Linear Algebra

Report 5

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Definition 1: Linear transformation

Let W, V be two vector spaces and T is a function that its domain is V and its codomain is W. We call T a linear transformation iff for any $c \in \mathbb{R}$ and for every vector $\vec{v}, \vec{u} \in V$ the following conditions hold:

- $T(\vec{v} + \vec{u}) = T(\vec{v}) + T(\vec{u})$
- $T(c\vec{v}) = cT(\vec{v})$.

Example

The linear transformation $T : V \to V$ such that $T(x, y) = (x + y, 3x - 7y)$. We see it is a linear transformation as it is closed under vector addition and scalar multiplication as follows Pick two vectors $\vec{u} = (x, y), \vec{v} = (a, b) \in V$ and scalar $c \in \mathbb{R}$, we see

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T((x + a, y + b)) \\ &= (x + a + y + b, 3x + 3a - 7y - 7b) \\ &= (x + y, 3x - 7y) + (a + b, 3a - 7b) = T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(c\vec{v}) &= T((ca, cb)) = (ca + cb, 3ca - 7cb) \\ &= (c(a + b), c(3a - 7b)) = c(a + b, 3a - 7b) = cT(\vec{v}). \end{aligned}$$

Non-example

The function $T(x, y) = (x + 1, y - 1)$ is not a linear transformation as it is not closed under vector addition. We pick two vectors $\vec{v} = (x, y), \vec{u} = (a, b) \in V$ and see

$$T(\vec{v} + \vec{u}) = T(a + x, b + y) = (a + b + 1, b + y - 1)$$

Which clearly does not satisfy the closure of addition.

Definition 2: Kernel of a linear transformation

Let $T : V \to W$ be a linear transformation. We define the kernel of the linear transformation as the set of all vectors in V that are mapped to the zero vector of W. that is,

$$kernel(T) = \{\vec{v} \in V | T(\vec{v}) = \vec{\mathbf{0}}_W\}$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection of a vector into the $yz - plane$, such that $T(x, y, z) = (0, y, z)$. Then $ker(T) = \{(t, 0, 0) | t \in \mathbb{R}.\}$

Non-example

A non example is any vector in V that its image is not $\vec{\mathbf{0}}_W$. An example is $S = \{(1, 1, 1)\}$, where $T(1, 1, 1) = (0, 1, 1)$.

Definition 3: Vector space isomorphism

Let V, W be two vector spaces. An *isomorphism* from V to W is a bijective linear transformation $T : V \to W$. We say V is isomorphic to W iff we can find at least one isomorphism from V to W and we write that $V \cong W$.

Example

The vector spaces

$$R^5 \cong M_{5\times1} \cong M_{1\times5} \cong P_4$$

are all isomorphic to each other since they all are of dimension 5.

Non-example

The vector spaces

$$R^3 \ncong M_{2\times 2}$$

are not isomorphic since they are of different dimensions, the first of dimension 3 and the latter of dimension 4.

Definition 4: Eigenvector of a matrix

A nonzero vector $\vec{x} \in R^n$ is called an eigenvector of a square matrix A of size $n \times n$ if and only if there exists a real number $\lambda \in R$ such that

$$A\vec{x} = \lambda\vec{x}$$

. The number λ is called an eigenvalue of A, and we say in such situation that $\vec{x}$ is an eigenvector of A corresponding to the eigenvalue λ.

Example

let A be the 2×2 matrix

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix}$$

An eigenvector is

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

As we can see that

$$A\vec{v} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\vec{v}.$$

Non-example

For the previous matrix A let

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And observe

$$A\vec{v} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix},$$

which is not an eigenvector as there is no $\lambda \in \mathbb{R}$ such that $A\vec{v} = \lambda\vec{v}$.