Linear Algebra

Assignment 7

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4.4 Spanning Sets and Linear Independence

Problem 2

d)

$$
\begin{aligned}
&(1,-5,-5) = a(1,2,-2) + b(2,-1,1) \\
\iff &1 = a + 2b, -5 = 2a - b, -5 = -2a + b
\end{aligned}
$$

We solve this system of equations:

$$
\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -5 \\ -2 & 1 & -5 \end{bmatrix} \xrightarrow{(R_2-2R_1)\to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -7 \\ -2 & 1 & -5 \end{bmatrix} \xrightarrow{(R_3+2R_1)\to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -7 \\ 0 & 5 & -3 \end{bmatrix}
$$

$$
\xrightarrow{(\frac{-1}{5})R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{7}{5} \\ 0 & 5 & -3 \end{bmatrix} \xrightarrow{(R_3-5R_2)\to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{7}{5} \\ 0 & 0 & -10 \end{bmatrix}.
$$

From the last step we get $0a + 0b = 10$. So the system has no solution and the vector u cannot be written as linear combination of vectors in S.

Problem 23

Let $\vec{v} = (a, b, c) \in R^3$.
We observe if $x(1, -2, 0) + y(0, 0, 1) + z(-1, 2, 0) = (a, b, c)$ has a unique solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{(R_2+2R_1)\to R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

It has a row of zeros, so it is not invertible and it does not have a unique solution, so this set does not span R^3.

Problem 46

We see:

$$2 + 3x + x^2 = 6 + 5x + x^2 + 2(-2 - x).$$

We could form one of the vectors by a linear combination of the other vector is the same set. So the set is linearly dependent.

Problem 50

We see that every 1 in each matrix corresponds to a zero in the other matrices. Hence, it is impossible to obtain one matrix by a linear combination by the other matrices. So the set is linearly independent.

Problem 57

b) By the equation

$$a(t, 1, 1) + b(1, 0, 1) + c(1, 1, 3t) = (0, 0, 0)$$

We construct this system of equations:

$$at + b + c = 0, a + c = 0, a + b + 3tc = 0.$$

We find the values of t that makes the following matrix invertible.

$$\begin{vmatrix} t & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 3t \end{vmatrix} = t(-1) - (3t - 1) + (1) = -4t + 2.$$

We see the determinant is not zero when $-4t + 2 \neq 0 \iff -4t \neq -2 \iff t \neq \frac{1}{2}$. So every other value for t makes this set linearly independent.

Problem 62

Attached in the mail (Written in paper.)

Problem 73

Proof. Since S is linearly Independent, $au + bv = \vec{0}$ has only the trivial solution. Pick two scalars $x, y \in \mathbb{R} - \{0\}$. Assume $x(u+v) + y(u-v) = (0,0)$ has other solution than the trivial one. Observe $xu + xv + yu - yv = (0,0) = (x+y)u + (x-y)v$. But we know u, v are linearly independent, so $x + y = 0, x - y = 0$ and clearly $x = 0, y = 0$. which is a contradiction. Hence, the set $\{u + v, u - v\}$ is linearly independent. ■

Problem 75

Proof. Let A be a nonsingular matrix of order 3, so A^{-1} exists. Pick scalars $a, b, c \in \mathbb{R}$. We want to prove $aAv_1 + bAv_2 + cAv_3 = \vec{0}$. only has the trivial solution. Observe that we can multiply both sides by A^{-1} and get $aA^{-1}Av_1 + bA^{-1}Av_2 + cA^{-1}Av_3 = A^{-1}\mathbf{0} = av_1 + bv_2 + cv_3 = \mathbf{0}$. But we know $\{v_1, v_2, v_3\}$ is linearly independent, so $a = 0, b = 0, c = 0$ and $\{Av_1, Av_2, Av_3\}$ is also linearly independent. ■

If A is a singular, the set is not linearly independent. A counter example is the zero matrix $\mathbf{0}_{3\times 3}$, which will satisfy the equation for any scalars $a, b, c \in \mathbb{R}$, so the equation will have other solutions than the trivial one and the set is linearly dependent.