Linear Algebra

Assignment 9

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June 2025

4.6 Rank of a matrix

Problem 11

a)

$$\begin{bmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix} \xrightarrow{\frac{-1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 2 & -2 & \frac{-5}{2} \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{(R_2 - 3R_1) \to R_2} \begin{bmatrix} 1 & 2 & -2 & \frac{-5}{2} \\ 0 & 0 & 0 & \frac{7}{2} \\ -2 & -4 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{(R_3 + 2R_1) \to R_3} \begin{bmatrix} 1 & 2 & -2 & \frac{-5}{2} \\ 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{7}{2}R_2 \to R_2} \begin{bmatrix} 1 & 2 & -2 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{(R_3 - 4R_2) \to R_3} \begin{bmatrix} 1 & 2 & -2 & \frac{-5}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the basis for the row space is the set $\{(1, 2, -2, \frac{-5}{2}), (0, 0, 0, 1)\}$.

b) The rank of the matrix is the cardinality of basis for the row space. So, the rank is 2.

Problem 14

We obtain a matrix A from the vectors in S.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

We keep applying the ERO till we reach the REF.

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 4 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{(R_2+R_1)\to R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{(R_3-2R_1)\to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ 0 & -1 & -7 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{5}R_2\to R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{8}{5} \\ 0 & -1 & -7 \end{bmatrix} \xrightarrow{(R_3+R_2)\to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & \frac{-27}{5} \end{bmatrix}$$
$$\xrightarrow{(\frac{-5}{27})R_3\to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rows of the matrix are exactly the vectors in S, we get that the supspace of $R^3 = Span(S) = R(A)$. Therefore, the set $\{(1, 2, 4), (0, 1, \frac{8}{5}), (0, 0, 1)\}$ forms a basis for the subspace.

Problem 25

a)

$$\begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix} \xrightarrow{(R_2-7R_1) \to R_2} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & \frac{9}{2} & 18 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{(R_3+2R_1) \to R_3} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & \frac{9}{2} & 18 \\ 0 & 0 & -2 & -8 \\ 2 & 4 & -2 & -2 \end{bmatrix} \xrightarrow{(R_4-2R_1) \to R_4} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & \frac{9}{2} & 18 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{2}{9}R_2 \to R_2} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -1 & 4 \end{bmatrix} \xrightarrow{(R_3+2R_2) \to R_3} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(R_3+R_2) \to R_3} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{8}R_3 \to R_3} \begin{bmatrix} 1 & 2 & \frac{-3}{2} & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the pivot columns of the REF matrix are the 1^{st}, 2^{nd}, and the 4^{th} column. Consequently, the set contains the 1^{st}, the 2^{nd}, and the 4^{th} columnns of matrix A:

$$\left\{ \begin{bmatrix} 2 \\ 7 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ -2 \\ -2 \end{bmatrix} \right\}$$

are the basis of the column space of A.

b) The rank of the matrix $A = dim(C(A)) = 3$.

Problem 35

We keep applying the EROs till we reach the REF:

$$A = \begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_1 \to R_1} \begin{bmatrix} 1 & \frac{2}{5} \\ 3 & -1 \\ 2 & 1 \end{bmatrix} \xrightarrow[(R_3 - 2R_1) \to R_3]{(R_2 - 3R_1) \to R_2} \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{-11}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$$
$$\xrightarrow{\frac{-5}{11}R_2 \to R_2} \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \\ 0 & \frac{1}{5} \end{bmatrix} \xrightarrow{(R_3 - \frac{1}{5}R_2) \to R_3} \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

We know that $N(B) = N(A)$. And we get the system

$$x + \frac{2}{5}y = 0y = 0.$$

This system clearly only has the trivial solution $(0,0)$. Hence, $N(A) = (0,0)$

Problem 42

Since we know that A, B are row equivalent, we can obtain B by applying EROs on A and they have the same row space, rank, nullity.

a) We can see that B is in the REF, and we know that the rank is the number of the pivot columnns and the nullity is the number of the non-pivot columns. It is clear that the $1^{st}, 2^{nd}, 4^{th}$ columns are pivot columns and the $3^{rd}, 5^{th}$ columns are the non-pivot columns. Hence, $rank(A) = 3, nullity(A) = 2$

b) Since B is in REF we obtain the follwoing homogeneous system of linear equations

$$\begin{aligned} x + z + l &= 0 \\ y - 2z + 3l &= 0 \\ w - 5l &= 0 \end{aligned}$$

, the non-pivot columns corresponds to free variables, so we choose $z = s, l = t$ and get that $w = 5t, y = 2s - 3t, x = -s - t$. and that is the solution of the homogeneous system. So we get that $N(A) = \left\{ \begin{bmatrix} -s-t \\ 2s-3t \\ s \\ 5t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$ and for every vector $\vec{v} \in N(A)$ we have:

$$\vec{v} = \begin{bmatrix} -s-t \\ 2s-3t \\ s \\ 5t \\ t \end{bmatrix} = \begin{bmatrix} -s \\ 2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ -3t \\ 0 \\ 5t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Thus, every vector $\vec{v} \in N(A)$ can be obtained by a linear combination of vectors in the set

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

So it is clearly a spanning set, it is also linearly independent as we have zeros in one vector that correspnds to numbers in the others, whihc implies we cannot get one by a linear combination of the other. Therefore, S is a basis of $N(A)$

c) The basis of the row space is the set that contains the pivot rows of the REF matrix. Hence, the basis $T = \{(1,0,1,0,1),(0,1,-2,0,3),(0,0,0,1,-5)\}$.

d) The basis of the column space of A is the set that contains the columns in A that correspnds to the pivot columns in B. Hence, the basis

$$W = \{(-2,1,3,1),(-5,3,11,7),(0,1,7,5)\}.$$

e) Since $dim(R(A)) = 3$, and the number of rows in A is 4, then the rows of A are not linearly independent.

f) i) We know that this set is the basis of $C(A)$ from (d). And we know that a basis is linearly independent. Hence, the set is linearly independent.

ii) The set is linearly dependent as we can get $a_3 = a_1 - a_2$.

iii) Since we have zero in a_3, a_1 that correspnds to number in a_5, then we cannot get any of them as linear combination of the others. Hence, the set is linearly independent.