 + 8R_1) \to R_2$.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}} = \overset{EA}{\begin{bmatrix} 1 & 0 & -1 \\ 6 & -2 & -5 \\ 0 & 4 & 5 \end{bmatrix}}.$$

Example 2.11. Consider the matrix A below. Find a matrix B in row-echelon form and elementary matrices $E_1, E_2, \dots, E_k$ such that $B = E_k \cdots E_2 E_1 A$.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

We will apply EROs on A until we obtain a matrix in REF, and then find the elementary matrices corresponding to the EROs we used.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$
$$\xrightarrow{(R_3 - 2R_1) \to R_3} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Each of the elementary row operations applied above corresponds to multiplying from the left by the elementary matrix obtained using the same operation from the identity matrix. It follows that:

$$\overset{B}{\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}} = \overset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}} \overset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \overset{E_1}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}}$$

Since applying EROs to a matrix yields the same matrix as multiplying it from the left by elementary matrices we get the following result.

Lemma 2.2.3.

A matrix A is row-equivalent to a matrix B if and only if there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

Theorem 2.2.4.

Any elementary matrix is invertible. Moreover, the inverse of an elementary matrix is elementary as well.

Proof. Given an elementary matrix E, we can transform it back to the identity matrix by a single application of an elementary row operation. Let F be the elementary matrix obtained by applying this elementary row operation to the identity matrix. It follows that $FE = I_n$, and so the inverse of E is F. Here are the details.

Case (i). The matrix E is obtained from I_n by interchanging two rows.
To transform E back to the identity matrix we simply need to interchange the same two rows. This can be done my multiply E from the left by itself, and so $EE = I_n$. Thus, E is the inverse of itself, i.e. $E^{-1} = E$.

Case (ii). The matrix E is obtained from I_n by multiply row R_i by $c \neq 0$.
To transform E back to the identity matrix we will multiply the same row by $\frac{1}{c}$. Thus, E^{-1} is the matrix obtained by applying the operation $\frac{1}{c}R_i \to R_i$ on I_n.

Case (iii). The matrix E is obtained from I_n using the operation $(R_i + cR_j) \to R_i$. It follows that E is the same as the identity matrix but it has the number c in the (i, j)-entry (instead of 0). To transform E back to the identity we will use the 1 in the (j, j)-entry to kill c get back I_n. Thus, E^{-1} is the matrix obtained by applying on I_n the operation $(R_i - cR_j) \to R_i$. ■

The example below clarifies the proof of the theorem above.

Example 2.12. Find the inverses of the elementary matrices below.

	Elementary E	Transform E to I_n	Inverse matrix E^{-1}
(i)	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
(ii)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$2R_3 \to R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
(iii)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$	$(R_3 + 2R_1) \to R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Next, we show that elementary matrices are the building blocks of invertible matrices

Theorem 2.2.5.

A matrix A is invertible if and only if A can be written as a product of elementary matrices.

Proof. ($\Rightarrow$) For the forward direction, assume that A is invertible. It follows that the matrix $[A \mid I_n]$ can be reduced to $[I_n \mid A^{-1}]$ using Gauss-Jordan elimination. Therefore, looking at the left sides, it shows that we can transform the matrix A to the identity matrix I by a sequence of elementary row operations. We know that an application of an ERO corresponds to multiplying from the left by the corresponding elementary matrix. Therefore, there are elementary matrices $E_1, E_2, \ldots, E_k$ corresponding to the EROs applied in transforming A to I_n such that $E_k E_{k-1} \cdots E_2 E_1 A = I_n$. We know that each E_i is invertible since it is elementary. We now multiply both sides of the equation of the the product of the inverses as follows:

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1})(E_k E_{k-1} \cdots E_2 E_1 A) = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) I_n.$$

Thus, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. Since the inverse of an elementary matrix is elementary, we know that each E_i^{-1} is elementary. Therefore, we managed to write A as a product of elementary matrices.

($\Leftarrow$) For the reverse direction, assume that $A = E_1E_2 \ldots E_k$, where each E_i is elementary. We know that each elementary matrix E_i is invertible. We also know, by Theorem 2.1.11, that the product of invertible matrices is invertible. Therefore, A is invertible as desired and its inverse is $A^{-1} = (E_1E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1}E_1^{-1}$. ■

Example 2.13. Check if the matrix A is invertible, and if so express A as a product of elementary matrices.

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

The determinant $\det(A) = -1(8) - (-2)3 = -8 + 6 = -2 \neq 0$. Thus, the matrix A is invertible. We now transform A to the identity matrix I_2.

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{-1R_1 \to R_1} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{(R_2 - 3R_1) \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{(R_1 - 2R_2) \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Rewriting the above applications of elementary row operations as multiplications from the left with the corresponding elementary matrices we get:

$$\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \overset{E_4}{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \overset{E_3}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}} \overset{E_2}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}} \overset{E_1}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}}.$$

Now solving for A by multiplying both sides by the inverses of the elementary matrices we can express A as a product of elementary matrices as desired.

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} = \overset{E_1^{-1}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \overset{E_2^{-1}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \overset{E_3^{-1}}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} \overset{E_4^{-1}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}.$$

We finish this section by collecting some of the important properties of invertible matrices we studied above.

Theorem 2.2.6.

Let A be an $n \times n$ matrix. The following statements are equivalent.

(i) The matrix A is invertible.

(ii) The system $AX = B$ has a unique solution for every $n \times 1$ column matrix B.

(iii) The homogeneous system $AX = \mathbf{0}_{n\times 1}$ has only the trivial solution.

(iv) The matrix A is row-equivalent to the identity matrix I_n.

(v) The matrix A can be written as a product of elementary matrices.

2.3 Determinant of a Matrix

We aim to associate every square matrix A with a special real number called its determinant, denoted by $\det(A)$ or $|A|$. So the determinant is function from the set of square matrices to the set $\mathbb{R}$ of real numbers. We have already defined the determinant of 2×2 matrices. Recall that the determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined as follows:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 2.14. Consider the matrices $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.
Then, $\det(A) = 2(2) - (-3)(1) = 7$, and $\det(B) = 2(2) - (1)(4) = 0$.

We will use determinants of 2×2 matrices to define the determinant of a 3×3 matrix. Then we will use determinants of 3×3 matrices to define the determinant of a 4×4 matrix, and so on and so forth. Towards this we need to define the minors and the cofactors of a matrix. For a square matrix A, we define the following.

- Let A_{ij} be the matrix obtained from A by deleting its i^{th} row and j^{th} column.
- Let $M_{ij} = \det(A_{ij})$. We call M_{ij} the (i, j)-*minor* of A.
- Let $C_{ij} = (-1)^{i+j} \det(A_{ij})$. We call C_{ij} the (i, j)-*cofactor* of A. Thus,

$$C_{ij} = \begin{cases} M_{ij} & \text{if } i + j \text{ is even,} \\ -M_{ij} & \text{if } i + j \text{ is odd.} \end{cases}$$

Below are sign patterns for cofactors in square matrices of order $3, 4, 5$, respectively.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \end{bmatrix}$$

5×5 matrix

Definition 2.3.1. (Determinant of a Matrix)

The *determinant* is a function that assigns to every square matrix A a real number denoted by $\det(A)$ or $|A|$. For a 1×1 matrix we define $\det([a]) = a$. The determinant of a matrix $A = [a_{ij}]$ of size $n \times n$ where $n \geq 2$ is the sum of the products of the entries in the first row with their corresponding cofactors.

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det(A_{1j}) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}\,.$$

Example 2.15. Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

To compute the determinant of A we need to calculate the cofactors of the first-row entries.

- $C_{11} = M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1.$
- $C_{12} = -M_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5.$
- $C_{13} = M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4.$

Therefore,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 0(-1) + 2(5) + 1(4) = 14.$$

Next, we compute the second-row cofactors.

- $C_{21} = -M_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2.$
- $C_{22} = M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4.$
- $C_{23} = -M_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = -(-8) = 8.$

The sum of products of second-row entries with their cofactors is also 14.

$$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 3(-2) + (-1)(-4) + 2(8) = 14.$$

This is not a coincidence as the next theorem states.

Theorem 2.3.2.

Let A be a square matrix of size $n \times n$. Then the sum of products of the i^{th} row entries with their cofactors (called the cofactor expansion along the i^{th} row) is equal to the determinant of A.

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Also, the sum of products of the j^{th} column entries with their cofactors (called the cofactor expansion along the j^{th} column) is equal to the determinant of A.

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Corollary 2.3.3.

Let A be a square matrix which has an entire row of zeros or an entire column of zeros. Then $\det(A) = 0$.

Example 2.16. Find the determinant of the matrix $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$.

The easy choice here is to use the third-column cofactor expansion of the determinant.

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} = 3C_{13} + 0C_{23} + 0C_{33} + 0C_{43} = 3C_{13}.$$

It remains to compute the cofactor $C_{13} = (-1)^4 M_{13} = M_{13}$. By definition, M_{13} is the determinant of the matrix obtained from A by deleting the 1^{st} row and 3^{rd} column.

$$\begin{bmatrix} 1 & \text{-2} & 3 & 0 \\ \mathbf{-1} & \mathbf{1} & 0 & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & 0 & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & 0 & \mathbf{-2} \end{bmatrix}$$

We will use the third-row cofactor expansion to compute the determinant of the resultant matrix. Thus,

$$C_{13} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ \mathbf{3} & \mathbf{4} & \mathbf{-2} \end{vmatrix} = (3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + (-1)(4)\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = -3+12+4 = 13.$$

Finally, we have that $\det(A) = 3C_{13} = 3 \times 13 = 39$.

Example 2.17 (Diagonals Method for 3×3 matrices)**.** Find a formula for the determinant of a general 3×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We need to compute the cofactors of the first row, namely, C_{11}, C_{12}, C_{13}.

- $C_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$.

- $C_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{23}a_{31} - a_{21}a_{33}$.

- $C_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$.

Therefore,

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}. \end{aligned}$$

To easily remember this formula we copy the first two columns of A and adjoin them to the right of A to get the 3×5 matrix below. The determinant of A is the sum of the products of the downward diagonals (black solid arrows) minus the sum of the products of the upward diagonals (dashed red arrows).

$$\begin{array}{ccccc} + & + & + & & \\ \end{array}$$
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$
$$\begin{array}{ccccc} - & - & - & & \\ \end{array}$$

Example 2.18. Find the determinant of the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$.

- Computing the determinant using the second-row cofactor expansion.

$$\begin{aligned} \det(A) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23} \\ &= -3\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 4 & -4 \end{vmatrix} \\ &= -3(6) + (-1)(-4) - 2(-8) - 2(-8) = -18 + 4 + 16 = 2. \end{aligned}$$

- Computing the determinant using third-column cofactor expansion.

$$\begin{aligned}\det(A) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= a_{13}M_{13} - a_{23}M_{23} + a_{33}M_{33} \\ &= \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 4 & -4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} \\ &= -8 + 16 - 6 = 2.\end{aligned}$$

- Using the diagonals method. We copy the first two columns and adjoin them to the right of A. Then "downward diagonals minus upward diagonals".

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 2 & 3 & -1 \\ 4 & -4 & 1 & 4 & -4 \end{bmatrix}$$

$$\begin{aligned}\det(A) &= 0 + (2)(2)(4) + (1)(3)(-4) - (4)(-1)(1) - 0 - (1)(3)(2) \\ &= 16 - 12 + 4 - 6 = 2.\end{aligned}$$

Example 2.19. Find the determinant of the lower-triangular matrix

$$A = \begin{bmatrix} \mathbf{2} & 0 & 0 & 0 \\ 4 & \mathbf{-2} & 0 & 0 \\ -5 & 6 & \mathbf{4} & 0 \\ 1 & 5 & 3 & \mathbf{3} \end{bmatrix}.$$

We will use the first-row cofactor expansion to compute the determinant.

$$\begin{aligned}|A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} \\ &= 2C_{11} + 0C_{12} + 0C_{13} + 0C_{14} \\ &= 2C_{11} = 2M_{11} = 2\begin{vmatrix} -2 & 0 & 0 \\ 6 & 4 & 0 \\ 5 & 3 & 3 \end{vmatrix} \\ &= 2\left(-2\begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} - 0\begin{vmatrix} 6 & 0 \\ 5 & 3 \end{vmatrix} + 0\begin{vmatrix} 6 & 4 \\ 5 & 3 \end{vmatrix}\right) \\ &= 2 \times -2 \times \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} \\ &= 2 \times -2 \times 4 \times 3 = -48.\end{aligned}$$

Observe that the determinant is the product of the main diagonal entries.

Example 2.20. Find the determinant of the upper-triangular matrix

$$B = \begin{bmatrix} \mathbf{2} & 3 & -1 \\ 0 & \mathbf{-4} & 3 \\ 0 & 0 & \mathbf{3} \end{bmatrix}.$$

We will use the first-column cofactor expansion to compute the determinant.

$$\begin{aligned}|B| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= 2C_{11} + 0C_{21} + 0C_{31} \\ &= 2C_{11} = 2M_{11} = 2\begin{vmatrix} -4 & 3 \\ 0 & 3 \end{vmatrix} \\ &= 2 \times -4 \times 3 = -24.\end{aligned}$$

Also here the determinant is the product of the main diagonal entries.

Theorem 2.3.4.

The determinant of a triangular matrix (lower or upper) is the product of the main diagonal entries. That is, for a triangular matrix $A = [a_{ij}]$ of size $n \times n$, we have that

$$\det(A) = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}.$$

Proof. We will prove the theorem by induction for lower triangular matrices. For the base case, suppose that $A = [a_{ij}]$ is a 2×2 lower triangular matrix. Then $a_{12} = 0$ and so

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - 0 \cdot a_{21} = a_{11}a_{22}.$$

Suppose the theorem holds for $n \times n$ lower triangular matrices. Let A be any $(n+1) \times (n+1)$ lower triangular matrix. Thus, A_{11} is an $n \times n$ lower triangular matrix whose diagonal entries are $a_{22}, a_{33}, \ldots, a_{nn}, a_{(n+1)(n+1)}$. We know by induction hypothesis that $\det(A_{11})$ is the product of its diagonal entries. Consequently,

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1(n+1)}C_{1(n+1)} \\ &= a_{11}C_{11} + 0 \cdot C_{12} + \cdots + 0 \cdot C_{1(n+1)} \\ &= a_{11}C_{11} = a_{11}M_{11} = a_{11}\det(A_{11}) \\ &\overset{IH}{=} a_{11}\, a_{22}\, a_{33} \cdots a_{nn}\, a_{(n+1)(n+1)}.\end{aligned}$$

This shows that $\det(A)$ is the product of its diagonal entries as desired. ■

Corollary 2.3.5.

For any positive integer n, we have $\det(I_n) = 1$.

2.4 Determinant and EROs

We now investigate the effect of applying elementary row operations on the determinant of a matrix.

Original matrix A	ERO	New matrix B
$\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = \mathbf{11}$	$R_1 \leftrightarrow R_2$	$\det(B) = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = \mathbf{-11}$
$\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = \mathbf{11}$	$3R_2 \to R_2$	$\det(B) = \begin{vmatrix} 2 & -3 \\ 3 & 12 \end{vmatrix} = \mathbf{33}$
$\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = \mathbf{11}$	$(R_2 - 2R_1) \to R_2$	$\det(B) = \begin{vmatrix} 2 & -3 \\ -3 & 10 \end{vmatrix} = \mathbf{11}$

The investigation above suggests the following theorem which we leave its proof as an exercise to the reader.

Theorem 2.4.1.

Suppose we apply an ERO on a matrix A to get a new matrix B.

(i) If A transforms to B by applying $R_i \leftrightarrow R_j$, then $\det(B) = -\det(A)$.

(ii) If A transforms to B by applying $cR_i \to R_i$, then $\det(B) = c\det(A)$.

(iii) If A transforms to B by applying $(R_i+cR_j) \to R_i$, then $\det(B) = \det(A)$.

Proof. Statement (i) follows from Statements (*ii*) and (*iii*) and the fact that swapping two rows can be obtained by applying three EROs of Type-III followed by one ERO of Type-II with $c = -1$ (see Lemma 1.3.6).

We show Statement (ii) by induction on n where A is of size $n \times n$. For $n = 1$, suppose $A = [a]$ and we obtain B by multiplying its only row by c. Then $B = [ca]$. So $\det(B) = \det([ca]) = ca = c\det([a]) = c\det(A)$. Suppose the statement holds for all square matrices of size $n \times n$. Let A be an arbitrary $(n+1) \times (n+1)$ matrix. Suppose $A = [a_{ij}]$ transforms to $B = [b_{ij}]$ by applying $cR_i \to R_i$. If $i = 1$, i.e. the first row of A is multiplied by c, then $b_{1j} = ca_{1j}$ and $B_{1j} = A_{1j}$ and so

$$\begin{aligned}\det(B) = \sum_{j=1}^{n}(-1)^{1+j}\, b_{1j}\det(B_{1j}) &= \sum_{j=1}^{n}(-1)^{1+j}\,(ca_{1j})\det(A_{1j}) \\ &= c\sum_{j=1}^{n}(-1)^{1+j}\, a_{1j}\det(A_{1j}) = c\det(A).\end{aligned}$$

If $2 \leq i \leq n+1$, i.e. a row other than the first row of A is multiplied by c, then $b_{1j} = a_{1j}$ and B_{1j} is an $n \times n$ matrix obtained by multiplying a row of A_{1j} by c. By induction hypothesis, we know that $\det(B_{ij}) = c\det(A_{ij})$. Thus,

$$\begin{aligned}\det(B) = \sum_{j=1}^{n}(-1)^{1+j}\, b_{1j}\det(B_{1j}) &= \sum_{j=1}^{n}(-1)^{1+j}\, a_{1j}(c\det(A_{1j})) \\ &= c\sum_{j=1}^{n}(-1)^{1+j}\, a_{1j}\det(A_{1j}) = c\det(A).\end{aligned}$$

■

Thus, Type-I ERO flips the sign of the determinant. Type-II ERO multiplies the old determinant by c. Type-III ERO does not change the determinant. We remark that the same theorem holds if we replace EROs with elementary *column* operations. The way to make use of this theorem is by transforming a matrix to a simpler matrix whose determinant is easy to compute, for instance, triangular matrices. The next example demonstrates this strategy. The table below is useful to keep track of the determinant.

$A \xRightarrow{ERO} B$	Determinant
Type-I	$\det(A) = -\det(B)$
Type-II	$\det(A) = \frac{1}{c}\det(B)$
Type-III	$\det(A) = \det(B)$

Example 2.21. Find $\det(A)$ using elementary row operations.

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

The strategy is to reduce A to a matrix in upper triangular form.

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \overset{(R_2 - 2R_1)\to R_2}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{\frac{-1}{7}R_2 \to R_2}{=} -(-7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{(R_3 - R_2)\to R_3}{=} 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = 7(1)(1)(-1) = -7.$$

Corollary 2.4.2.

Let A be a square matrix with at least one of the following properties.

(i) One row of A is a multiple of another row (this includes the case of two identical rows).

(ii) One column of A is a multiple of another column (this includes the case of two identical columns).

Then $\det(A) = 0$.

The next example give a sketch of a proof of the theorem above.

Example 2.22. Find the determinant of the matrices below.

(i) Using the first-row cofactor expansion we get that

$$\begin{vmatrix} 0 & 0 & 0 \\ 2 & 4 & -5 \\ 3 & -5 & 2 \end{vmatrix} = 0\begin{vmatrix} 4 & -5 \\ -5 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & -5 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = 0.$$

(ii) When a row is the multiple of another row we proceed as follows to get a row of zeros.

$$\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} \overset{(R_3 - R_1)\to R_3}{=} \begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

(iii) When a column is the multiple of another column we proceed as follows to get a column of zeros.

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -6 \\ -2 & 0 & 6 \end{vmatrix} \overset{(C_3 + 3C_1)\to C_3}{=} \begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -2 & 0 & 0 \end{vmatrix} = 0.$$

Example 2.23. Find the determinant of matrix A using EROs.

$$|A| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{vmatrix} \overset{(R_2 - 2R_1)\to R_2}{=} \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix} = 0, \text{ since } R_3 \text{ is a multiple of } R_2.$$

Example 2.24. Use elementary column operations to find the determinant of B.

$$|B| = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \overset{(C_3 + 2C_1)\to C_3}{=} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -3\begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = -3(15 - 16) = 3.$$

Example 2.25. Find the determinant of the matrix A below.

$$A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}.$$

We apply Type-III EROs to eliminate the entries below the leading 1 in the first column.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \\
&= (y-x)(z^2-x^2) - (y^2-x^2)(z-x) \\
&= (y-x)(z-x)(z+x) - (y-x)(y+x)(z-x) \\
&= (y-x)(z-x)(z+x-y-x) = (y-x)(z-x)(z-y).
\end{aligned}$$

Using the fact that $\det(I_n) = 1$ and by taking $A = I_n$ and B to be an elementary matrix we get the following result by applying Theorem 2.4.1.

Corollary 2.4.3. (Determinant of Elementary Matrices)

Suppose that E is an elementary matrix.

(i) If E is obtained by applying $R_i \leftrightarrow R_j$, then $\det(E) = -1$.

(ii) If E is obtained by applying $cR_i \to R_i$, then $\det(E) = c$ *where $c \neq 0$.*

(iii) If E is obtained by applying $(R_i + cR_j) \to R_i$, then $\det(E) = 1$.

Consequently, the determinant of an elementary matrix is always nonzero.

2.5 Multiplicativity of Determinant

The main objective of this section is to prove the remarkable theorem that a square matrix is invertible if and only if its determinant is nonzero. Consequently, the determinant of a matrix tells us whether the matrix is invertible or singular. Moreover, we will prove in this section another remarkable property of the determinant, namely, the determinant is a *multiplicative* function, which means that the determinant of a product of matrices is the product of the determinants of the matrices, that is, we have $\det(AB) = \det(A)\,\det(B)$ for any square matrices A and B of the same size.

Example 2.26. We illustrate the multiplicativity of the determinant function for matrices A and B below.

- $\det(A) = \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} = 3.$ $\qquad \det(B) = \begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix} = 9.$
- $AB = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 34 & 23 \end{bmatrix}.$
- $\det(AB) = 13(23) - 8(34) = 27 = 3 \times 9 = \det(A)\det(B).$

This shows that the determinant of AB is the product of their determinants.

Towards proving that the determinant function is multiplicative, we first prove a special case of the theorem where the first matrix is elementary and the second is arbitrary.

Lemma 2.5.1.

Let E be an elementary matrix and B be any square matrix, both of size $n \times n$. Then,

$$\det(EB) = \det(E)\,\det(B).$$

Proof. We have three cases to consider depending on the type of the elementary row operation that is used to obtain the elementary matrix E from the identity matrix I_n. We will be using below Theorem 2.2.2, Theorem 2.4.1, and Corollary 2.4.3.

Case (I). If E is obtained from I_n by interchanging two rows, then $\det(E) = -1$. Moreover, as EB is the matrix obtained from B by interchanging the same two rows, we get that $\det(EB) = (-1)\det(B) = \det(E)\det(B)$ as desired.

Case (II). If E is obtained from I_n by multiplying a row of I by a nonzero constant c, then $\det(E) = c$. Moreover, as EB is the matrix obtained from B by multiplying the same row by c, we get that $\det(EB) = c\det(B) = \det(E)\det(B)$ as desired.

Case (III). If E is obtained by adding a multiple of one row of I_n to another row, then $\det(E) = 1$. Then EB is the matrix obtained by applying the same ERO on

B. Moreover, we know that Type-III EROs do not change the determinant. Thus, $\det(EB) = \det(B) = 1 \cdot \det(B) = \det(E)\det(B)$ as desired.

Therefore, in all cases we have shown that $\det(EB) = \det(E)\det(B)$. ■

Using mathematical induction we can generalize the result above to a more general statement.

Corollary 2.5.2.

Let $E_1, E_2, \ldots, E_k$ be elementary matrices, and let B be any square matrix, all of the same size. Then,

$$\det(E_1E_2 \cdots E_kB) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(B).$$

We now can prove that the determinant of a matrix determines whether the matrix is invertible or not. If the determinant is nonzero, then the matrix is invertible. If the determinant is 0, then the matrix is noninvertible. Here is a proof.

Theorem 2.5.3.

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. For the forward direction, suppose A is invertible. By Theorem 2.2.5 we know that A is a product of elementary matrices. Thus, $A = E_1E_2 \cdots E_k$ for some elementary matrices E_i. By the previous lemma, $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_k)$. By Corollary 2.4.3 we know that $\det(E_i) \neq 0$ since E_i is an elementary matrix. It follows that $\det(A)$ is a product of nonzero numbers, and so it is itself nonzero.

For the reverse direction, suppose $\det(A) \neq 0$. Apply the Gauss-Jordan elimination method to A to obtain a matrix B in reduced row-echelon form. Therefore, there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k \cdots E_2E_1A$. By the corollary above, we get that $\det(B) = \det(E_k)\cdots\det(E_2)\det(E_1)\det(A)$. As all values on the right-hand side are nonzero, it follows that $\det(B) \neq 0$. As B is a square matrix in RREF, either B is the identity matrix, or B contains a row of zeros. The latter case is impossible because if so, we get $\det(B) = 0$, however, $\det(B) \neq 0$. Thus, it must be that $B = I_n$, and so A is row-equivalent to I_n. By Theorem 2.1.5, we get that A is invertible. ■

Corollary 2.5.4.

A square matrix A is not invertible if and only if $\det(A) = 0$.

Corollary 2.5.5.

Let A be a square matrix. A system of linear equations $AX = B$ has a unique solution if and only if $\det(A) \neq 0$.

Example 2.27. Which of the following matrices is invertible?

(a) We use the first-row cofactor expansion of the determinant.

$$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0 - 2\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 3 & 2 \end{vmatrix} = -2(-6) - 1(12) = 12 - 12 = 0.$$

So the matrix A is noninvertible (or singular).

(b) We use the second-row cofactor expansion of the determinant.

$$|B| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 0 & -3 \\ 4 & 4 & 4 \end{vmatrix} = -(-3)\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} = 3(4) = 12 \neq 0.$$ So the matrix B is invertible.

We now have all the information needed to prove that the determinant function is multiplicative.

Theorem 2.5.6.

Let A and B be square matrices of same size. Then

$$\det(AB) = \det(A)\det(B).$$

Proof. Let A and B be any square matrices of size $n \times n$. We split the proof into two cases, whether A is invertible or not.

Case 1. If A is noninvertible.
In this case, by Lemma 2.1.14, we know that AB must be noninvertible as well. By Theorem 2.5.3 we get that $\det(A) = 0$ and $\det(AB) = 0$. Therefore,

$$\det(AB) = 0 = 0 \cdot \det(B) = \det(A)\det(B).$$

Case 2. If A is invertible.
By Theorem 2.2.5, we it follows that A is a product of elementary matrices, say, $A = E_1E_2\cdots E_k$ where each E_i is an elementary matrix. By two applications of Corollary 2.5.2 we get that

$$\begin{aligned} \det(AB) &= \det(E_1E_2\cdots E_kB) \\ &= \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) \\ &= \det(E_1E_2\cdots E_k)\det(B) \\ &= \det(A)\det(B). \end{aligned}$$

This completes the proof. ■

Corollary 2.5.7.

Let A be a square matrix of size $n \times n$ and let $c \in \mathbb{R}$. Then

$$\det(cA) = c^n \det(A).$$

Proof. First, observe that the matrix cI_n is the $n \times n$ diagonal matrix where every entry on the main diagonal is c and every other entry is 0. It follows that the determinant of cI_n is the product of the diagonal entries, so $\det(cI_n) = c^n$. Next, using the fact that the determinant function is multiplicative we proceed to establish the desired result.

$$\det(cA) = \det(c(I_nA)) = \det((cI_n)A) = \det(cI_n)\det(A) = c^n \det(A).$$

■

Example 2.28. Find the determinant of $A = \begin{bmatrix} 15 & 0 & 0 \\ 25 & 20 & 5 \\ 30 & 10 & 5 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} 15 & 0 & 0 \\ 25 & 20 & 5 \\ 30 & 10 & 5 \end{bmatrix}\right) = \det\left(5\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 1 \\ 6 & 2 & 1 \end{bmatrix}\right) \\ &= 5^3 \det\left(\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 1 \\ 6 & 2 & 1 \end{bmatrix}\right) = 5^3\left(3\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}\right) \\ &= 5^3 \cdot 3 \cdot 2 = 750. \end{aligned}$$

Corollary 2.5.8.

Suppose A is an invertible matrix. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof. Since A is invertible we know that A^{-1} exists and $\det(A) \neq 0$. Using the fact that the determinant is multiplicative, it follows that

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1.$$

Thus, $\det(A^{-1}) = \frac{1}{\det(A)}$ as desired. ■

Example 2.29. Find $\det(A^{-1})$ where $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$.

$$\det(A) = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} \overset{(R_3-2R_1)\to R_3}{=} \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 1 & -6 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -6 \end{vmatrix} = 4.$$

Thus, $\det(A^{-1}) = \frac{1}{4}$.

Confirm this answer by directly computing the determinant of $A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & \frac{-3}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$.

Example 2.30. Find the determinants of matrix A below and its transpose A^T.

$$\det(A) = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{vmatrix} = -2 \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} = -2(3) = -6. \qquad \text{(Using second row)}$$

$$\det(A^T) = \begin{vmatrix} 3 & 2 & -4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} = -2(3) = -6. \qquad \text{(Using second column)}$$

Exercise. Prove that $\det(A) = \det(A^T)$ for any square matrix A.

Example 2.31. Suppose that A and B are 3×3 matrices such that $\det(A) = 3$ and $\det(B) = 4$. Let $C = 7A^5 B^T A^{-3} B^4$. Find $\det(C)$.

$$\begin{aligned} \det(C) &= \det(7A^5 B^T A^{-3} B^4) \\ &= 7^3 \det(A^5 B^T A^{-3} B^4) \\ &= 7^3 \det(A^5) \det(B^T) \det(A^{-3}) \det(B^4) \\ &= 7^3\, |A|^5\, |B|\, |A^{-1}|^3\, |B|^4 \\ &= 7^3\, |A|^5\, |B|\, \frac{1}{|A|^3}\, |B|^4 \\ &= 7^3\, |A|^2\, |B|^5 = 7^3\, 3^2\, 4^5 = 3161088. \end{aligned}$$

Example 2.32. Find all values of x which make the matrix below noninvertible.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & -1 & 2 & 3 \\ x^2 & 1 & 4 & 9 \\ x^3 & -1 & 8 & 27 \end{bmatrix}.$$

We know that A is noninvertible if and only if $\det(A) = 0$. We compute the determinant via the first-column cofactor expansion.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & -1 & 2 & 3 \\ x^2 & 1 & 4 & 9 \\ x^3 & -1 & 8 & 27 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & 9 \\ -1 & 8 & 27 \end{vmatrix} - x \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 9 \\ -1 & 8 & 27 \end{vmatrix} + x^2 \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ -1 & 8 & 27 \end{vmatrix} - x^3 \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} \\
&= -72 - 12x + 48x^2 - 12x^3 \\
&= -12(6 + x - 4x^2 + x^3) \\
&= -12(x-2)(x-3)(x+1).
\end{aligned}$$

Therefore, $\det(A) = 0$ when $x = 2$, $x = 3$, and $x = -1$.

Let us collect all the properties of the determinant function we studied in one place.

Theorem 2.5.9.

Let A, B be square matrices of size $n \times n$.

(i) $\det(AB) = \det(A)\det(B)$.

(ii) A is invertible if and only if $\det(A) \neq 0$.

(iii) A is noninvertible if and only if $\det(A) = 0$.

(iv) $\det(cA) = c^n \det(A)$.

(v) $\det(A^k) = (\det(A))^k$ *for any integer $k \geq 1$.*

(vi) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

(vii) $\det(A) = \det(A^T)$.

(viii) $\det(I_n) = 1$.

(ix) If A has a row of zeros, then $\det(A) = 0$.

(x) If $A = [a_{ij}]$ is a triangular matrix, then $\det(A) = a_{11}\, a_{22} \cdots a_{nn}$.

Theorem 2.5.10.

Let F be a function that assigns to every square matrix A a real number $F(A)$. Suppose that F satisfies the following properties for any matrix A:

- $F(I_n) = 1$ *for every* $n \geq 1$.
- *If A transforms to B by applying $cR_i \to R_i$, then $F(B) = c\,F(A)$.*
- *If A transforms to B by applying $(R_i + cR_j) \to R_i$, then $F(B) = F(A)$.*

Then $F(A) = \det(A)$ for every square matrix A. Consequently, the determinant is the unique function which satisfies these three properties.

Proof. First, using Lemma 1.3.6 and the properties of F, it follows that if A transforms to B by interchanging two rows, then $F(B) = -F(A)$. With this we can easily show that $F(E) = \det(E)$ for any elementary matrix E (see Corollary 2.4.3), in particular:

$$F(E) = \begin{cases} -1 & E \text{ is of Type-I;} \\ c & E \text{ is of Type-II;} \\ 1 & E \text{ is of Type-III.} \end{cases}$$

Now suppose that A and B are square matrices such that $B = E_k \cdots E_2 E_1 A$ where each E_i is elementary. Since multiplying from the left by an elementary matrix corresponds to applying an ERO, it follows from the properties of F that $F(B) = F(E_k) \cdots F(E_2) F(E_1) F(A)$.

Next, we will show that if a square matrix A has a row of zeros, then $F(A) = 0$. Suppose a square matrix A has a row of zeros, say row R_i consists entirely of zeros. Let B be the matrix obtained from A by multiplying the i^{th} row R_i of A by 2. By the second property of the function F, we get that $F(B) = 2F(A)$. As all entries of R_i are zeros, we get that $B = A$. Thus, $F(A) = 2F(A)$, and so $F(A) = 0$.

Finally, pick an arbitrary square matrix B. If B is invertible, then B is row-equivalent to I_n, and so $B = E_k \cdots E_2 E_1 I_n$ for some elementary matrices E_i. It follows by the multiplicativity of the determinant that

$$\begin{aligned} F(B) &= F(E_k) \cdots F(E_2) F(E_1) F(I_n) \\ &= \det(E_k) \cdots \det(E_2) \det(E_1) \det(I_n) \\ &= \det(E_k \cdots E_2 E_1 I_n) = \det(B). \end{aligned}$$

Otherwise, if B is noninvertible, then B is row-equivalent to a matrix A which has a row of zeros, and so $B = E_m \cdots E_2 E_1 A$ for some elementary matrices E_i. Therefore,

$$F(B) = F(E_m) \cdots F(E_2) F(E_1) F(A) = F(E_k) \cdots F(E_2) F(E_1) \cdot 0 = 0 = \det(B).$$

Therefore, for any square matrix B, we have that $F(B) = \det(B)$ as desired. ∎

We can use the theorem above to show that the cofactor expansion along the i^{th} row, and the cofactor expansion along the j^{th} column are both equal to the determinant by showing that they satisfy the three properties in the theorem.

Chapter 3

Vector Spaces

In this chapter we will study algebraic structures called vector spaces. An example of a vector space is the set $\mathbb{M}_{3\times 5}$ of all 3×5 matrices equipped with matrix addition and scalar multiplication. Observe that this set together with these two operations satisfy many nice properties that were studied in Chapter 1. For instance, the set $\mathbb{M}_{3\times 5}$ is closed under matrix addition and scalar multiplication. Also, matrix addition is commutative and scalar multiplication distributes over matrix addition, and many other nice properties. There are many structures in mathematics consisting of a set equipped with addition and scalar multiplication that satisfy the same nice properties. The n-dimensional spaces are among the first examples.

3.1 The n-Dimensional Space $\mathbb{R}^n$

A *vector* in the n-dimensional space $\mathbb{R}^n$ is an ordered n-tuple $\vec{v} = (x_1, x_2, x_3, \ldots, x_n)$ where each x_i is a real number. The real numbers $x_1, x_2, \ldots, x_n$ are called the *components* or the *coordinates* of the vector $\vec{v}$.

- $\mathbb{R}$ is the 1-dimensional space; the set of all real numbers.
- $\mathbb{R}^2$ is the 2-dimensional space; the set of all ordered *pairs* of real numbers.

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

- $\mathbb{R}^3$ is the 3-dimensional space; the set of all ordered *triples* of real numbers.

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

- $\mathbb{R}^4$ is the 4-dimensional space; the set of all ordered *quadruples* of real numbers.

$$\mathbb{R}^4 = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{R}\}$$

- $\mathbb{R}^n$ is the n-dimensional space; the set of all ordered n-tuples of real numbers.

$$\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in \mathbb{R}\}$$

Definition 3.1.1. (Equality of Vectors in $\mathbb{R}^n$)

Two vectors $\vec{u} = (x_1, x_2, x_3, \ldots, x_n)$ and $\vec{v} = (y_1, y_2, y_3, \ldots, y_n)$ in $\mathbb{R}^n$ are *equal* if and only if $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$. That is, their corresponding components are equal.

We next define the standard operations of vector addition and scalar multiplication in the n-dimensional space $\mathbb{R}^n$. Let $\vec{u} = (x_1, x_2, \ldots, x_n)$ and $\vec{v} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$ be a scalar.

Vector addition.

$$\vec{u} + \vec{v} = (x_1 + y_1\, ,\, x_2 + y_2\, ,\, x_3 + y_3\, ,\, \ldots\, ,\, x_n + y_n).$$

Scalar multiplication.

$$c\,\vec{u} = (cx_1\, ,\, cx_2\, ,\, cx_3\, ,\, \ldots\, ,\, cx_n).$$

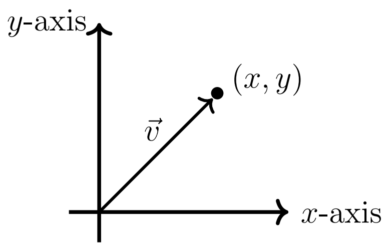
Moreover, we have that:

- The *zero vector* in $\mathbb{R}^n$ is $\vec{\mathbf{0}} = (0, 0, 0, \ldots, 0)$, it is the additive identity.
- $-\vec{u} = (-1)\vec{u} = (-x_1\, ,\, -x_2\, ,\, -x_3\, ,\, \ldots\, ,\, -x_n)$ is the additive inverse of $\vec{u}$.
- $\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = (x_1 - y_1\, ,\, x_2 - y_2\, ,\, x_3 - y_3\, ,\, \ldots\, ,\, x_n - y_n)$.

We now discuss, in more detail, vectors in the 2-dimensional space (also called the cartesian plane):

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

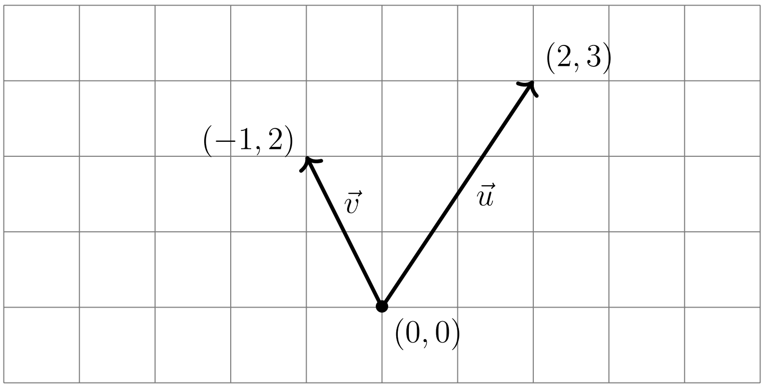
A vector $\vec{v} = (x, y)$ in the cartesian plane $\mathbb{R}^2$ is represented geometrically by a directed line segment whose initial point is the origin and whose terminal point is the point (x, y).



The *zero vector* is the vector $\vec{\mathbf{0}} = (0, 0)$. Two vectors $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ are declared *equal* if and only if $x_1 = x_2$ and $y_1 = y_2$.

The term "vector" derives from the Latin word "vectus", meaning "to carry". The idea is that if one was to carry an object from the origin to the point (x, y), the trip could be represented by the vector $\vec{v} = (x, y)$. In physics and engineering, a vector is characterised by two quantities: length and direction.

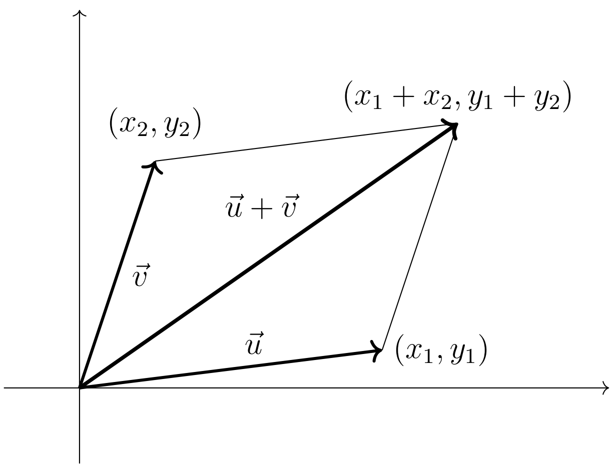
Example 3.1. Represent the vectors $\vec{u} = (2, 3)$ and $\vec{v} = (-1, 2)$ in the plane.



Let $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ be vectors in the plane $\mathbb{R}^2$. Their sum is the vector:

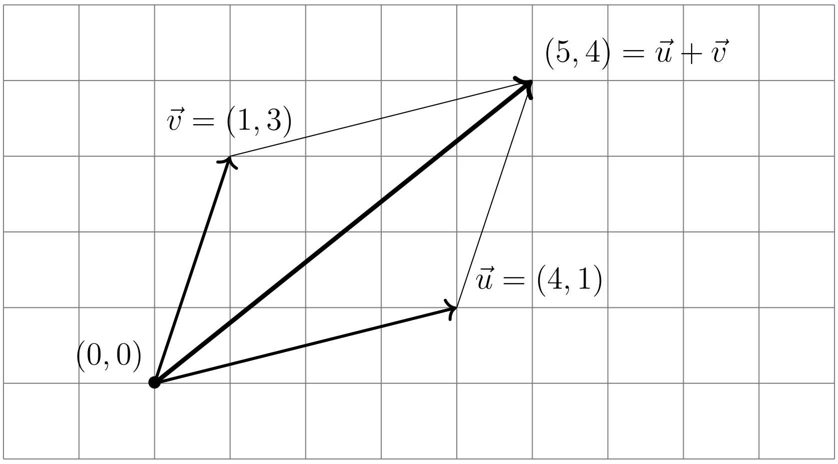
$$\vec{u} + \vec{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Geometrically, the sum $\vec{u}+\vec{v}$ is the diagonal of the parallelogram having the vectors $\vec{u}$ and $\vec{v}$ as its sides. One may think of $\vec{u} + \vec{v}$ as the trip taken in the plane starting at the origin, moving in the direction indicated by vector $\vec{u}$, and then continues in the direction indicated by vector $\vec{v}$.



Example 3.2. Consider the following vectors in the cartesian plane: $\vec{u} = (4, 1)$ and $\vec{v} = (1, 3)$. Find their sum and sketch the three vectors in the cartesian plane.

$$\vec{u} + \vec{v} = (4, 1) + (1, 3) = (5, 4).$$



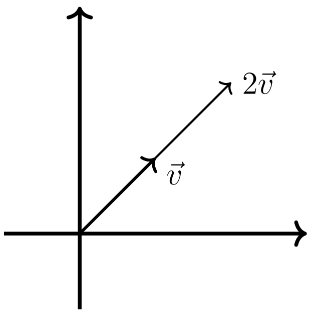
Example 3.3. Let $\vec{s} = (3, -2)$, $\vec{t} = (-3, 2)$, and $\vec{u} = (2, 1)$.

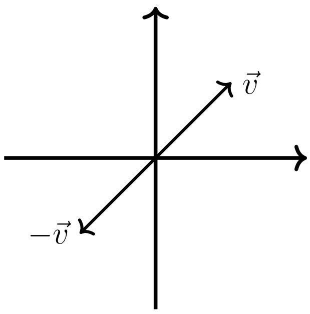
- $\vec{s} + \vec{t} = (3, -2) + (-3, 2) = (0, 0) = \vec{\mathbf{0}}$.
- $\vec{u} + \vec{\mathbf{0}} = (2, 1) + (0, 0) = (2, 1) = \vec{u}$.
- $\vec{s} + \vec{u} = (3, -2) + (2, 1) = (5, -1)$.

Let $\vec{v} = (x, y)$ be a vector and $c \in \mathbb{R}$ be a scalar. The scalar multiplication of c by $\vec{v}$ is the vector:

$$c\,\vec{v} = (cx, cy).$$

Historically, c is called a scalar because multiplying a vector by a real number changes its "scale" or length.





Example 3.4. Let $\vec{u} = (3, 4)$ and $\vec{v} = (-2, 6)$.

(i) $\dfrac{1}{2}\vec{v} = \dfrac{1}{2}(-2, 6) = (-1, 3)$.

(ii) $\vec{u} - \vec{v} = (3, 4) - (-2, 6) = (3, 4) + (2, -6) = (5, -2)$.

(iii) $\dfrac{1}{2}\vec{v} + \vec{u} = \dfrac{1}{2}(-2, 6) + (3, 4) = (-1, 3) + (3, 4) = (2, 7)$.

Example 3.5. Let $\vec{u} = (-1, 0, 1)$ and $\vec{v} = (2, -1, 5)$ be vectors in $\mathbb{R}^3$.

- $\vec{u} + \vec{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$.
- $2\vec{u} = 2(-1, 0, 1) = (-2, 0, 2)$.
- $\vec{v} - 2\vec{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3)$.

Example 3.6. Let $\vec{u} = (2, -1, 5, 0), \vec{v} = (4, 3, 1, -1), \vec{w} = (-6, 2, 0, 3)$ be vectors in $\mathbb{R}^4$.

$$\begin{aligned}
2\vec{u} - (\vec{v} + 3\vec{w}) &= 2(2, -1, 5, 0) - ((4, 3, 1, -1) + 3(-6, 2, 0, 3)) \\
&= (4, -2, 10, 0) - ((4, 3, 1, -1) + (-18, 6, 0, 9)) \\
&= (4, -2, 10, 0) - (-14, 9, 1, 8) \\
&= (18, -11, 9, -8).
\end{aligned}$$

We now list 10 important properties that are satisfied by vectors in the n-dimensional space $\mathbb{R}^n$ equipped with vector addition and scalar multiplication.

Theorem 3.1.2. (Vector addition and Scalar Multiplication in $\mathbb{R}^n$)

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in the n-dimensional space, and let $c, d \in \mathbb{R}$ be scalars. The following statements are true.

1. $\vec{u} + \vec{v}$ *is a vector* $\mathbb{R}^n$. *(Closure under addition)*
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. *(Vector addition is Commutative)*
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$. *(Vector addition is associative)*
4. $\vec{u} + \vec{\mathbf{0}} = \vec{u}$. *(Additive identity)*
5. $\vec{u} + (-\vec{u}) = \vec{\mathbf{0}}$. *(Additive inverse)*
6. $c\vec{u}$ *is a vector in* $\mathbb{R}^n$. *(Closure under scalar multiplication)*
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$. *(Distributive property)*
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$. *(Distributive property)*
9. $c(d\vec{u}) = (cd)\vec{u}$. *(Scalar multiplication is associative)*
10. $1\vec{u} = \vec{u}$. *(Multiplication identity)*

Proof. We prove property (8). For simplicity, let $\vec{u} = (x, y)$ be any vector in $\mathbb{R}^2$ and let $c, d \in \mathbb{R}$ be any scalars. We start with the left side and proceed towards the right side.

$$\begin{aligned}
(c + d)\vec{u} &= (c + d)(x\,,\, y) = ((c + d)x\,,\, (c + d)y) \\
&= (cx + dx\,,\, cy + dy) = (cx, cy) + (dx, dy) \\
&= c(x, y) + d(x, y) = c\vec{u} + d\vec{y}.
\end{aligned}$$

We leave the proof of the remaining statements for exercise. ■

Theorem 3.1.3.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and $c \in \mathbb{R}$ be a scalar. Then the following proprieties are true.

1. *If $\vec{v} + \vec{u} = \vec{v}$, then $\vec{u} = \vec{\mathbf{0}}$.* *(Additive identity is unique)*
2. *If $\vec{v} + \vec{u} = \vec{\mathbf{0}}$, then $\vec{u} = -\vec{v}$.* *(Additive inverse is unique)*
3. $0\,\vec{v} = \vec{\mathbf{0}}$.
4. $c\,\vec{\mathbf{0}} = \vec{\mathbf{0}}$.
5. *If $c\vec{v} = \vec{\mathbf{0}}$, then $c = 0$ or $\vec{v} = \vec{\mathbf{0}}$.*
6. $-(-\vec{v}) = \vec{v}$.

Proof. We prove Property (1). Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^n$. Suppose $\vec{v} + \vec{u} = \vec{v}$.

$$\begin{aligned}
\vec{v} + \vec{u} &= \vec{v} && \\
-\vec{v} + (\vec{v} + \vec{u}) &= -\vec{v} + \vec{v} && \text{(Add } -\vec{v} \text{ to both sides)} \\
(-\vec{v} + \vec{v}) + \vec{u} &= -\vec{v} + \vec{v} && \text{(Associativity)} \\
\vec{\mathbf{0}} + \vec{u} &= \vec{\mathbf{0}} && \text{(Additive inverse)} \\
\vec{u} &= \vec{\mathbf{0}} && \text{(Additive identity)}
\end{aligned}$$

We leave the proof of the remaining properties as an exercise for the reader. ■

It is very useful to represent a vector $\vec{v} = (x_1, x_2, x_3, \ldots, x_n)$ in the n-dimensional space $\mathbb{R}^n$ as a row matrix or a column matrix.

1. A $1 \times n$ row matrix $\vec{v} = [x_1\ x_2\ x_3\ \ldots\ x_n]$.
2. An $n \times 1$ column matrix $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Note that in this case *matrix* addition and scalar multiplication will coincide with the corresponding *vector* operations.

3.2 Vector Spaces

The properties of vector addition and scalar multiplication in $\mathbb{R}^n$ are also shared with many other mathematical structures like matrices, polynomials, and functions. Any algebraic structure that satisfies these properties is called a *vector space*, and its members are called *vectors*. Let us define vector spaces in an abstract way.

Definition 3.2.1. (Vector Space)

A *vector space* over the field $\mathbb{R}$ of real numbers is a set V, whose members are called *vectors*, together with two operations:

- Vector addition that takes two vectors $\vec{u}$ and $\vec{v}$ from V and produce a third vector denoted by $\vec{u}+\vec{v}$.
- Scalar multiplication that takes a scalar $c \in \mathbb{R}$ and a vector $\vec{v} \in V$, and produces a new vector denoted by $c\vec{v}$.

which satisfy the following axioms, called *vector space axioms*:

(1) Closure of vector addition.
For every $\vec{u}, \vec{v}$ in V, we have $\vec{u}+\vec{v}$ is also in V.

(2) Commutativity of vector addition.
For every $\vec{u}, \vec{v}$ in V, we have $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

(3) Associativity of vector addition.
For every $\vec{u}, \vec{v}, \vec{w}$ in V, we have $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

(4) Existence of additive identity.
There exists a vector $\vec{\mathbf{0}}$ in V, called the *zero vector*, such that for every $\vec{v}$ in V, we have $\vec{v}+\vec{\mathbf{0}} = \vec{v}$.

(5) Existence of additive inverse.
For every $\vec{v}$ in V, there exists a vector $-\vec{v}$ in V, called the *additive inverse* of v, such that $\vec{v}+(-\vec{v}) = \vec{\mathbf{0}}$.

(6) Closure of scalar multiplication.
For every $\vec{v}$ in V and scalar c in $\mathbb{R}$, we have $c\,\vec{v}$ belongs to V.

(7) Distributivity of scalar multiplication over vector addition.
For every $\vec{u}, \vec{v}$ in V and scalar c in $\mathbb{R}$, we have $c\,(\vec{u}+\vec{v}) = c\,\vec{u}+c\,\vec{v}$.

(8) Distributivity of scalar multiplication over field addition.
For every $\vec{v}$ in V and scalars c, d in $\mathbb{R}$, we have $(c+d)\vec{v} = c\vec{v}+d\vec{v}$.

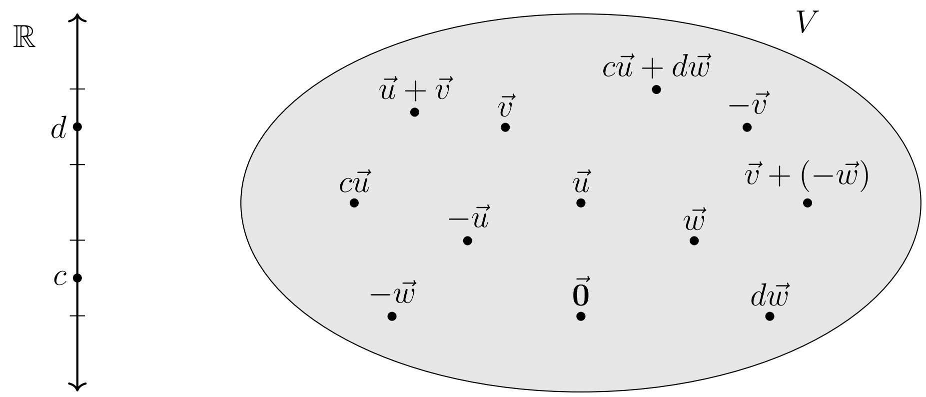
(9) Compatibility of scalar multiplication with field multiplication.
For every $\vec{v}$ in V and scalars c, d in $\mathbb{R}$, we have $(cd)\,\vec{v} = c\,(d\,\vec{v})$.

(10) Unitarity. For every $\vec{v}$ in V, we have $1\,\vec{v} = \vec{v}$.

It is important to realize that a vector space consists of 4 entities:

- The set V of vectors.
- The set of scalars. It could be any field, however, here we always choose the field of real numbers $\mathbb{R}$.
- The operation of vector addition: a function from $V \times V \to V$.
- The operation of scalar multiplication: a function from $\mathbb{R} \times V \to V$.

Here is a picture of an abstract vector space V.



We now give a list of examples of vector spaces. The reader is encouraged to check that all of the 10 vector space axioms are satisfied in the examples below.

Example 3.7 (The n-Dimensional Space)**.** The set $\mathbb{R}^n$ of all ordered n-tuples of real numbers with their addition and scalar multiplication is a vector space as we have seen in the previous section. Vectors in the space $\mathbb{R}^n$ are of the form

$$\vec{v} = (x_1, x_2, x_3, \ldots, x_n)$$

where each $x_i \in \mathbb{R}$ for each $1 \le i \le n$.

Example 3.8. We describe a vector space structure on the set $\mathbb{R}^2$ different than the standard one. We define vector addition and scalar multiplication as follows. Let $\vec{u} = (a, b)$ and $\vec{v} = (x, y)$ be vectors in $\mathbb{R}^2$, and let $c \in \mathbb{R}$ be a scalar.

- $\vec{u} \oplus \vec{v} = (a + x - 3,\ b + y + 8)$.
- $c \cdot \vec{v} = (cx - 3c + 3,\ cy + 8c - 8)$.

Show that the set $\mathbb{R}^2$ together with these operations form a vector space. The zero vector is $\vec{\mathbf{0}} = (3,\ -8)$ and the additive inverse of $\vec{v}$ is $-\vec{v} = (-x + 6,\ -y - 16)$.

Example 3.9 (Matrices of a fixed size)**.** For instance, take the vector space $\mathbb{M}_{2\times3}$ of all 2×3 matrices. Here the operations are matrix addition and scalar multiplication. Suppose that A and B are 2×3 matrices, and $c \in \mathbb{R}$. Then $A + B$ is also a 2×3

matrix, and so is cA. The zero vector is the zero matrix $\mathbf{0}_{2\times 3}$ It is easy to see that the set of 2×3 matrices satisfies the other 8 axioms (see the chapter on matrices). Vectors in this space have the following form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In general, fix some positive integers m and n. Then the set of all $m \times n$ matrices is a vector space denoted by $\mathbb{M}_{m\times n}$ under matrix addition and scalar multiplication as previously defined.

Example 3.10 (Polynomials with real coefficients)**.** The vector space of all polynomial of degree 2 or less. Let $\mathcal{P}_2$ be the set of all polynomials of the form:

$$p(x) = a_0 + a_1 x + a_2 x^2$$

where $a_0, a_1, a_2 \in \mathbb{R}$.

Let $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$ be two polynomials in $\mathcal{P}_2$. Then we define their addition as the polynomial whose coefficients are the sum of the corresponding coefficients in the two polynomials being added. More precisely,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2.$$

Moreover, for a scalar $c \in \mathbb{R}$, we define scalar multiplication in the natural way:

$$cp(x) = ca_0 + ca_1 x + ca_2 x^2.$$

Let us show that $\mathcal{P}_2$ is a vector space. So, we need to show that $\mathcal{P}_2$ satisfies that 10 vector space axioms. As the set of real numbers is closed under addition and multiplication, we have that both $p(x) + q(x)$ and $cp(x)$ are in $\mathcal{P}_2$ whenever $p(x)$ and $q(x)$ are. Let us now verify the commutativity axiom.

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1 x + b_2 x^2) + (a_0 + a_1 x + a_2 x^2) \\ &= q(x) + p(x). \end{aligned}$$

The zero vector of $\mathcal{P}_2$ is the *zero polynomial* is $\mathbf{0}(x) = 0 + 0x + 0x^2$.

In general, we have that the set $\mathcal{P}_n$ of all polynomials of degree n or less is a vector space under polynomial addition and scalar multiplication. Vectors in $\mathcal{P}_n$ have the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

Example 3.11 (The vector space of continuous real valued functions)**.** Let $\mathcal{C}(-\infty, \infty)$ be the set all continuous functions $f : \mathbb{R} \to \mathbb{R}$. This set includes all polynomial functions, trigonometric functions such as $f(x) = \sin(x)$, $g(x) = \cos(x)$, and exponential functions such as $h(x) = e^x$, and many other continuous functions.

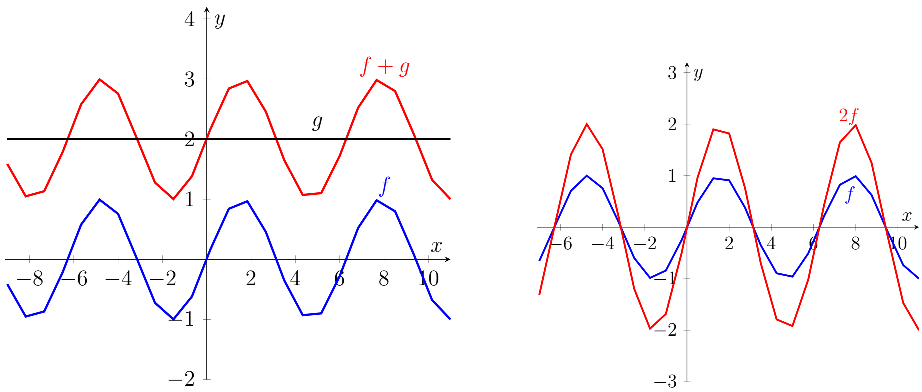
Let $f(x)$ and $g(x)$ be functions in $\mathcal{C}(-\infty, \infty)$. We define their addition as follows:

$$(f + g)(x) = f(x) + g(x)$$

and scalar multiplication as follows:

$$(cf)(x) = c(f(x)).$$

Below, on the left side we have the graphs of $f(x) = \sin(x)$ and $g(x) = 2$, and their addition $(f + g)(x) = \sin(x) + 2$. On the right side, we have the graphs of $f(x)$ and its scalar multiplication $(2f)(x) = 2\sin(x)$.



From Calculus, we know that sum of two continuous functions is a continuous function, so $(f + g)(x)$ is in $\mathcal{C}(-\infty, \infty)$. Also, we know that $(cf)(x)$ is a continuous function as well, and so $(cf)(x)$ belong to $C(-\infty, \infty)$. This shows that the set $C(-\infty, \infty)$ is closed under the operations of function addition and scalar multiplication just defined. The zero vector in the space $\mathcal{C}(-\infty, \infty)$ is the *zero function* $z : \mathbb{R} \to \mathbb{R}$ where $z(x) = 0$ for all $x \in \mathbb{R}$. We leave it for the reader to check that $\mathcal{C}(-\infty, \infty)$ satisfies the remaining axioms of vector spaces.

The same above treatment shows that the set $\mathcal{C}[a, b]$ of all continuous functions $f : [a, b] \to \mathbb{R}$ defined on the closed interval $[a, b]$ is a vector space.

Important Vector Spaces over $\mathbb{R}$

- $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers.
- $\mathbb{M}_{m \times n}$ is the set of all $m \times n$ matrices with real entries.
- $\mathcal{P}_n$ is set of all polynomials of degree n or less with real coefficients.

- $\mathcal{P}$ is the set of all polynomials with real coefficients.
- $\mathcal{C}(-\infty, \infty)$ is the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.
- $\mathcal{C}[a, b]$ is the set of all continuous functions $f : [a, b] \to \mathbb{R}$.
- $\mathcal{D}(-\infty, \infty)$ is the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

We now give examples of sets with addition and scalar multiplication that do not form vector spaces because at least one of the 10 vector space axioms is not satisfied.

Example 3.12 (Non-example)**.** The set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ under the usual integer addition and scalar multiplication is not a vector space over $\mathbb{R}$ because it is not closed under scalar multiplication. For example, 3 is an integer, and $\frac{1}{2}$ is a scalar. But $\frac{1}{2}(3) = 1.5$ which is not an integer.

Example 3.13 (Non-example)**.** The set of all polynomials of degree equal to 2 is not a vector space as it is not closed under addition. Take $p(x) = x^2$ and $q(x) = -x^2 + x + 1$. Then $p(x) + q(x) = x + 1$ which is of degree 1.

Example 3.14 (Non-example)**.** Consider $\mathbb{R}^2$ with standard addition, but scalar multiplication is defined by

$$c\vec{v} = c(x, y) = (cx, 0)$$

for any vector $\vec{v} \in \mathbb{R}^2$ and scalar $c \in \mathbb{R}$. Here both operation are closed. However, the 10^{th} axiom of the vector space axioms is not satisfied since for $\vec{u} = (2, 3)$, we have that $1\vec{u} = 1(2, 3) = (2, 0) \neq \vec{u}$.

From the vector space axioms we know that there is a special vector called the zero vector which has the additive identity property. Is it possible in a vector space to have another vector with same property? Similarly, the axioms tell us that every vector has an additive inverse. Is it possible for some vectors to have more than one additive inverse?

Theorem 3.2.2.

Let V be a vector space over $\mathbb{R}$. The following hold.

(i) The zero vector $\vec{\mathbf{0}}$ of V is unique.

(ii) The additive inverse $-\vec{v}$ of any vector $\vec{v}$ is unique.

Proof. (i) Suppose that $\vec{z}$ is a vector in V satisfies the additive identity property, that is, $\vec{v} + \vec{z} = \vec{v}$ for any vector $\vec{v} \in V$. Since $\vec{z}$ has the additive identity property, we have $\vec{\mathbf{0}} + \vec{z} = \vec{\mathbf{0}}$. Also, since $\vec{\mathbf{0}}$ has the additive identity property and vector addition is commutative, we have $\vec{\mathbf{0}} + \vec{z} = \vec{z} + \vec{\mathbf{0}} = \vec{z}$. Thus, $\vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{z} = \vec{z}$. This shows that any vector which satisfies the additive identity axiom must be the zero vector $\vec{\mathbf{0}}$.

(ii) Choose any vector $\vec{v}$ in V. Suppose that $\vec{u}$ is also an additive inverse of $\vec{v}$, that is, $\vec{v}+\vec{u}=\vec{\mathbf{0}}$. Next we add $-\vec{v}$ to both sides to get $-\vec{v}+(\vec{v}+\vec{u})=-\vec{v}+\vec{\mathbf{0}}$. By associativity of addition, we get $(-\vec{v}+\vec{v})+\vec{u}=-\vec{v}$. And so $\vec{\mathbf{0}}+\vec{u}=-\vec{v}$, and thus $\vec{u}=-\vec{v}$ as desired. ■

We next establish further results that hold true in any vector space.

Theorem 3.2.3.

Let V be a vector space over $\mathbb{R}$, let $\vec{v}$ be a vector in V, and let $c\in\mathbb{R}$ be a scalar. Then the following are true.

(i) $0\,\vec{v}=\vec{\mathbf{0}}$.

(ii) $c\,\vec{\mathbf{0}}=\vec{\mathbf{0}}$.

(iii) $(-1)\vec{v}=-\vec{v}$.

(iv) If $c\,\vec{v}=\vec{\mathbf{0}}$, *then* $c=0$ *or* $\vec{v}=\vec{\mathbf{0}}$.

Proof. (i) We start with the equality $0\,\vec{v}=0\,\vec{v}$.

$$\begin{aligned}
0\,\vec{v} &= 0\,\vec{v} && \\
(0+0)\vec{v} &= 0\vec{v} && (0=0+0 \text{ in } \mathbb{R})\\
0\vec{v}+0\vec{v} &= 0\vec{v} && \text{Distributive Property}\\
(0\vec{v}+0\vec{v})+-(0\vec{v}) &= 0\vec{v}+-(0\vec{v}) && \text{Add } -(0\vec{v}) \text{ to both sides}\\
(0\vec{v}+0\vec{v})+-(0\vec{v}) &= \vec{\mathbf{0}} && \text{Additive Inverse}\\
0\vec{v}+(0\vec{v}+-(0\vec{v})) &= \vec{\mathbf{0}} && \text{Associative Property}\\
0\vec{v}+\vec{\mathbf{0}} &= \vec{\mathbf{0}} && \text{Additive Inverse}\\
0\vec{v} &= \vec{\mathbf{0}} && \text{Additive Identity}
\end{aligned}$$

(ii) We start with the equality $c\,\vec{\mathbf{0}}=c\,\vec{\mathbf{0}}$.

$$\begin{aligned}
c\,\vec{\mathbf{0}} &= c\,\vec{\mathbf{0}} && \\
c\,(\vec{\mathbf{0}}+\vec{\mathbf{0}}) &= c\,\vec{\mathbf{0}} && \text{Additive Identity}\\
c\,\vec{\mathbf{0}}+c\,\vec{\mathbf{0}} &= c\,\vec{\mathbf{0}} && \text{Distributive Property}\\
(c\,\vec{\mathbf{0}}+c\,\vec{\mathbf{0}})+-(c\vec{\mathbf{0}}) &= c\,\vec{\mathbf{0}}+-(c\,\vec{\mathbf{0}}) && \text{Add } -(c\,\vec{\mathbf{0}}) \text{ to both sides}\\
(c\,\vec{\mathbf{0}}+c\,\vec{\mathbf{0}})+-(c\vec{\mathbf{0}}) &= \vec{\mathbf{0}} && \text{Additive Inverse}\\
c\,\vec{\mathbf{0}}+(c\,\vec{\mathbf{0}}+-(c\,\vec{\mathbf{0}})) &= \vec{\mathbf{0}} && \text{Associative Property}\\
c\,\vec{\mathbf{0}}+\vec{\mathbf{0}} &= \vec{\mathbf{0}} && \text{Additive Inverse}\\
c\,\vec{\mathbf{0}} &= \vec{\mathbf{0}} && \text{Additive Identity}
\end{aligned}$$

(iii) We will show that $(-1)\,\vec{v}$ is the additive inverse of $\vec{v}$.

$$\vec{v} + (-1)\,\vec{v} = 1\,\vec{v} + (-1)\,\vec{v} = (1 + (-1))\,\vec{v} = 0\,\vec{v} = \vec{\mathbf{0}}.$$

Therefore, the vector $(-1)\,\vec{v}$ is the additive inverse of $\vec{v}$ since their sum is the zero vector, in other words, $(-1)\,\vec{v} = -\vec{v}$ as desired.

(iv) Assume that $c\,\vec{v} = \vec{\mathbf{0}}$. If $c = 0$, then we are done. Otherwise, suppose that $c \neq 0$. We need to show that $\vec{v} = \vec{\mathbf{0}}$. Using Unitarity (Axiom 10) and Part (ii) we get the following.

$$\vec{v} = 1\,\vec{v} = \left(\frac{1}{c} \cdot c\right)\vec{v} = \frac{1}{c}(c\,\vec{v}) = \frac{1}{c}\,\vec{\mathbf{0}} = \vec{\mathbf{0}}.$$

This completes the proof. ■

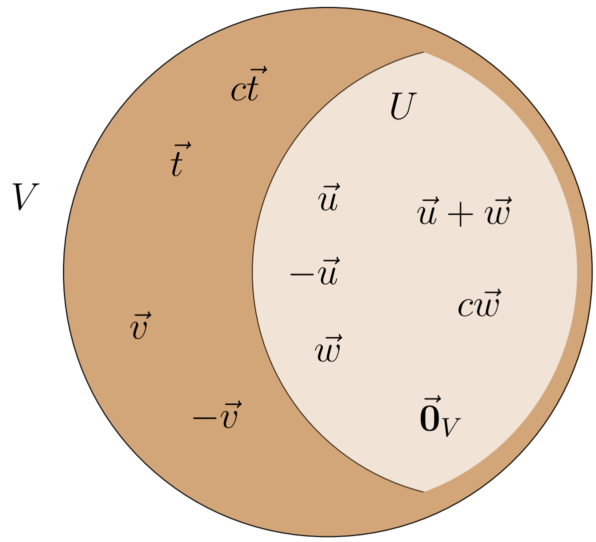
3.3 Linear Subspaces

A set is a collection of objects, called its *elements* or its *members*. Given sets A and B, we say that A is a *subset* of B, and write $A \subseteq B$, if every element of A is also an element of B. Given a vector space V, there will be special subsets of V; those subsets who are vector spaces on their own with respect to the operations of V.

Definition 3.3.1. (Subspace)

Let V be a vector space. A subset $U \subseteq V$ is called a *subspace* of V if and only if U itself is a vector space under the operations of vector addition and scalar multiplication inherited from V.

The first examples of subspaces of a vector space V are the *zero subspace* $\{\vec{\mathbf{0}}\}$ and the whole space V. In general, here is a picture of how a subspace would look like.



Thus, subspaces are subsets of a vector space which satisfy the vector space axioms on their own. How to quickly test whether a given subset of V is a subspace or not?

Theorem 3.3.2. (Subspace Test)

Let V be a vector space, and let U be any subset of V. Then U is a subspace of V if and only if

(1) U is a nonempty subset.

(2) U is closed under vector addition.
Whenever $\vec{u}$ and $\vec{w}$ are in U, then $\vec{u} + \vec{w}$ is also in U.

(3) U is closed under scalar multiplication.
Whenever $\vec{w} \in U$ and $c \in \mathbb{R}$, then $c\vec{w}$ is in U.

Proof. Let V be a vector space and let $U \subseteq V$.

($\Rightarrow$) This direction is straight forward. To see this, suppose that $U \subseteq V$ is a vector space on its own with the operations of vector addition and scalar multiplication of V. Clearly, U is nonempty because the additive identity axiom tells us that there exists a zero vector in U. Also, the axioms say that U is closed under vector addition and scalar multiplication.

($\Leftarrow$) Assume that a subset $U \subseteq V$ is nonempty and closed under vector addition and scalar multiplication. We need to show that U is a vector space, in other words, we need to show that U with vector addition and scalar multiplication of V satisfy the vector space axioms. We already assume that U is closed under vector addition and scalar multiplication, these are Axiom (1) and Axiom (6).

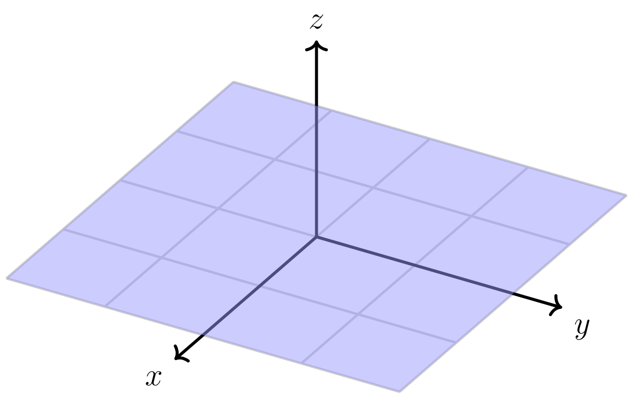
Since U is nonempty, there is some vector $\vec{w}$ in U. Since U is closed under scalar multiplication, it follows that $0\vec{w} \in U$ as well, but $\vec{\mathbf{0}} = 0\vec{w}$ and so $\vec{\mathbf{0}} \in U$. Thus, U contains the zero vector and so Axiom (4) is satisfied by U.

If $\vec{w}$ is in U, then as U is closed under scalar multiplication we get that $(-1)\vec{w} = -\vec{w}$ is in U. This shows that every $\vec{w}$ in U has an additive inverse in U. This show that U satisfied Axiom (5).

The rest of the vector space axioms are universal statements describing properties which hold for all members of V, and as they are true in V and $U \subseteq V$, they are also true in U. For instance, Axiom (2) says that addition of vectors from V is a commutative operation, and as vectors of U are also vectors of V, it follows that addition of vectors from U is commutative as well, thus, U satisfies commutativity of addition. This shows that U itself is a vector space. ∎

Remark. A vector space V is clearly a subspace of itself. A subspace of V which is not V is called a *proper subspace*. Also the set $\{\vec{\mathbf{0}}\}$, which contains only the zero vector, is a subspace of V called the *zero subspace* or the *trivial subspace*. Subspaces other than V and $\{\vec{\mathbf{0}}\}$ are called *proper nontrivial* subspaces.

Example 3.15. Show that $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. Observe that U is the set of all vectors in $\mathbb{R}^3$ whose third component is 0, that is, U is the xy-plane.



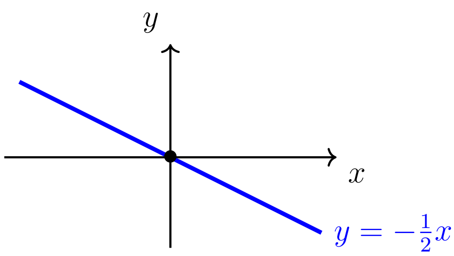
- First U is a nonempty as it contains the zero vector $\vec{\mathbf{0}} = (0, 0, 0)$.

- The set U is closed under addition.
  Let $\vec{u}$ and $\vec{w}$ be vectors in U. It follows that $\vec{u} = (x_1, x_2, 0)$ and $\vec{w} = (y_1, y_2, 0)$. Then,
  $$\vec{u} + \vec{w} = (x_1, x_2, 0) + (y_1, y_2, 0) = (x_1 + y_1\ ,\ x_2 + y_2\ ,\ 0).$$
  Clearly, $\vec{u} + \vec{w}$ belongs to U since its third component is 0.

- The set U is closed under scalar multiplication.
  Let $c \in \mathbb{R}$ and Let $\vec{u} \in U$. Then $c\vec{u} = c(x_1, x_2, 0) = (cx_1, cx_2, 0)$ which is in U.

Thus, by the subspace test, the xy-plane is a subspace of the 3-dimensional space $\mathbb{R}^3$.

Example 3.16. Show that the set $U = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ is a subspace of $\mathbb{R}^2$. Observe that U is the set of points lying on the line $y = -\frac{1}{2}x$.



Clearly, the zero vector $(0, 0)$ is in U because $0 + 2(0) = 0$, so U is a nonempty set. We now need to show that U is closed under vector addition and scalar multiplication. Let $\vec{u} = (a, b)$ and $\vec{w} = (c, d)$ be any two vectors in U. Their sum is $\vec{u} + \vec{w} = (a, b) + (c, d) = (a + c, b + d)$. We have to show that $\vec{u} + \vec{w}$ is in U. Since $\vec{u}, \vec{w} \in U$, we know that $a + 2b = 0$ and $c + 2d = 0$. Now,

$$(a + c) + 2(b + d) = a + c + 2b + 2d = (a + 2b) + (c + 2d) = 0 + 0 = 0.$$

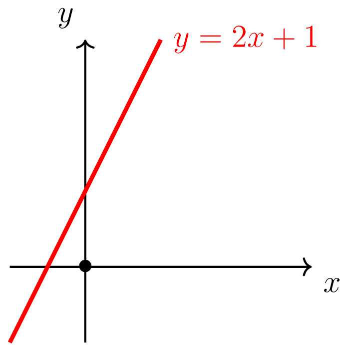
This shows that $\vec{u} + \vec{w}$ is in U.

Next, let $\alpha \in \mathbb{R}$. We need to show that $\alpha\vec{u} \in U$. But $\alpha\vec{u} = \alpha(a, b) = (\alpha a, \alpha b)$. We now check that

$$\alpha a + 2\alpha b = \alpha(a + 2b) = \alpha 0 = 0.$$

Thus, $\alpha\vec{u} \in U$ and so U is closed under scalar multiplication. By the subspace test, this shows that the line $y = -\frac{1}{2}x$ is a subspace of the cartesian plane $\mathbb{R}^2$.

Example 3.17. Is the set of points on the line $y = 2x + 1$ a subspace of $\mathbb{R}^2$?

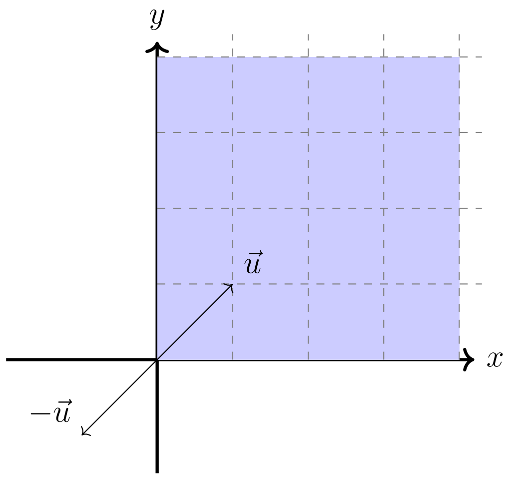


No, because the zero vector $(0, 0)$ is not on this line, and so this line is not a vector space on its own. Another failure is that this line is not closed under scalar multiplication. For instance, $\vec{u} = (0, 1)$ lies on the line, but $2\vec{u} = 2(0, 1) = (0, 2)$ does not lie on the line because $2 \neq 2(0) + 1$.

Example 3.18. The first quadrant of the cartesian plane

$$U = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\, y \geq 0\}$$

is *not* a subspace of $\mathbb{R}^2$.



The set U is not closed under scalar multiplication, as the vector $\vec{u} = (1, 1)$ is in U, however, the vector $-1\vec{u} = (-1)(1, 1) = (-1, -1)$ is not in U. Therefore, U is not a subspace.

In the previous examples we have seen subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ which are subspaces and others which are not subspaces. The next two theorems describe precisely how subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$ look like.

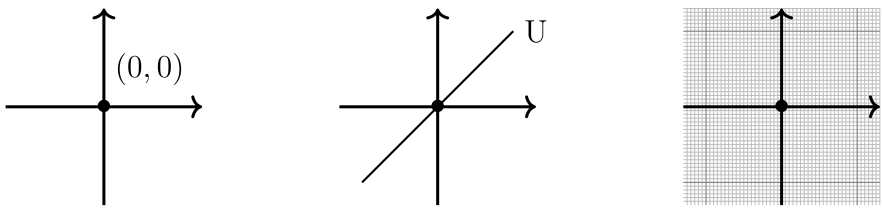
Theorem 3.3.3.

Any subspace of $\mathbb{R}^2$ must be one of the following three possibilities.

(i) *The zero subspace $\{(0,0)\}$.*

(ii) *The set of all points on a straight line that passes through the origin.*

(iii) *The whole space $\mathbb{R}^2$.*



Theorem 3.3.4.

A subset U of the 3-dimensional space $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ if and only if U is one of the following subsets:

(i) *$U = \{(0,0,0)\}$ the zero subspace.*

(ii) *U is the set of all points on a line that pass through the origin.*

(iii) *U consists of all points in a plane that pass through origin.*

(iv) *$U = \mathbb{R}^3$.*

Example 3.19. Let U be the set of 2×2 symmetric matrices. Show that U is a subspace of the vector space $\mathbb{M}_{2\times 2}$ of all 2×2 matrices.

Clearly, U is a nonempty set since the zero matrix $\mathbf{0}_{2\times 2}$ is symmetric. Next, let A and B be any two matrices in U. This means that A and B are symmetric matrices and so $A^T = A$ and $B^T = B$. We need to show that $A + B$ is in U too. So we need to show that $A + B$ is symmetric. Observe that

$$(A + B)^T = A^T + B^T = A + B.$$

Thus, $A + B$ is symmetric and so $A + B \in U$.

It remains to show U is closed under scalar multiplication. So choose any scalar $c \in \mathbb{R}$. We need to show that $cA \in U$. To show that cA is symmetric we proceed as follows:

$$(cA)^T = cA^T = cA.$$

Thus, cA is symmetric and so $cA \in U$. Therefore, by the subspace test, U is a subspace of $\mathbb{M}_{2\times 2}$.

Example 3.20. Let U be the set of all 2×2 noninvertible matrices. Show that U is *not* a subspace of $\mathbb{M}_{2\times 2}$.

One reason why U is not a subspace is because U does not satisfy Axiom (1) of vector spaces. To see this, consider the matrices

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

Observe that $\det(A) = 0$ and $\det(B) = 0$ and so both are noninvertible and so they belong to U. However, their sum is

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Observe that $\det(A + B) = 1$ implying that $A + B$ is invertible and so $A + B$ is not in U. So U is not closed under addition and so it is not a subspace of $\mathbb{M}_{2\times 2}$.

Example 3.21. Recall that $\mathcal{P}_n$ is the vector space of polynomials of degree n or less. Clearly, $\mathcal{P}_1$ is a subspace of $\mathcal{P}_2$ and $\mathcal{P}_2$ is a subspace of $\mathcal{P}_3$, and so on. Moreover, each $\mathcal{P}_n$ is a subspace of the vector space $\mathcal{P}$ of all polynomials.

$$\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \mathcal{P}_4 \subseteq \cdots \subseteq \mathcal{P}_n \subseteq \mathcal{P}_{n+1} \subseteq \cdots \subseteq \mathcal{P}.$$

Example 3.22. In the vector space $\mathcal{P}_2$ of all polynomials of degree 2 or less, let U be the subset of all polynomials of degree exactly 2 together with the zero polynomial $\mathbf{0}(x)$. The subset U is not a subspace of $\mathcal{P}_2$ because it is not closed under addition. To see this, take the polynomials $p(x) = 3x^2 + 5x - 3$ and $q(x) = -3x^2 - x + 5$ in the subset U. Their sum is

$$p(x) + q(x) = (3x^2 + 5x - 3) + (-3x^2 - x + 5) = 4x + 2,$$

which is a polynomial of degree 1, and so $p(x) + q(x)$ does not belong to U.

Example 3.23. Recall that $\mathcal{C}(-\infty, \infty)$ is the vector space of all continuous real-valued functions defined on $\mathbb{R}$. Consider the subset

$$\mathcal{E} = \{\, f \in \mathcal{C}(-\infty, \infty) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R} \,\}.$$

Thus, $\mathcal{E}$ is the set of all continuous even functions from $\mathbb{R}$ to $\mathbb{R}$. Clearly, $\mathcal{E}$ is nonempty because it contains the function $h(x) = \cos(x)$. Moreover, choose any functions f and g from $\mathcal{E}$. We claim that their sum is also an even function. Let $x \in \mathbb{R}$ and observe that

$$(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x).$$

It follows that $f + g$ belong to $\mathcal{E}$. Similarly, for a scalar $c \in \mathbb{R}$ we have that

$$(cf)(x) = cf(x) = cf(-x) = (cf)(-x),$$

for any $x \in \mathbb{R}$, showing that cf is also an even function. Thus, $\mathcal{E}$ is closed under function addition and scalar multiplication. By the subspace test, we get that $\mathcal{E}$ is a subspace of $\mathcal{C}(-\infty, \infty)$.

Example 3.24. From Calculus we know that if a real-valued function is differentiable at a point then it is continuous at that point. Recall that a function is differentiable if its derivative exists everywhere on its domain. It follows that the the set $\mathcal{D}(-\infty, \infty)$ of all differentiable functions defined on $\mathbb{R}$ is a subset of the set $\mathcal{C}(-\infty, \infty)$ of all continuous functions defined on $\mathbb{R}$. Clearly, $\mathcal{D}(-\infty, \infty)$ is nonempty as it contains all polynomials because polynomials are differentiable functions. Moreover, if we pick any functions $f, g \in \mathcal{D}(-\infty, \infty)$, then their sum $f+g$ is also a differentiable function and its derivative is $(f+g)'(x) = f'(x) + g'(x)$, and so $\mathcal{D}(-\infty, \infty)$ is closed under function addition. Moreover, for $c \in \mathbb{R}$, the function cf is also differentiable and its derivative is $(cf)'(x) = cf'(x)$, and so $cf \in \mathcal{D}(-\infty, \infty)$ showing that $\mathcal{D}(-\infty, \infty)$ is closed under scalar multiplication. By the subspace test, we get that $\mathcal{D}(-\infty, \infty)$ is a subspace of the vector space $\mathcal{C}(-\infty, \infty)$.

Theorem 3.3.5.

Let V be a vector space, and let U and W be subspaces of V. Then their intersection $U \cap W$ is also a subspace of V.

Proof. Let U and W be subspaces of a vector space V. First, the intersection $U \cap W = \{\vec{a} \mid \vec{a} \in U \text{ and } \vec{a} \in W\}$ contains the zero vector since $\vec{\mathbf{0}} \in U$ and $\vec{\mathbf{0}} \in W$, and so $U \cap W$ is a nonempty set. It remains to show that that $U \cap W$ is closed under vector addition and scalar multiplication. Pick any two vectors $\vec{a}$ and $\vec{b}$ in $U \cap W$ and $c \in \mathbb{R}$. We will show that $\vec{a} + \vec{b} \in U \cap W$ and $c\vec{a} \in U \cap W$. By choice of $\vec{a}$ and $\vec{b}$, it follows that $\vec{a}, \vec{b} \in U$ and $\vec{a}, \vec{b} \in W$. Since U is a subspace, we get that $\vec{a}+\vec{b} \in U$, and as W is a subspace, we get that $\vec{a}+\vec{b} \in W$. Thus, $\vec{a}+\vec{b} \in U \cap W$, and so $U \cap W$ is closed under vector addition. Moreover, as both U and W are closed under scalar multiplication, it follows that $c\vec{a} \in U$ and $c\vec{a} \in W$ yielding $c\vec{a} \in U \cap W$. Thus, $U \cap W$ is closed under scalar multiplication. Therefore, by the subspace test, we get that $U \cap W$ is a subspace of V. ■

3.4 Spanning Sets

Definition 3.4.1. (Linear Combination)

Let V be a vector space. We say a vector $\vec{v}$ is a *linear combination* of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ if and only if $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$ where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ are scalars. This means that $\vec{v}$ is the sum of scalar multiples of the vectors $\vec{u}_1, \ldots, \vec{u}_k$.

Example 3.25. Let $\vec{v} = (1,3,1)$, $\vec{u}_1 = (0,1,2)$, and $\vec{u}_2 = (1,0,-5)$ be vectors in $\mathbb{R}^3$. The vector $\vec{v}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

$$\begin{aligned} 3\vec{u}_1 + \vec{u}_2 &= 3(0,1,2) + (1,0,-5) \\ &= (0,3,6) + (1,0,-5) \\ &= (1,3,1) = \vec{v}. \end{aligned}$$

Thus, we have $\vec{v} = 3\vec{u}_1 + \vec{u}_2$.

Example 3.26. Consider the vectors below from the vector space $\mathbb{M}_{2\times 2}$.

$$\vec{v} = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}.$$

Observe that $\vec{v}$ is a linear combination of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

$$\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix} = \vec{v}.$$

So we can obtain the matrix $\vec{v}$ starting with the matrices $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

Staring with a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ we will collect all the vectors that can be obtained from S as linear combinations in one set, called the span of S.

Definition 3.4.2. (Span of Vectors)

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space V. The *span* of S is defined to be the set of all linear combinations of vectors in S.

$$\operatorname{Span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_k\vec{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

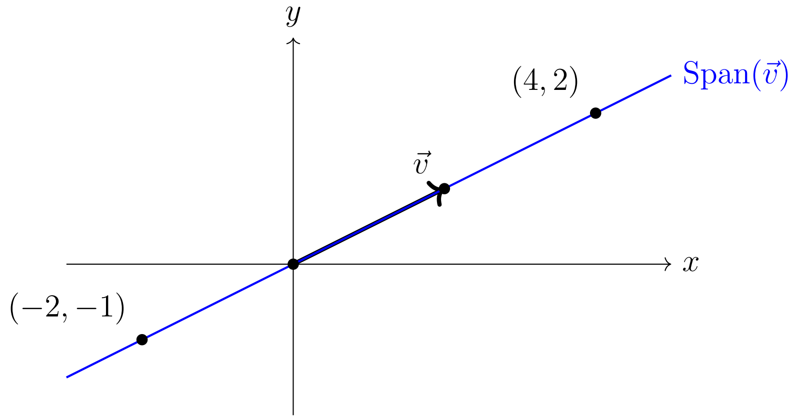
We also define $\operatorname{Span}(\emptyset) = \{\vec{\mathbf{0}}\}$.

The definition says that a vector $\vec{w} \in \operatorname{Span}(S)$ if and only if we can find scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_k\vec{v}_k$.

Example 3.27. Working in the vector space $\mathbb{R}^2$, find the span of $S = \{\vec{v}\}$ where $\vec{v} = (2, 1)$.

$$\text{Span}(S) = \{c\vec{v} \mid c \in \mathbb{R}\} = \{c(2,1) \mid c \in \mathbb{R}\} = \{(2c, c) \mid c \in \mathbb{R}\}.$$

When we plot the vectors in Span(S) we get the straight line $y = \frac{1}{2}x$.



Remark. If we think of the vectors in S as ingredients and we think of vector addition and scalar multiplication as our cooking tools, then Span(S) will be the collection of all the dishes (vectors) that we can cook starting from the ingredients from S. So Span(S) is the menu of the restaurant whose cooking ingredients are from the set S.

In $\mathbb{R}^2$, we may also think of Span(S) as the set of all the points in the cartesian plane we can reach starting from the origin and proceeding by taking trips described by the vectors in S.

Theorem 3.4.3.

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ be a set of vectors in some vector space V. Then Span(S) *is a subspace of V.*

Proof. It is clear that Span(S) is a subset of V by Axiom 1 (closure of addition) and Axiom 6 (closure of scalar multiplication). We will use the subspace test. First, observe that the zero vector is a linear combination of the vectors in S since $\vec{\mathbf{0}} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k$, and so $\vec{\mathbf{0}} \in$ Span(S) showing that Span(S) is a nonempty set. Next, we need to show that Span(S) is closed under addition and scalar multiplication. Let $\vec{u}, \vec{v}$ be vectors in Span(S). This means that there exists scalars $b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k$ from $\mathbb{R}$ such that

$$\begin{aligned}
\vec{u} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k, \\
\vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k.
\end{aligned}$$

Then, using vector space axioms (which axioms precisely?) we can show that

$$\vec{u} + \vec{v} = (b_1 + c_1)\vec{v}_1 + (b_2 + c_2)\vec{v}_2 + \cdots + (b_k + c_k)\vec{v}_k.$$

Also for any scalar $\alpha \in \mathbb{R}$, we get that

$$\alpha\,\vec{u} = (\alpha b_1)\vec{v}_1 + (\alpha b_2)\vec{v}_2 + \cdots + (\alpha b_k)\vec{v}_k.$$

This shows that both $\vec{u} + \vec{v}$ and $c\vec{u}$ are linear combinations of the vectors in S and so they are both in Span(S). Therefore, the subset Span(S) is a subspace of V. ■

Example 3.28. Working in the vector space $\mathbb{R}^3$, is the vector $\vec{v} = (1, 1, 1)$ a linear combination of the vectors in the set S below? That is, does $\vec{v}$ belong to Span(S)?

$$S = \{(1, \overset{\vec{u}_1}{2}, 3), (0, \overset{\vec{u}_2}{1}, 2), (-1, \overset{\vec{u}_3}{0}, 1)\}.$$

To accomplish the task we need to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$.

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ (1, 1, 1) &= (c_1 - c_3\,,\ 2c_1 + c_2\,,\ 3c_1 + 2c_2 + c_3).\end{aligned}$$

This yields the following system of linear equations in the variables c_1, c_2, c_3.

$$\begin{aligned} c_1 - c_3 &= 1\\ 2c_1 + c_2 &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix we get:

$$\begin{bmatrix} 1 & 0 & -1 & 1\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 1\end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & -1 & 1\\ 0 & \mathbf{1} & 2 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus, the system has infinitely many solutions where c_3 is a free variable. For any $t \in \mathbb{R}$, we get that $c_1 = 1 + t,\ \ c_2 = -1 - 2t,\ \ c_3 = t$ is a solution. We just need one solution, say $t = 1$ gives the solution $c_1 = 2,\ \ c_2 = -3,\ \ c_3 = 1$. Thus,

$$\vec{v} = (1, 1, 1) = 2\vec{u}_1 - 3\vec{u}_2 + \vec{u}_3.$$

Note that $\vec{v}$ can be written in infinitely many ways as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$. For every $t \in \mathbb{R}$, we get that

$$\vec{v} = (1 + t)\vec{u}_1 - (1 + 2t)\vec{u}_2 + t\vec{u}_3.$$

Therefore, $\vec{v} \in$ Span(S).

Example 3.29. Working in $\mathbb{R}^3$, is $\vec{w} = (1, -2, 2)$ a linear combination of the vectors in the set S below? That is, does $\vec{w}$ belong to Span(S)?

$$S = \{\overset{\vec{u}_1}{(1,2,3)}, \overset{\vec{u}_2}{(0,1,2)}, \overset{\vec{u}_3}{(-1,0,1)}\}.$$

We need to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$(1, -2, 2) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1).$$

This equation in $\mathbb{R}^3$ gives the following system of linear equations.

$$\begin{aligned} c_1 - c_3 &= 1 \\ 2c_1 + c_2 &= -2 \\ 3c_1 + 2c_2 + c_3 &= 2 \end{aligned}$$

Applying Gauss-Jordan elimination to the augmented matrix of this system we get:

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

From the third row we see that this system is inconsistent (has no solution), and thus, $\vec{w}$ cannot be written as a linear combination of the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Therefore, we conclude that $\vec{w} \notin$ Span(S).

In the previous example, we have seen that not every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors in the set S. We now focus on sets of vectors which contain enough vectors so that every vector in the ambient vector space can be expressed as a linear combination of the vectors in the set.

Definition 3.4.4. (Spanning Set)

A subset $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of a vector space V is a *spanning set* of V if and only if every vector in V can be written as a linear combination of the vectors in S. In such case, we say "S spans V".

Example 3.30 (Non-example)**.** The set $S = \{(1, 2, 3)\,, (0, 1, 2)\,, (-1, 0, 1)\}$ is not a spanning set of the vector space $\mathbb{R}^3$ because $\vec{w} = (1, -2, 2)$ cannot be expressed as a linear combination of vectors in S as shown in the previous example.

Lemma 3.4.5.

Let S be a set of vectors from a vector space V. Then Span$(S) = V$ *if and only if S is a spanning set of V.*

Example 3.31. The set $\{(1,0)\,,\,(0,1)\}$ is a spanning set for the vector space $\mathbb{R}^2$. To see this, let $\vec{v} = (x, y)$ be any vector in $\mathbb{R}^2$. Then

$$x(1,0) + y(0,1) = (x,0) + (0,y) = (x,y) = \vec{v}.$$

So any vector in $\mathbb{R}^2$ can be expressed as a linear combination of the vector in the set $\{(1,0)\,,\,(0,1)\}$ and so it is a spanning set for $\mathbb{R}^2$.

Example 3.32. The set $S = \{(1,0,0)\,,\,(0,1,0)\,,\,(0,0,1)\}$ is a spanning set for the vector space $\mathbb{R}^3$. To see this, let $\vec{v} = (x, y, z)$ be any vector in $\mathbb{R}^3$. Then

$$x(1,0,0) + y(0,1,0) + z(0,0,1) = (x,0,0) + (0,y,0) + (0,0,z) = (x,y,z) = \vec{v}.$$

So $\{(1,0,0)\,,\,(0,1,0)\,,\,(0,0,1)\}$ is a spanning set for $\mathbb{R}^3$.

Example 3.33. The set $S = \{1\,,\,x\,,\,x^2\}$ of polynomials spans the vector space $\mathcal{P}_2$. Take any polynomial $p(x) = a + bx + cx^2$ in $\mathcal{P}_2$, and observe that $p(x)$ can be written as

$$\begin{aligned}
&a(1 + 0x + 0x^2) + b(0 + x + 0x^2) + c(0 + 0x + x^2)\\
&= (a + 0x + 0x^2) + (0 + bx + 0x^2) + (0 + 0x + cx^2)\\
&= a + bx + cx^2 = p(x).
\end{aligned}$$

So $\{1\,,\,x\,,\,x^2\}$ is a spanning set of $\mathcal{P}_2$.

Example 3.34. Show that the set $S = \{\overset{\vec{u}_1}{(1,2,3)}\,,\,\overset{\vec{u}_2}{(0,1,2)}\,,\,\overset{\vec{u}_3}{(-2,0,1)}\}$ spans the vector space $\mathbb{R}^3$. Let $\vec{v} = (a, b, c)$ be any vector in $\mathbb{R}^3$. We need to find scalars $x, y, z \in \mathbb{R}$ such that

$$\begin{aligned}
\vec{v} = (a,b,c) &= x\vec{u}_1 + y\vec{u}_2 + z\vec{u}_3\\
&= x(1,2,3) + y(0,1,2) + z(-2,0,1)\\
&= (x - 2z\,,\, 2x + y\,,\, 3x + 2y + z).
\end{aligned}$$

This vector equation produces the following system of linear equations.

$$\begin{aligned}
x - 2z &= a\\
2x + y &= b\\
3x + 2y + z &= c
\end{aligned}$$

The coefficient matrix of the system is $A = \begin{bmatrix} 1 & 0 & -2\\ 2 & 1 & 0\\ 3 & 2 & 1 \end{bmatrix}$. Also, A is invertible as

$$\det(A) = \begin{vmatrix} 1 & 0\\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1\\ 3 & 2 \end{vmatrix} = 1 - 2 = -1 \neq 0.$$

Thus, the system has a unique solution and so any vector in $\mathbb{R}^3$ is a linear combination of the vector in S, that is, the set S spans $\mathbb{R}^3$. For the details on how to find the desired linear combination, recall that the unique solution of a system $AX = B$ with invertible coefficient matrix is given by $X = A^{-1}B$. Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 4b - 2c \\ 2a - 7b + 4c \\ -a + 2b - 1c \end{bmatrix}.$$

Therefore, any vector $\vec{v} = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination of vectors is S as

$$\vec{v} = (a, b, c) = (-a + 4b - 2c)\vec{u}_1 + (2a - 7b + 4c)\vec{u}_2 + (-a + 2b - 1c)\vec{u}_3.$$

For instance,

$$(3, 3, 1) = 7\vec{u}_1 - 11\vec{u}_2 + 2\vec{u}_3.$$

Therefore, the set $S = \{(1, 2, 3)\,, (0, 1, 2)\,, (-2, 0, 1)\}$ is a spanning set of $\mathbb{R}^3$.

3.5 Linear Independence

Working in the vector space $\mathbb{R}^3$, consider the vectors:

$$\vec{v}_1 = (1,3,1), \quad \vec{v}_2 = (0,1,2), \quad \vec{v}_3 = (1,0,-5).$$

Can we express the zero vector $\vec{\mathbf{0}}$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$? In other words, can we find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{\mathbf{0}}.$$

The answer is indeed yes, just take the trivial solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Is there a nontrivial solution? That is, a choice of the scalars where at least one of them is nonzero. The answer here is also yes, take $c_1 = 1, c_2 = -3, c_3 = -1$ and observe that

$$\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3 = (1,3,1) - 3(0,1,2) - (1,0,-5) = (0,0,0) = \vec{\mathbf{0}}.$$

With this nontrivial solution, we can write $\vec{v}_1$ as a linear combination of the other vectors $\vec{v}_2$ and $\vec{v}_3$ as follows: $\vec{v}_1 = 3\vec{v}_2 + \vec{v}_3$, meaning that we can obtain $\vec{v}_1$ from the vectors $\vec{v}_2$ and $\vec{v}_3$ using the operations of vector addition and scalar multiplication. In other scenarios, we can find vectors where the *only* way to express the zero vector as a linear combination of them is the trivial way; by taking all the scalars to be zeros. This motivates the following concept.

Definition 3.5.1. (Linear Independence)

A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors in a vector space V is called *linearly independent* if and only if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{\mathbf{0}}$$

has only the trivial solution: $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$. Otherwise, if there is also a nontrivial solution, then we call S a *linearly dependent* set.

Example 3.35. The set $S = \{(1,3,1),\ (0,1,2),\ (1,0,-5)\}$ from the discussion above is a linearly dependent set of $\mathbb{R}^3$ because we managed to express $\vec{\mathbf{0}}$ as a linear combination of the vectors in S in a nontrivial way, where $\vec{\mathbf{0}} = \vec{v}_1 - 3\vec{v}_2 - \vec{v}_3$.

Example 3.36. Working in the vector space $\mathbb{R}^2$.

(a) The set $S = \{(1,2),\ (2,4)\}$ is linearly dependent because

$$2(1,2) - (2,4) = (0,0).$$

(b) The set $S = \{(1,0),\ (0,1),\ (-2,5)\}$ is linearly dependent because

$$2(1,0) - 5(0,1) + (-2,5) = (0,0).$$

(c) The set $S = \{(0,0),\ (1,2)\}$ is linearly dependent because

$$1(0,0) + 0(1,2) = (0,0).$$

Lemma 3.5.2.

In a vector space, any set containing $\vec{\mathbf{0}}$ is linearly dependent.

Proof. Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a set of vectors in a vector space V, and suppose that $\vec{\mathbf{0}} \in S$. Without loss of generality, assume that $\vec{v}_1 = \vec{\mathbf{0}}$. Then we have that

$$2\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k = \vec{\mathbf{0}} + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} = \vec{\mathbf{0}}.$$

This shows that we can express the zero vector as a linear combination of the vectors in S in a nontrivial way and so S is linearly dependent. ∎

Lemma 3.5.3.

Suppose that $\vec{v}$ is a nonzero vector of a vector space V. Then the set $\{\vec{v}\}$ is linearly independent.

Proof. Suppose $\vec{v}$ is a nonzero vector. We examine the vector equation $c\vec{v} = \vec{\mathbf{0}}$. As we are working in a vector space, we know that if $c\vec{v} = \vec{\mathbf{0}}$, then $c = 0$ or $\vec{v} = \vec{\mathbf{0}}$. But we assumed $\vec{v}$ is a nonzero vector, and so it must be $c = 0$. It follows that the equation has only the trivial solution and so $\{\vec{v}\}$ is linearly independent. ∎

Example 3.37. Working in the vector space $\mathbb{R}^3$, determine whether S is linearly independent or linearly dependent.

$$S = \{\overset{\vec{v}_1}{(1,2,3)}, \overset{\vec{v}_2}{(0,1,2)}, \overset{\vec{v}_3}{(-2,0,1)}\}.$$

We need to figure out whether the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{\mathbf{0}}$ has a nontrivial solution or not.

$$c_1(1,2,3) + c_2(0,1,2) + c_3(-2,0,1) = (0,0,0)$$

The vector equation above gives the following *homogeneous* system of linear equations.

$$\begin{aligned} c_1 - 2c_3 &= 0 \\ 2c_1 + c_2 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

Computing the determinant of the coefficient matrix A we get:

$$\det(A) = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 - 2(1) = -1 \neq 0.$$

So the coefficient matrix is invertible, meaning that the system has a unique solution, namely, the trivial solution: $c_1 = 0,\ c_2 = 0,\ c_3 = 0$. This means that the zero vector of $\mathbb{R}^3$ can only be expressed in the trivial way as a linear combination of the vectors in S, and so the set S is linearly independent.

Example 3.38. Working in the vector space $\mathbb{M}_{2\times 2}$, determine whether S is linearly independent or linearly dependent.

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

We examine the vector equation: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{\mathbf{0}}$.

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Computing the left hand side to get the matrix equation below:

$$\begin{bmatrix} 2c_1 + 3c_2 + c_3 & c_1 \\ 2c_2 + 2c_3 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives the following homogeneous system of linear equations.

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix we get:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the system has a unique solution, the trivial solution $c_1 = 0, c_2 = 0, c_3 = 0$. Thus, the set S is linearly independent. This means that there is only one linear combination of the matrices in S which gives the zero matrix $\mathbf{0}_{2\times 2}$, it is the trivial linear combination.

Example 3.39. Working in the vector space $\mathcal{P}_2$ of all polynomials of degree 2 or less. Determine whether S is linearly independent or linearly dependent.

$$S = \{\overset{p_1(x)}{1 + x - 2x^2},\quad \overset{p_2(x)}{2 + 5x - x^2},\quad \overset{p_3(x)}{x + x^2}\}.$$

We need to find out whether the vector equation below has a nontrivial solution.

$$\begin{aligned} c_1p_1(x) + c_2p_2(x) + c_3p_3(x) &= \mathbf{0}(x) \\ c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) &= 0 + 0x + 0x^2 \\ (c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 &= 0 + 0x + 0x^2 \end{aligned}$$

This gives the following homogeneous system of linear equations.

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

Next, we compute the determinant of the coefficient matrix A.

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 6 - 6 = 0.$$

Thus, the homogeneous system has infinitely many solutions yielding that S is a linearly dependent set. We apply Gaussian elimination to the augmented matrix to find a nontrivial solution to the vector equation above.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} \mathbf{1} & 2 & 0 & 0 \\ 0 & \mathbf{1} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The third column of the REF matrix is a non-pivot column and so we can take c_3 as a free variable. So let $c_3 = t$ where $t \in \mathbb{R}$. Then $c_2 = -\frac{1}{3}c_3 = -\frac{1}{3}t$ and $c_1 = -2c_2 = \frac{2}{3}t$. Therefore, the system has infinitely many solutions and its solution set is

$$\left\{ \left(\frac{2}{3}t,\ -\frac{1}{3}t,\ t \right) \mid t \in \mathbb{R} \right\}.$$

For instance, take $t = 3$, then $c_1 = 2, c_2 = -1, c_3 = 3$ is a nontrivial solution to the system that yields a nontrivial linear combination of the polynomials in S producing the zero polynomial.

$$2(1 + x + 2x^2) + (-1)(2 + 5x - x^2) + 3(x + x^2) = 0 + 0x + 0x^2 = \mathbf{0}(x).$$

Therefore, S is linearly dependent. In this case, observe that we can express one of the vectors in S as a linear combination of the others.

$$(2 + 5x - x^2) = 2(1 + x + 2x^2) + 3(x + x^2).$$

Thus, $p_2(x) = 2p_1(x) + 3p_3(x)$.

The previous example is not a special case. Whenever we have a linearly dependent set, then at least one of the vectors in it can be expressed as a linear combination of the others. That is, if a subset S of a vector space is linearly dependent, then there is at least one vector $\vec{u} \in S$ such that $\vec{u} \in \text{Span}(S \setminus \{u\})$. The converse is also true as we show in the next theorem.

Theorem 3.5.4.

A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ where $k \geq 2$ of vectors in a vector space V is linearly dependent if and only if at least one vector in S can be written as a linear combination of the other vectors in S.

Proof. ($\Rightarrow$) For the forward direction, assume that S is linearly dependent. Then there are scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ not all zeros such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_k = \vec{\mathbf{0}}.$$

Without loss of generality assume that $c_1 \neq 0$. Then, using the vector space axioms, we get:

$$\begin{aligned}
c_1\vec{v}_1 &= -(c_2\vec{v}_2) + -(c_3\vec{v}_3) + \cdots + -(c_k\vec{v}_k) \\
c_1\vec{v}_1 &= (-1)(c_2\vec{v}_2) + (-1)(c_3\vec{v}_3) + \cdots + (-1)(c_k\vec{v}_k) \\
c_1\vec{v}_1 &= (-c_2)\vec{v}_2 + (-c_3)\vec{v}_3 + \cdots + (-c_k)\vec{v}_k \\
\frac{1}{c_1}(c_1\vec{v}_1) &= \frac{1}{c_1}\Big((-c_2)\vec{v}_2 + (-c_3)\vec{v}_3 + \cdots + (-c_k)\vec{v}_k\Big) \\
(\frac{1}{c_1}c_1)\vec{v}_1 &= \frac{1}{c_1}(-c_2\vec{v}_2) + \frac{1}{c_1}(-c_3\vec{v}_3) + \cdots + \frac{1}{c_1}(-c_k\vec{v}_k) \\
(1)\,\vec{v}_1 &= \left(\frac{-c_2}{c_1}\right)\vec{v}_2 + \left(\frac{-c_3}{c_1}\right)\vec{v}_3 + \cdots + \left(\frac{-c_k}{c_1}\right)\vec{v}_k \\
\vec{v}_1 &= \left(\frac{-c_2}{c_1}\right)\vec{v}_2 + \left(\frac{-c_3}{c_1}\right)\vec{v}_3 + \cdots + \left(\frac{-c_k}{c_1}\right)\vec{v}_k
\end{aligned}$$

Thus, the vector $\vec{v}_1$ is a linear combination of the vectors $\vec{v}_2, \ldots, \vec{v}_k$.

($\Leftarrow$) For the reverse direction, suppose we can express $\vec{v}_1$ as a linear combination of the other vectors, say, $\vec{v}_1 = c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_k\vec{v}_k$. Then, using the vector space axioms we get:

$$(1)\vec{v}_1 + (-c_2)\,\vec{v}_2 + (-c_3)\,\vec{v}_3 + \cdots + (-c_k)\,\vec{v}_k = \vec{\mathbf{0}}.$$

As not all the scalars in this vector equation are zeros (the first scalar is 1), we get that S is linearly dependent. ■

Example 3.40. In the vector space $\mathcal{P}_2$, the set

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{1 + x - 2x^2, \quad 2 + 5x - x^2, \quad x + x^2\}$$

is linearly dependent as $2\vec{v}_1 - \vec{v}_2 + 3\vec{v}_3 = \vec{\mathbf{0}}$. It follows that $\vec{v}_2 = 2\vec{v}_1 + 3\vec{v}_3$ and so

$$2 + 5x - x^2 = 2(1 + x - 2x^2) + 3(x + x^2).$$

The polynomial $\vec{v}_2$ is a linear combination of the other polynomials $\vec{v}_1$ and $\vec{v}_3$.

Chapter 4

The Dimension of a Vector Space

The dimension of a vector space is the least number of vectors needed to span the entire space. Surprisingly, this number is independent of the particular choice of the spanning vectors! The dimension of a vector space is an intrinsic information of the space that completely describes its entire structure in the sense that any two vector spaces who have the same dimension are essentially the same, more precisely, we say they are “isomorphic”.

4.1 Basis

Definition 4.1.1. (Basis of a Vector Space)

A *basis* of a vector space V is a subset $S \subseteq V$ of vectors which is both a spanning set of V and a linearly independent set.

The definition says that a basis S of a vector space must contain enough vectors to span the whole of the space, but not so many vectors that one of them can be written as a linear combination of other vectors in S, that is, S has no redundant vectors. The number of vectors in a basis is optimal.

Example 4.1. In the vector space $\mathbb{R}^3$ the set B below is a basis for $\mathbb{R}^3$.

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

It is obvious that S spans $\mathbb{R}^3$. Moreover, it is easy to see that the vector equation:

$$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$$

has only the trivial solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, and so B is a linearly independent set. We call B the standard basis for $\mathbb{R}^3$.

In general, the *standard basis* of the n-dimensional space $\mathbb{R}^n$ is the set

$$B = \{\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3,\ \ldots,\ \vec{e}_n\}$$

where $\vec{e}_i$ is the n-tuple which has 1 in the i^{th} component and 0s in all other components. The standard basis for $\mathbb{R}^2$ is

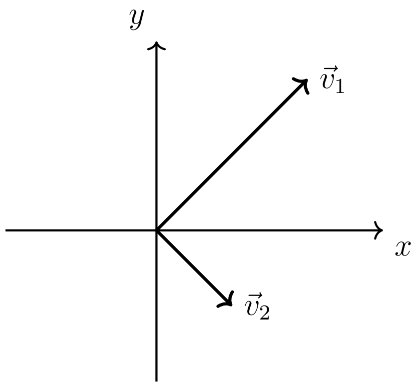
$$\{(1,0),\ (0,1)\}.$$

The standard basis for $\mathbb{R}^4$ is

$$\{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)\}.$$

Can you find basis for $\mathbb{R}^n$ other than the standard basis?

Example 4.2. The set $S = \{\overset{\vec{v}_1}{(2,2)}, \overset{\vec{v}_2}{(1,-1)}\}$ is a basis for the vector space $\mathbb{R}^2$.



We first show that S spans $\mathbb{R}^2$. To verify this, let $\vec{u} = (a,b)$ be any vector in $\mathbb{R}^2$. We need to find scalars $x, y \in \mathbb{R}$ such that

$$\begin{aligned} x\,\vec{v}_1 + y\,\vec{v}_2 &= \vec{u} \\ x(2,2) + y(1,-1) &= (a,b) \\ (2x,2x) + (y,-y) &= (a,b) \\ (2x+y, 2x-y) &= (a,b) \end{aligned}$$

This vector equation yields the following system of linear equations.

$$\begin{aligned} 2x + y &= a \\ 2x - y &= b \end{aligned}$$

The coefficient matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$ of this system is the matrix whose columns are the vectors in S. The determinant of A is $\det(A) = -2 - 2 = -4 \neq 0$. Thus, A is invertible and so the system has a unique solution meaning that any arbitrary vector in $\mathbb{R}^2$ can be expressed as a linear combination of the vectors $(2,2)$ and $(1,-1)$. Thus, S spans $\mathbb{R}^2$ as desired.

Next, we aim to show that S is a linearly independent set. We need to show that the vector equation $x\,\vec{v}_1 + y\,\vec{v}_2 = \vec{\mathbf{0}}$ has only the trivial solution. Using the computations above, it follows that $(2x+y,\, 2x-y) = (0,0)$ yielding the homogeneous system of linear equations below.

$$\begin{aligned} 2x + y &= 0 \\ 2x - y &= 0 \end{aligned}$$

We already checked that the coefficient matrix A is invertible and so this system has only the trivial solution: $x = 0$ and $y = 0$, meaning that the only way to express the zero vector $(0,0)$ as a linear combination of the vectors in S is the trivial linear combination. Therefore, the set S is a linearly independent set. Since S is both a spanning set of $\mathbb{R}^2$ and linearly independent, we get that S is a basis of $\mathbb{R}^2$.

Example 4.3. The set $S = \{(1,2,3),\ (0,1,2),\ (-2,0,1)\}$ is a basis for the vector space $\mathbb{R}^3$. Inspired by the previous example, we will check if the matrix A whose columns are the vectors in S is invertible.

$$\det(A) = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 - 2 = -1 \neq 0.$$

This implies that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors in S and also the vector equation

$$x(1,2,3) + y(0,1,2) + z(-2,0,1) = (0,0,0)$$

has only the trivial solution and so S is linearly independent.

Example 4.4. We work in the vector space $\mathcal{P}_3$ of all polynomials of degree 3 or less with real coefficients. The set

$$\{1,\ x,\ x^2,\ x^3\}$$

is a basis for $\mathcal{P}_3$ called the *standard basis*. Clearly, the set S spans $\mathcal{P}_3$. Moreover, examining the vector equation:

$$c_0(1) + c_1(x) + c_2(x^2) + c_3(x^3) = \vec{\mathbf{0}} = 0 + 0x + 0x^2 + 0x^3,$$

we see that it has only the trivial solution: $c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0$, showing that S is linearly independent.

Example 4.5. The *standard basis* for the vector space $\mathbb{M}_{2\times 2}$ of all 2×2 matrices is the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The standard basis of the vector space $\mathbb{M}_{m\times n}$ is the set which contains $m \cdot n$ many matrices of size $m \times n$ each having one entry equal to 1 and all other entries equal to 0. More precisely, for every $1 \leq k \leq m$ and $1 \leq l \leq n$, we have an $m \times n$ matrix E_{kl} whose (k,l)-entry is 1 and all other entries are 0s.

Theorem 4.1.2.

Suppose that B is a basis for a vector space V. Then every vector in V can be written as a linear combination of the vectors in B in a unique way.

Proof. Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of a vector space V. Choose any vector $\vec{u}$ in V. Since B spans V, there are scalars $b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that

$$\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

To show that this is a unique representation, assume we can represent the vector $\vec{u}$ in some other way, and so assume there are scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Both of these linear combinations are equal to $\vec{u}$, and so equal to each other. Thus,

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Using the vector space axioms we get the following vector equation:

$$(c_1 - b_1)\vec{v}_1 + (c_2 - b_2)\vec{v}_2 + \cdots + (c_n - b_n)\vec{v}_n = \vec{\mathbf{0}}.$$

Observe that this is a linear combination of the vectors in the basis B giving the zero vector, and as B is linearly independent, this last equation has only the trivial solution, meaning that $c_1 - b_1 = 0$, $c_2 - b_2 = 0$, $\ldots$, $c_n - b_n = 0$ which yields that $c_1 = b_1$, $c_2 = b_2$, $\ldots$, $c_n = b_n$. Therefore, every vector in V has a unique representation as a linear combination of the vectors in the basis B. ■

Let V be a vector space and let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V. Let $\vec{u}$ be a vector in V. We just proved that $\vec{u}$ can be expressed in one and only one way as a linear combination of vectors in B. Therefore, there are unique scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n.$$

The n-tuple $(c_1, c_2, \ldots, c_n)$ of these scalars is called the *coordinates of $\vec{u}$ relative to the basis B* and is denoted by $[\vec{u}]_B$. We also express the coordinates $[\vec{u}]_B$ as an $n \times 1$ column matrix written as follows.

$$[\vec{u}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Example 4.6. In $\mathbb{R}^3$, the standard basis is $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. Take the vector $\vec{u} = (-2,1,3)$ in $\mathbb{R}^3$. Then $\vec{u} = -2(1,0,0) + 1(0,1,0) + 3(0,0,1)$. Thus,

$$[\vec{u}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Example 4.7. Consider the basis $B = \{(1,0),(1,2)\}$ for $\mathbb{R}^2$. Let $\vec{u} = (5,4)$. Find the coordinates of $\vec{x}$ relative to B.

We first express $\vec{x}$ as a linear combination of vectors in B. One can check that $\vec{u} = (5,4) = 3(1,0) + 2(1,2)$. Thus,

$$[\vec{u}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Example 4.8. The standard basis for $\mathcal{P}_3$ is $S = \{1, x, x^2, x^3\}$. Consider the polynomial $p(x) = 4 - 2x^2 + 3x^3$ in $\mathcal{P}_3$. Then $p(x) = 4(1) + 0(x) - 2(x^2) + 3(x^3)$. So

$$[p(x)]_S = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}.$$

Example 4.9. Consider the basis $B = \{(1,0,1),(0,-1,2),(2,3,-5)\}$ of the space $\mathbb{R}^3$. Find the coordinates of $\vec{u} = (1,2,-1)$ relative to B.

We need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\vec{u} = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5).$$

This gives us the following system of linear equations

$$\begin{aligned} c_1 + 2c_3 &= 1 \\ -c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned}$$

In matrix form, we get the following matrix equation.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

As the system has a unique solution, the coefficient matrix A is invertible and the solution is $A^{-1}\vec{u}$.

$$[\vec{u}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

Theorem 4.1.3.

Suppose that B is a basis for a vector space V containing exactly n vectors. Then any subset of V containing more than n vectors must be linearly dependent.

Proof. Let V be a vector space, and let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V which contains n vectors. Choose any subset $S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_k\}$ of vectors from V containing k vectors where $k > n$. Since B is a basis, it spans V and so every vector in S is a linear combination of the vectors of B. Thus, we can find scalars such that the following equations hold.

$$\begin{aligned}
\vec{w}_1 &= c_{11}\vec{v}_1 + c_{12}\vec{v}_2 + \cdots + c_{1n}\vec{v}_n \\
\vec{w}_2 &= c_{21}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{2n}\vec{v}_n \\
&\vdots \\
\vec{w}_k &= c_{k1}\vec{v}_1 + c_{k2}\vec{v}_2 + \cdots + c_{kn}\vec{v}_n
\end{aligned}$$

To show that the set S is linearly dependent we need to find a nontrivial solution to the following vector equation.

$$x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3 + \cdots + x_k\vec{w}_k = \vec{\mathbf{0}}.$$

We next substitute the representations above of each $\vec{w}_i$ in this equation to get:

$$\begin{aligned}
&x_1(c_{11}\vec{v}_1 + c_{12}\vec{v}_2 + \cdots + c_{1n}\vec{v}_n)\ + \\
&x_2(c_{21}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{2n}\vec{v}_n)\ + \\
&\qquad\vdots \\
&x_k(c_{k1}\vec{v}_1 + c_{k2}\vec{v}_2 + \cdots + c_{kn}\vec{v}_n) = \vec{\mathbf{0}}.
\end{aligned}$$

Using the vector space axioms we can rearrange this equation to get a linear combination of the vectors in the basis B.

$$\begin{aligned}
(c_{11}x_1 + c_{21}x_2 + \cdots + c_{k1}x_k)\vec{v}_1 + (c_{12}x_1 + c_{22}x_2 + \cdots + c_{k2}x_k)\vec{v}_2 \\
+ \cdots + (c_{1n}x_1 + c_{2n}x_2 + \cdots + c_{kn}x_k)\vec{v}_n = \vec{\mathbf{0}}.
\end{aligned}$$

We know that B being a basis is a linearly independent set and so any linear combination of the vectors in B giving the zero vector (as the one we just obtained) must be the trivial linear combination, that is, all of its scalars are zeros. Therefore, we get a homogeneous system of linear equations in the variables $x_1, x_2, \ldots, x_k$.

$$\begin{aligned}
c_{11}x_1 + c_{21}x_2 + \cdots + c_{k1}x_k &= 0 \\
c_{12}x_1 + c_{22}x_2 + \cdots + c_{k2}x_k &= 0 \\
&\vdots \\
c_{1n}x_1 + c_{2n}x_2 + \cdots + c_{kn}x_k &= 0
\end{aligned}$$

This homogeneous system has n equations and k variables, moreover, we assumed that $k > n$. Thus, we have a homogeneous system with fewer equations than variables, and so such system has infinitely many solutions by Theorem 1.4.8. Thus, any nontrivial solution $(x_1, x_2, \ldots, x_k)$ of the system is a nontrivial solution to the equation:

$$x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3 + \cdots + x_k\vec{w}_k = \vec{\mathbf{0}}.$$

It follows that there is a nontrivial linear combination of the vectors in S which gives the zero vector, and so the set S is linearly dependent as we wanted to establish. ■

Example 4.10. The set $\{(1, 3, -1),\ (2, 3, 7),\ (0, -1, 6),\ (5, 7, -2)\}$ is a linearly dependent subset of $\mathbb{R}^3$ because it contains 4 vectors and we already know a basis of $\mathbb{R}^3$ which has 3 vectors, namely, the standard basis.

Example 4.11. The set $\{3,\ 2+4x,\ x^2+4x^3,\ 1-x^2, 6x+x^3\}$ is a linearly dependent subset of $\mathcal{P}_3$ because it contains 5 vectors and we already know a basis of $\mathcal{P}_3$ which has 4 vectors, namely, the standard basis $\{1,\ x,\ x^2,\ x^3\}$.

Theorem 4.1.4.

Suppose that V is a vector space which has a basis with n vectors. Then every basis of V has also n vectors.

Proof. Suppose that $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of V. Choose any other basis $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ of V. We now use Theorem 4.1.3 twice. Since B is a basis and S is linearly independent, we must have that $m \leq n$. Otherwise, if $m > n$, then S is linearly dependent which is not the case because S is a basis. Similarly, since S is a basis and B is linearly independent, we must have that $n \leq m$. Therefore, we have $m \leq n$ and $n \leq m$ implying that $n = m$ as desired. ■

4.2 Dimension

Our last result above states that all bases of a vector space contain the same number of vectors. This number is an important property of a vector space.

Definition 4.2.1. (Dimension of a Vector Space)

The *dimension* of a vector space V is the number of vectors in a basis of V. The dimension of V is denoted by $\dim(V)$. We also set $\dim(\{\vec{\mathbf{0}}\}) = 0$.

Using the standard bases discussed earlier we get the following results.

- $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathcal{P}_n) = n + 1$.
- $\dim(\mathbb{M}_{m\times n}) = mn$.

Remark. Let V be a vector space. When the number of vectors in a basis of V is finite, we say that V is a *finite dimensional* vector space. If V has no finite basis, we say that V is an *infinite dimensional* vector space. The vector space $\mathcal{P}$ of all polynomials has no finite basis. A basis for $\mathcal{P}$ is the set $\{1, x, x^2, x^3, \ldots\}$ and so $\mathcal{P}$ is an infinite dimensional vector space.

The following is an easy consequence of Theorem 4.1.3.

Corollary 4.2.2.

Let V be a vector space with $\dim(V) = n$. Then any linearly independent subset has at most n vectors. That is, if $S \subseteq V$ is linearly independent, then $|S| \leq n$.

Corollary 4.2.3.

Suppose that V is a finite dimensional vector space. If $U \subseteq V$ is a subspace, then $\dim(U) \leq \dim(V)$.

Proof. Suppose that $\dim(V) = n$ and $\dim(U) = m$. Let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be a basis of U. Therefore, S is a linearly independent subset of V containing m vectors. By the previous corollary we must have that $m \leq n$ as desired. ■

Corollary 4.2.4.

Suppose that U is a subspace of a finite dimensional vector space V. If $\dim(U) = \dim(V)$, then $U = V$.

Proof. Suppose that $\dim(U) = n = \dim(V)$. Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be a basis for U, and so $U = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$. For the sake of contradiction, assume that $U \neq V$. So there is a vector $\vec{v} \in V$ which is not in U and so $\vec{v} \notin \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$. This implies that the set $S' = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{v}\}$ is a linearly independent set in the vector space V (why?). Since $\dim(V) = n$, we know that the maximum cardinality of a linearly independent set is n, however, S' is linearly independent and has $n+1$ vectors! We got a contradiction, and so $U = V$ as desired. ■

Example 4.12. Recall that all bases have the same number of vectors.

(a) The set $\{(4, -8, 2),\ (3, 6, 1)\}$ is not a basis for $\mathbb{R}^3$ because $\dim(\mathbb{R}^3) = 3$, and so any basis of $\mathbb{R}^3$ must contain 3 vectors.

(b) The set $\{2x + 3,\ x^3,\ x^2 - 5,\ 5x + 2,\ 4 + x + x^3\}$ is not a basis for $\mathcal{P}_3$ because $\dim(\mathcal{P}_3) = 4$, and so any basis of $\mathcal{P}_3$ must contain 4 polynomials.

Example 4.13. Determine the dimension of the following subspaces of $\mathbb{R}^3$.

Recall that subspaces are themselves vector spaces, and so it makes sense to find the dimension of a subspace. To find the dimension of a vector space it is sufficient to find a basis of it.

(a) The subspace $U = \{(2c, c, 0) \mid c \in \mathbb{R}\}$.

Consider the vector $\vec{v} = (2, 1, 0)$. Clearly $\vec{v} \in U$. Moreover,

$$U = \{(2c, c, 0) \mid c \in \mathbb{R}\} = \{c(2, 1, 0) \mid c \in \mathbb{R}\} = \{c\vec{v} \mid c \in \mathbb{R}\} = \text{Span}(\{\vec{v}\}).$$

Therefore, the set $\{\vec{v}\}$ is a spanning set for U. Furthermore, a set which contains exactly one nonzero vector is linearly independent. Therefore, the set $\{(2, 1, 0)\}$ is a basis of U, and thus, $\dim(U) = 1$.

(b) The subspace $W = \{(a, b - a, b) \mid a, b \in \mathbb{R}\}$.

Choose any vector $\vec{w} \in W$. Then there are scalars $a, b \in \mathbb{R}$ such that

$$\vec{w} = (a, b - a, b) = (a, -a, 0) + (0, b, b) = a(1, -1, 0) + b(0, 1, 1) = a\vec{u} + b\vec{v},$$

where $\vec{u} = (1, -1, 0)$ and $\vec{v} = (0, 1, 1)$. It follows that any vector $\vec{w} \in W$ is a linear combination of the vectors $\vec{u}$ and $\vec{v}$. Clearly, both $\vec{u}, \vec{v} \in W$. It follows that $\text{Span}(\{\vec{u}, \vec{v}\}) = W$, and so $\{\vec{u}, \vec{v}\}$ is a spanning set of W. Moreover, one can check that $\{\vec{u}, \vec{v}\}$ is linearly independent because no one of them is a scalar multiple of the other, and so it is a basis for W. Therefore, $\dim(W) = 2$.

Example 4.14. Consider the subset $S \subseteq \mathbb{R}^4$ below. Let $W = \text{Span}(S)$. So W is the subspace of $\mathbb{R}^4$ spanned by S. Find the dimension of W.

$$S = \{(-1, 2, 5, 0),\ (3, 0, 1, -2),\ (-5, 4, 9, 2)\}.$$

Clearly, by definition of W, we know that S is a spanning set of W. However, S is not a basis of W since it is not linearly independent because one of its vectors is a linear combination of the other two vectors.

$$(-5, 4, 9, 2) = 2(-1, 2, 5, 0) - (3, 0, 1, -2).$$

So one of the vectors in S is redundant and we can get rid of it without changing its span. Consequently, let $B = \{(-1, 2, 5, 0), (3, 0, 1, -2)\}$. Therefore,

$$\text{Span}(B) = \text{Span}(S) = W.$$

So B is a spanning set of W. Moreover, it is obvious that no vector in B is a scalar multiple of the other, and so B is linearly independent implying that B is a basis of W. Therefore, $\dim(W) = 2$.

Exercise. Suppose that S is a spanning set of a finite dimensional vector space V. Show that there is a subset $B \subseteq S$ which is a basis for V. In other words, any spanning set of V contains a basis for V.

Theorem 4.2.5.

Let V be a finite dimensional vector space and consider a subset $S \subseteq V$. If $|S| = \dim(V)$ and S is linearly independent, then S is a basis of V.

Proof. Suppose that $S \subseteq V$, $|S| = \dim(V) = n$, and S is linearly independent. For the sake of contradiction, suppose that S is not a spanning set of V. Therefore, there is a vector $\vec{v} \in V$ such that $\vec{v} \notin \text{Span}(S)$. This implies that the set $S' = S \cup \{\vec{v}\}$ is still linearly independent (why?). So we got a linearly independent set S' which has $n + 1$ vectors. Since $\dim(V) = n$, we know that any linearly independent set must have at most n vectors, thus we got a contradiction, meaning that S must be a spanning set. Therefore, the set S is a basis for V. ■

Theorem 4.2.6.

Let V be a finite dimensional vector space and consider a subset $S \subseteq V$. If $|S| = \dim(V)$ and S spans V, then S is a basis of V.

Proof. Suppose that $S \subseteq V$, $|S| = \dim(V) = n$, and S is a spanning set for V. The case when $n = 1$ is easy. So suppose that $n \geq 2$. For the sake of contradiction, assume that S is linearly dependent. Therefore, there is a vector $\vec{u} \in S$ which is a linear combination of the other vectors in S. Let S^- be the set of all the vectors in S without the vector $\vec{u}$. Then, S^- is still a spanning set of V (why?). If S^- is linearly independent, we stop. Otherwise, we can remove one more vector which can be expressed as a linear combination of the other vectors and staying a spanning set. Eventually, we will obtain a subset of S which is a spanning set and linearly independent with less vectors than the number of vectors in S. This means that we found a basis for V which has less than n vectors, a contradiction! As we know that all bases have the same cardinality. Therefore, S must be linearly independent, and so S is a basis for V. ■

4.3 Rank and Nullity

Consider the 3×4 matrix below. It has 3 rows and 4 columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

In this section, we think of the rows of A as vectors in $\mathbb{R}^4$, and we sometimes call them the *row vectors* of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}, \quad \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}, \quad \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Similarly, we think of the columns of A as vectors in $\mathbb{R}^3$ calling them occasionally the *column vectors* of A.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \quad \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}.$$

In general, an $m \times n$ matrix has m row vectors which belong to $\mathbb{R}^n$ and n column vectors which belong to $\mathbb{R}^m$.

Remark. We will not differentiate between row matrices (matrices of size $1 \times n$) and vectors in $\mathbb{R}^n$. Similarly, we will not differentiate between column matrices (matrices of size $n \times 1$) and vectors in $\mathbb{R}^n$.

Definition 4.3.1. (Row and Column Spaces)

Let $A_{m\times n}$ be a matrix with rows $\vec{r}_1, \ldots, \vec{r}_m$ and columns $\vec{c}_1, \ldots, \vec{c}_n$.

- The *row space* $R(A)$ of matrix A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A. In symbols, $R(A) = \text{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m)$.
- The *column space* $C(A)$ of matrix A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A. In symbols, $C(A) = \text{Span}(\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n)$.

Remark. Let A be a matrix. Then $C(A) = R(A^T)$.

Lemma 4.3.2.

Suppose that matrix A is transformed to matrix B by applying a single elementary row operation. Then, the row space of A is equal to the row space of B.

Proof. Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m$ be the row vectors of A. We will split our proof into three cases based on the type of the elementary row operation.

Type-I operation: $R_i \leftrightarrow R_j$. Without loss of generality, suppose we interchanged the first two rows of of A to obtain B. Thus, the rows of B are $\vec{r}_2, \vec{r}_1, \vec{r}_3, \dots, \vec{r}_m$. Then, as vector addition is commutative, we get that

$$\begin{aligned} R(A) &= \mathrm{Span}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m) \\ &= \{c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \{c_2\vec{r}_2 + c_1\vec{r}_1 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \mathrm{Span}(\vec{r}_2, \vec{r}_1, \vec{r}_3, \dots, \vec{r}_m) = R(B). \end{aligned}$$

Type-II operation: $cR_i \to R_i$. Without loss of generality, suppose we multiplied the first row of of A by a nonzero $\alpha \in \mathbb{R}$ to obtain B. Thus, the rows of B are $\alpha\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m$. Then, by compatability of scalar multiplication, we get that

$$\begin{aligned} R(A) &= \mathrm{Span}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m) \\ &= \{c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \left\{\frac{c_1}{\alpha}(\alpha\vec{r}_1) + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\right\} \\ &= \mathrm{Span}(\alpha\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m) = R(B). \end{aligned}$$

Type-III operation: $(R_i + cR_j) \to R_i$. Without loss of generality, suppose that the first row of of A is replaced by the result of adding it to a multiple of the second row. Thus, the rows of B are $(\vec{r}_1 + \alpha\vec{r}_2), \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m$. Then, by compatibility of scalar multiplication, we get that

$$\begin{aligned} R(B) &= \mathrm{Span}(\,\vec{r}_1 + \alpha\vec{r}_2\,,\ \vec{r}_2\,,\ \vec{r}_3\,,\ \dots,\ \vec{r}_m\,) \\ &= \{c_1(\vec{r}_1 + \alpha\vec{r}_2) + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \{c_1\vec{r}_1 + c_1(\alpha\vec{r}_2) + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \{c_1\vec{r}_1 + (c_1\alpha)\vec{r}_2 + c_2\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \{c_1\vec{r}_1 + (c_1\alpha + c_2)\vec{r}_2 + c_3\vec{r}_3 + \cdots + c_m\vec{r}_m \mid c_i \in \mathbb{R}\} \\ &= \mathrm{Span}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_m) = R(A). \end{aligned}$$

Therefore, we have shown that applying an ERO will not change the row space of a matrix. ■

Corollary 4.3.3.

If matrices A and B are row-equivalent, then the row space of A is equal to the row space of B.

Proof. Suppose that matrices A and B are row-equivalent. Then B is obtained from A by applying finitely many EROs.

$$A \implies A_1 \implies A_2 \implies \cdots \implies A_k \implies B$$

By the lemma above, we know that $R(A) = R(A_1) = R(A_2) = \cdots = R(A_k) = R(B)$. Therefore, $R(A) = R(B)$. ■

Lemma 4.3.4.

Let $B_{m\times n}$ be a matrix in row-echelon form.

(i) The nonzero rows of B form a basis for the row space $R(B)$.

(ii) The pivot columns of B form a basis for the column space $C(B)$.

Proof. For simplicity, we may assume that $B_{m\times n}$ is in reduced row-echelon form (RREF). Suppose that $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ are the rows of B and that $\vec{r}_k$ for some $k \le m$ is the last nonzero row. By definition, the row space $R(B)$ is the span of all the rows of B. Observe that a zero row is the zero vector of $\mathbb{R}^n$, and so a linear combination of all the rows of B is equal to the same linear combination without the zero rows. More precisely, for scalars $c_1, c_2, \ldots, c_m$ we have

$$\begin{aligned} c_1\vec{r}_1 + \cdots + c_k\vec{r}_k + c_{k+1}\vec{r}_{k+1} + \cdots + c_m\vec{r}_m &= c_1\vec{r}_1 + \cdots + c_k\vec{r}_k + c_{k+1}\vec{\mathbf{0}} + \cdots + c_m\vec{\mathbf{0}} \\ &= c_1\vec{r}_1 + \cdots + c_k\vec{r}_k + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} \\ &= c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_k\vec{r}_k. \end{aligned}$$

Therefore, $\mathrm{Span}(\vec{r}_1, \vec{r}_2, \ldots, , \vec{r}_k, \ldots, \vec{r}_m) = \mathrm{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k)$.

Next, the set of all nonzero rows of B form a linearly independent set in $\mathbb{R}^n$. To see this, choose any nonzero row $\vec{r}_i$ of B. As B is in RREF, we know that all the entries above and below the leading 1 of $\vec{r}_i$ are zeros. It follows that it is impossible to express $\vec{r}_i$ as a linear combination of the other nonzero rows because any linear combination of them will have a 0 in the component of the leading 1 of $\vec{r}_i$, however, $\vec{r}_i$ has 1 in that component. Thus, no nonzero row of B is a linear combination of the other nonzero rows implying that the nonzero rows form a linearly independent set. Therefore, the set of all nonzero rows of B spans the row space of B and is linearly independent. In other words, the nonzero rows form a basis for $R(B)$.

We leave the proof of the second statement as an exercise. Show that if B is in reduced row-echelon form, then each non-pivot column is a linear combination of the previous pivot columns where the entries in the non-pivot column are exactly the scalars needed to write it as a linear combination of the previous pivot columns. ■

Our work above provides an algorithm for extracting a basis for the row space of a matrix A as described in the next corollary.

Corollary 4.3.5.

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space $R(A)$. Consequently, $\dim(R(A))$ *is the number of nonzero rows of B.*

The next results provide an algorithm for extracting a basis for the column space of a matrix A.

Theorem 4.3.6.

Let A and B be $m \times n$ matrices. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be the columns of A and let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ be the columns of B. Suppose there is an $m \times m$ matrix F such that $B = FA$. Fix some indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

(i) If the columns $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$ are linearly independent, then also the columns $\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}$ are linearly independent.

(ii) For any $1 \leq j \leq n$ and scalars $c_1, c_2, \ldots, c_k$, if $\vec{a}_j = c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k}$, then $\vec{b}_j = c_1\vec{b}_{i_1} + c_2\vec{b}_{i_2} + \cdots + c_k\vec{b}_{i_k}$.

In words, if the column $\vec{a}_j$ of A is a linear combination of the columns $\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}$, then the corresponding column $\vec{b}_j$ of B is a linear combination of the columns $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$, with the same scalars.

(iii) For any $1 \leq j \leq n$, if the column $\vec{a}_j \in \operatorname{Span}(\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k})$, then the column $\vec{b}_j \in \operatorname{Span}(\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k})$.

Proof. Since $B = FA$, it follows by definition of matrix multiplication that $\vec{b}_i = F\vec{a}_i$ for every $1 \leq i \leq n$.

(i) Suppose that the columns $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$ are linearly independent. To show that $\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}$ are linearly independent we need to show that the vector equation $c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k} = \vec{\mathbf{0}}$ has only the trivial solution.

$$\begin{aligned} c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k} = \vec{\mathbf{0}} &\implies F(c_1\vec{a}_{i_1}) + F(c_2\vec{a}_{i_2}) + \cdots + f(c_k\vec{a}_{i_k}) = F\vec{\mathbf{0}} \\ &\implies c_1(F\vec{a}_{i_1}) + c_2(F\vec{a}_{i_2}) + \cdots + c_k(F\vec{a}_{i_k}) = \vec{\mathbf{0}} \\ &\implies c_1\vec{b}_{i_1} + c_2\vec{b}_{i_2} + \cdots + c_k\vec{b}_{i_k} = \vec{\mathbf{0}}. \end{aligned}$$

Since we assumed that $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$ are linearly independent, we must have that $c_1 = 0, c_2 = 0, \ldots, c_k = 0$ as desired. Therefore, $\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}$ are linearly independent as well.

(ii) Suppose that $\vec{a}_j = c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k}$. From this linear combination we proceed as follows.

$$\begin{aligned}\vec{a}_j = c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k} &\implies F\vec{a}_j = F(c_1\vec{a}_{i_1}) + F(c_2\vec{a}_{i_2}) + \cdots + f(c_k\vec{a}_{i_k}) \\ &\implies \vec{b}_j = c_1(F\vec{a}_{i_1}) + c_2(F\vec{a}_{i_2}) + \cdots + c_k(F\vec{a}_{i_k}) \\ &\implies \vec{b}_j = c_1\vec{b}_{i_1} + c_2\vec{b}_{i_2} + \cdots + c_k\vec{b}_{i_k}.\end{aligned}$$

(iii) Assume that $\vec{a}_j \in \text{Span}(\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k})$. Then there are scalars $c_1, c_2, \ldots, c_k$ such that $\vec{a}_j = c_1\vec{a}_{i_1} + c_2\vec{a}_{i_2} + \cdots + c_k\vec{a}_{i_k}$. By Part (ii), it follows that the same linear combination holds for the corresponding columns of B, and so we get that $\vec{b}_j = c_1\vec{b}_{i_1} + c_2\vec{b}_{i_2} + \cdots + c_k\vec{b}_{i_k}$. Therefore, $\vec{b}_j \in \text{Span}(\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k})$. ■

Theorem 4.3.7.

Suppose a matrix A is reduced by applying EROs to a matrix B in row-echelon form. Then the columns of A which correspond to the pivot columns of B form a basis for the column space $C(A)$. Consequently, $\dim(C(A))$ *is the number of pivot columns of B.*

Proof. Let A be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. Suppose that A is reduced by applying EROs to a matrix B in row-echelon form where the columns of B are $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$. It follows that there are elementary matrices $E_1, E_2, \ldots, E_t$ such that $B = (E_t \cdots E_2E_1)A$. Let $F = E_t \cdots E_2E_1$ and so $B = FA$. Since each E_i is invertible, their product F is also invertible as the product of invertible matrices is also invertible. We also get that $A = F^{-1}B$.

Let $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$ be the pivot columns of B where $1 \le i_1 < i_2 < \cdots < i_k \le n$. We know that $\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}$ form a basis for the column space $C(B)$. Thus, they are linearly independent, and so by Theorem 4.3.6, the corresponding columns $\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}$ of A are also linearly independent.

Now, let $\vec{b}_j$ be a non-pivot column of B, so $j \notin \{i_1, i_2, \ldots, i_k\}$. Since $\vec{b}_j$ belongs to the column space $C(B)$ and $\{\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k}\}$ is a spanning set for $C(B)$ we get that $\vec{b}_j \in \text{Span}(\vec{b}_{i_1}, \vec{b}_{i_2}, \ldots, \vec{b}_{i_k})$. Since $A = F^{-1}B$, by Theorem 4.3.6, we get that the corresponding column $\vec{a}_j \in \text{Span}(\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k})$. Therefore,

$$C(A) = \text{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) = \text{Span}(\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}).$$

Therefore, the set $\{\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}\}$ is a spanning set for the column space $C(A)$. This shows that $\{\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k}\}$ form a basis for the column space of A as we wanted to demonstrate. ■

We summarize our findings below.

♣ **Finding a basis for $R(A)$ and for $C(A)$.**

- Start with any matrix A.

- Apply EROs to transform A to a matrix B in REF.
- The nonzero rows of B form a basis for the row space of A.
- The columns of A which correspond to the pivot columns of B form a basis for the column space of A.

Example 4.15. Find a basis for the row space of A.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{EROs} B = \begin{bmatrix} \mathbf{1} & 3 & 1 & 3 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the set $S = \{(1,3,1,3),(0,1,1,0),(0,0,0,1)\}$ forms a basis for the row space of A. It follows that every vector in $R(A)$ can be written as a linear combination of the vectors in S in a unique way. We get that $\dim(R(A)) = 3$. Moreover, observe that the pivot columns of the REF matrix B are the 1^{st}, 2^{nd}, and 4^{th} columns. Consequently, the 1^{st}, 2^{nd}, and 4^{th} columns of A for a basis for $C(A)$. Thus, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

is a basis for the column space of A. Therefore, $\dim(C(A)) = 3$ as well.

Example 4.16. Find a basis for the subspace $U = \mathrm{Span}(S)$ of $\mathbb{R}^3$ spanned by

$$S = \{(-1,2,5),(3,0,3),(5,1,8)\}.$$

Let A be the matrix whose rows are the vectors in S.

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \xrightarrow{EROs} B = \begin{bmatrix} \mathbf{1} & -2 & -5 \\ 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the vectors in S are exactly the rows of A be get that $U = \mathrm{Span}(S) = R(A)$. Thus, the set $\{(1,-2,-5),(0,1,3)\}$ of nonzero rows of B forms a basis for the subspace $U = \mathrm{Span}(S)$.

Example 4.17. Find a basis for the column space of A using the transpose of A.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

Note that the column space of a matrix A is the row space of A^T.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & -3 & 3 & 2 \\ 0 & \mathbf{1} & 9 & -5 & -6 \\ 0 & 0 & \mathbf{1} & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1,0,-3,3,2),(0,1,9,-5,-6),(0,0,1,-1,-1)\}$ forms a basis for the row space of A^T. Since $C(A) = R(A^T)$ this set is a basis for the column space of A as well. Written vertically, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

forms a basis for the column space of A. Thus, $\dim(C(A)) = 3$.

Theorem 4.3.8.

Let A be an $m \times n$ matrix. Then the row space and column space of A have the same dimension. In symbols, $\dim(R(A)) = \dim(C(A))$.

Proof. Let A be an $m \times n$ matrix. Reduce A to a matrix B in REF by applying EROs. Then

$$\begin{aligned} \dim(R(A)) &= \text{number of nonzero rows of } B \\ &= \text{number of leading 1s in } B \\ &= \text{number of pivot columns of } B \\ &= \dim(C(A)). \end{aligned}$$

This completes the proof. ■

Definition 4.3.9. (Rank of a Matrix)

The *rank* of a matrix A, denoted by $\operatorname{rank}(A)$, is the dimension of the row space of A (or column space of A).

An easy observation is that $\operatorname{rank}(A_{m\times n}) \leq \min(m,n)$.

Example 4.18. Find the rank of the matrix A below.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \xrightarrow{EROs} B = \begin{bmatrix} \mathbf{1} & -2 & 0 & 1 \\ 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & \mathbf{1} & 3 \end{bmatrix}$$

As the matrix B in row-echelon form has 3 nonzero rows which form a basis for $R(A)$, it follows that $\operatorname{rank}(A) = \dim(R(A)) = 3$.

Lemma 4.3.10.

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\operatorname{rank}(A) = n$.

The next theorem gives a nice characterization for the members of the column space of a matrix A.

Theorem 4.3.11.

Let A be an $m \times n$ matrix. Let $\vec{b}$ be an $m \times 1$ column matrix. Then $\vec{b}$ is a linear combination of the columns of A if and only if the system $A\vec{x} = \vec{b}$ is consistent. Consequently,

$$C(A) = \left\{ \vec{b} \in \mathbb{R}^m \mid \textit{the system } A\vec{x} = \vec{b} \textit{ is consistent} \right\}.$$

Proof. Let $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ be the columns of the matrix $A_{m\times n} = [a_{ij}]$. Let $\vec{b}$ be any $m \times 1$ column matrix. Then,

$$\begin{aligned}
\vec{b} \in C(A) &\iff \vec{b} \in \operatorname{Span}(\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n) \\
&\iff \vec{b} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \cdots + \alpha_n\vec{c}_n \text{ where each } \alpha_i \in \mathbb{R} \\
&\iff \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\
&\iff \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11}\alpha_1 \\ a_{21}\alpha_1 \\ \vdots \\ a_{m1}\alpha_1 \end{bmatrix} + \begin{bmatrix} a_{12}\alpha_2 \\ a_{22}\alpha_2 \\ \vdots \\ a_{m2}\alpha_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}\alpha_n \\ a_{2n}\alpha_n \\ \vdots \\ a_{mn}\alpha_n \end{bmatrix} \\
&\iff \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{m2}\alpha_n \\ \vdots \\ a_{m1}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{mn}\alpha_n \end{bmatrix} \\
&\iff \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \\
&\iff \vec{b} = A\vec{s} \quad (\text{so } \vec{s} = (\alpha_1, \dots, \alpha_n) \text{ is a solution to } \vec{b} = A\vec{x}) \\
&\iff \vec{b} = A\vec{x} \ \text{ is consistent.}
\end{aligned}$$

We have shown that the members of the column space of A are exactly the column matrices of constant terms of consistent systems with coefficient matrix A. ∎

We can extract the following consequence from the proof of the result above.

Corollary 4.3.12.

Let A be a matrix of size $m \times n$ with columns $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$, and let $\vec{b}$ an $m \times 1$ column matrix. Then the tuple $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ is a solution of the system $A\vec{x} = \vec{b}$ if and only if $\vec{b} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \cdots + \alpha_n\vec{c}_n$.

In addition to the row space and column space of a matrix A, we associate a third vector space to A, called the nullspace.

Definition 4.3.13. (Nullspace of a Matrix)

The *nullspace* of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ given by

$$N(A) = \{\, \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{\mathbf{0}}_{m\times 1} \,\}.$$

The *nullity* of A is the dimension of the nullspace of A.

For an $m \times n$ matrix A, the members of the nullspace $N(A)$ are $n \times 1$ matrices which we think of them as vectors in $\mathbb{R}^n$. Observe that the nullspace $N(A)$ is the solution set of the homogeneous system $A\vec{x} = \vec{\mathbf{0}}$, where $\vec{\mathbf{0}}$ here is the $m \times 1$ zero matrix. Since EROs do not change the solution set of a system, we have the following easy consequence.

Corollary 4.3.14.

If A and B are row-equivalent matrices, then they have the same nullspace, that is, $N(A) = N(B)$.

We still need to verify that $N(A)$ is indeed a subspace of $\mathbb{R}^n$.

Lemma 4.3.15.

The nullspace of a matrix $A_{m\times n}$ is a subspace of $\mathbb{R}^n$.

Proof. We will use the subspace test. Let A be an $m \times n$ matrix. Remember that a homogeneous system is always consistent since substituting zeros for all variables satisfies all equations. In other words, $A\vec{\mathbf{0}}_{n\times 1} = \vec{\mathbf{0}}_{m\times 1}$ and so $\vec{\mathbf{0}}_{n\times 1} \in N(A)$ implying the nullspace is nonempty.

Next, we need to show that $N(A)$ is closed under vector addition and scalar multiplication. Choose any vectors $\vec{u}$, $\vec{v} \in N(A)$ and we think we them as $n \times 1$ matrices. This means that $A\vec{u} = \vec{\mathbf{0}}_{m\times 1}$ and $A\vec{v} = \vec{\mathbf{0}}_{m\times 1}$. And choose any scalar $c \in \mathbb{R}$. We proceed as follows.

- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{\mathbf{0}}_{m\times 1} + \vec{\mathbf{0}}_{m\times 1} = \vec{\mathbf{0}}_{m\times 1}$.

- $A(c\vec{v}) = c(A\vec{v}) = c\,\vec{\mathbf{0}}_{m\times 1} = \vec{\mathbf{0}}_{m\times 1}$.

This yields that both $\vec{u}+\vec{v}$ and $c\vec{v}$ are solutions to the system $A\vec{x} = \vec{\mathbf{0}}_{m\times 1}$ and so they belong to the nullspace of A as desired. So, $N(A)$ is a subspace of $\mathbb{R}^n$. ■

Example 4.19. Find the nullspace of A, that is, find the solution set for the homogeneous system $A\vec{x} = \vec{\mathbf{0}}$.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}_{3x4} \xrightarrow{EROs} B = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From B we obtain the following homogeneous system of linear equations.

$$\begin{aligned} x + 2y + 3w &= 0 \\ z + w &= 0 \end{aligned}$$

The 2^{nd} and 4^{th} columns of B are the non-pivot columns and they correspond to the variables y and w which we will take as the free variables. Choose any real numbers $s, t \in \mathbb{R}$ to obtain the solution below for the system:

$$w = t,\ z = -t,\ y = s,\ x = -2s - 3t.$$

Therefore, the nullspace of A is

$$N(A) = \left\{ \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Now observe that for any vector $\vec{v} \in N(A)$ we have that

$$\vec{v} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ 0 \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

This observation shows that any vector in the nullspace $N(A)$ is a linear combination of these two vectors meaning that they span the nullspace. Moreover, no one of them is a scalar multiple of the other and so they form a linearly independent set. Therefore, the set

$$S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

forms a basis for the nullspace $N(A)$. So the nullity of A is $\dim(N(A)) = 2$. Here, 2 is the number of the free variables s, t which is the number of the non-pivot columns of B. Moreover, we have that $\text{rank}(A) = 2$, it is the number of pivot columns in the REF matrix B. Note that the sum of rank and nullity is the number of pivot columns plus the number of non-pivot columns which is the number of all columns of B (or A).

$$\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4.$$

We have seen above that A is row-equivalent to the matrix B below in REF:

$$\begin{bmatrix} \boxed{\begin{matrix} \mathbf{1} \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ \mathbf{1} \\ 0 \end{matrix}} & \begin{matrix} 3 \\ 1 \\ 0 \end{matrix} \end{bmatrix}$$

The two pivot columns (1^{st} and 3^{rd}) determine $\text{rank}(A)$, and the two non-pivot columns (2^{nd} and 4^{th}) determine $\text{nullity}(A)$.

Theorem 4.3.16. (Rank-Nullity Theorem)

Let A be a matrix with m rows and n columns. Then

$$n = \text{rank}(A) + \text{nullity}(A).$$

Corollary 4.3.17.

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\text{nullity}(A) = 0$.

Example 4.20. Find the rank and nullity of A.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{EROs} B = \begin{bmatrix} \mathbf{1} & 0 & -2 & 0 & 1 \\ 0 & \mathbf{1} & 3 & 0 & -4 \\ 0 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three nonzero rows in B which form a basis for the row space of A. So, $\text{rank}(A) = \dim(R(A)) = 3$. Moreover, by the rank-nullity theorem we get that $\text{nullity}(A) =$ number of columns $- \text{rank}(A) = 5 - 3 = 2$.

In B, the first, second, and fourth columns are the pivot columns. So the first, second, and fourth columns of A form a basis for the column space $C(A)$, that is,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

is a linearly independent set that spans column space $C(A)$.

We remark that by Theorem 4.3.6, the same relations between the columns of B transfer to the columns of A. For instance, the third column of B is a linear combination of the first and second columns: $\vec{b}_3 = -2\vec{b}_1 + 3\vec{b}_2$. This same relation holds for columns of A: $\vec{a}_3 = -2\vec{a}_1 + 3\vec{a}_2$ meaning that:

$$\begin{bmatrix} -2 \\ -3 \\ 1 \\ 9 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}.$$

Theorem 4.3.18.

Let A be an $n \times n$ matrix. Then A is invertible if and only if the rows of A form a basis of $\mathbb{R}^n$.

Proof. Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ be the rows of A. Moreover, notice that the rows of the identity matrix are the standard basis of $\mathbb{R}^n$, namely, the vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$, where $\vec{e}_i$ is n-tuple which has 1 in the i^{th} position and 0s elsewhere.

$$\begin{aligned} A \text{ is invertible } &\Leftrightarrow A \text{ is row-equivalent to } I_n \\ &\Leftrightarrow R(A) = \operatorname{Span}(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) \\ &\Leftrightarrow R(A) = \mathbb{R}^n \\ &\Leftrightarrow \operatorname{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) = \mathbb{R}^n \\ &\Leftrightarrow \text{The set } \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \text{ forms a basis for } \mathbb{R}^n. \end{aligned}$$

In the last equivalence, we are using the fact that if a set S of n vectors spans a vector space V of dimension n, then S is linearly independent. ■

Corollary 4.3.19.

Let A be an $n \times n$ matrix. Then A is invertible if and only if the columns of A form a basis of $\mathbb{R}^n$.

Proof. A matrix A is invertible if and only if its transpose A^T is invertible if and only if the rows of A^T form a basis for $\mathbb{R}^n$ if and only if the columns of A form a basis for $\mathbb{R}^n$. ■

Example 4.21. Let A be an $m \times n$ matrix. Then, $\operatorname{rank}(A) = \operatorname{rank}(A^T)$ because

$$\operatorname{rank}(A) = \dim(R(A)) = \dim(C(A^T)) = \operatorname{rank}(A^T).$$

However, nullity$(A) \neq$ nullity(A^T) in general. For instance, consider the matrix A and its transpose below.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

As both are in REF, we know the nullity is the number of the nonpivot columns. So we get that nullity$(A) = 1$, however, the nullity$(A^T) = 0$.

Putting our results together, we finish this section be stating several conditions equivalent to a square matrix being invertible.

Theorem 4.3.20.

Let A be an $n \times n$ matrix. The following conditions are equivalent.

(1) *A is invertible.*

(2) *A is row equivalent to I_n.*

(3) $\det(A) \neq 0$.

(4) *$A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.*

(5) *$A\vec{x} = \vec{\mathbf{0}}$ has only the trivial solution.*

(6) $N(A) = \{\vec{\mathbf{0}}\}$.

(7) $\text{nullity}(A) = 0$.

(8) $\text{rank}(A) = n$.

(9) $R(A) = \mathbb{R}^n$.

(10) $C(A) = \mathbb{R}^n$

(11) *The rows of A form a basis for $\mathbb{R}^n$, in particular, the rows of A are linearly independent.*

(12) *The columns of A form a basis for $\mathbb{R}^n$, in particular, the columns of A are linearly independent.*

4.4 Change of Basis

Let V be a vector space and let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for V. Recall that for a vector $\vec{u}$ in V, the coordinates $[\vec{u}]_B$ of $\vec{u}$ relative to the basis B is the unique n-tuple of scalars $(c_1, c_2, \dots, c_n)$ such that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n.$$

Our goal in this section is to learn how to change basis in $\mathbb{R}^n$. Given two bases B and C of $\mathbb{R}^n$, how are $[\vec{x}]_B$ and $[\vec{x}]_C$ related?

Let $B = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ and $C = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be two bases of a vector space V. Moreover, let

$$[\vec{v}_1]_C = \begin{bmatrix} c_{11} \\ c_{22} \\ \vdots \\ c_{n1} \end{bmatrix}, \quad [\vec{v}_2]_C = \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}, \quad \dots, \quad [\vec{v}_n]_C = \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix}.$$

Then the *transition matrix* from B to C is given by

$$Q = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}.$$

That is, for any vector $\vec{x} \in V$, we have that

$$[\vec{x}]_C = Q[\vec{x}]_B.$$

We use Gauss-Jordan elimination to find the transition matrix Q from basis B to basis C. First, we define a matrix B whose columns are the vectors in the basis B. Similarly, define a matrix C whose columns are the vectors in the basis C. Then, to find Q, we do the following:

$$[C \mid B] \xrightarrow{EROs} [I_n \mid Q].$$

Example 4.22. Find the transition matrix from B to C for the following bases for $\mathbb{R}^2$: $B = \{(-3, 2), (4, -2)\}$ and $C = \{(-1, 2), (2, -2)\}$. Begin with forming the matrix $[C \mid B]$ and proceed to reduce it to $[I_n : Q]$.

$$\left[\begin{array}{cc|cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{EROs} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

Therefore, the transition matrix from B to C is

$$Q = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}.$$

Let $\vec{x} = (5, -2) \in \mathbb{R}^2$. Then, $\vec{x} = 1(-3, 2) + 2(4, -2)$. It follows that $[\vec{x}]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. And so the coordinates of $\vec{x}$ relative to C are

$$[\vec{x}]_C = Q\,[\vec{x}]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

We may verify this by checking that $\vec{x} = 3(-1, 2) + 4(2, -2)$.

The transition matrix from C to B is

$$Q^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}.$$

Example 4.23. Find the transition matrix from basis B to basis C in $\mathbb{R}^3$ where we are given $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $C = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$.

$$[C \mid B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textit{EROs}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right].$$

So the transition matrix from B to C is

$$Q = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

Find $[\vec{x}]_C$ where $\vec{x} = (1, 2, -1)$. As B is the standard basis we have that $[\vec{x}]_B = \vec{x}$.

$$[\vec{x}]_C = Q\,[\vec{x}]_B = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

One may check that $\vec{x} = 5(1, 0, 1) - 8(0, -1, 2) - 2(2, 3, -5)$.

Chapter 5

Linear Transformations

5.1 Linear Transformations

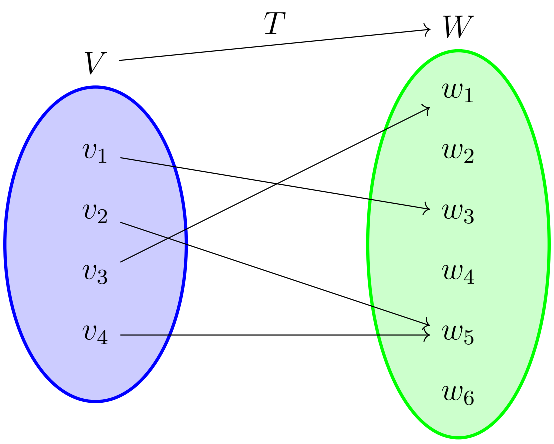
Let V and W be two sets. A *function* $T : V \to W$ from V to W is an assignment which assigns to every element $v \in V$ one and only one element $w \in W$. The set V is called the *domain* of T, and the set W is called the *codomain* of T. When T assigns to $v \in V$ the element $w \in W$, we write $T(v) = w$ and we say that w is the *image* of v, and that v is a *preimage* of w. The *range* of T is the set of images of all elements in the domain V. Thus,

$$\text{range}(T) = \{\, T(v) \mid v \in V \,\} .$$

For an element $w \in W$, the *preimage* of w, denoted by $T^{-1}(w)$, is the set of all $v \in V$ such that $T(v) = w$. Observe that $T^{-1}(w)$ is a subset of the domain V. In symbols,

$$T^{-1}(w) = \{v \in V \mid T(v) = w\} .$$

Below is a diagram for a function $T : V \to W$ from a set V to a set W. Observe that $T(v_1) = w_3$, $T(v_2) = w_5$, $T(v_3) = w_1$, and $T(v_4) = w_5$.



- $\text{dom}(T) = V = \{v_1, v_2, v_3, v_4\}$.
- $\text{codomain}(T) = W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$.
- $\text{range}(T) = \{T(x) \mid x \in V\} = \{T(v_1), T(v_2), T(v_3), T(v_4)\} = \{w_1, w_3, w_5\}$.
- The preimage of w_1 is the set $T^{-1}(w_1) = \{v_3\}$.
- The preimage of w_2 is the set $T^{-1}(w_2) = \emptyset$; the empty set.
- The preimage of w_5 is the set $T^{-1}(w_5) = \{v_2, v_4\}$.

Example 5.1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $T(x, y) = (x - y,\ x + 2y)$.
(a) Let $\vec{v} = (-1, 2) \in \mathbb{R}^2$ be a member of the domain of T, then its image is

$$T(\vec{v}) = T(-1, 2) = (-1 - 2,\ -1 + 2 \cdot 2) = (-3, 3).$$

(b) Let $\vec{w} = (-1, 11) \in \mathbb{R}^2$ be a member of the codomain of T. Find the preimage of $\vec{w}$. Thus, We need to find all $\vec{v} \in \mathbb{R}^2$ in the domain of T such that $T(\vec{v}) = \vec{w}$.

$$T(\vec{v}) = T(x, y) = (x - y,\ x + 2y) = (-1, 11)$$

So, we need to solve the following system of linear equations

$$\begin{aligned} x - y &= -1 \\ x + 2y &= 11 \end{aligned}$$

The coefficient matrix is invertible and so the system has a unique solution, namely, $x = 3$ and $y = 4$. So, the preimage of $\vec{w}$ is $T^{-1}(\vec{w}) = \{(3, 4)\}$.

In this section, we will study a special kind of functions from one vector space to another vector space. These functions preserve the operations of vector addition and scalar multiplication.

Definition 5.1.1. (Linear Transformation)

Let V and W be vector spaces. A *linear transformation* from V to W is a function $T : V \to W$ such that for all vectors $\vec{u}, \vec{v} \in V$, and scalar $c \in \mathbb{R}$ the following conditions hold:

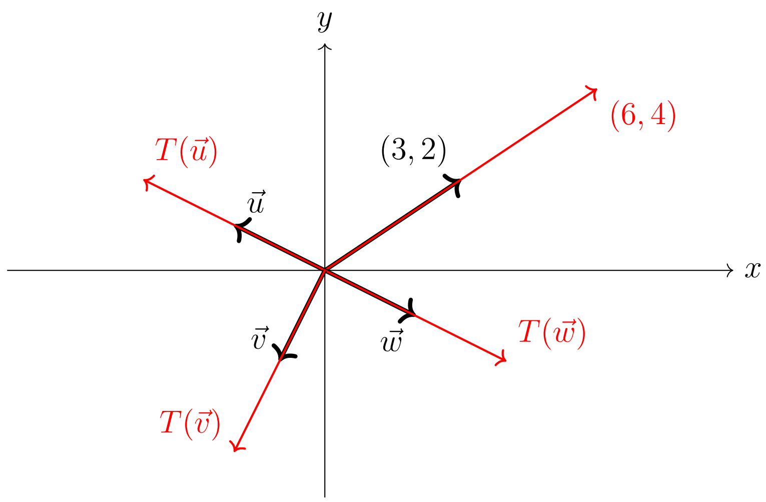
- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\,\vec{u}) = c\,T(\vec{u})$

Remark. Note that on the left side of these two conditions, the operations act on vectors in V, while they act on vectors in W on the right side.

The first condition says "the image of the sum is the sum of the images". It says, if you add vectors in V, and then apply T to their sum, you will get the same vector in W if you first apply T to the two vectors $\vec{u}$ and $\vec{v}$, and then add their images in W. The second condition says that the image of a scalar multiple of a vector in V is the scalar multiple of its image in W.

It is useful to think of a linear transformations $T : V \to V$ as a form of motion which moves vectors within the vector space in a smooth way like rotating vectors or scaling their length.

Example 5.2. Consider the motion which stretches every vector in the cartesian plane by a factor of 2. In other words, we are talking about the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = 2\vec{v}$ for every $\vec{v} \in \mathbb{R}^2$. For instance, $T(3, 2) = (6, 4)$.



Show that T is indeed a linear transformation. Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^2 = \mathrm{dom}(T)$, and let $c \in \mathbb{R}$.

(1) First, we check that T preserves vector addition.

$$T(\vec{u} + \vec{v}) = 2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v} = T(\vec{u}) + T(\vec{v}).$$

(2) Second, we check that T preserves scalar multiplication.

$$T(c\,\vec{u}) = 2(c\vec{u}) = c(2\vec{u}) = c\,T(\vec{u}).$$

Therefore, T is a linear transformation.

Lemma 5.1.2.

The following two functions are linear transformations.

- *The zero transformation* $Z : V \to W$*, given by* $Z(\vec{v}) = \vec{\mathbf{0}}_W$*, for every* $\vec{v} \in V$*.*
- *The identity transformation* $I : V \to V$*, given by* $I(\vec{v}) = \vec{v}$*, for every* $\vec{v} \in V$*.*

Example 5.3. The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x - y,\, x + 2y)$ is a linear transformation. Let $\vec{u} = (a, b)$ and $\vec{v} = (x, y)$ be two vectors in $\mathbb{R}^2 = \operatorname{dom}(T)$, and let $c \in \mathbb{R}$.

(1) First, we check that T preserves vector addition.

$$\begin{aligned}
T\,(\vec{u} + \vec{v}) = T((a, b) + (x, y)) &= T\,(a + x,\, b + y)\\
&= ((a + x) - (b + y),\, (a + x) + 2\,(b + y))\\
&= ((a - b) + (x - y),\, (a + 2b) + (x + 2y))\\
&= (a - b,\, a + 2b) + (x - y,\, x + 2y)\\
&= T(a, b) + T(x, y) = T(\vec{u}) + T(\vec{v}).
\end{aligned}$$

(2) Second, we check that T preserves scalar multiplication.

$$\begin{aligned}
T(c\,\vec{u}) &= T(c\,(a, b)) = T(ca, cb)\\
&= (ca - cb,\, ca + 2cb) = (c(a - b),\, c(a + 2b))\\
&= c(a - b,\, a + 2b) = c\,T(\vec{u}).
\end{aligned}$$

Therefore, T is a linear transformation.

Exercise. Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x^2\,,\, y^2)$ is not a linear transformation.

Linear transformations satisfy several nice properties as the theorem below shows.

Theorem 5.1.3.

Let $T : V \to W$ *be a linear transformation from vector space* V *to vector space* W*. Then, the following statements are true.*

(i) $T(\vec{\mathbf{0}}_V) = \vec{\mathbf{0}}_W$.

(ii) $T(-\vec{v}) = -T(\vec{v})$*, for every* $\vec{v} \in V$.

(iii) $T\,(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$*, for every* $\vec{u}, \vec{v} \in V$.

(iv) $T\,(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_kT(\vec{v}_k)$*, for every* $\vec{v}_i \in V$ *and* $c_i \in \mathbb{R}$.

Proof. (i) Choose any $\vec{v} \in V$. Then, $T(\vec{\mathbf{0}}_V) = T(0\,\vec{v}) = 0\,T(\vec{v}) = \vec{\mathbf{0}}_W$.

(ii) $T(-\vec{v}) = T((-1)\vec{v}) = (-1)T(\vec{v}) = -T(\vec{v})$.

(iii) $T(\vec{u} - \vec{v}) = T(\vec{u} + (-\vec{v})) = T(\vec{u}) + T(-\vec{v}) = T(\vec{u}) + (-T(\vec{v})) = T(\vec{u}) - T(\vec{v})$.

(iv) By mathematical induction on $k \geq 1$. ■

Example 5.4 (Non-example)**.** The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x+1, y+1)$ is not a linear transformation because it does not preserve scalar multiplication. To see this, take $c = 3$ and $\vec{v} = (2, 1)$. Then $T(c\vec{v}) = T(3(2, 1)) = T(6, 3) = (7, 4)$. However, $c\,T(\vec{v}) = 3T(2, 1) = 3(3, 2) = (9, 6)$, thus, $T(c\vec{v}) \neq cT(\vec{v})$. Another quicker way to see that T cannot be a linear transformation is that it does not send the zero vector to itself since $T(\vec{\mathbf{0}}) = T(0, 0) = (1, 1) \neq (0, 0)$.

Example 5.5 (Non-example)**.** The determinant function det : $\mathbb{M}_{2\times 2} \to \mathbb{R}$ is not a linear transformation because it does not preserve vector addition as the determinant of the sum of matrices is not equal to the sum of their determinants in general. To see this, consider the matrices:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}, \qquad A + B = \begin{bmatrix} 8 & 4 \\ 5 & 4 \end{bmatrix}.$$

We have $\det(A) + \det(B) = (4(2) - 1(2)) + (4(2) - 3(3)) = 6 - 1 = 5$, however, $\det(A + B) = 8(4) - 4(5) = 12$.

Example 5.6. Consider the function $F : \mathbb{M}_{m\times n} \to \mathbb{M}_{n\times m}$ which maps an $m \times n$ matrix A to its transpose, that is, $F(A) = A^T$. Then, the function F is a linear transformation. To see this, choose any $m \times n$ matrices A and B, and any scalar $c \in \mathbb{R}$.

(i) $F(A + B) = (A + B)^T = A^T + B^T = F(A) + F(B)$.

(ii) $F(c\,A) = (c\,A)^T = c\,A^T = c\,F(A)$.

Thus, the transpose function is a linear transformation.

Example 5.7. The differential operator $D_x : \mathcal{P}_n \to \mathcal{P}_{n-1}$ is the function given by

$$D_x(f) = \frac{d}{dx}(f)$$

for every polynomial $f(x) \in \mathcal{P}_n$. In particular, if $f(x) = a_0 + a_1x_1 + a_2x^2 + \cdots + a_nx_n$, then $D_x(f) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$. Show that D_x is a linear transformation.

We know that for D_x to be a linear transformation, it must preserve vector addition and scalar multiplication. Choose any polynomials f and g in $\mathcal{P}_n$, and any scalar $c \in \mathbb{R}$.

(i) $D_x(f + g) = \frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g) = D_x(f) + D_x(g)$.

(ii) $D_x(c\,f) = \frac{d}{dx}(cf) = c\frac{d}{dx}(f) = c\,D_x(f)$.

Thus, we conclude that the differential operator D_x is a linear transformation.

Example 5.8 (Linear Transformation and Basis)**.** Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ where its action on the standard basis is as follows: $T(1,0,0) = (2,-1,4)$, $T(0,1,0) = (1,5,-2)$, and $T(0,0,1) = (0,3,1)$. Find $T(2,3,-2)$.

We can see that we can write the vector $(2,3,-2)$ as a linear combination of the given three vectors (standard basis) as

$$(2,3,-2) = 2\,(1,0,0) + 3\,(0,1,0) - 2\,(0,0,1)$$

Using Property (iv) of the theorem above, we have that

$$\begin{aligned} T(2,-3,2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1) \\ &= (4,2,-8) + (3,15,-6) - (0,6,2) = (7,7,0). \end{aligned}$$

We learn from the previous example that if we know the images of the vectors in a basis of V under a linear transformation $T : V \to W$, then we know everything about T. Here are the details.

- Suppose that $T : V \to W$ is a linear transformation between vector spaces V and W.
- Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the domain V.
- Assume we know the images $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$.
- Choose any vector $\vec{u}$ in the domain V.
- We know that $\vec{u}$ is a linear combination of vectors in the basis B. So

  $$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

  for some scalars $c_1, c_2, \ldots, c_n$.
- Finally, the image of $\vec{u}$ under the linear transformation T is

  $$T(\vec{u}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n).$$

5.2 Linear Transformations and Matrices

Our next theorem shows that any matrix gives rise to a linear transformation!

Lemma 5.2.1.

Let A be an $m \times n$ matrix. The function $T : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T(\vec{v}) = A\vec{v}$$

for every $\vec{v} \in \mathbb{R}^n$, is a linear transformation.

Proof. As A is an $m \times n$ matrix and $\vec{v}$ is an $n \times 1$ matrix, their product $A\vec{v}$ is an $m \times 1$ matrix that we think of as a vector in $\mathbb{R}^m$. So T is a function that sends $\vec{v} \in \mathbb{R}^n$ to $A\vec{v} \in \mathbb{R}^m$. Next, we need to check that T preserves vector addition and scalar multiplication. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then,

(i) $T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$.

(ii) $T(c\vec{u}) = A(c\vec{u}) = c(A\vec{u}) = c\,T(\vec{u})$.

This shows that $T(\vec{v}) = A\vec{v}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. ■

Remark. If $A_{m\times n}$ is the zero matrix, then $T(\vec{v}) = A\vec{v}$ is the zero transformation $Z : \mathbb{R}^n \to \mathbb{R}^m$. And if A is the identity matrix I_n, then $T(\vec{v}) = A\vec{v}$ is the identity transformation: $I : \mathbb{R}^n \to \mathbb{R}^n$.

Example 5.9. Let $A_{3\times 2}$ be the matrix below.

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$$

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ where $T(\vec{v}) = A\vec{v} = A \begin{bmatrix} x \\ y \end{bmatrix}$.

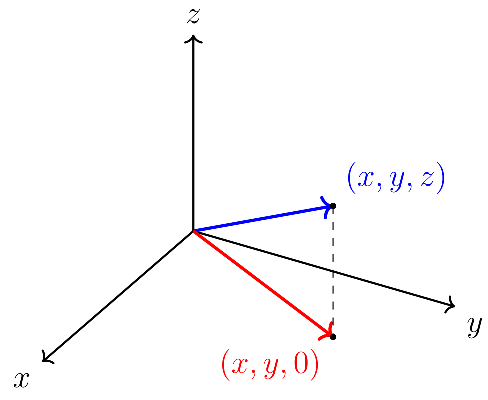
(a) $T(\vec{\mathbf{0}}) = T(0,0) = A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathbf{0}}$.

(b) $T(2,-1) = A \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$.

(c) $T(-2,1) = A \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix} = -T(2,-1)$.

Example 5.10 (Projection in 3D Space)**.** The function $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps every vector in $\mathbb{R}^3$ to its projection on the xy-plane is a linear transformation.

$$T(x, y, z) = (x, y, 0).$$



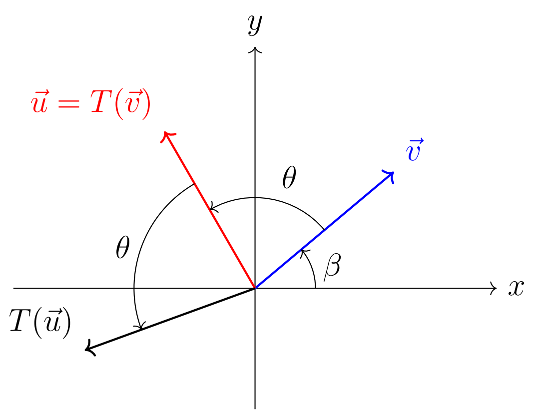
One way to see T is a linear transformation is to notice that it is of the form $T(\vec{v}) = A\vec{v}$ where

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example 5.11 (Rotation in 2D Space)**.** The function T which rotates every point in the cartesian plane counter clockwise by some angle θ is a linear transformation. It is a linear transformation because it is of the form $T(\vec{v}) = A\vec{v}$ where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We now verify our claim that T indeed rotates every vector in the plane by angle θ. Choose any vector $\vec{v} = (x, y)$ in $\mathbb{R}^2$. It follows that $x = r\cos\beta$ and $y = r\sin\beta$ where $r = \sqrt{x^2 + y^2}$ and $\beta = \tan^{-1}(y/x)$, that is, r is the length of the vector $\vec{v}$ and β is the angle between $\vec{v}$ and the positive x-axis.



$$
\begin{aligned}
T(\vec{v}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\beta \\ r\sin\beta \end{bmatrix} \\
&= \begin{bmatrix} r(\cos\beta\cos\theta - \sin\beta\sin\theta) \\ r(\sin\theta\cos\beta + \sin\beta\cos\theta) \end{bmatrix} \\
&= \begin{bmatrix} r\cos(\beta+\theta) \\ r\sin(\beta+\theta) \end{bmatrix} = \vec{u}.
\end{aligned}
$$

So the image of $\vec{v}$ under T is the vector $\vec{u}$, and the vector $\vec{u}$ has the same length r as that of vector $\vec{v}$, however, it makes an angle of $\beta + \theta$ with the positive x-axis. Thus, $\vec{u}$ is obtained by rotating $\vec{v}$ counter clockwise by angle θ.

We have shown above that matrices give rise to linear transformations, more precisely, that the function $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{v}) = A\vec{v}$ where A is an $m \times n$ matrix is a linear transformation. What about the other way around? Given any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, will we always be able to find some matrix which induces the transformation T? The answer of this question is positive!

Recall that the standard basis for $\mathbb{R}^n$ is the set $\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$, where $\vec{e_i}$ is the $n \times 1$ column matrix whose $(i, 1)$-entry is 1, and has zeros elsewhere. We will show below that if we know the action of some linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ on the standard basis of $\mathbb{R}^n$, then we know how T acts on all of $\mathbb{R}^n$. Moreover, the theorem below also says that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(\vec{v}) = A\vec{v}$ for some suitable matrix A. This gives a complete description for all linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$.

Theorem 5.2.2.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation where the action of T on the standard basis $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ is as follows:

$$T\left(\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}\right) = \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right) = \begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}, \quad \ldots, \quad T\left(\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}\right) = \begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}.$$

Let A be the $m \times n$ matrix whose columns are $T(\vec{e_1})$, $T(\vec{e_2})$, $\ldots$, $T(\vec{e_n})$, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Then, we have that $T(\vec{v}) = A\vec{v}$ for every $\vec{v}$ in $\mathbb{R}^n$. (The matrix A is called the standard matrix for T.)

Proof. Let $\vec{v}$ be any vector in $\mathbb{R}^n$. So,

$$\vec{v} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = x_1\,\vec{e_1} + x_2\,\vec{e_2} + \cdots + x_n\,\vec{e_n}.$$

Since T is a linear transformation, it preserves addition and scalar multiplication, and so

$$\begin{aligned} T(\vec{v}) &= T(x_1\,\vec{e}_1 + x_2\,\vec{e}_2 + \cdots + x_n\,\vec{e}_n) = x_1\,T(\vec{e}_1) + x_2\,T(\vec{e}_2) + \cdots + x_n\,T(\vec{e}_n) \\ &= x_1 \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix} + x_2 \begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix} + \cdots + x_n \begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix} = \begin{bmatrix}a_{11}x_1\\a_{21}x_1\\\vdots\\a_{m1}x_1\end{bmatrix} + \cdots + \begin{bmatrix}a_{1n}x_n\\a_{2n}x_n\\\vdots\\a_{mn}x_n\end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = A\,\vec{v}. \end{aligned}$$

Therefore, $T(\vec{v}) = A\vec{v}$ as desired. ■

Example 5.12. Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x, y, z) = (x - 2y,\ 2x + y).$$

We need to find the images of the standard basis of $\mathbb{R}^3$, namely, $T(\vec{e_1})$, $T(\vec{e_2})$, and $T(\vec{e_3})$.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-2\\1\end{bmatrix},\quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}.$$

Thus, the standard matrix of $T:\mathbb{R}^3\to\mathbb{R}^2$ is $\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}$ and so for any $\vec{x}\in\mathbb{R}^3$ we have that

$$T(\vec{x})=A\vec{x}=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

We next state a more general theorem for matrices representing linear transformations relative to any choice of bases. Recall that if B is a basis for a vector space V and $\vec{u}\in V$, then the coordinates of $\vec{u}$ relative to basis B is denoted by $[\vec{u}]_B$.

Theorem 5.2.3.

Let $T:V\to W$ be a linear transformation between vector spaces. Suppose that $B=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is a basis for V and let S be a basis for W. Suppose further that

$$\big[T(\vec{v}_1)\big]_S=\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix},\quad \big[T(\vec{v}_2)\big]_S=\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix},\quad\ldots,\quad \big[T(\vec{v}_k)\big]_S=\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}.$$

Then, the $m\times n$ matrix A whose columns are $\big[T(\vec{v}_i)\big]_S$, that is,

$$A=\begin{bmatrix}a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix}$$

is such that for any $\vec{u}$ in V we have that

$$\big[T(\vec{u})\big]_S=A\,\big[\vec{u}\big]_B.$$

(This matrix A is called the matrix of T relative to the bases B and S.)

Example 5.13. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation given by the assignment $T(\vec{v})=T(x,y)=(x+y,2x-y)$. Find the matrix of T relative to the bases B and S where $B=\{(1,2),(-1,1)\}$ and $S=\{(1,0),(0,1)\}$. Then find $T(\vec{v})$ where $\vec{v}=(2,1)$.

- $T(1,2) = (3,0) = 3\,(1,0) + 0\,(0,1)$. So, $\big[T(1,2)\big]_S = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.
- $T(-1,1) = (0,-3) = 0\,(1,0) - 3\,(0,1)$. So, $\big[T(-1,1)\big]_S = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$.

Thus, the matrix of T relative to the bases B and S is

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}.$$

Now, we find the image of $\vec{v}$ by T. Since $\vec{v} = (2,1) = 1(1,2) + (-1)(-1,1)$, we get that $[\vec{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Now,

$$[T(\vec{v})]_S = A\,\big[\vec{v}\big]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Therefore, $T(\vec{v}) = 3\,(1,0) + 3\,(0,1) = (3,3)$. We can check our answer by directly finding the image using the definition of T as follows.

$$T(\vec{v}) = T(2,1) = (2+1,\ 2(2)-1) = (3,3).$$

Example 5.14. Let V be a vector space with basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Find the matrix of the identity transformation $I : V \to V$ relative to the basis B and B itself.

$$\big[I(\vec{v}_1)\big]_B = \big[\vec{v}_1\big]_B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \big[I(\vec{v}_2)\big]_B = \big[\vec{v}_2\big]_B = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \big[I(\vec{v}_n)\big]_B = \big[\vec{v}_n\big]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $A = I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$. So, $\big[I(\vec{u})\big]_B = I_n\,\big[\vec{u}\big]_B$ for any $\vec{u}$ in V.

5.3 Kernel and Range

Definition 5.3.1. (Kernel of a Linear Transformation)

Let $T : V \to W$ be a linear transformation. The *kernel* of T is the set of all vectors $\vec{v}$ in V that are mapped to the zero vector $\vec{\mathbf{0}}_W$. That is,

$$\ker(T) = \left\{ \vec{v} \in V \mid T(\vec{v}) = \vec{\mathbf{0}}_W \right\}.$$

Example 5.15. Find the kernel of the linear transformations below.

- Let $Z : V \to W$ be the zero transformation. Then, $\ker(Z) = V$ since every vector in V is mapped to $\vec{\mathbf{0}}_W$.
- Let $I : V \to V$ be the identity transformation. Then, $\ker(T) = \{\vec{\mathbf{0}}_V\}$ since the only vector that is mapped to $\vec{\mathbf{0}}_V$ is $\vec{\mathbf{0}}_V$.

Notice that in both cases the kernel is a subspace of the domain V.

Example 5.16. Let $F : \mathbb{M}_{3\times 2} \to \mathbb{M}_{2\times 3}$ be the transpose linear transformation given by $F(A) = A^T$ for any 3×2 matrix A. We can see that the zero matrix $\mathbf{0}_{3\times 2}$ is the only matrix in the domain of F whose transpose is equal to the $\mathbf{0}_{2\times 3}$; the zero vector of the codomain of F. Therefore,

$$\ker(F) = \{\mathbf{0}_{3\times 2}\} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

So $\ker(F)$ is the trivial subspace.

Example 5.17. Find the kernel of the projection $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x, y, z) = (x, y, 0)$. This linear transformation projects every vector (x, y, z) in $\mathbb{R}^3$ to the vector $(x, y, 0)$ in the xy-plane. We can see that $\ker(T)$ is the set of all points on the z-axis since every such point will be mapped to $(0, 0, 0)$ by T. Equivalently speaking,

$$\begin{aligned} \ker(T) &= \left\{ \vec{v} \in \mathbb{R}^3 \mid T(\vec{v}) = \vec{\mathbf{0}} \right\} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0, 0, 0) \right\} \\ &= \left\{ (x, y, z) \in \mathbb{R}^3 \mid (x, y, 0) = (0, 0, 0) \right\} \\ &= \left\{ (x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } y = 0 \right\} = \{(0, 0, z) \mid z \in \mathbb{R}\}. \end{aligned}$$

This shows that $\ker(T)$ is the set of all points on the z-axis in $\mathbb{R}^3$. Observe that $\ker(T)$ is a subspace of $\mathbb{R}^3$ as we know that lines passing through the origin are subspaces. Another way to see it is that notice that $\ker(T) = \text{Span}\{(0, 0, 1)\}$, and we know that the span of a set of vectors is a subspace.

Example 5.18. Consider the matrix $A_{2\times 3}$ below. Find the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$ where $\vec{x}$ is a 3×1 column matrix.

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

To find the kernel, we need to find which 3×1 matrices $\vec{x}$ are mapped to the zero vector $\mathbf{0}_{2\times 1}$ of $\mathbb{R}^2$ by T. So, find $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = \vec{\mathbf{0}}$. In other words, we need to find the solution set of the system $A\vec{x} = \vec{\mathbf{0}}$. This means that $\ker(T)$ is the nullspace of A, that is, $\ker(T) = N(A)$.

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix equation above yields the following homogeneous system of linear equations which has fewer equations than variables.

$$\begin{aligned} x - y - 2z &= 0 \\ -x + 2y + 3z &= 0 \end{aligned}$$

Using Gaussian elimination, the augmented matrix of the system above is reduced to a matrix in row-echelon form shown below.

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 1 & 0 \end{bmatrix}$$

The third column of the REF matrix is a non-pivot column and so we take the variable z to be the free variable. Choose any parameter $t \in \mathbb{R}$ for the value of z and, consequently, we express the solution as $x = t$, $y = -t$, $z = t$. Thus, the kernel of the linear transformation T is the solution set of the system above.

$$\ker(T) = N(A) = \left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

We have shown that any vector $\vec{v} \in \mathbb{R}^3$ of the form $\vec{v} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$ is mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by the transformation T and so it belongs to the kernel of T. A final remark is that $\ker(T)$ is clearly a subspace of $\mathbb{R}^3$ since it is the span of a vector and we know that the span of a set of vectors is always a subspace.

In all of the previous examples, $\ker(T)$ happened to be a subspace. This is not a coincidence! It is always a subspace.

Theorem 5.3.2.

The kernel of any linear transformation $T : V \to W$ between vector spaces is a subspace of the domain V.

Proof. We will use the subspace test. The kernel of T is always a nonempty set since T is a linear transformation and so $T(\vec{\mathbf{0}}_V) = \vec{\mathbf{0}}_W$ yielding $\vec{\mathbf{0}}_V \in \ker(T)$. It remains to show that $\ker(T)$ is closed under vector addition and scalar multiplication. Let $\vec{u}, \vec{v}$ be vectors in $\ker(T)$. This means that $T(\vec{u}) = \vec{\mathbf{0}}_W$ and $T(\vec{v}) = \vec{\mathbf{0}}_W$. We need to show that $\vec{u} + \vec{v}$ is also in $\ker(T)$. We can see that

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{\mathbf{0}}_W + \vec{\mathbf{0}}_W = \vec{\mathbf{0}}_W.$$

Since T maps $\vec{u} + \vec{v}$ to $\vec{\mathbf{0}}_W$, we get that $\vec{u} + \vec{v}$ belongs to $\ker(T)$, and so $\ker(T)$ is closed under addition. It remains to show the closure under scalar multiplication as follows. So pick any scalar $c \in \mathbb{R}$.

$$T(c\vec{u}) = cT(\vec{u}) = c\,\vec{\mathbf{0}}_W = \vec{\mathbf{0}}_W.$$

Thus, $c\vec{u}$ is in $\ker(T)$ as well. Thus, $\ker(T)$ is closed under scalar multiplication. Therefore, $\ker(T)$ is a subspace of V. ■

There is another subspace associated to a linear transformation $T : V \to W$ which is the range of T. Recall that the range of T is the set of images of all vectors in V. That is,

$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

Theorem 5.3.3.

The range of any linear transformation $T : V \to W$ is a subspace of the codomain W.

Proof. We will use the subspace test. Since T is a linear transformation, we know that $T(\vec{\mathbf{0}}_V) = \vec{\mathbf{0}}_W$, and thus, $\vec{\mathbf{0}}_W$ belongs to $\text{range}(T)$ showing that $\text{range}(T)$ is nonempty. Next, we will show that $\text{range}(T)$ is closed under vector addition and scalar multiplication. So, let $\vec{u}, \vec{v}$ be in $\text{range}(T)$. Thus, there are $\vec{x}, \vec{y}$ in V ($\text{dom}(T)$) such that $T(\vec{x}) = \vec{u}$ and $T(\vec{y}) = \vec{v}$. Clearly, their sum $\vec{x} + \vec{y}$ is in V as V is a vector space and moreover,

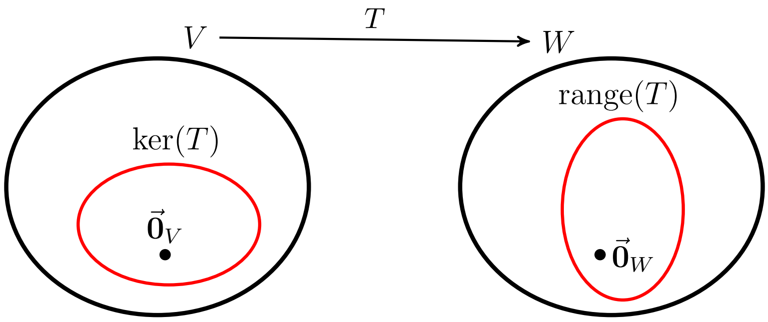
$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{u} + \vec{v}.$$

This shows that $\vec{u} + \vec{v}$ is in $\text{range}(T)$ because it is the image of the vector $\vec{x} + \vec{y}$ from V. So, $\text{range}(T)$ is closed under vector addition. Next, choose any scalar $c \in \mathbb{R}$ and observe that

$$T(c\vec{x}) = c\,T(\vec{x}) = c\vec{u}.$$

Thus, $c\vec{u}$ is in $\text{range}(T)$ because $c\vec{u}$ is the image of the vector $c\vec{x}$ from V. So, $\text{range}(T)$ is closed under scalar multiplication. Since $\text{range}(T)$ is closed under vector addition and scalar multiplication, we can conclude that $\text{range}(T)$ is a subspace of the codomain W. ■

Let $T : V \to W$ be a linear transformation between vector spaces. The following picture shows the kernel and range as subspaces of V and W, respectively.



Let us now examine the case where the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$ for a fixed $m \times n$ matrix A and any $n \times 1$ matrix $\vec{x}$.

Lemma 5.3.4.

Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$. Then,

- $\ker(T) =$ *nullspace*(A).
- $\text{range}(T) =$ *columnspace*(A).

Proof. For the kernel of T we proceed as follows.

$$\ker(T) = \left\{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{\mathbf{0}}\right\} = \left\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{\mathbf{0}}\right\} = N(A).$$

Thus, $\ker(T)$ is the solution set of the homogeneous system $A\vec{x} = \vec{\mathbf{0}}$. So, in this case, $\ker(T)$ is precisely the nullspace of A.

For the range of T we proceed as follows.

$$\begin{aligned} \text{range}(T) &= \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\} = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \\ &= \left\{\vec{b} \in \mathbb{R}^m \mid \text{there exists } \vec{x} \in \mathbb{R}^n \text{ such that } \vec{b} = A\vec{x}\right\} \\ &= \left\{\vec{b} \in \mathbb{R}^m \mid \text{the system } A\vec{x} = \vec{b} \text{ is consistent}\right\} \\ &= C(A). \end{aligned}$$

The last equality follows from Theorem 4.3.11. Therefore, $\text{range}(T)$ is the column space of the matrix A. ■

Example 5.19. Consider the matrix $A_{4\times 5}$ below. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x} \in \mathbb{R}^5$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

- Find $\ker(T)$ and find a basis for it as a subspace of $\mathbb{R}^5$.
- Find range(T) and find a basis for it as a subspace of $\mathbb{R}^4$.

Since $\ker(T)$ is the nullspace of A, we need to solve the homogeneous system $A\vec{x} = \vec{\mathbf{0}}$. Applying Gaussian-elimination method on the augmented matrix to reduce it to a row-equivalent matrix in row-echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} \mathbf{1} & 0 & 2 & 0 & -1 & 0 \\ 0 & \mathbf{1} & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The 3^{rd} and 5^{th} columns are non-pivot columns and they correspond to free variables x_3 and x_5. To express the solution of the system we choose parameters $s, t \in \mathbb{R}$. A solution has the form: $x_1 = -2s + t$, $x_2 = s + 2t$, $x_3 = s$, $x_4 = -4t$, $x_5 = t$. From this, we can conclude that

$$\ker(T) = N(A) = \left\{ \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -2s \\ s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 2t \\ 0 \\ -4t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$
$$= \left\{ s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Thus, a basis for $\ker(T)$ is the set $B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$ and, consequently, $\dim(\ker(T)) = 2$.

To find the range of the linear transformation we will use the fact that range(T) is the column space of A. From above we see that the matrix A is row-equivalent to the matrix below in row-echelon form .

$$\begin{bmatrix} \mathbf{1} & 0 & 2 & 0 & -1 \\ 0 & \mathbf{1} & -1 & 0 & -2 \\ 0 & 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the $1^{st}, 2^{nd}$, and 4^{th} columns of the REF matrix are pivot columns, the corresponding columns of A form a basis for the column space of A. Thus, one basis for the column space of A is the set $S = \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$. Since range$(T) = C(A)$, we get $\dim(\text{range}(T)) = \text{rank}(A) = 3$.

Motivated by the example above, we present the following definition.

Definition 5.3.5. (Nullity and Rank)

Let $T : V \to W$ be a linear transformation between vector spaces. We define:

- $\text{nullity}(T) = \dim(\ker(T))$.
- $\text{rank}(T) = \dim(\text{range}(T))$.

Theorem 5.3.6.

Let $T : V \to W$ be a linear transformation between vector spaces, and suppose that $\text{dom}(T) = V$ is a finite dimensional vector space. Then,

$$\text{rank}(T) + \text{nullity}(T) = \dim(\text{dom}(T)).$$

Proof. For simplicity, we will prove the theorem when $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. We have seen previously that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by some suitable matrix. So, it is enough to prove the theorem for $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A. Note that $\dim(\mathbb{R}^n) = n =$ number of columns of A. First,

$$\text{rank}(T) = \dim(\text{range}(T)) = \dim(C(A)) = \text{rank}(A).$$

Second, we have the following,

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(N(A)) = \text{nullity}(A).$$

By the rank-nullity theorem for the matrix A we know that $\text{rank}(A) + \text{nullity}(A) =$ number of columns of A which is n. We can conclude that

$$\text{rank}(T) + \text{nullity}(T) = \text{rank}(A) + \text{nullity}(A) = n = \dim(\mathbb{R}^n),$$

which was to be demonstrated. ∎

Example 5.20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be represented by the matrix A below.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x} \in \mathbb{R}^3$. The matrix A is in row-echelon form and it has two nonzero rows, so $\text{rank}(A) = 2$. Thus, $\text{rank}(T) = \text{rank}(A) = 2$. Using the previous theorem, we get that $\text{nullity}(T) = \dim(\mathbb{R}^3) - \text{rank}(T) = 3 - 2 = 1$.

Example 5.21. Let $T : \mathbb{R}^5 \to \mathbb{R}^7$ be some linear transformation.

(a) If $\dim(\text{range}(T)) = 2$, find $\dim(\ker(T))$.
$\dim(\ker(T)) = \text{nullity}(T) = \dim(\mathbb{R}^5) - \text{rank}(T) = 5 - 2 = 3$.

(b) If $\text{nullity}(T) = 4$, find $\text{rank}(T)$.
$\text{rank}(T) = \dim(\mathbb{R}^5) - \text{nullity}(T) = 5 - 4 = 1$.

(c) If $\ker(T) = \{\vec{\mathbf{0}}\}$, find $\text{rank}(T)$.
$\text{rank}(T) = \dim(\mathbb{R}^5) - \text{nullity}(T) = 5 - \dim(\ker(T)) = 5 - 0 = 5$.

Theorem 5.3.7.

Let $T : V \to W$ be a linear transformation between vector spaces V and W. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V. Then we have that

$$\text{range}(T) = \text{Span}\big(T(\vec{v}_1),\ T(\vec{v}_2),\ \ldots,\ T(\vec{v}_n)\big).$$

5.4 Isomorphisms of Vector Spaces

Definition 5.4.1. (Injective and Surjective Functions)

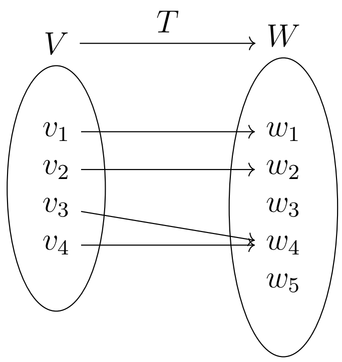
Let $T : V \to W$ be a function from set V to set W.

(i) We say T is *injective* (or *one-to-one*) if and only if every element in the codomain W has at most one preimage in V.

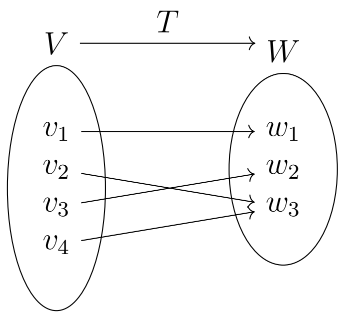
(ii) We say T is *surjective* (or *onto*) if and only if every element in the codomain W has at least one preimage in V.

(iii) We say T is *bijective* if and only if T is both injective and surjective. A bijective function is also called a *bijection*.

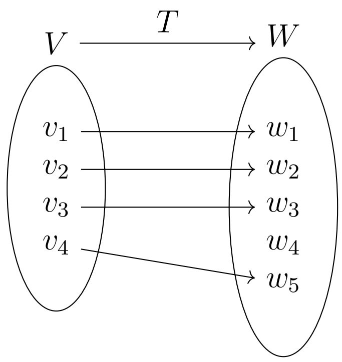
Not injective and not surjective



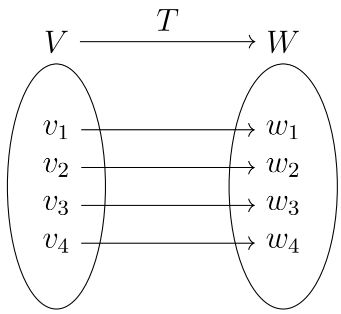
Not injective but surjective



Injective but not surjective



Injective and surjective (bijection)



A function $T : V \to W$ is surjective if and only if $\text{range}(T) = \text{codom}(T)$.

There are other equivalent ways to say that a function $T : V \to W$ is injective. All of the following conditions are equivalent.

- Every $w \in \text{codom}(T)$ has at most one preimage.
- No two distinct elements of the domain V have the same image.

- Every $w \in \text{range}(T)$ has exactly one preimage.
- For all $u, v \in V$, if $T(u) = T(v)$, then $u = v$.
- For all $u, v \in V$, if $u \neq v$, then $T(u) \neq T(v)$.

Example 5.22.

- Suppose V is a nontrivial vector space. The zero transformation $Z : V \to W$, given by $Z(\vec{v}) = \vec{\mathbf{0}}_W$, is not injective because there are two distinct vectors in V which are mapped to the zero vector in W.
- The identity transformation $I : V \to V$ given by $I(\vec{v}) = \vec{v}$ is injective because for any vectors $\vec{u}, \vec{v} \in V$, if $I(\vec{u}) = I(\vec{v})$, then by definition of the transformation I we get $\vec{u} = \vec{v}$, showing that I is injective

Theorem 5.4.2.

Let $T : V \to W$ be a linear transformation between vector spaces. Then, T is injective if and only if $\ker(T) = \{\vec{\mathbf{0}}_V\}$.

Proof. ($\Rightarrow$) Suppose T is a injective linear transformation. We already know that $\vec{\mathbf{0}}_V$ is in $\ker(T)$ because $T(\vec{\mathbf{0}}_V) = \vec{\mathbf{0}}_W$. Now, for the sake of contradiction, suppose that there is a nonzero vector $\vec{v} \in V$ which is in $\ker(T)$. Then, $T(\vec{v}) = \vec{\mathbf{0}}_W$ and so $\vec{\mathbf{0}}_W$ has at least two different preimages namely $\vec{\mathbf{0}}_V$ and $\vec{v}$ which contradicts the fact that T is injective. Thus, $\ker(T) = \{\vec{\mathbf{0}}_V\}$.

($\Leftarrow$) Conversely, suppose that $\ker(T) = \{\vec{\mathbf{0}}_V\}$. Let $\vec{u}, \vec{v}$ be vectors in V and assume that $T(\vec{u}) = T(\vec{v})$. Using the fact that T is a linear transformation, it follows that

$$T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{\mathbf{0}}_W.$$

Thus, the vector $\vec{u} - \vec{v}$ is in $\ker(T)$. But, $\ker(T)$ contains only the zero vector of V. This means that $\vec{u} - \vec{v} = \vec{\mathbf{0}}_V$ and so $\vec{u} = \vec{v}$. So, we showed that whenever $T(\vec{u}) = T(\vec{v})$, then $\vec{u} = \vec{v}$. Therefore, T is injective and this completes the proof. ∎

Lemma 5.4.3.

Let $T : V \to W$ be a linear transformation, where W is a finite dimensional vector space. Then, T is surjective if and only if $\text{rank}(T) = \dim(W)$.

Proof. Let $T : V \to W$ be a linear transformation as stated above.

$$\begin{aligned} T \text{ is surjective } &\Leftrightarrow \text{range}(T) = \text{codom}(T) \\ &\Leftrightarrow \dim(\text{range}(T)) = \dim(\text{codom}(T)) \\ &\Leftrightarrow \text{rank}(T) = \dim(W). \end{aligned}$$

In the backward direction of the second equivalence above we use the fact that $\text{range}(T)$ is a subspace of the codomain of T together with Corollary 4.2.4. ∎

Theorem 5.4.4.

Suppose that V and W are finite dimensional vector spaces where $\dim(V) = \dim(W)$. Let $T : V \to W$ be a linear transformation between them. Then, T is injective if and only if T is surjective.

Proof. Let $\dim(V) = \dim(W) = n$ for some nonnegative integer n.

($\Rightarrow$) Assume that T is injective. It follows that $\ker(T) = \{\vec{\mathbf{0}}_V\}$ and so $\text{nullity}(T) = 0$. We then proceed as follows,

$$\dim(\text{range}(T)) = \text{rank}(T) = \dim(V) - \text{nullity}(T) = n - 0 = n = \dim(W).$$

By Corollary 4.2.4, we get that $\text{range}(T) = \text{codom}(T)$, and so T is surjective.

($\Leftarrow$) Conversely, suppose T is surjective. Then,

$$\text{rank}(T) = \dim(\text{range}(T)) = \dim(W) = n.$$

Thus, $\dim(\ker(T)) = \text{nullity}(T) = n - \text{rank}(T) = n - n = 0$. So, $\ker(T) = \{\vec{\mathbf{0}}_V\}$, and it follows that T is injective. ∎

Example 5.23. Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{x}) = A\vec{x}$ where the matrix A is given below. Note that all the matrices below are in row-echelon form.

(a) $A_{3\times 3} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A_{3\times 2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $A_{2\times 3} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(d) $A_{3\times 3} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

	$T : \mathbb{R}^n \to \mathbb{R}^m$	$\dim(\text{dom}(T))$	$\text{rank}(T)$	$\text{nullity}(T)$	Injective	Surjective
(a)	$T : \mathbb{R}^3 \to \mathbb{R}^3$	3	3	0	Yes	Yes
(b)	$T : \mathbb{R}^2 \to \mathbb{R}^3$	2	2	0	Yes	No
(c)	$T : \mathbb{R}^3 \to \mathbb{R}^2$	3	2	1	No	Yes
(d)	$T : \mathbb{R}^3 \to \mathbb{R}^3$	3	2	1	No	No

In the table above we used the following facts.

- $\text{rank}(T) = \dim(C(A)) =$ number of nonzero rows in A.
- $\text{nullity}(T) = \dim(\text{dom}(T)) - \text{rank}(T)$.
- T is injective if and only if $\text{nullity}(T) = 0$.

- T is surjective if and only if $\dim(\text{range}(T)) = \dim(\text{codom}(T))$.

Sometimes, we can think of different vector spaces as being essentially the same object. For example, think of $\mathbb{R}^3$ and $\mathbb{M}_{3\times 1}$. Think about the vectors in both of these vector spaces and also think about vector addition and scalar multiplication. When two vector spaces are essentially the same they are said to be isomorphic to each other.

Definition 5.4.5. (Vector Space Isomorphism)

An *isomorphism* from vector space V to vector space W is a bijective linear transformation $T : V \to W$. We say that V is *isomorphic* to W if there exists at least one isomorphism from V to W. We write $V \cong W$ when V and W are isomorphic.

The next theorem is an important result which tells us when vector spaces are isomorphic to each other.

Theorem 5.4.6.

Two finite-dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Proof. Let V and W be finite-dimensional vector spaces.

($\Rightarrow$) Suppose that V and W are isomorphic. Then, there exists some isomorphism $T : V \to W$. As T is injective, we get that $\ker(T) = \{\vec{\mathbf{0}}_V\}$ and consequently, $\text{nullity}(T) = \dim(\ker(T)) = \dim(\{\vec{\mathbf{0}}\}) = 0$. Thus, using the rank nullity theorem, we get $\dim(\text{range}(T)) = \text{rank}(T) = \dim(V) - 0 = \dim(V)$. As T is surjective, we know that $\text{range}(T) = \text{codom}(T) = W$, and therefore, $\dim(W) = \dim(\text{range}(T)) = \dim(V)$, which was to be demonstrated.

($\Leftarrow$) Conversely, assume that $\dim(V) = \dim(W) = n$ for some positive integer n. Choose some basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for vector space V, and some basis $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ for vector space W. We will use these two bases to define an isomorphism T from V to W. Pick any vector $\vec{u}$ in V. Since B is a basis for V, there are unique scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + + \cdots + c_n\vec{v}_n$. So the coordinates of $\vec{u}$ relative to B is $[\vec{u}]_B = (c_1, c_2, \ldots, c_n)$. Now, define the image of $\vec{u}$ under the function T to be $T(\vec{u}) = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n$.

$$(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) \overset{T}{\longmapsto} (c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n)$$

Observe that the uniqueness of the scalars $c_1, c_2, \ldots, c_n$ is needed for T to be a function, otherwise, T could map a vector to two or more vectors violating the definition of a function. With this mapping we get a function $T : V \to W$. We now verify that this function $T : V \to W$ is an isomorphism (bijective linear transformation). Towards this aim, choose any vectors $\vec{x}$ and $\vec{y}$ in V. We first express $\vec{x}$ and $\vec{y}$ as a linear combination of the vectors in the basis B, let us say, $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + + \cdots + c_n\vec{v}_n$ and $\vec{y} = d_1\vec{v}_1 + d_2\vec{v}_2 + + \cdots + d_n\vec{v}_n$. Using the vector space axioms we show that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$.

$$\begin{aligned}
T(\vec{x} + \vec{y}) &= T\big((c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n)\big) \\
&= T\big((c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_n + d_n)\vec{v}_n)\big) \\
&= (c_1 + d_1)\,\vec{w}_1 + (c_2 + d_2)\,\vec{w}_2 + \cdots + (c_n + d_n)\,\vec{w}_n \\
&= c_1\vec{w}_1 + d_1\vec{w}_1 + c_2\vec{w}_2 + d_2\vec{w}_2 + \cdots + c_n\vec{w}_n + d_n\vec{w}_n \\
&= (c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n) + (d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n) \\
&= T(c_1\vec{v}_1 + c_2\vec{v}_2 + + \cdots + c_n\vec{v}_n) + T(d_1\vec{v}_1 + d_2\vec{v}_2 + + \cdots + d_n\vec{v}_n) \\
&= T(\vec{x}) + T(\vec{y}).
\end{aligned}$$

Next, choose any scalar $a \in \mathbb{R}$, we aim to show that $T(a\,\vec{x}) = a\,T(\vec{x})$.

$$\begin{aligned}
T(a\,\vec{x}) &= T\big(a(c_1\vec{v}_1 + c_2v_2 + \cdots + c_n\vec{v}_n)\big) = T\big((ac_1)\vec{v}_1 + (ac_2)\vec{v}_2 + \cdots + (ac_n)\vec{v}_n\big) \\
&= (ac_1)\vec{w}_1 + (ac_2)\vec{w}_2 + \cdots + (ac_n)\vec{w}_n = a(c_1\vec{w}_1) + a(c_2\vec{w}_2) + \cdots + a(c_n\vec{w}_n) \\
&= a(c_1\vec{w}_1 + c_2w_2 + \cdots + c_n\vec{w}_n) = aT(c_1\vec{v}_1 + c_2\vec{v}_2 + + \cdots + c_n\vec{v}_n) = a\,T(\vec{x}).
\end{aligned}$$

Next, we show that T which is injective. Towards this goal, assume that $T(\vec{x}) = T(\vec{y})$. We need to show that $\vec{x} = \vec{y}$.

$$\begin{aligned}
T(\vec{x}) = T(\vec{y}) &\implies T\big(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n\big) = T\big(d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n\big) \\
&\implies c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n \\
&\implies (c_1 - d_1)\vec{w}_1 + (c_2 - d_2)\vec{w}_2 + \cdots + (c_n - d_n)\vec{w}_n = \vec{\mathbf{0}}.
\end{aligned}$$

We arrived to a linear combination of the vectors in the basis S equal to the zero vector. As the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent, it follows that $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\ldots$, $c_n - d_n = 0$, and so, $c_1 = d_1$, $c_2 = d_2$, $\ldots$, $c_n = d_n$, implying that $\vec{x} = \vec{y}$ as desired. Therefore, the linear transformation T is injective. Furthermore, since $\dim(V) = \dim(W)$ and T is injective, we must have that T is surjective as well by Theorem 5.4.4. We now know that T is an isomorphism.

Therefore, we found one isomorphism from V to W, and consequently, V is isomorphic to W as desired. ∎

For example, the vector spaces

$$\mathbb{R}^4 \cong \mathbb{M}_{4\times 1} \cong \mathbb{M}_{2\times 2} \cong \mathcal{P}_3$$

are all isomorphic to each other since all of them are of dimension 4. The n-dimensional vector space $\mathbb{R}^n$ is a perfect representative for vector spaces of dimension n. Thus, using the theorem above we get that once we know the dimension of a vector space, say of dimension n, then we know everything about it; it looks like the vector space $\mathbb{R}^n$.

Corollary 5.4.7.

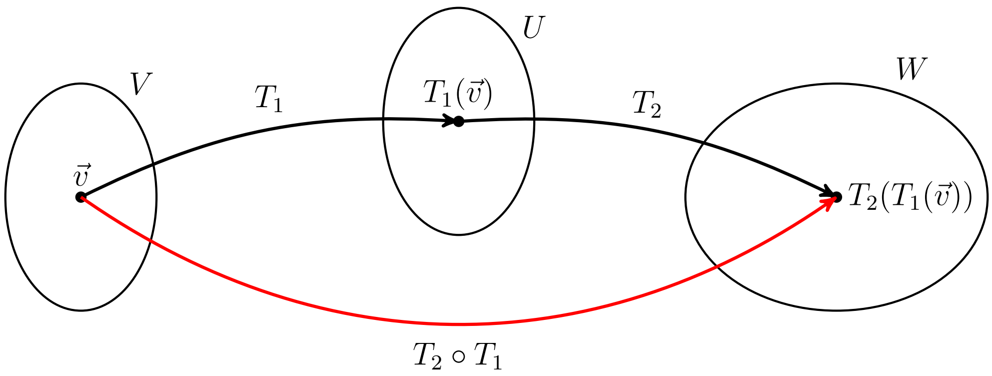
Any vector space over $\mathbb{R}$ of dimension n is isomorphic to $\mathbb{R}^n$.

5.5 Composition of Linear Transformations

Let V, U, W be vector spaces and let $T_1 : V \to U$ and $T_2 = U \to W$ be linear transformations. We may form a new linear transformation, denoted by $T_2 \circ T_1$, from V to W called the *composition* of T_2 after T_1, which is defined as follows

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v}))$$

for all $\vec{v}$ in V.



Let us confirm that the composition $T = T_2 \circ T_1$ is indeed a linear transformation. So we check that T preserves vector addition and scalar multiplication. Let $\vec{u}, \vec{v}$ be two vectors in $V = \mathrm{dom}(T_1)$.

$T(\vec{u}+\vec{v}) = T_2\left(T_1\left(\vec{u}+\vec{v}\right)\right) = T_2\left(T_1(\vec{u})+T_1(\vec{v})\right) = T_2\left(T_1(\vec{u})\right)+T_2\left(T_1(\vec{v})\right) = T(\vec{u})+T(\vec{v}).$

Next, choose any scalar $c \in \mathbb{R}$ and observe that,

$T(c\,\vec{u}) = T_2\left(T_1(c\,\vec{u})\right) = T_2\left(c\,T_1(\vec{u})\right) = c\,T_2\left(T_1(\vec{u})\right) = c\,T(\vec{u}).$

We have demonstrated that the composition of linear transformations is a linear transformation.

Lemma 5.5.1.

Consider linear transformations $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ given by $T_1(\vec{x}) = A_1\vec{x}$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ given by $T_2(\vec{x}) = A_2\vec{x}$. Their composition $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the matrix A_2A_1, that is, $(T_2 \circ T_1)(\vec{x}) = (A_2A_1)\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$.

Proof. Let $T = T_2 \circ T_1$ and let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then, the image of the vector $\vec{v}$ under the composition T is

$$T(\vec{v}) = (T_2 \circ T_1)(\vec{v}) = T_2\left(T_1(\vec{v})\right) = T_2(A_1\vec{v}) = A_2(A_1\vec{v}) = (A_2A_1)\vec{v}.$$

In the last equality we used the fact that matrix multiplication is associative. ■

Example 5.24. Consider the linear transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ given by the rule $T_1(x, y, z) = (2x + y, 0, x + z)$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2(x, y, z) = (x - y, z, y)$. Find the standard matrices for their compositions $T_2 \circ T_1$ and $T_1 \circ T_2$.

We first compute the standard matrices A_1 and A_2 for T_1 and T_2, respectively.

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The standard matrix for $T_2 \circ T_1$ is $A_2A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The standard matrix for $T_1 \circ T_2$ is $A_1A_2 = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Definition 5.5.2.

Let $T : V \to V$ be a linear transformation. The *inverse* of T, if it exists, is a linear transformation $F : V \to V$ such that $(F \circ T)(\vec{v}) = \vec{v}$ and $(T \circ F)(\vec{v}) = \vec{v}$ for all $\vec{v}$ in V. If the inverse of T exists, we say that T is *invertible* and denote the inverse by $T^{-1} : V \to V$.

Theorem 5.5.3.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$ for some $n \times n$ matrix A. Then the following are equivalent:

(i) T is invertible.

(ii) T is an isomorphism.

(iii) A is invertible.

Lemma 5.5.4.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$ is an invertible linear transformation. Then the standard matrix representing the inverse T^{-1} is A^{-1}, and thus, $T^{-1}(\vec{x}) = A^{-1}\,\vec{x}$.

Example 5.25. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (2x + 3y + z, 3x + 3y + z, 2x + 4y + z).$$

Find the inverse $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$, if it exists.

We know that the matrix representing T has columns $T(\vec{e_1})$, $T(\vec{e_2})$, and $T(\vec{e_3})$. Thus, we can see that

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Note that A is invertible since $\det(A) \neq 0$. Then the matrix representing T^{-1} is A^{-1} and so

$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Chapter 6

Eigenvalues and Eigenvectors

6.1 Eigenvalues and Eigenvectors

Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. We imagine the action of T on an arbitrary vector $\vec{x}$ as moving $\vec{x}$ to another vector $T(\vec{x})$ in the plane. Remember that $\mathrm{Span}(\vec{x})$ is the straight line passing through the origin and through the vector $\vec{x}$. One would expect seeing the transformation T moving vectors in the plane pushing them out their own span. However, imagine that there is a special vector $\vec{x}$ which T moves within its own span, what a coincidence! In other words, $\vec{x}$ is a vector for which there is a scalar λ such that $T(\vec{x}) = \lambda\,\vec{x}$. Geometrically, this says that the effect of T on the vector $\vec{x}$ is stretching it or squeezing it by a scalar λ. Such a vector is very special for the linear transformation T and it is called an *eigenvector* of T and its partner scalar λ is called an *eigenvalue* of T. Clearly, as T preserves scalar multiplication, if $\vec{x}$ is an eigenvector of T with eigenvalue λ, then any vector $\vec{v}$ in the span of $\vec{x}$ is also an eigenvector for T that is scaled by the same factor λ. To see this, pick any vector $\vec{v} \in \mathrm{Span}(\vec{x})$, and so $\vec{v} = c\vec{x}$ for some scalar c. To see that $\vec{v}$ is also an eigenvector we check that $T(\vec{v}) = \lambda\vec{v}$ as shown below.

$$T(\vec{v}) = T(c\vec{x}) = c\,T(\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x}) = \lambda\vec{v}.$$

We also require eigenvectors to be nonzero vectors. To put it in a nutshell, an eigenvector of a linear transformation $T : V \to V$ is a nonzero vector which stays in its own span after the action of T. That is, a nonzero vector $\vec{x}$ is an eigenvector of T if and only if $T(\vec{x}) \in \mathrm{Span}(\vec{x})$.

For an example in the 3D space, suppose that we have a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates vectors in the 3D space by an angle θ about a straight line passing through the origin (we call this line the axis of rotation). Then any nonzero vector on the axis of rotation is an eigenvector because such a vector $\vec{x}$ stays in its own span under the rotating action of T. Furthermore, eigenvectors of a rotation transformation are neither stretched nor squeezed, their length stay unchanged, and so the corresponding eigenvalues are 1 and -1.

In the case that the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is represented by a square matrix A, that is, $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$, then an eigenvector of T is a nonzero vector $\vec{x}$ which satisfies $A\vec{x} = \lambda\vec{x}$ for some scalar λ. We say in such a case that $\vec{x}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ. Our aim in this section is to search for eigenvectors of square matrices and their corresponding eigenvalues. It is the time now to define eigenvalues and their eigenvectors for a square matrix (or for the linear transformation represented by that matrix).

Definition 6.1.1. (Eigenvectors and Eigenvalues)

A nonzero vector $\vec{x} \in \mathbb{R}^n$ is called an *eigenvector* of a square matrix A of size $n \times n$ if and only if there exists a real number $\lambda \in \mathbb{R}$ such that

$$A\,\vec{x} = \lambda\,\vec{x}.$$

The number λ is called an *eigenvalue* of A, and we say in such situation that $\vec{x}$ is an eigenvector of A corresponding to the eigenvalue λ.

Example 6.1. (a) One can check that

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$. Observe that by multiplying both sides of the matrix equation above by 4, say, we get that $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is also an eigenvector corresponding to the eigenvalue 5 as well. One can see that all nonzero vectors in $\text{Span}(\vec{x})$ are eigenvectors corresponding to eigenvalue 5.

(b) Here is another example.

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Therefore, $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue $\lambda = -1$. Also, the column matrix $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ is also an eigenvector for the same eigenvalue.

If $\vec{x}$ is an eigenvector of a square matrix A corresponding to eigenvalue λ, then $\vec{y} = c\vec{x}$ for any nonzero $c \in \mathbb{R}$ is also an eigenvector of A corresponding to λ. To show this, we need to show that $A\vec{y} = \lambda\vec{y}$.

$$A\vec{y} = A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x}) = \lambda\vec{y}.$$

Therefore, $A\vec{y} = \lambda\vec{y}$, and so $\vec{y}$ is an eigenvector of A corresponding to eigenvalue λ.

How can we find eigenvalues and eigenvectors of a square matrix $A_{n\times n}$? We need to search for a nonzero $n \times 1$ column matrix and a real number $\lambda \in \mathbb{R}$ such that $A\,\vec{x} = \lambda\,\vec{x}$. We proceed as follows.

$$\begin{aligned} A\,\vec{x} = \lambda\,\vec{x} &\iff \lambda\vec{x} - A\,\vec{x} = \vec{\mathbf{0}} \\ &\iff \lambda(I_n\,\vec{x}) - A\,\vec{x} = \vec{\mathbf{0}} \\ &\iff (\lambda I_n)\,\vec{x} - A\,\vec{x} = \vec{\mathbf{0}} \\ &\iff (\lambda I_n - A)\,\vec{x} = \vec{\mathbf{0}}. \end{aligned}$$

Observe that $(\lambda I_n - A)$ is an $n \times n$ matrix, and $\vec{x}$ is an $n \times 1$ matrix. For example, when $n = 3$, rewriting the equation $(\lambda I_n - A)\,\vec{x} = \vec{\mathbf{0}}$ in details produces the following matrix equation.

$$\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In general, we can see that the matrix equation $(\lambda I_n - A)\,\vec{x} = \vec{\mathbf{0}}$ represents a homogeneous system of n linear equations in n variables with coefficient matrix $\lambda I_n - A$. The variables of the system correspond to the components of the eigenvectors, and as eigenvectors are nonzero vectors we are interested in nontrivial solutions of this homogeneous system. We know that such a homogeneous system has nontrivial solutions if and only if its coefficient matrix is noninvertible. Therefore, the quest for eigenvectors boils down to finding which values of λ make the coefficient matrix $\lambda I_n - A$ noninvertible?

We know that the matrix $\lambda I - A$ is noninvertible if and only if $\det(\lambda I - A) = 0$. Thus, we need to find the values of λ which make the determinant $\det(\lambda I - A)$ equal to 0. The determinant $\det(\lambda I - A)$ is a function in λ, actually, it is a polynomial in λ of degree n called the *characteristic polynomial* of the matrix A. The polynomial equation $\det(\lambda I - A) = 0$ is called the *characteristic equation* of the matrix A. As our goal is to find the values of λ for which $\det(\lambda I - A) = 0$, we may say instead that our aim is to find the roots of the characteristic polynomial of A. These roots are exactly the eigenvalues of the matrix A.

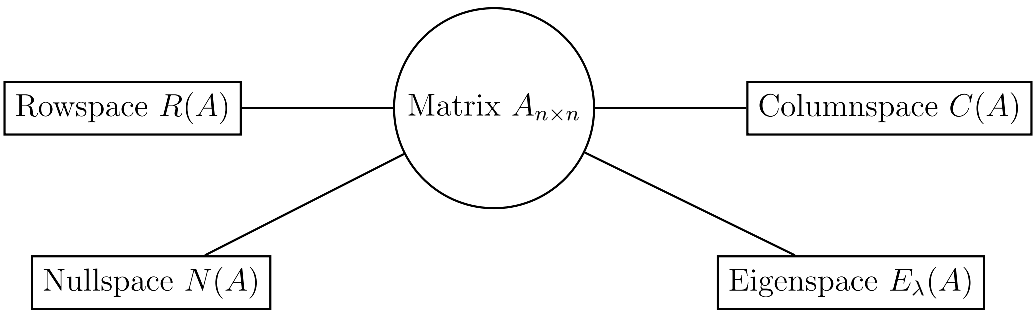
To find the eigenvectors corresponding to a particular eigenvalue λ we need to solve the homogeneous system $(\lambda I_n - A)\,\vec{x} = \vec{\mathbf{0}}$, that is, we need to find the nullspace $N(\lambda I_n - A)$. All solutions of this system except the trivial solution are eigenvectors corresponding to λ.

Definition 6.1.2. (Characteristic Polynomial of a Matrix)

Let A be a square matrix of size $n \times n$.

- The *characteristic polynomial* of A is $\chi_A(\lambda) = \det(\lambda I - A)$.
- For an eigenvalue λ of A we define the *eigenspace* $E_\lambda(A)$ corresponding to λ is the solution set of the homogeneous system $(\lambda I - A)\,\vec{x} = \vec{\mathbf{0}}$. Equivalently, $E_\lambda(A)$ is the nullspace of the matrix $\lambda I - A$.
- The *geometric multiplicity* of an eigenvalue λ of A is the dimension of its eigenspace.

Clearly, the eigenvectors corresponding to eigenvalue λ are the vectors in $E_\lambda(A)$ except the zero vector. Also, if a vector $\vec{u} \in E_\lambda(A)$, then $A\vec{u} = \lambda\vec{u}$. Furthermore, the eigenspace $E_\lambda(A)$, being the nullspace of some matrix, is clearly a subspace of $\mathbb{R}^n$. At this point, the vector spaces we associate to a square matrix $A_{n\times n}$ are its rowspace, columnspace, nullspace, and all of its eigenspaces, all of them are subspaces of the n-dimensional space $\mathbb{R}^n$.



We sum up our discussion in the following points.

- Choose a square matrix A.
- Find its characteristic polynomial,

$$\chi_A(\lambda) = \det(\lambda I_n - A).$$

- The eigenvalues of A are the roots of its characteristic polynomial.
- The eigenspace of $E_A(\lambda)$ of an eigenvalue λ is the solution set of the homogeneous system $(\lambda I - A)\,\vec{x} = \vec{\mathbf{0}}$, thus, $E_A(\lambda) = N(\lambda I - A)$.
- The eigenvectors corresponding to eigenvalue λ are all the vectors in the eigenspace $E_A(\lambda)$ except the zero vector.
- The geometric multiplicity of eigenvalue λ is $\dim(E_\lambda(A))$.

Exercise. Let A be a square matrix of size $n \times n$. Show that the characteristic polynomial $\chi_A(\lambda)$ is always a monic polynomial; meaning that the nonzero coefficient of the highest power of λ is equal to 1. Therefore,

$$\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$.

Example 6.2. Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

- We first find the characteristic polynomial of A:

$$\begin{aligned}
\chi_A(\lambda) = \det(\lambda I - A) &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\
&= (\lambda - 1)(\lambda - 3) - 8 \\
&= \lambda^2 - 3\lambda - \lambda + 3 - 8 \\
&= \lambda^2 - 4\lambda - 5 \\
&= (\lambda - 5)(\lambda + 1).
\end{aligned}$$

The roots of the characteristic polynomial

$$\chi_A(\lambda) = \lambda^2 - 4\lambda - 5$$

are $\lambda_1 = 5$ and $\lambda_2 = -1$. These are the two eigenvalues of A.

- Next, we find the eigenspace corresponding to $\lambda_1 = 5$ by solving the homogeneous system $(5I - A)\vec{x} = \vec{\mathbf{0}}$ by Gaussian elimination.

$$5I - A = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The homogeneous system corresponding to the REF matrix is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, a general solution is $x = t$ and $y = t$, where $t \in \mathbb{R}$. Thus, the eigenspace of A corresponding to 5 is

$$E_5(A) = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

So the eigenvectors of $\lambda_1 = 5$ are vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ where $t \in \mathbb{R}$ and $t \neq 0$.

• Finally, the eigenspace corresponding to eigenvalue $\lambda_2 = -1$ are found by solving the system $(-I - A)\vec{x} = \vec{\mathbf{0}}$.

$$-I - A = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors of $\lambda_2 = -1$ are the nonzero solutions to the system

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A general solution of the system is given by $x = -2t$ and $y = t$ for any $t \in \mathbb{R}$. Therefore, the eigenspace of A corresponding to eigenvalue -1 is

$$E_{-1}(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right).$$

Consequently, the eigenvectors corresponding to eigenvalue -1 have the form $\begin{bmatrix} -2t \\ t \end{bmatrix}$ where $t \in \mathbb{R}$ and $t \neq 0$. For example, check that

$$A \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{ and } A \begin{bmatrix} 4 \\ -2 \end{bmatrix} = - \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

That is, the vector $(3, 3)$ is an eigenvector for eigenvalue 5 and the vector $(4, -2)$ is an eigenvector for eigenvalue -1, both of them stay in their own span after the action of the linear transformation $T(\vec{x}) = A\vec{x}$. Both eigenvalues of A have geometric multiplicity 1 as the dimension of each of their corresponding eigenspaces is 1.

Example 6.3. Find the eigenvalues and corresponding eigenvectors of the matrix below

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

• First, we find the characteristic polynomial $\chi_A(\lambda)$ of the matrix A.

$$\begin{aligned} \chi_A(\lambda) = \det(\lambda I - A) &= \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 4 & 0 & -2 \\ 0 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 4 \end{vmatrix} \\ &= (\lambda - 4)^3. \end{aligned}$$

The characteristic polynomial

$$\chi_A(\lambda) = \lambda^3 - 12\lambda^2 + 48\lambda - 64$$

has one root $\lambda = 4$ with algebraic multiplicity 3. Thus, we have one eigenvalue $\lambda = 4$ for matrix A.

- Eigenspace for $\lambda = 4$. We need to solve the system $(\lambda I - A)\,\vec{x} = \vec{\mathbf{0}}$.

$$\lambda I - A = I - A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace corresponding to eigenvalue 4 is the solution set of this homogeneous system.

$$E_4(A) = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

Therefore, the geometric multiplicity of eigenvalue 4 is $\dim(E_4(A)) = 2$.

Example 6.4. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

- First, we find the characteristic polynomial $\chi_A(\lambda)$ of the matrix A.

$$\begin{aligned} \chi_A(\lambda) = \det(\lambda I - A) &= \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -1 & \lambda - 2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & -1 \\ 1 & \lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & \lambda - 2 \\ 1 & 1 \end{vmatrix} \\ &= (\lambda - 1)(\lambda^2 - 2\lambda + 1) + 2(-\lambda + 1) + 2(-1 - \lambda + 2) \\ &= \lambda^3 - 3\lambda^2 - \lambda + 3 \\ &= (\lambda^2 - 1)(\lambda - 3) = (\lambda - 1)(\lambda + 1)(\lambda - 3). \end{aligned}$$

The roots of the characteristic polynomial

$$\chi_A(\lambda) = \lambda^3 - 3\lambda^2 - \lambda + 3$$

are the eigenvalues of A. We have three eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 3$.

- Eigenspace for $\lambda_1 = 1$. We need to solve the system $(\lambda_1 I - A)\,\vec{x} = \vec{\mathbf{0}}$.

$$\lambda_1 I - A = I - A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xRightarrow{EROs} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace corresponding to eigenvalue 1 is the solution set of this homogeneous system.

$$E_1(A) = \left\{ \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right).$$

- Eigenspace for $\lambda_2 = -1$.

$$\lambda_2 I - A = -I - A = \begin{bmatrix} -2 & -2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace corresponding to eigenvalue -1 is

$$E_{-1}(A) = \left\{ \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right).$$

- Eigenspace for $\lambda_3 = 3$.

$$\lambda_3 I - A = 3I - A = \begin{bmatrix} 2 & -2 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{EROs} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace corresponding to eigenvalue 3 is

$$E_3(A) = \left\{ \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right).$$

For example, the vector $(-6, -3, 3)$ is an eigenvector of A corresponding to eigenvalue 3, in other words,

$$A \begin{bmatrix} -6 \\ -3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -6 \\ -3 \\ 3 \end{bmatrix}.$$

Each eigenvalue in this example has geometric multiplicity 1 as the dimension of each of their corresponding eigenspaces is 1.

Lemma 6.1.3.

The number 0 is an eigenvalue of a square matrix A if and only if A is noninvertible.

Proof. Let A be a square matrix of size $n \times n$. Let $\chi_A(\lambda)$ be its characteristic polynomial.

$$\begin{aligned}
\text{The number 0 is an eigenvalue of } A \quad & \text{if and only if} \quad 0 \text{ is a root of } \chi_A(\lambda) \\
& \text{if and only if} \quad \chi_A(0) = 0 \\
& \text{if and only if} \quad \det(0I - A) = 0 \\
& \text{if and only if} \quad \det(-A) = 0 \\
& \text{if and only if} \quad \det(A) = 0 \\
& \text{if and only if} \quad A \text{ is noninvertible.}
\end{aligned}$$

This completes the proof. ■

Lemma 6.1.4.

Let A be a square matrix. Let λ_1 and λ_2 be two distinct eigenvalues of A. Then the intersection of their eigenspaces contains only the zero vector, in symbols, $E_{\lambda_1}(A) \cap E_{\lambda_2}(A) = \{\vec{\mathbf{0}}\}$.

Proof. Let λ_1 and λ_2 be two distinct eigenvalues of A. Clearly, $\vec{\mathbf{0}} \in E_{\lambda_1}(A) \cap E_{\lambda_2}(A)$ as both sets are subspaces. Next, choose any vector $\vec{x}$ in $E_{\lambda_1}(A) \cap E_{\lambda_2}(A)$. Thus, $\vec{x} \in E_{\lambda_1}(A)$ and $\vec{x} \in E_{\lambda_2}(A)$. This yields that $A\vec{x} = \lambda_1\vec{x}$ and $A\vec{x} = \lambda_2\vec{x}$. So, $\lambda_1\vec{x} = \lambda_2\vec{x}$, and so $(\lambda_1 - \lambda_2)\,\vec{x} = \vec{\mathbf{0}}$. This implies $\lambda_1 - \lambda_2 = 0$ or $\vec{x} = \vec{\mathbf{0}}$. Since the first one is not possible as λ_1 and λ_2 are distinct values, it follows that $\vec{x} = \vec{\mathbf{0}}$. We have shown that if $\vec{x} \in E_{\lambda_1}(A) \cap E_{\lambda_2}(A)$, then $\vec{x} = \vec{\mathbf{0}}$, meaning that the only vector in $E_{\lambda_1}(A) \cap E_{\lambda_2}(A)$ is the zero vector. ■

Definition 6.1.5. (Similar Matrices)

A matrix A is said to be *similar* to matrix B if and only if there exists an invertible matrix M such that $B = M^{-1}AM$.

Lemma 6.1.6.

If matrices A and B are similar, then they have the same eigenvalues.

Proof. Suppose that A and B are similar matrices. Then $B = M^{-1}AM$ for some invertible matrix M. Choose any $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}
\lambda I - A &= \lambda I - MBM^{-1} \\
&= \lambda(MIM^{-1}) - MBM^{-1} \\
&= M(\lambda I)M^{-1} - MBM^{-1} \\
&= M(\lambda I - B)M^{-1}.
\end{aligned}$$

Next, we proceed to show that A and B have the same eigenvalues using the fact the the determinant function is multiplicative.

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A &\iff \det(\lambda I - A) = 0 \\
&\iff \det(M(\lambda I - B)M^{-1}) = 0 \\
&\iff \det(M)\det(\lambda I - B)\det(M^{-1}) = 0 \\
&\iff \det(M)\det(M^{-1})\det(\lambda I - B) = 0 \\
&\iff \det(M)\frac{1}{\det(M)}\det(\lambda I - B) = 0 \\
&\iff \det(\lambda I - B) = 0 \\
&\iff \lambda \text{ is an eigenvalue of } B.
\end{aligned}$$

Therefore, similar matrices have the same eigenvalues. ■

6.2 Cayley-Hamilton Theorem

Let $p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients. For a square matrix $A_{n\times n}$, we mean by $p(A)$ the matrix obtained by substituting A for the variable x in the polynomial $p(x)$. More precisely,

$$p(A) = a_n A^n + \cdots + a_2 A^2 + a_1 A + a_0 I_n.$$

Theorem 6.2.1. (Cayley-Hamilton Theorem)

Any square matrix A satisfies its own characteristic equation. More precisely, let $\chi_A(\lambda)$ be the characteristic polynomial of a matrix A. Then, $\chi_A(A) = \mathbf{0}_{n\times n}$.

The next examples demonstrates how we can use the Cayley-Hamilton Theorem to compute the inverse A^{-1} and powers A^k of a square matrix A.

Example 6.5. We previously found the characteristic polynomial of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ to be

$$\chi_A(\lambda) = \lambda^2 - 4\lambda - 5$$

We now verify Cayley-Hamilton Theorem for A.

$$\begin{aligned} \chi_A(A) &= A^2 - 4A - 5I_2 \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}^2 - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}_{2\times 2}. \end{aligned}$$

This shows that A satisfies its own characteristic equation, that is,

$$A^2 - 4A - 5I_2 = \mathbf{0}_{2\times 2}.$$

We now use this fact to find the inverse A^{-1}.

$$\begin{aligned} A^2 - 4A - 5I_2 = \mathbf{0}_{2\times 2} &\iff A^2 - 4A = 5I_2 \\ &\iff A(A - 4I_2) = 5I_2 \\ &\iff A\left(\frac{1}{5}(A - 4I_2)\right) = I_2. \end{aligned}$$

This implies that $A^{-1} = \frac{1}{5}(A - 4I_2)$.

$$A^{-1} = \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}.$$

We now proceed to use $A^2 - 4A - 5I_2 = \mathbf{0}_{2\times 2}$ to compute powers of A.

- $A^2 = 4A + 5I$.
- $A^3 = AA^2 = A(4A + 5I) = 4A^2 + 5A = 4(4A + 5I) + 5A = 21A + 20I$.
- $A^4 = AA^3 = A(21A + 20I) = 21A^2 + 20A = 21(4A + 5I) + 20A = 104A + 105I$.

In general, for each $k \geq 2$ we can express $A^k = cA + dI$ for some scalars $c, d \in \mathbb{R}$.

Example 6.6. The characteristic polynomial of matrix A below was previously found to be $\chi_A(\lambda) = \lambda^3 - 3\lambda^2 - \lambda + 3$.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

By Cayley-Hamilton Theorem we know that $A^3 - 3A^2 - A + 3I_3 = \mathbf{0}_{3\times 3}$. We use this to find the inverse A^{-1}.

$$\begin{aligned} A^3 - 3A^2 - A + 3I = \mathbf{0} &\implies A^3 - 3A^2 - A = -3I \\ &\implies A(A^2 - 3A - I) = -3I \\ &\implies A\left(\frac{-1}{3}(A^2 - 3A - I)\right) = I. \end{aligned}$$

Therefore, $A^{-1} = \frac{-1}{3}(A^2 - 3A - I_3)$.

We now proceed to use $A^3 - 3A^2 - A + 3I_3 = \mathbf{0}_{3\times 3}$ to compute powers of A.

- $A^3 = 3A^2 + A - 3I$.
- $A^4 = AA^3 = A(3A^2 + A - 3I) = 3(3A^2 + A - 3I) + A^2 - 3A = 10A^2 - 9I$.
- $A^5 = AA^4 = A(10A^2 - 9I) = 10A^3 - 9A = 10(3A^2 + A - 3I) - 9A = 30A^2 + A - 30I$.

In general, for each $k \geq 3$ we can express $A^k = aA^2 + bA + cI$ for some scalars $a, b, c \in \mathbb{R}$.

Chapter 7

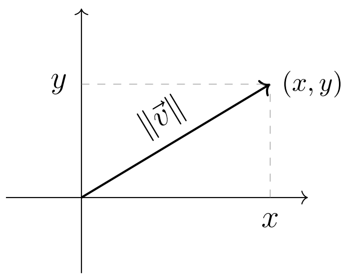
Inner Product Spaces

Inner product spaces are vector spaces enriched with a new operation, called an *inner product*, which takes two vectors and produces a scalar (e.g. a real number or a complex number). The inner product of vectors $\vec{u}$ and $\vec{v}$ is usually denoted by $\langle \vec{u}, \vec{v} \rangle$. Of course, we expect the inner product to behave nicely with the vector space operations: vector addition and scalar multiplication. With an inner product we can talk about geometric notions such as length of a vector, distance between vectors, angle between vectors, and orthogonality of vectors. A special kind of inner product spaces are the Hilbert spaces which are widely used in the area of functional analysis.

The very first example of an inner product is the dot product operation defined on the n-dimensional space $\mathbb{R}^n$. Inner product spaces obtained from $\mathbb{R}^n$ together with the dot product are called Euclidean vector spaces.

7.1 Euclidean Spaces $\mathbb{R}^n$

Using the Pythagorean theorem we can find the lengths of vectors in the cartesian plane. The *length* of a vector $\vec{v} = (x, y)$ in $\mathbb{R}^2$ is given by $\|\vec{v}\| = \sqrt{x^2 + y^2}$.



Definition 7.1.1. (Length of a Vector in $\mathbb{R}^n$)

Let $\vec{v} = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{R}^n$. The *length* (or *norm*) of vector $\vec{v}$ is defined to be

$$\|\vec{v}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$$

It follows directly from the definition of the norm that for any $\vec{v} \in \mathbb{R}^n$ we have:

- $\|\vec{v}\| \in \mathbb{R}$ and $\|\vec{v}\| \geq 0$.
- $\|\vec{v}\| = 0$ if and only if $\vec{v}$ is the zero vector $\vec{\mathbf{0}}$.

Example 7.1. Let $\vec{v} = (0, -2, 1, 4, -2)$ be a vector in $\mathbb{R}^n$. Then its length is

$$\|\vec{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5.$$

Note that each vector in the standard basis of $\mathbb{R}^n$ has length 1, and a member of the standard basis is called a *standard unit vector*. In physics, the standard unit vectors of $\mathbb{R}^3$ are denoted by:

$$\hat{\imath} = (1, 0, 0), \qquad \hat{\jmath} = (0, 1, 0), \qquad \hat{k} = (0, 0, 1).$$

We say that two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are *parallel* if one of them is a scalar multiple of the other, that is, $\vec{u} = c\,\vec{v}$ for some $c \in \mathbb{R}$.

- If $c > 0$, then we say that $\vec{u}$ and $\vec{v}$ have the same direction.
- If $c < 0$, then we say that $\vec{u}$ and $\vec{v}$ have opposite directions.

Theorem 7.1.2.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and $c \in \mathbb{R}$ be a scalar. Then,

$$\|c\,\vec{v}\| = |c| \cdot \|\vec{v}\|.$$

Proof. Let $\vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Thus, $\vec{v} = (x_1, x_2, \ldots, x_n)$ for some real numbers x_i.

$$\begin{aligned} \|c\,\vec{v}\| &= \|c(x_1, x_2, \ldots, x_n)\| = \|(cx_1, cx_2, \ldots, cx_n)\| \\ &= \sqrt{(cx_1)^2 + (cx_2)^2 + \cdots + (cx_n)^2} = \sqrt{c^2(x_1^2 + x_2^2 + \cdots + x_n^2)} \\ &= \sqrt{c^2} \cdot \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |c| \cdot \|\vec{v}\|. \end{aligned}$$

∎

Let $\vec{v}$ be a nonzero vector in $\mathbb{R}^n$. We define the *unit vector* in the direction of $\vec{v}$ to be the vector

$$\frac{1}{\|\vec{v}\|}\vec{v}.$$

Note that both vectors $\vec{v}$ and $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ have same direction since $\dfrac{1}{\|\vec{v}\|} > 0$. Moreover, the length of $\vec{u}$ is

$$\|\vec{u}\| = \left|\frac{1}{\|\vec{v}\|}\right| \|\vec{v}\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1.$$

Example 7.2. Find the unit vector in the direction of $\vec{v} = (3, -1, 2)$.

$$\|\vec{v}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

So, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{14}}(3, -1, 2) = \left(\dfrac{3}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}\right)$. The length of $\vec{u}$ is indeed 1.

$$\|\vec{u}\| = \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{-1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{9}{14} + \frac{1}{14} + \frac{4}{14}} = \sqrt{\frac{14}{14}} = 1.$$

Using the length function we can define the distance between two vectors. We define the distance between vectors $\vec{u}$ and $\vec{v}$ to be the length of the vector $\vec{u} + (-\vec{v})$.

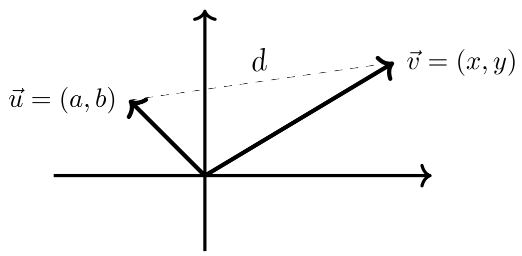
Definition 7.1.3. (Distance between Vectors in $\mathbb{R}^n$)

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. The *distance* between $\vec{u}$ and $\vec{v}$ is defined to be

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

For vectors $\vec{v} = (x, y)$ and $\vec{u} = (a, b)$ in the cartesian plane $\mathbb{R}^2$, the distance between them is given by

$$d(\vec{v}, \vec{u}) = \|\vec{v} - \vec{u}\| = \sqrt{(x - a)^2 + (y - b)^2}.$$



Example 7.3. Let $\vec{u} = (0, 2, 2)$ and $\vec{v} = (2, 0, 1)$ be vectors in $\mathbb{R}^3$. The distance between them is

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|(0, 2, 2) - (2, 0, 1)\| = \|(-2, 2, 1)\| = \sqrt{4 + 4 + 1} = \sqrt{9} = 3.$$

The following result is a direct consequence of the definition of distance.

Lemma 7.1.4.

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$.

- $d(\vec{u}, \vec{v}) \in \mathbb{R}$ *and* $d(\vec{u}, \vec{v}) \geq 0$.
- $d(\vec{u}, \vec{v}) = 0 \iff \vec{u} = \vec{v}$.
- $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.

Definition 7.1.5. (Dot Product)

We define a function called the *dot product* from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ as follows. For any vectors $\vec{u} = (x_1, x_2, \ldots, x_n)$ and $\vec{v} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, their dot product is given by

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Example 7.4. Given $\vec{u} = (1, 2, 0, -3)$ and $\vec{v} = (3, -2, 4, 2)$ in $\mathbb{R}^4$, their dot product is

$\vec{u} \cdot \vec{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7.$

Remark. If we think of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ as column matrices, their dot product can be represented using matrix multiplication as $\vec{u} \cdot \vec{v} = (\vec{u})^T \vec{v}$.

The next result follows directly from the definition of the dot product in $\mathbb{R}^n$. The second point tells us that we can obtain the length of a vector using the dot product.

Theorem 7.1.6.

Let $\vec{u}, \vec{v}$ be any vectors in $\mathbb{R}^n$. Then, the following properties hold.

(i) The dot product is commutative:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}.$$

(ii) The dot product of a vector with itself is the square of its length:

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

The next theorem describes how the dot product interacts with vector addition and scalar multiplication in $\mathbb{R}^n$.

Theorem 7.1.7.

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$ be a scalar. Then, the following properties are true.

(i) The dot product distributes over vector addition:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

(ii) The dot product is compatible with scalar multiplication:

$$c\,(\vec{u} \cdot \vec{v}) = (c\,\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\,\vec{v}).$$

Example 7.5. Consider the vectors $\vec{u} = (2, -2)$, $\vec{v} = (5, 8)$, $\vec{t} = (6, 8)$, and $\vec{w} = (-4, 3)$ in the cartesian plane $\mathbb{R}^2$.

(a) $\vec{u} \cdot \vec{v} = (2)(5) + (-2)(8) = -6$,

(b) $\vec{v} \cdot \vec{w} = (5)(-4) + (8)(3) = 4$,

(c) $\vec{t} \cdot \vec{w} = (6)(-4) + (8)(3) = 0$,

(d) $(\vec{u} \cdot \vec{v})\vec{w} = -6\vec{w} = (24, -18)$,

(e) $\vec{u} \cdot (2\vec{v}) = 2(\vec{u} \cdot \vec{v}) = 2(-6) = -12$,

(f) $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} = 4^2 + 3^2 = 25$, and so $\|\vec{w}\| = 5$.

(g) $\vec{u} \cdot (\vec{v} - 2\vec{w}) = \vec{u} \cdot \vec{v} - 2(\vec{u} \cdot \vec{w}) = -6 - 2(-8 - 6) = 22$.

(h) $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| = \|(9, 5)\| = \sqrt{9^2 + 5^2} \approx 10.295$.

Lemma 7.1.8.

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.$$

Proof. Let $\vec{u}$ and $\vec{v}$ be any vectors in $\mathbb{R}^n$. We proceed by using the definition of the dot product.

Let $\vec{u} = (x_1, x_2, \ldots, x_n)$ and $\vec{v} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$.

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= \|(x_1 + y_1, \ldots, x_n + y_n)\|^2 = (x_1 + y_1)^2 + \cdots + (x_n + y_n)^2 \\
&= \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} (x_i^2 + 2x_i y_i + y_i^2) \\
&= \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 = \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.
\end{aligned}$$

This completes the proof. ■

7.2 Inner Product Spaces

We will equip a vector space with an operation called an *"inner product"*. The dot product defined on the Euclidean space $\mathbb{R}^n$ in the previous section is one example of an inner product on a vector space. What axioms should an inner product satisfy to generalise the dot product in arbitrary vector spaces?

Definition 7.2.1. (Inner Product)

Let V be a vector space. An *inner product* on V is an operation $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ that associates to each pair of vectors $\vec{u}$ and $\vec{v}$ in V one real number denoted by $\langle\vec{u},\vec{v}\rangle$. Moreover, such operation must also satisfy the following axioms.

(1) The inner product is commutative.
For every $\vec{u}$, $\vec{v}$ in V, we have $\langle\vec{u},\vec{v}\rangle = \langle\vec{v},\vec{u}\rangle$.

(2) The inner product distributes over vector addition.
For every $\vec{u}$, $\vec{v}$, $\vec{w}$ in V, we have $\langle\vec{u},\ \vec{v}+\vec{w}\rangle = \langle\vec{u},\vec{v}\rangle + \langle\vec{u},\vec{w}\rangle$.

(3) The inner product is compatible with scalar multiplication.
For every $\vec{u}$, $\vec{v}$ in V, and any $c \in \mathbb{R}$, we have $c\,\langle\vec{u},\vec{v}\rangle = \langle c\,\vec{u},\vec{v}\rangle$.

(4) Positive-Definiteness.
For any $\vec{v}$ in V, if $\vec{v} \neq \vec{\mathbf{0}}$, then $\langle\vec{v},\vec{v}\rangle > 0$.

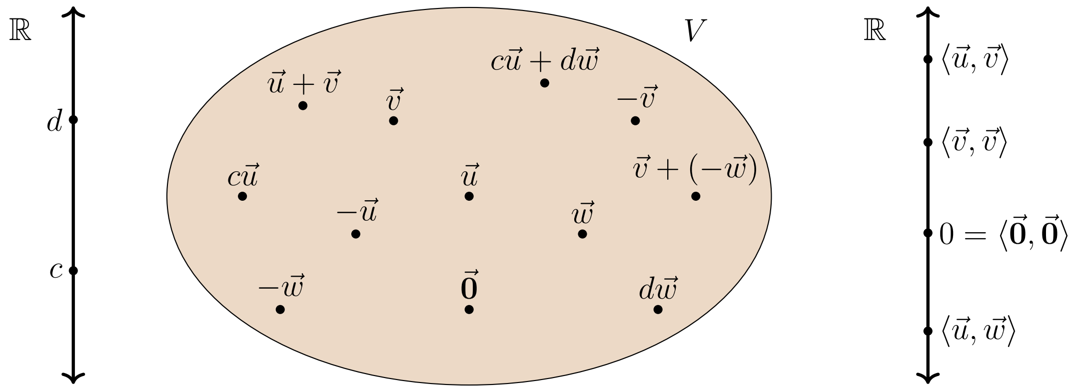
(5) $\langle\,\vec{\mathbf{0}},\vec{\mathbf{0}}\,\rangle = 0$.

An *inner product space* (IPS) is a vector space with an inner product.

Notice that an inner product space consists of 5 entities which together satisfy 15 axioms (5 axioms for for vector addition, 5 axioms for the scalar multiplication, 5 axioms for the inner product). The entities are:

- The set V of vectors.
- The set of scalars (any field such as $\mathbb{R}$).
- The operation of vector addition: a function from $V \times V \to V$.
- The operation of scalar multiplication: a function from $\mathbb{R} \times V \to V$.
- The operation of inner product: a function from $V \times V \to \mathbb{R}$.

Here is a picture of an inner product space V.



Example 7.6. The Euclidean inner product for $\mathbb{R}^n$. This is the inner product given by the dot product. Let $\vec{u} = (x_1, x_2, \ldots, x_n)$ and $\vec{v} = (y_1, y_2, \ldots, y_n)$. Then

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We already know that the dot product $\vec{u} \cdot \vec{v}$ satisfies the axioms of the inner product. Thus, $\mathbb{R}^n$ with the dot product is an inner product space.

Example 7.7 (Another inner product on $\mathbb{R}^2$)**.** Let $\vec{u} = (x, y)$ and $\vec{v} = (a, b)$ be vectors in $\mathbb{R}^2$. Define their inner product to be

$$\langle \vec{u}, \vec{v} \rangle = xa + 2yb.$$

We check that this operation satisfies the axioms of an inner product.

(1) $\langle \vec{u}, \vec{v} \rangle = xa + 2yb = ax + 2by = \langle \vec{v}, \vec{u} \rangle$.

(2) Pick a third vector $\vec{w} = (r, s)$. Then, $\vec{v} + \vec{w} = (a + r,\ b + s)$.

$$\begin{aligned} \langle \vec{u}, \vec{v} + \vec{w} \rangle &= x(a + r) + 2y(b + s) = xa + xr + 2yb + 2ys \\ &= (xa + 2yb) + (xr + 2ys) = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle. \end{aligned}$$

(3) $c\langle \vec{u}, \vec{v} \rangle = c(xa + 2yb) = (cx)a + 2(cy)b = \langle c\vec{u}, \vec{v} \rangle$.

(4) If $\vec{u} \neq \vec{\mathbf{0}}$, then $x \neq 0$ or $y \neq 0$, in any case, we get that $\langle \vec{u}, \vec{u} \rangle = x^2 + 2y^2 > 0$.

(5) $\langle \vec{\mathbf{0}}, \vec{\mathbf{0}} \rangle = 0(0) + 2(0)(0) = 0$.

Example 7.8 (Non-example)**.** Let $\vec{u} = (x, y)$ and $\vec{v} = (a, b)$ be vectors in $\mathbb{R}^2$. Define $\langle \vec{u}, \vec{v} \rangle = xa - yb$. Show that this operation is not an inner product on $\mathbb{R}^2$.

Consider the vector $\vec{v} = (2, 3)$. Then

$$\langle \vec{v}, \vec{v} \rangle = 2(2) - 3(3) = 4 - 9 = -5 \leq 0.$$

So, it violates positive-definiteness, and so it is not an inner product.

Example 7.9 (An inner product on $\mathcal{P}_n$). Given two polynomials

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \text{and} \quad q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$

in the vector space $\mathcal{P}_n$, we define their (standard) inner product to be

$$\langle p, q \rangle = \sum_{i=0}^{n} a_ib_i = a_0b_0 + a_1b_1 + \cdots + a_nb_n.$$

Verify that this operation satisfies the five axioms of an inner product.

Example 7.10 (An inner product on $\mathbb{M}_2$). Consider 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Define their (standard) inner product to be

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

Then, $\langle A, B \rangle$ satisfies the five axioms of an inner product. In general, the standard inner product of two matrices A and B in $\mathbb{M}_{m\times n}$ is the sum of the product of the corresponding entries of A and B.

Example 7.11 (An inner product on continuous functions). The vector space $C[a, b]$ is the vector space of all real-valued continuous functions defined on the interval $[a, b]$. Let f and g be functions in $C[a, b]$. Define their inner product to be

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

This operation satisfies the inner product axioms.

Theorem 7.2.2.

Let V be an inner product space. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in V and let $c \in \mathbb{R}$ be a scalar. The following statements are true.

(i) $\langle \vec{\mathbf{0}}, \vec{v} \rangle = 0$.

(ii) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.

(iii) $\langle \vec{u}, c\vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$.

We now use the inner product to define the length of a vector and the distance between vectors in any inner product space.

Definition 7.2.3. (Norm of a Vector)

Let V be an inner product space.

- The *norm* (or length) of a vector $\vec{v}$ is
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$
- A vector $\vec{v}$ is called a *unit vector* if and only if $\|\vec{v}\| = 1$.
- If $\vec{v}$ is a nonzero vector, then the vector $\dfrac{1}{\|\vec{v}\|}\vec{v}$ is called the *unit vector in the direction of* $\vec{v}$.

Lemma 7.2.4. (Properties of the Norm)

Let $\vec{u}$ be a vector in an inner product space and let $c \in \mathbb{R}$. Then the following hold.

(i) $\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle$.

(ii) $\|\vec{u}\| \geq 0$.

(iii) $\|\vec{u}\| = 0$ *if and only if* $\vec{u} = \vec{\mathbf{0}}$.

(iv) $\|c\,\vec{u}\| = |c|\,\|\vec{u}\|$.

Example 7.12. Consider the inner product space $C[-1,1]$ of real-valued continuous functions defined on the closed interval $[-1,1]$. Let $f(x) = x$ and let $h(x) = \cos(\pi x)$. Find the norm of f and h.

- $\|f\|^2 = \langle f, f \rangle = \displaystyle\int_{-1}^{1} f(x)\,f(x)\,dx = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$. Thus, $\|f\| = \sqrt{\frac{2}{3}}$.
- To find $\|h\|$ we will use the formula $\cos^2\theta = \dfrac{\cos(2\theta)+1}{2}$.

$$\begin{aligned}
\|h\|^2 = \langle h, h \rangle &= \int_{-1}^{1} h(x)\,h(x)\,dx = \int_{-1}^{1} \cos^2(\pi x)\,dx \\
&= \int_{-1}^{1} \frac{\cos(2\pi x)+1}{2}\,dx = \frac{1}{2}\left(\int_{-1}^{1} \cos(2\pi x)\,dx + \int_{-1}^{1} 1\,dx\right) \\
&= \frac{1}{2}\left[\frac{\sin(2\pi x)}{2\pi} + x\right]_{-1}^{1} = \frac{1}{2}(1-(-1)) = 1.
\end{aligned}$$

Thus, $\|h\| = \sqrt{1} = 1$.

Definition 7.2.5. (Distance between Vectors)

Let V be an inner product space. The *distance* between vectors $\vec{u}$ and $\vec{v}$ is

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|.$$

Lemma 7.2.6. (Properties of the Distance)

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space. Then the following hold.

(i) $d(\vec{u},\vec{v}) \geq 0$.

(ii) $d(\vec{u},\vec{v}) = 0$ *if and only if* $\vec{u}=\vec{v}$.

(iii) $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$.

Example 7.13 (Inner Product for $\mathcal{P}_2$)**.** For polynomials $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ in $\mathcal{P}_2$, we define their inner product to be

$$\langle p,\, q\rangle = a_0b_0 + a_1b_1 + a_2b_2.$$

Let $p(x) = 1-2x^2,\;\; q(x) = 4-2x+x^2$ and $r(x) = x+2x^2$.

(a) $\langle p, q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$.

(b) $\langle q, r\rangle = (4)(0) + (-2)(1) + (1)(2) = 0$.

(c) $\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$.

(d) $d(p,q) = \|p-q\| = \|-3+2x-3x^2\| = \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$.

Example 7.14. Use the inner product previously defined on $C[0,1]$.

Let $f(x) = x$ and $g(x) = x^2$ be function in $C[0,1]$.

(a) Find the norm of f. $\|f\|^2 = \langle f, f\rangle = \displaystyle\int_0^1 f(x)f(x)\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$.

Thus, $\|f\| = \frac{1}{\sqrt{3}}$.

(b) Find the distance between f and g.

$$\begin{aligned}\langle f-g, f-g\rangle &= \langle x-x^2, x-x^2\rangle \\ &= \int_0^1 (x-x^2)^2\,dx = \int_0^1 (x^2-2x^3+x^4)\,dx \\ &= \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{30}\,.\end{aligned}$$

Therefore, $d(f,g) = \|f-g\| = \sqrt{\langle f-g, f-g\rangle} = \dfrac{1}{\sqrt{30}}\,.$

Lemma 7.2.7.

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space. Then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2 .$$

Proof. Let $\vec{u}$ and $\vec{v}$ be any vectors in an inner product space. We proceed by using the properties of the inner product stated above.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \langle\, (\vec{u}+\vec{v})\,,\, (\vec{u}+\vec{v})\,\rangle \\
&= \langle\, \vec{u}\,,\, \vec{u}+\vec{v}\,\rangle + \langle\, \vec{v}\,,\, \vec{u}+\vec{v}\,\rangle \\
&= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle + \langle \vec{v}, \vec{u}\rangle + \langle \vec{v}, \vec{v}\rangle \\
&= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle + \langle \vec{u}, \vec{v}\rangle + \langle \vec{v}, \vec{v}\rangle \\
&= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2 .
\end{aligned}$$

Therefore, $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2$ as desired. ■

Substitute $c\vec{v}$ for $\vec{v}$ in the lemma above to obtain the following corollary.

Corollary 7.2.8.

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space V, and let $c \in \mathbb{R}$ be a scalar. Then

$$\|\vec{u} + c\,\vec{v}\|^2 = \|\vec{u}\|^2 + 2c\,\langle \vec{u}, \vec{v}\rangle + c^2\,\|\vec{v}\|^2.$$

We now have all the tools needed to establish the Cauchy–Schwarz inequality (also called Cauchy–Bunyakovsky–Schwarz inequality). The famous inequality gives an upper bound on the inner product between two vectors in an inner product space in terms of the norms the vectors. More precisely, it states that the absolute value of the inner product of two vectors is at most the product of their norms. The Cauchy-Schwarz inequality is considered one of the most significant and widely used inequalities in mathematics.

Theorem 7.2.9. (Cauchy-Schwarz Inequality)

Let V be an inner product space, and let $\vec{u}, \vec{v}$ be vectors in V. Then

$$|\langle \vec{u}, \vec{v}\rangle| \leq \|\vec{u}\|\,\|\vec{v}\|.$$

Proof. If $\vec{v} = \vec{\mathbf{0}}$, then both sides are zero and we are done. Otherwise, assume that

$\vec{v} \neq \vec{\mathbf{0}}$. It follows that $\|\vec{v}\| \neq 0$. Take the scalar $c = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$. Then,

$$\begin{aligned}
0 \leq \|\vec{u} - c\vec{v}\|^2 &= \|\vec{u} + (-c)\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - 2c \langle \vec{u}, \vec{v} \rangle + c^2 \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - 2\frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} + \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} \\
&= \|\vec{u}\|^2 - \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} .
\end{aligned}$$

Thus,

$$\frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} \leq \|\vec{u}\|^2 ,$$

which leads to

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2 ,$$

and by taking the square root of both sides of the inequality, we get

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$$

as desired. ■

The Cauchy-Schwarz Inequality for the Euclidean space $\mathbb{R}^n$ with the dot product states that for any vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ we have that $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$.

Example 7.15. Let $\vec{u} = (1, -1, 3)$ and $\vec{v} = (2, 0, -1)$ be vectors in $\mathbb{R}^3$. Verify Cauchy-Schwarz Inequality for vectors $\vec{u}$ and $\vec{v}$.

The absolute value of their dot product is $|\vec{u} \cdot \vec{v}| = |2 + 0 - 3| = |-1| = 1$. The norm of $\vec{u}$ is $\|\vec{u}\| = \sqrt{1 + 1 + 9} = \sqrt{11}$, and the norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{4 + 0 + 1} = \sqrt{5}$. To verify the Cauchy-Schwarz Inequality we need to check that $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$. Observe that

$$1 \leq \sqrt{11} \cdot \sqrt{5} = \sqrt{55} \approx 7.416 .$$

Example 7.16. Verify the Cauchy-Schwarz Inequality with $f(x) = 1$ and $g(x) = x$ in the vector space $C[0, 1]$. We need to show that $|\langle f, g \rangle| \leq \|f\| \, \|g\|$.

- $\langle f, g \rangle = \displaystyle\int_0^1 f(x)g(x)\, dx = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.
- $\|f\|^2 = \langle f, f \rangle = \displaystyle\int_0^1 f(x)f(x)\, dx = \int_0^1 1dx = [x]_0^1 = 1$. So, $\|f\| = \sqrt{1} = 1$.
- $\|g\|^2 = \langle g, g \rangle = \displaystyle\int_0^1 g(x)g(x)\, dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$. So, $\|g\| = \frac{1}{\sqrt{3}}$.

We now check that:

$$|\langle f, g\rangle| = \frac{1}{2} \le \|f\|\,\|g\| = 1 \cdot \frac{1}{\sqrt{3}} \approx 0.577\,.$$

Theorem 7.2.10. (Triangle Inequality)

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space. Then

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

Proof. We will now prove the Triangle Inequality

$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2 \\ &\le \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 && \text{(since } \vec{u}\cdot\vec{v} \le |\vec{u}\cdot\vec{v}|) \\ &\le \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 && \text{(By Cauchy-Schwarz Inequality)} \\ &\le (\|\vec{u}\| + \|\vec{v}\|)^2\,. \end{aligned}$$

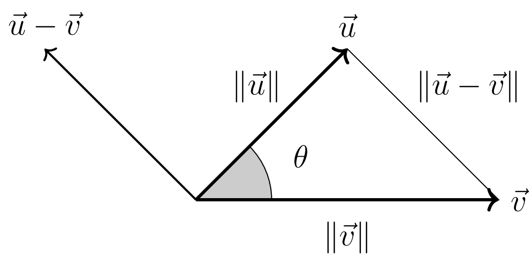
By taking the square root on both sides, we will get the Triangle Inequality:

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

■

♣ The Angle between Vectors

As we have seen, the inner product was used to define the norm (length) of vectors and also the distance between vectors. We aim to use the inner product to define the angle between vectors in inner product spaces. As usual we start our investigation in the cartesian plane $\mathbb{R}^2$. Consider two vectors $\vec{u}$ and $\vec{v}$ in the plane.



The cosine rule relates the lengths of the sides of a triangle to one of its angles, it states that:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\,\cos\theta.$$

Therefore,

$$\|\vec{u}\|^2 - 2\langle\vec{u},\vec{v}\rangle + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\,\cos\theta.$$

Thus,

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \, \cos\theta.$$

We will use this equation to define the angle between vectors in an arbitrary inner product space. Since $-1 \leq \cos\theta \leq 1$, it follows that to be able to find the angle θ from the equation above we need that

$$-1 \leq \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1.$$

Luckily, this is guaranteed by Cauchy-Schwartz Inequality. We now can define the angle between two vectors in any inner product space using the inner product operation.

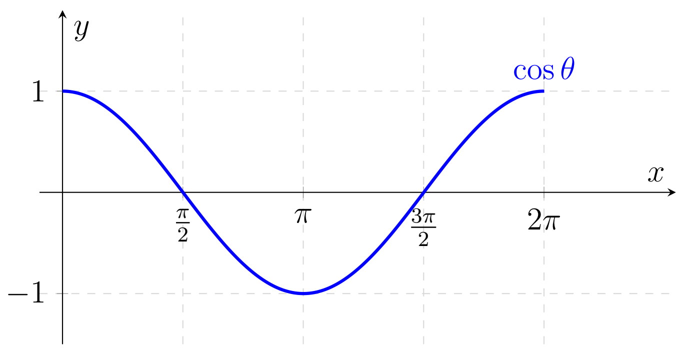
Definition 7.2.11. (Angle between Vectors)

Let V be an inner product space. The *angle* between nonzero vectors $\vec{u}$ and $\vec{v}$ in V is the angle θ with

$$\cos\theta = \frac{\langle \vec{u}\,, \, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}\,, \qquad (\text{where } 0 \leq \theta \leq \pi).$$

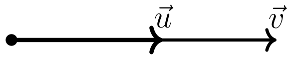
We call vectors $\vec{u}$ and $\vec{v}$ *orthogonal* when $\theta = \frac{\pi}{2}$, equivalently, when $\langle \vec{u}\,, \, \vec{v} \rangle = 0$. We write $\vec{u} \perp \vec{v}$ for orthogonal vectors.

Below is the graph of the cosine function.

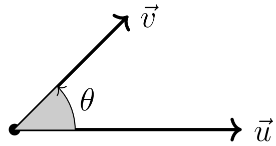


In the plane $\mathbb{R}^2$ we have the following cases for the angle between two vectors.

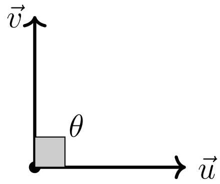
(1) Vectors in same direction, $\theta = 0$



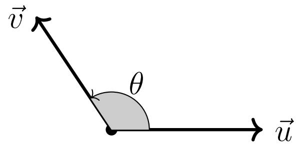
(2) $0 < \vec{u} \cdot \vec{v}$ and $0 < \theta < \dfrac{\pi}{2}$



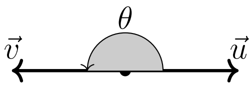
(3) Orthogonal vectors, $\vec{u} \cdot \vec{v} = 0$ and $\theta = \dfrac{\pi}{2}$



(4) $\vec{u} \cdot \vec{v} < 0$ and $\dfrac{\pi}{2} < \theta < \pi$



(5) Vectors have opposite direction, $\theta = \pi$



Example 7.17. Let $\vec{u} = (3, 0)$ and $\vec{v} = (2,\, 2\sqrt{3})$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{(3)(2) + (0)(2\sqrt{3})}{\sqrt{9+0}\,\sqrt{4+12}} = \frac{6}{12} = \frac{1}{2}.$$

So, $\theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$.

Example 7.18. Let $\vec{u} = (-4, 0, 2, -2)$ and $\vec{v} = (2, 0, -1, 1)$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{(-4)(2) + (0)(0) + (2)(-1) + (-2)(1)}{\sqrt{16+4+4}\,\sqrt{4+1+1}} = \frac{-12}{\sqrt{144}} = -1.$$

So, $\theta = \cos^{-1}(-1) = \pi$, hence $\vec{u}$ and $\vec{v}$ have opposite directions, clearly, $\vec{u} = -2\vec{v}$.

Example 7.19. Determine all vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u} = (4, 2)$.

Let $\vec{v} = (x, y)$. We want $\vec{u} \cdot \vec{v} = 0$.

$$\vec{u} \cdot \vec{v} = (4, 2) \cdot (x, y) = 4x + 2y = 0.$$

Let $x = t$, then $y = -2t$, so $\vec{v} = (t, -2t)$ where $t \in \mathbb{R}$.

Theorem 7.2.12. (Pythagorean Theorem)

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space. Then $\vec{u}$ and $\vec{v}$ are orthogonal if and only if

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 &\iff \|\vec{u}\|^2 + 2\langle\vec{u},\vec{v}\rangle + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \\
&\iff 2\langle\vec{u},\vec{v}\rangle = 0 \\
&\iff \langle\vec{u},\vec{v}\rangle = 0 \\
&\iff \vec{u} \perp \vec{v}.
\end{aligned}$$

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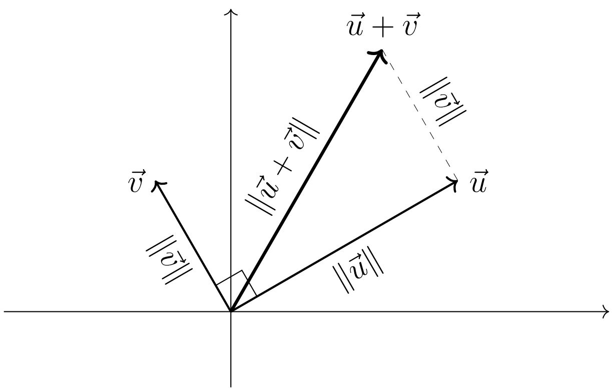
The diagram below shows two orthogonal vectors $\vec{u}$ and $\vec{v}$ in the cartesian plane.



Example 7.20. We work in the inner product space $\mathbb{M}_2$. Consider the two vectors:

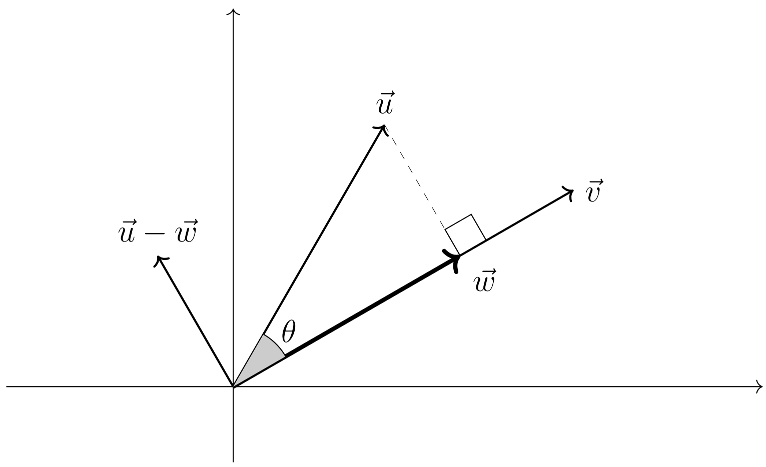
$$A = \begin{bmatrix} 1 & \sqrt{3} \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \sqrt{3} \\ 4 & \sqrt{5} \end{bmatrix}.$$

Find the angle between matrices A and B.

- $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{(1^2 + (\sqrt{3})^2 + 2^2 + (-1)^2)} = \sqrt{9} = 3.$
- $\|B\| = \sqrt{\langle A, A\rangle} = \sqrt{(1^2 + (\sqrt{3})^2 + 4^2 + (\sqrt{5})^2)} = \sqrt{25} = 5.$
- $\langle A, B\rangle = 1(1) + \sqrt{3}(\sqrt{3}) + 2(4) + (-1)(\sqrt{5}) = 12 - \sqrt{5}.$
- $\cos\theta = \dfrac{\langle A\,,\, B\rangle}{\|A\|\,\|B\|} = \dfrac{12-\sqrt{5}}{3\cdot 5} = 0.8 - \dfrac{1}{3\sqrt{5}}.$
- $\theta = \cos^{-1}\left(0.8 - \dfrac{1}{3\sqrt{5}}\right) \approx \cos^{-1}(0.651) \approx 0.862$ radians.

7.3 Orthogonal Projection

Given nonzero vectors $\vec{u}$ and $\vec{v}$ in an inner product space, we want to find a vector $\vec{w}$ in the direction of $\vec{v}$ which has the shortest distance from the other vector $\vec{u}$; we call such vector the *orthogonal projection of* $\vec{u}$ *onto* $\vec{v}$ and denote it by $\text{proj}_{\vec{v}}(\vec{u})$. So, here $\vec{w} \in \text{Span}(\vec{v})$. Let us work in the cartesian plane $\mathbb{R}^2$. For $\vec{w}$ of to be of shortest distance from $\vec{u}$ we need the vector $\vec{u} - \vec{w}$ to be orthogonal to $\vec{v}$.



We need to find the orthogonal projection $\vec{w}$ in terms of the vectors $\vec{u}$ and $\vec{v}$. Since the orthogonal projection $\vec{w}$ is parallel to $\vec{v}$, we have that $\vec{w}$ is a scalar multiple of the vector $\vec{v}$. We can obtain $\vec{w}$ by scalaing the unit vector in the diection of $\vec{v}$ by the length $\|\vec{w}\|$. From the diagram above, the length of $\vec{w}$ is $\|\vec{w}\| = \|\vec{u}\| \cos\theta$. Recall that $\dfrac{1}{\|\vec{v}\|}\vec{v}$ is the unit vector in the direction of $\vec{v}$. Therefore,

$$\text{proj}_{\vec{v}}(\vec{u}) = \vec{w} = \|\vec{w}\| \left(\frac{1}{\|\vec{v}\|}\vec{v} \right) = \left(\frac{\|\vec{u}\| \cos\theta}{\|\vec{v}\|} \right) \vec{v} = \left(\frac{\|\vec{u}\| \, \|\vec{v}\| \cos\theta}{\|\vec{v}\|^2} \right) \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}.$$

Thus, $\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$. For an arbitrary inner product space we have the following definition.

Definition 7.3.1. (Orthogonal Projection)

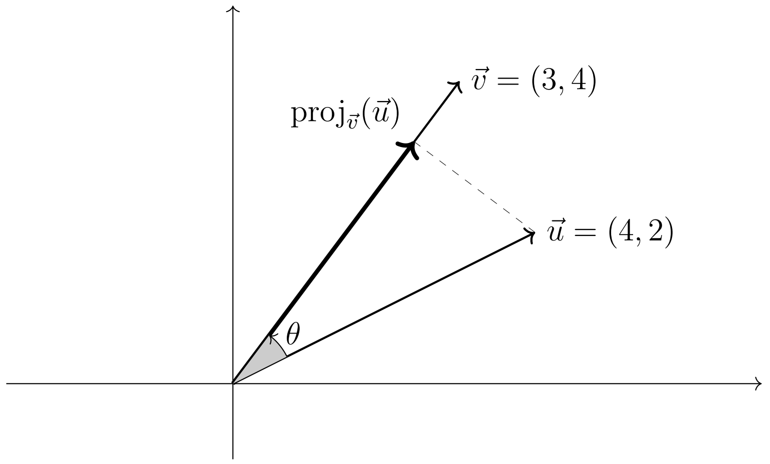
Let V be an inner product space. Let $\vec{u}, \vec{v}$ be vectors in V with $\vec{v} \neq \vec{\mathbf{0}}$. The *orthogonal projection* of $\vec{u}$ onto $\vec{v}$ is the vector

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}\,,\, \vec{v} \rangle}{\langle \vec{v}\,,\, \vec{v} \rangle} \vec{v}.$$

Observe that in an inner product space V, the orthogonal projection $\text{proj}_{\vec{v}}(\vec{u})$ is a function from V to the $\text{Span}(\vec{v})$ which sends any vector $\vec{u} \in V$ to a vector that is a scalar multiple of $\vec{v}$.

Example 7.21. Let $\vec{u} = (4,2)$ and $\vec{v} = (3,4)$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v} = \left(\frac{(4)(3) + (2)(4)}{3^2 + 4^2} \right)(3,4) = \frac{4}{5}(3,4) = (2.4, 3.2).$$



Exercise. Show that in an inner product space the vector $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ is orthogonal to the vector $\vec{v}$. You need to show that their inner product is 0.

Theorem 7.3.2.

Among all scalar multiples of $\vec{v}$, the projection of $\vec{u}$ onto $\vec{v}$ is the one closest to $\vec{u}$. That is, for all scalars $c \in \mathbb{R}$ we have that

$$d(\,\vec{u}\,,\, \text{proj}_{\vec{v}}(\vec{u})\,) \leq d(\vec{u}, c\vec{v}).$$

Example 7.22. Working in the inner product space $C[0,1]$ of continuous functions, consider the functions

$$f(x) = 1, \qquad g(x) = x, \qquad h(x) = \sin\left(\frac{\pi}{4}x\right).$$

- Find the norm of f.

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 1\,dx = 1. \quad \text{Thus, } \|f\| = \sqrt{1} = 1.$$

- Find the norm of g.

$$\|g\|^2 = \langle g, g \rangle = \int_0^1 x^2\,dx = 1/3. \quad \text{Thus, } \|g\| = \frac{1}{\sqrt{3}}.$$

- Find the norm of h. We use below the double angle formula for cosine.

$$\begin{aligned}
\|h\|^2 = \langle h, h \rangle &= \int_0^1 \sin^2\left(\frac{\pi}{4}x\right)\,dx \\
&= \int_0^1 \frac{1}{2}\left(1 - \cos\left(2\left(\frac{\pi}{4}x\right)\right)\right) = \frac{1}{2}\int_0^1 1\,dx - \frac{1}{2}\int_0^1 \cos\left(\frac{\pi}{2}x\right)\,dx \\
&= \frac{1}{2} - \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}x\right)\right]_0^1 = \frac{1}{2} - \frac{1}{\pi} = \frac{\pi - 2}{2\pi}.
\end{aligned}$$

Thus, $\|h\| = \dfrac{\sqrt{\pi - 2}}{\sqrt{2\pi}}$.

- Find the angle between f and g.

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Thus,

$$\cos\theta = \frac{\langle f\,,\,g\rangle}{\|f\|\,\|g\|} = \frac{1/2}{1\cdot 1/\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

Therefore, the angle $\theta = \cos^{-1}\left(\frac{1}{2\sqrt{3}}\right) \approx 1.27795$ radians.

- Find the orthogonal projection of f onto g.

$$\mathrm{proj}_g(f) = \frac{\langle f, g\rangle}{\langle g, g\rangle}g(x) = \frac{\int_0^1 x\,dx}{\int_0^1 x^2\,dx}g(x) = \frac{1/2}{1/3}g(x) = \frac{3}{2}g(x) = 1.5x.$$

- Find the distance between f and g.

$$\begin{aligned}\langle f-g\,,\,f-g\rangle &= \langle 1-x, 1-x\rangle\\ &= \int_0^1 (1-x)^2\,dx = \int_0^1 (1-2x+x^2)\,dx\\ &= \left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{3}\,.\end{aligned}$$

Therefore, $d(f,g) = \|f-g\| = \sqrt{\langle f-g\,,\,f-g\rangle} = \dfrac{1}{\sqrt{3}}$.

7.4 Orthonormal Basis

We have seen that a vector space has many different bases (plural of basis). In this section, we will study a special kind of bases.

Definition 7.4.1. (Orthonormal Vectors)

Let V be an inner product space. A set $S = \{v_1, v_2, \cdots, v_n\}$ of vectors in V is called *orthonormal* if and only if

- Every pair of distinct vectors in S is orthogonal, that is, if $i \neq j$, then $\langle v_i, v_j \rangle = 0$. In this case, we say S is an *orthogonal* set.
- Every vector in S is a unit vector, that is, $\|v_i\| = 1$ for $i = 1, 2, 3, \ldots, n$.

Example 7.23. The standard basis $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ is an orthonormal basis.

Example 7.24. Consider the inner product space $\mathbb{R}^3$.

$$S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right) \right\}.$$

Call the vectors in S: $\vec{v}_1, \vec{v}_2, \vec{v}_3$, respectively.

- $\vec{v}_1 \cdot \vec{v}_2 = \frac{-1}{6} + \frac{1}{6} + 0 = 0.$
- $\vec{v}_1 \cdot \vec{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0.$
- $\vec{v}_2 \cdot \vec{v}_3 = \frac{-\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0.$
- $\|\vec{v}_1\| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1.$
- $\|\vec{v}_2\| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{\frac{1}{18} + \frac{1}{18} + \frac{16}{18}} = 1.$
- $\|\vec{v}_3\| = \sqrt{\vec{v}_3 \cdot \vec{v}_3} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1.$

Thus, the set S is an orthonormal basis. Check that S is a basis for $\mathbb{R}^3$, that is, S spans $\mathbb{R}^3$ and S is linearly independent. Therefore, S is an orthonormal basis for the inner product space $\mathbb{R}^3$.

Theorem 7.4.2.

Let V be an inner product space, and $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an orthogonal set of nonzero vectors in V. Then S is linearly independent.

Proof. Consider the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{\mathbf{0}}$$

We need to show that $c_i = 0$ for all $i = 1, 2, \ldots, n$. Fix one of the vectors $\vec{v}_i$ in S. Form the inner product of the left side of the equation with $\vec{v}_i$, this should be equal to the inner product of the right side with $\vec{v}_i$.

$$\langle\ (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_i\vec{v}_i + \cdots + c_n\vec{v}_n)\ ,\ \vec{v}_i\ \rangle = \langle\ \vec{\mathbf{0}}, \vec{v}_i\ \rangle$$

Using the inner product axioms we get:

$$c_1\langle\vec{v}_1, \vec{v}_i\rangle + c_2\langle\vec{v}_2, \vec{v}_i\rangle + \cdots + c_i\langle\vec{v}_i, \vec{v}_i\rangle + \cdots + c_n\langle\vec{v}_n, \vec{v}_i\rangle = 0.$$

As S is orthogonal, $\langle\vec{v}_j, \vec{v}_i\rangle = 0$ when $j \neq i$. It follows that

$$0 + 0 + \cdots + c_i\langle\vec{v}_i, \vec{v}_i\rangle + 0 + \cdots + 0 = 0.$$

Therefore, $c_i\langle\vec{v}_i, \vec{v}_i\rangle = 0$. Since $\vec{v}_i \neq \vec{\mathbf{0}}$, by positive-definiteness, we get that $\langle\vec{v}_i, \vec{v}_i\rangle > 0$, and thus it must be that $c_i = 0$ as desired. Since the chosen vector $\vec{v}_i$ was arbitrary, we get that $c_k = 0$ for all $k = 1, 2, \ldots, n$, proving that S is linearly independent. ■

Corollary 7.4.3.

Let V be an inner product space with $\dim(V) = n$. Then any orthogonal set containing n nonzero vectors is a basis for V.

Example 7.25. Use the corollary above to show that the following set is a basis for $\mathbb{R}^3$.

$$S = \{(2, 2, -2), (1, 0, 1), (-1, 2, 1)\}$$

Call the vectors in S: $\vec{v}_1, \vec{v}_2, \vec{v}_3$, respectively. We will show that S is orthogonal.

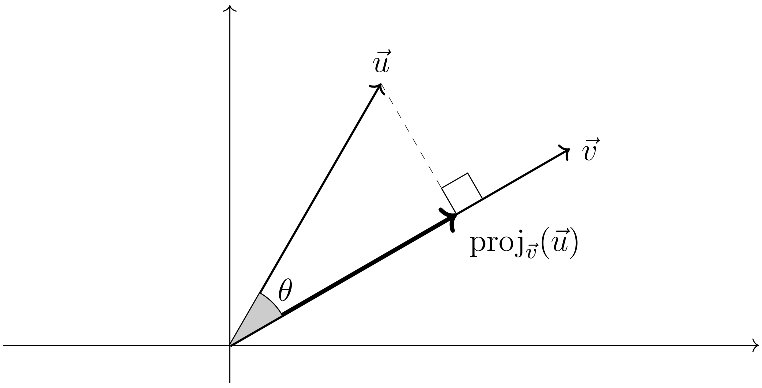
$$\vec{v}_1 \cdot \vec{v}_2 = 2 + 0 - 2 = 0.$$
$$\vec{v}_1 \cdot \vec{v}_3 = -2 + 4 - 2 = 0.$$
$$\vec{v}_2 \cdot \vec{v}_3 = -1 + 0 + 1 = 0.$$

Therefore, the set S is an orthogonal set containing 3 nonzero vectors of $\mathbb{R}^3$, and so it forms a basis for $\mathbb{R}^3$.

We will now discuss an important characteristic of an orthonormal basis of an inner product space. Recall the orthogonal projection of a vector $\vec{u}$ onto a vector $\vec{v}$ is the vector

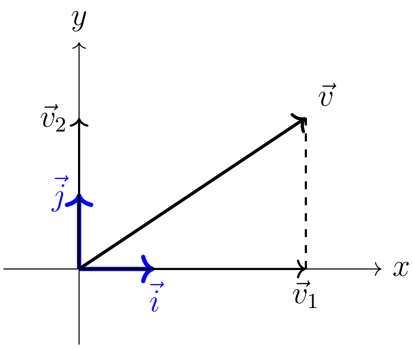
$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle\vec{u}, \vec{v}\rangle}{\langle\vec{v}, \vec{v}\rangle}\vec{v}.$$

Moreover, the orthogonal projection $\text{proj}_{\vec{v}}(\vec{u})$ is the closest vector to $\vec{u}$ which is parallel to $\vec{v}$.



Consider the inner product space $\mathbb{R}^2$ and its orthonormal basis $\{\vec{i}, \vec{j}\}$ where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Next, choose any arbitrary vector $\vec{v}$ in $\mathbb{R}^2$. Let $\vec{v}_1 = \text{proj}_{\vec{i}}(\vec{v})$ and $\vec{v}_2 = \text{proj}_{\vec{j}}(\vec{v})$. Then, $\vec{v}_1 = \dfrac{\langle \vec{v}, \vec{i} \rangle}{\langle \vec{i}, \vec{i} \rangle} \vec{i} = \langle \vec{v}, \vec{i} \rangle \vec{i}$ and $\vec{v}_2 = \dfrac{\langle \vec{v}, \vec{j} \rangle}{\langle \vec{j}, \vec{j} \rangle} \vec{j} = \langle \vec{v}, \vec{j} \rangle \vec{j}$. Therefore,

$$\vec{v} = \vec{v}_1 + \vec{v}_2 = (\vec{v} \cdot \vec{i}) \vec{i} + (\vec{v} \cdot \vec{j}) \vec{j} = c_1 \vec{i} + c_2 \vec{j}.$$



So the coordinates of $\vec{v}$ relative to the basis $\{\vec{i}, \vec{j}\}$ is $[\vec{v}] = \begin{bmatrix} \vec{v} \cdot \vec{i} \\ \vec{v} \cdot \vec{j} \end{bmatrix}$.

Theorem 7.4.4.

Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an orthonormal basis for an inner product space V. Then the coordinate representation of any vector $\vec{u}$ with respect to the orthonormal basis B is

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n.$$

Consequently, $[\vec{u}]_B = (\langle \vec{u}, \vec{v}_1 \rangle, \langle \vec{u}, \vec{v}_2 \rangle, \ldots, \langle \vec{u}, \vec{v}_n \rangle)$.

Proof. As B is a basis for V, a vector $\vec{u}$ can be represented uniquely as

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

for some scalars $c_1, c_2, \cdots, c_n$ in $\mathbb{R}$. Take the inner product with $\vec{v}_i$ of both sides of the equation. As B is an orthogonal set, we know that $\langle \vec{v}_k, \vec{v}_i \rangle = 0$ when $k \neq i$. Thus, we get that

$$\begin{aligned}
\langle \vec{u}, \vec{v}_i \rangle &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + \cdots + c_i \langle \vec{v}_i, \vec{v}_i \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_i \rangle \\
\langle \vec{u}, \vec{v}_i \rangle &= 0 + 0 + \cdots + c_i \langle \vec{v}_i, \vec{v}_i \rangle + 0 + \cdots + 0 \\
\langle \vec{u}, \vec{v}_i \rangle &= c_i \, \langle \vec{v}_i, \vec{v}_i \rangle \\
\langle \vec{u}, \vec{v}_i \rangle &= c_i \cdot \|\vec{v}_i\|^2 \\
\langle \vec{u}, \vec{v}_i \rangle &= c_i \cdot 1 \\
\langle \vec{u}, \vec{v}_i \rangle &= c_i.
\end{aligned}$$

Therefore, the unique scalar c_i of the vector $\vec{v}_i$ in the representation of $\vec{u}$ is the number $\langle \vec{u}, \vec{v}_i \rangle$ which was to be shown. ∎

Example 7.26. Find the coordinates of $\vec{u} = (5, -5, 2)$ relative to the orthonormal basis B of $\mathbb{R}^3$ where

$$B = \left\{ \left(\frac{3}{5}, \frac{4}{5}, 0 \right), \left(\frac{-4}{5}, \frac{3}{5}, 0 \right), (0, 0, 1) \right\}.$$

- $\vec{u} \cdot \vec{v}_1 = 3 - 4 + 0 = -1.$
- $\vec{u} \cdot \vec{v}_2 = -4 - 3 + 0 = -7.$
- $\vec{u} \cdot \vec{v}_3 = 0 + 0 + 2 = 2.$

Therefore, $[\vec{u}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}.$

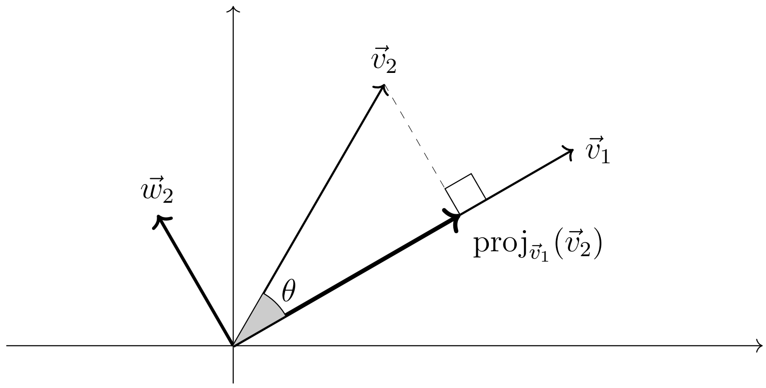
7.5 Gram-Schmidt Process

We will now present an algorithm which transforms a given basis of an inner product space V to an orthonormal basis. In a nutshell, here are the steps. We begin with any basis for an inner product space V.

(1) Convert the given basis to an orthogonal basis.

(2) Normalize each vector.

For instance, start with two vectors $\vec{v}_1$ and $\vec{v}_2$. Then, replace $\vec{v}_2$ with $\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{v}_1}(\vec{v}_2)$. Now $\{\vec{v}_1, \vec{w}_2\}$ is an orthogonal set.



Algorithm 7.5.1. (Gram-Schmidt Orthonormalisation Process)

Given a basis $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ for an inner product space V.

(1) Start with $\vec{w}_1 = \vec{v}_1$.

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 \,.$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}\vec{w}_2 \,.$$

$$\vdots$$

$$\vec{w}_n = \vec{v}_n - \frac{\langle \vec{v}_n, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 - \frac{\langle \vec{v}_n, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}\vec{w}_2 - \cdots - \frac{\langle \vec{v}_n, \vec{w}_{n-1} \rangle}{\langle \vec{w}_{n-1}, \vec{w}_{n-1} \rangle}\vec{w}_{n-1} \,.$$

The set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ forms an orthogonal basis for V.

(2) Let $\vec{u}_i = \dfrac{1}{\|\vec{w}_i\|}\,\vec{w}_i$ for each $i = 1, 2, \ldots, n$.

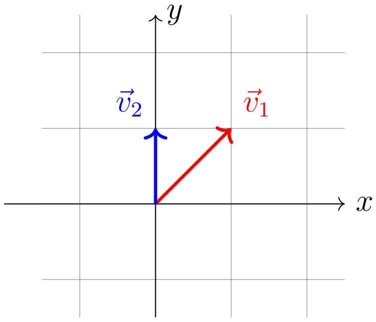
Then the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ forms an orthonormal basis for V.

Example 7.27. Apply the Gram-Schmidt process to the basis $B = \{\overset{\vec{v}_1}{(1,1)}, \overset{\vec{v}_2}{(0,1)}\}$ of $\mathbb{R}^2$.

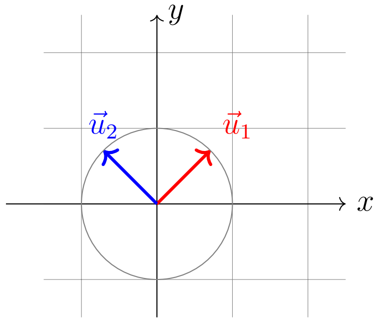
- $\vec{w}_1 = \vec{v}_1 = (1, 1)$.
- $\vec{w}_2 = \vec{v}_2 = \dfrac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1$.
- $= (0, 1) - \dfrac{[0 + 1]}{[1 + 1]}(1, 1) = (0, 1) - \dfrac{1}{2}(1, 1) = \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$.
- $\vec{u}_1 = \dfrac{\vec{w}_1}{\|\vec{w}_1\|} = \dfrac{1}{\sqrt{1 + 1}}(1, 1) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.
- $\vec{u}_2 = \dfrac{\vec{w}_2}{\|\vec{w}_2\|} = \dfrac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}}(\dfrac{-1}{2}, \dfrac{1}{2}) = \left(\dfrac{-\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

Thus, the set $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

Input vectors



Output vectors (orthonormal basis)



Example 7.28. Apply the Gram-Schmidt process to the basis of $\mathbb{R}^3$.

$$B = \{(1, \overset{\vec{v}_1}{1}, 0), (1, \overset{\vec{v}_2}{2}, 0), (0, \overset{\vec{v}_3}{1}, 2)\}.$$

The output will be an orthonomal basis for the inner product space $\mathbb{R}^3$.

- $\vec{w}_1 = \vec{v}_1 = (1, 1, 0)$.
- $\vec{w}_2 = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = (\dfrac{-1}{2}, \dfrac{1}{2}, 0)$.
- $\vec{w}_3 = \vec{v}_3 - \dfrac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 - \dfrac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 = (0, 1, 2) - \dfrac{1}{2}(1, 1, 0) - \dfrac{1/2}{1/2}(\dfrac{-1}{2}, \dfrac{1}{2}, 0) = (0, 0, 2)$.
- $\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0)$.
- $\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \sqrt{2}(\frac{-1}{2}, \frac{1}{2}, 0)$.
- $\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1)$.

The set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.