Linear Algebra

Assignment 4

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3.1 The Determinant of a Matrix

Problem 12

$\begin{vmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4) - 4 \times 0 = \lambda^2 - 6\lambda + 8.$

Problem 25

$x \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - y \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = x(2-0) - y(3-0) - (3-2) = 2x - 3y - 1.$

Problem 51

$\lambda \begin{vmatrix} \lambda + 1 & 2 \\ 1 & \lambda \end{vmatrix} = \lambda(\lambda(\lambda+1) - 2) = \lambda(\lambda^2 + \lambda - 2) = \lambda((\lambda+2)(\lambda-1)) = 0$. Since the product is zero, then $\lambda = 0$ or $\lambda = 1$ or $\lambda = 0$.

Problem 68

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^3-a^3 & c^3-a^3 \end{vmatrix} \qquad ((R_3 - a^3R_1) \to R_3 \text{ and } (R_2 - aR_1) \to R_2)$$

$$\begin{aligned}
&= (b-a)(c^3-a^3) - (c-a)(b^3-a^3) \\
&= (b-a)(c-a)(c^2+a^2+ca) - (c-a)(b-a)(b^2+a^2+ba) \\
&= (c-a)(b-a)(c^2-b^2+a(c-b)) \\
&= (c-a)(b-a)((c-b)(c+b)+a(c-b)) \\
&= (c-a)(b-a)(c-b)(a+b+c) \\
&= (a-b)(b-c)(c-a)(a+b+c)
\end{aligned}$$

3.2 Determinants and Elementary Operations

Problem 23

$$det\begin{pmatrix} 5 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & 1 & 2 \\ -1 & 0 & 3 & 1 \end{pmatrix} = -\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= -1 + 6 - 2 - 2 + 6 + 1 = 8.$$

Problem 44

Proof. We compute the determinant using cofactor expansion as follows:

$$det\begin{pmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{pmatrix} = (1+a)\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1+b & 1 \end{vmatrix}$$

$$\begin{aligned}
&= (1+a)((1+b)(1+c)-1) - (1+c-1) + (1-1-b) \\
&= (1+b+a+ab)(1+c) - (1+a) - c - b \\
&= 1+b+a+ab+c+bc+ac+abc-1-a-c-b \\
&= ab+bc+ac+abc = \frac{abc}{abc}(ab+bc+ac+abc) \\
&= abc(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}+1) = abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})
\end{aligned}$$

That satisfies the proof. ∎

3.3 Properties of Determinants

Problem 45

a) We know $|A^T| = |A|$, so $|A^T| = 5\begin{vmatrix} -3 & 0 \\ -1 & 2 \end{vmatrix} = 5(-6) = -30$.

b) By multiplicativity of a determinant, $det(A^2) = det(AA) = det(A)det(A) = -30 \times -30 = 900$.

c) By multiplicativity of a determinant, $det(A^T A) = det(A^T)det(A)$. And we know $det(A^T) = det(A)$. So, $|A^T A| = -30 \times -30 = 900$.

d) A is a matrix of size 3×3, so $|2A| = 2^3|A| = 8 \times -30 = -240$.

e) $|A^{-1}| = \frac{1}{|A|} = \frac{-1}{30}$.

Problem 59

Proof. By Multiplicativity of Determinant, $det(AB) = det(A) \times det(B) = det(I) = 1$. We clearly see that if either $det(A), det(B)$ is zero, then it is impossible for their product to be 1. Therefore, we showed that $|A| \neq 0, |B| \neq 0$. ■

Problem 68

Proof. We know that if a matrix is invertible then its determinant cannot be zero. And we know, by multiplicativity of determinant, that $det(A^{10}) = (det(A))^{10}$. Suppose A is invertible, then $|A| \neq 0$ and clearly $|A|^{10} \neq 0$. A contradiction. Therefore, A must be singular if $det(A^{10}) = \mathbf{0}$. ■

Problem 69

Proof. Suppose A a square, skew-symmetric matrix. We know $A^T = -A$. By multiplicativity of determinant, $|A^T| = |-A| = (-1)^n|A|$. But we know that $|A^T| = |A|$, so $|A| = (-1)^n|A|$, which satisfies the proof. ■

Problem 83

Proof. Suppose A is an idempotent matrix and $A^2 = A$. By multiplicativity of determinant, $det(A^2) = det(A)$ and $det(A) \times det(A) = det(A)$. If $det(A) \neq 0$, then we can devide booth sides by $det(A)$ and get $det(A) = 1$. In case $det(A) = 0$, it is clearly obvious that it satisfies $|A^2| = |A|$. Hence, that satisfies the proof. ■