Linear Algebra

Assignment 11

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6.2 The Kernel and Range of a Linear Transformation

Problem 7

By definition, the zero vector of codomain is the polynomial

$$0 + 0x$$

. Comparing the zero vector of the codomain with the linear transformation we get

$$0 + 0x = a_1 + 2a_2x$$

. Thus,

$$a_1 = 0, a2 = 0$$

. But we see that a_0 is not in the image so it can be a free variable and we let it to be a real number t. Therefore,

$$ker(T) = t + 0x + 0x^2 | t \in \mathbb{R}$$

Problem 22

a) To find the kernel we need to solve the system

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To solve this homogenous system, we start applying the EROs

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 2 & -3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 4 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & \frac{1}{4} \\ 2 & -3 \\ 0 & 0 \end{bmatrix} \xrightarrow{(R_2 - 2R_1) \to R_2} \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & \frac{-7}{2} \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{-2}{7}R_2 \to R_2} \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We have that $y = 0, x = 0$. So the kernel is the trivial space

$$ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

b) Using the fact that $nullity(T) = dim(ker(T))$. Then $nullity(T) = 0$, since the kernel only contains the trivial space.

c) $Range(T) = C(A)$, the span of the columns in A that correspond to the pivot columns, so

$$Range(T) = span \left\{ \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

d) $Rank(T) = dim(Range(T)) = dim(C(A)) = 2.$

Problem 40

Observe that $T(x, y, z) = (x, y, 0)$ can be written in the form $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

A is clearly in REF, and has two pivot columns which are the $1^{st}, 2^{nd}$ columns, so $rank(A) = 2$. Using the fact that $rank(T) + nullity(T) = rank(A) + nullity(A) = n$ we deduce that $nullity(T) = 3 - 2 = 1$.
To find the kernel, we solve the system $A\vec{x} = \vec{b}$ when $\vec{b} = \vec{\mathbf{0}}_{3\times 3}$. We see that A is already in REF, so we don't need to apply the Gaussian elimination method. We get the system

: $x = 0, y = 0$ and z is a free varible since it corresponds to the third column which is a non-pivot column. So

$$ker(T) = N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

Clearly, the geometric representation of the kernel is the set of all points in the $z - axis$ in $\mathbb{R}^3$.

To find the range, we use that $range(T) = C(A)$. Since A is in REF, we pick the columns in A that are pivot columns to get the basis of the column space.

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = range(T)$$

Which is interpretted geometrically as the set of every point in $xy - plane$ in $\mathbb{R}^3$.

Problem 45

We use the theorem that states

$$rank(T) + nullity(T) = dim(domain(T)).$$

We already have that $rank(T) = 4$, and $dim(domain(T)) = 2 \times 4 = 8$. Hence,

$$nullity(T) = dim(domain(T)) - rank(T) = 8 - 4 = 4.$$

Problem 57

By definition of the linear transformation, we have

$$T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$

The zero vector of the codomain is $0 + 0x + 0x^2 + 0x^3$. We compare the zero vector to the image of the linear transformation and get that in the kernal $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$. The ramaining is a_0 which we take as a free variable as its value does not affect the linear transformation. Thus,

$$ker(T) = \{t + 0x + 0x^2 + 0x^3 + 0x^4 | t \in \mathbb{R}\}.$$

Problem 58

We solve the following integral

$$\int_0^1 a_0 + a_1 x + a_2 x^2 = \left[a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3\right]_0^1$$
$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3}.$$

By definition of the kernel, the zero vector in $\mathbb{R}$ is $\mathbf{0}$. Then, we need to solve

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0.$$

Take a_1, a_2 as free variables s, t respectivily. Hence,

$$a_0 = \frac{-1}{2}s - \frac{1}{3}t.$$

and

$$p(x) = \frac{-1}{2}s - \frac{1}{3}t + sx + tx^2$$
$$= s(\frac{-1}{2} + x) + t(-\frac{1}{3} + x^2)$$

We see that we get a set of linear combinations of polynomials which form the kernel. Therefore,

$$ker(T) = span((\frac{-1}{2} + x), (-\frac{1}{3} + x^2))$$