Linear Algebra

Assignment 8

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4.5 Basis and Dimension

Problem 20

Proof. We know that the basis must be linearly independent. But we see $(-1,0,0) = 0(1,0,0) + (-1)(1,0,0)$, clearly the set is not linearly independent. Therefore, we proved the set is not a basis for R^3. ■

Problem 30

Proof. We know that the basis must be linearly independent. But we see $3x^2 = 3(1-2x-x^2) + (-3+6x)$, clearly the set is not linearly independent. Therefore, we proved the set is not a basis for P_2. ■

Problem 34

Proof. We know that the basis of any $\mathbb{M}_{m\times n}$ matrix is a set contains $m \times n$ many matrices of size $m \times n$. So the basis for $\mathbb{M}_{2\times 2}$ will have $2 \times 2 = 4$ elements, but the set contains only three elements. Therefore, it is not a basis for $\mathbb{M}_{2\times 2}$. ■

Problem 44

Proof. We know that the basis of any R^n vector space is a set contains n many vectors of size n and it must be linearly independent. The given set, $S = (0,0,0),(1,5,6),(6,2,1)$, which has three elements but it has the zero vector, so it is linearly dependent and it cannot be the basis for R^3. Therefore, we proved S is not a basis for R^3. ■

Problem 48

Proof. First, we need to determine whether the set S is a spanning set. Let (a,b,c,d) any vector that contains the coffectients of any polynomial in P_3. Pick scalars $x,y,z,w \in \mathbb{R}$. We need to show that $x(0,4,0,-1,0) + y(5,0,0,1) + z(1,3,0,0) + w(0,0,-3,2) = (a,b,c,d)$ has a unique solution (x,y,z,w) for every choice of (a,b,c,d). We get the system:

$$\begin{aligned} 5y + z &= a \\ 4x + 3z &= b \\ -3w &= c \\ y + 2w &= d \end{aligned}$$

We construct the matrix of the coffectients and compute its determinant:

$$A = \begin{bmatrix} 0 & 5 & 1 & 0 \\ 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{bmatrix},$$

$$det(A) = -4 \begin{vmatrix} 5 & 1 & 0 \\ 0 & 0 & -3 \\ 1 & 0 & 2 \end{vmatrix} = -4 * 3(5 * 0 - 1) = 12 \neq 0.$$

So the matrix is invertible and the system has a unique solution. Hence, the set is a spanning set of P_3.(1) If we choose (a, b, c, d) to be $(0, 0, 0, 0)$, and we know from (1) that the coffectient matrix is invertible and the system has a unique solution for every choice of (a, b, c, d). Then the trivial solution is the only solution. Hence, the set is linearly Independent. (2) Therefore, from (1),(2), we proved that S is a basis for P_3. ■

Problem 70

We know that a basis for R^3 must has three vectors. We find one example of that vector: $(0, 0, -2)$. To see that it is a basis we show it is a linearly independent spanning set. Pick scalars $x, y, z \in \mathbb{R}$. We need to show that $x(1, 0, 2) + y(0, 1, 1) + z(0, 0, -2) = (a, b, c)$ has a unique solution. We get the system:

$$\begin{aligned} x + 2z &= a \\ y + z &= b \\ -2z &= c \end{aligned}$$

We compute the determinant of the coffectient matrix of this system:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -2(1 - 0) = -2 \neq 0.$$

So the matrix is invertible and the system has a unique solution. If we choose $(a, b, c) = (0, 0, 0)$ it will also have one solution -as we proved the coffectient matrix is invertible- which is the trivial one. Therefore, we found a basis for R^3.

Problem 71

a) The line represented by the equation $y = \frac{1}{2}x$.

b) A basis for W is the set $(2, 1)$.
For every $(2t, t)$, we can represent it by $t(2, 1)$. Clearly the set spans W. The set has one vector so it is linearly Independent.

c) Since the basis consists of one vector, the dimension of W is 1.

Problem 78

a) A valid choice of a basis is $(1, 0, 1, 2), (4, 1, 0, -1)$.
We see it spans W as we can construct any vector in W by: $s(1, 0, 1, 2)+ t(4, 1, 0, -1) = (s + 4t, t, s, 2s - t)$. It is obviously linearly independent as if we multiplied the first vector by any scalar we cannot get the second vector as we have zero in the first vector that corresponds to 1 in the second vector.

b) Since the basis consists of two vectors, the dimension of W is 2.