Linear Algebra

Assignment 5

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4.1 Vectors in R^n

Problem 24

$u = (1, 2, 3), v = (2, -2, -1), w = (4, 0, -4)$.
$2u + v - w + 3z = \mathbf{0} = (0, 0, 0)$.

$$\begin{aligned}
2u + v - w &\iff \mathbf{0} - 3z \\
&\iff \mathbf{0} + (-3z) \\
&\iff -3z. \\
&\iff \frac{-1}{3}(2u + v - w) = \frac{-1}{3}(-3z) \\
&\iff \frac{-1}{3}(2u + v - w) = z. \\
&\iff z = \frac{-1}{3}(2(1, 2, 3) + (2, -2, 1) + (-1)(4, 0, -4)) \\
&\quad = \frac{-1}{3}((2, 4, 6) + (2, -2, -1) - (4, 0, -4)) \\
&\quad = \frac{-1}{3}(2 + 2 + 4, 4 - 2, 6 - 1 - 4) \\
&\quad = \frac{-1}{3}(10, 2, 1) = (\frac{-10}{3}, \frac{-2}{3}, \frac{-1}{3}).
\end{aligned}$$

Problem 27

a) It is a scalar multiple of z. $v = \frac{3}{2}z = \frac{3}{2}(3, 2, -5) = (\frac{9}{2}, 3, \frac{-15}{2})$.

b) No, it is not a scalar multiple of z. The ratio is different between $\frac{9}{3} = 3, \frac{-6}{2} = -3$.

Problem 34

b) $2w - \frac{1}{2}u = 2(2, -2, 1, 3) - \frac{1}{2}(1, 2, -3, 1) = (4, -4, 2, 6) - (\frac{1}{2}, 1, \frac{-3}{2}, \frac{1}{2}) = (\frac{7}{2}, -5, \frac{7}{2}, \frac{11}{2})$.

4.2 Vector Spaces

Problem 13

Proof. Suppose A, B are matrices of size 4×6. By definition of matrix addition, $A + B = C$ is also of size 4×6, so the addition is closed (1). We also know $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$, clearly commutative (2). Suppose another matrix D of the same size. By associativity of real numbers, $A + B + D = [a_{ij}] + [b_{ij}] + [d_{ij}] = [a_{ij} + b_{ij} + d_{ij}] = [a_{ij} + (b_{ij} + d_{ij})] = [(a_{ij} + b_{ij}) + d_{ij}] = A + (B + D) = (A + B) + D$, we can see addition here is associative (3). We observe that $\mathbf{0}_{4\times 6}$ is the additive identity as $A + \mathbf{0}_{4\times 6} = A$. (4). The fifth axioms holds as $A - A = A + (-A) = [a_{ij} - a_{ij}] = \mathbf{0}_{4\times 6}$. (5). By definition of matrix scalar multiplication, suppose $c, k \in \mathbb{R}$, and A, B are matrices of size 4×6, then (cA) is a matrix of size 4×6 and scalar multiplication is closed (6), it also implies that $c(A + B) = c[a_{ij} + b_{ij}] = [ca_{ij} + cb_{ij}] = [ca_{ij}] + [cb_{ij}] = cA + cB$ (7), and $(c + k)A = (c + k)[a_{ij}] = [(c + k)a_{ij}] = [ca_{ij} + ka_{ij}] = cA + kA$ (8). We also have $cdA = c(dA) = (cd)A$ by scalar multiplication of matrices (9). It is obvious that $1A = A$ (10). Hence, It is a vector space. ■

Problem 15

It is not a vector space as $x^3 + 5x^2 + (-1)(x^3) = x^3 + 5x^2 - x^3 = 5x^2$, clearly not closed under addition as we can see the result is a polynomial of degree 2.

Problem 17

It is not a vector space, take $p(x) = x$ and take $q(x) = -x$, then $p(x) + q(x) = x - x = 0$ wich is not in the set. Hence, the set is not closed under addition.

Problem 19

As stated in the lecture notes, the set $\mathcal{P}_n$ of all polynomials of degree n or less is a vector space under polynomial addition and scalar multiplication. Therefore, the set of all polynomials of degree four or less is a vector space.

Problem 22

It is not a vector space, take the pair $(1, 1)$. Observe $(-1)(1, 1) = (-1, -1)$, clearly not closed under scalar multiplication.

Problem 24

It is a vector space.

(1),(2) Choose x to be a, b, where $a, b \in \mathbb{R}$. We observe $(a, \frac{1}{2}a)+(b, \frac{1}{2}b) = (a+b, \frac{1}{2}a+\frac{1}{2}b) = (a+b, \frac{1}{2}(a+b)) = (b+a, \frac{1}{2}(b+a))$, clearly closed under addition and it satisfies the commutativity of addition.

(3) Choose a, b, c, where $a, b, c \in \mathbb{R}$. Then $(a, \frac{1}{2}a) + (b, \frac{1}{2}b) + (c, \frac{1}{2}c) = (a, \frac{1}{2}a) + (b+c, \frac{1}{2}(b+c)) = (a+b, \frac{1}{2}(a+b)) + (c, \frac{1}{2}c)$, satisfies the addition associativity.

(4),(5) It is clear that $(0, \frac{1}{2}(0)) = (0, 0)$ is the additive identity, and $(-a, \frac{1}{2}(-a))$ is the additive inverse, where $a \in \mathbb{R}$.

(6),(7),(8) Choose k, d to be real numbers scalars. See that $c(a, \frac{1}{2}) = (ca, \frac{c}{2}a) = (ca, \frac{1}{2}(ca))$, and $c((a, \frac{1}{2}a) + (b, \frac{1}{2}b)) = c(a+b, \frac{1}{2}(a+b)) = (c(a+b), \frac{1}{2}c(a+b)) = (ca+cb, \frac{1}{2}(ca+cb)) = (ca, \frac{c}{2}a)+(cb, \frac{c}{2}b)$. We can also see $(c+d)(a, \frac{1}{2}a) = ((c+d)a, \frac{(c+d)}{2}a) = (ca+da, \frac{c}{2}a+\frac{d}{2}a) = (ca, \frac{c}{2}a) + (da, \frac{d}{2}a)$.

(9) It is clear that $(cd)(a, \frac{1}{2}a) = c(da, \frac{d}{2}a)$, and $1(a, \frac{1}{2}a) = (a, \frac{1}{2}a)$.

Problem 26

It is not a vector space.

Suppose A is a matrix of the form $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$, we get a matrix $\begin{bmatrix} a+a & b+b \\ c+c & 2 \end{bmatrix}$ when we add A to itself, clearly it is not closed ander addition.

Problem 31

It is not a vector space.

Choose two singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Adding them, we get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the identity matrix, and we know the identity matrix is invertible, clearly not closed under addition.

Problem 32

It is not a vector space.

Choose two invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Adding them, we get $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, which is a singular matrix, clearly not closed under addition.

Problem 34

It is a vector space.
Suppose A, B are upper triangular square matrices of size 3, and c, k are real numbers.
(1),(2),(6) From matrix addition and scalar multiplication, we know $A + B$ and cA are upper triangular square matrices of size 3, clearly closed under addition and scalar multiplication. We also know $A + B = B + A$, which satisfies addition commutativity.
(3),(7) Suppose D an upper triangular square matrix of size 3. By matrix addtion, we know $(A + B) + D = A + (B + D)$. We also know that $c(A + B) = cA + cB$.
(4),(5),(10) We clearly see that the zero square matrix of size 3 is an additive identity in this set, and the additive inverse of any matrix A is $-A$ such that $A + (-A) = \mathbf{0}_{3\times 3}$. $1A = A$ is obviously satisfied for any matrix.
(8) We observe $(c + k) = A = (c + k)[a_{ij}] = [(c + k)a_{ij}] = [ca_{ij} + ka_{ij}] = c[a_{ij}] + k[a_{ij}] = cA + kA$.
(9) Finally, observe $(ck)A = ck[a_{ij}] = c[ka_{ij}] = c(kA)$.

Problem 36

It is a vector space.
Suppose $f(x), g(x)$ to be a continous functions defined on the interval [-1,1]. It is clear that this set is a subset of the set of continous real valued functions, which we proved it is a vector space in class. Since booth f, g are defined on the interval, and by what we defined in class, $(f+g)(x) = f(x)+g(x)$ and $(cf)(x) = c(f(x))$. It is clear that it is closed under addition and scalar multiplication. It is also non empty as this set includes the zero function. So, it passes the subspace test. Hence, the set of all continous functions defined on interval $[-1, 1]$ is a vector space.

Problem 37

It is a vector space. Suppose a, b are positive real numbers. By definition, their addition is ab, which is a positive real number, clearly closed under addition (1). We can also see it satisfies commutativity of addition as $b + a = ba = ab = a + b$. (2).
Suppose another real number c. By definition $c + a + b = c(ab) = c + (a + b) = (ca)b = (c + a) + b$, clearly satisfies the associativity (3).
We find $a + 1 = a(1) = a$, clearly the additive identity (4). And the additive inverse is clearly 0 as $0 + a = (0)a = 0$ (5).
(6) By definition of scalar multiplication, suppose c a real number, then $ca = a^c$, clearly a positive real number. Hence, it is closed under scalar multiplication. For (7), suppose c, d are scalars and a, b are positive real nubmers, then $c(a + b) = (a + b)^c = (ab)^c = a^c b^c =$

$ca + cb$. And for (8), suppose c, d are scalars, then $(c+d)a = a^{c+d} = a^c * a^d = ca + da$. (9) Suppose c, d are real scalars, then $(cd)a = a^{(cd)} = (a^c)^d = d(a^c) = d(ca)$. The last one is also satisfied as a^1 is a.

Problem 38

Suppose the pairs $(x, y), (a, b)$ where x, y are real numbers. We can see it is clearly closed under addition and scalar multiplication as $(x, y)+(a, b) = (xa, yb)$, and $c(x, y) = (cx, cy)$. Its additiion is also commutative as $(a, b) + (x, y) = (ax, by) = (xa, yb) = (x, y) + (a, b)$. Suppose another pair (n, m), where n,m are real numbers. We observe that $(a, b)+(n, m)+(x, y) = (anx, bmy) = (a(nx), b(my)) = (a, b) + (nx, my) = (a, b) + ((n, m) + (x, y)) = ((an)x, (bm)y) = (an, bm) + (x, y) = ((a, b) + (n, m)) + (x, y)$. We notice that $(1, 1)$ is the additive identity (The zero vector) as $(1, 1) + (a, b) = (a(1), b(1)) = (a, b)$. For the additive inverse, we need to find a pair (x,y) such that $(a, b) + (x, y) = (1, 1)$ so $x = \frac{1}{a}$ and $y = \frac{1}{b}$, which will not hold if a, b are zeros. So, it does not satisfy the axiom of additive inverse. It fails for distributivity of scalar multiplication of vector addition as we observe $c((a, b) + (x, y)) = (cax, cby)$ will equal $(ca, cb) + (cx, cy) = (c^2xa, c^2by)$. only if $c = c^2$, which will only hold if $c = 0, c = 1$ but we defined c to be any real number. It also fails for axiom (8) as $(c + d)(x, y) = ((c + d)x, (c + d)y)$, but $c(x, y) + d(x, y) = (cdx^2, cdy^2)$, clearly they are not equal, for example, let $c = 2, d = 3, (1, 1)$. After substituting in the previous equations, we get that $(5, 5) \neq (6, 6)$.

Problem 41

a) Suppose c, d are real numbers, and the pair (x, y) such that $x, y \in \mathbb{R}$. We see $(c + d)(x, y) = ((c + d)x, y)$ but $c(x, y) + d(x, y) = ((c + d)x, 2y)$, clearly it fails to satisfy axiom (8). Hence, it is not a vector space.