Enhancing Algorithm Efficiency for the Vehicle Routing Problem: A Heuristic Approach Based on Column Generation.

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Abstract

This paper addresses the computational challenges associated with solving the Vehicle Routing Problem (VRP) through column generation, specifically focusing on improving the efficiency of the subproblem. Typically, the pricing subproblem of the VRP is cast as a Resource-Constrained Shortest Path Problem (RCSPP) and is frequently addressed through label-setting algorithms. Three well established exact algorithms—the Elementary Labeling Algorithm (ELA), the State-Space Relaxed ELA (SSRELA), and its K-best variant—are implemented in this study, alongside the introduction of two novel heuristic methods: I-SSRELA and I-M-SSRELA.

Through computational experiments on generated graph instances, I evaluate each algorithm's performance by comparing runtime, the number of iterations executed, and the number of generated columns. The results indicate that exact algorithms impose significant computational burdens when the underlying graph contains a high proportion of negative-cost edges and scale poorly as the problem size increases. In contrast, the proposed heuristic approaches employ a more conservative exploration strategy, mitigating this issue and achieving significant performance improvements. These findings indicate that the proposed heuristics provide a promising approach for accelerating column generation procedures in the context of the VRP.

Keywords: Combinatorial Optimization, Operations Research, Mixed-Integer Programming, Heuristic, Shortest Path Problem with Resource Constraints, Column Generation.

Contents

1 Introduction **3**

2 Related work **5**

3 Problem setup **6**

3.1 Master problem: set-covering formulation of a vehicle routing problem 6

3.2 Subproblem: resource-constrained shortest path problem 8

4 Algorithms to solve the subproblem **10**

5 Algorithm efficiency testing **13**

5.1 Results for the elementary labeling algorithm(ELA) 14

5.2 Results for the State-space relaxed ELA(SSRELA) 14

5.3 Results for the K-best SSRELA 15

5.4 Results for the ”ignoring” algorithm(I-SSRELA) 16

5.5 Result for the ”Ignoring” algorithm with multiplicity(I-M-SSRELA) 16

6 Conclusions and Future Work **17**

List of Tables

1 Test Results for the ELA 19

2 Summary Statistics for the Test Results of ELA 20

3 Results for the SSRELA 21

4 Summary Statistics for the results of SSRELA 22

5 Results for the ”Ignoring” Algorithm(I-SSRELA) 23

6 Summary Statistics for the results of the ”Ignoring” Algorithm(I-SSRELA) 24

7 Result of the ”Ignoring” algorithm with multiplicity(I-M-SSRELA) 25

8 Summary Statistics for the results of the ”Ignoring” algorithm with multiplicity(I-M-SSRELA) 26

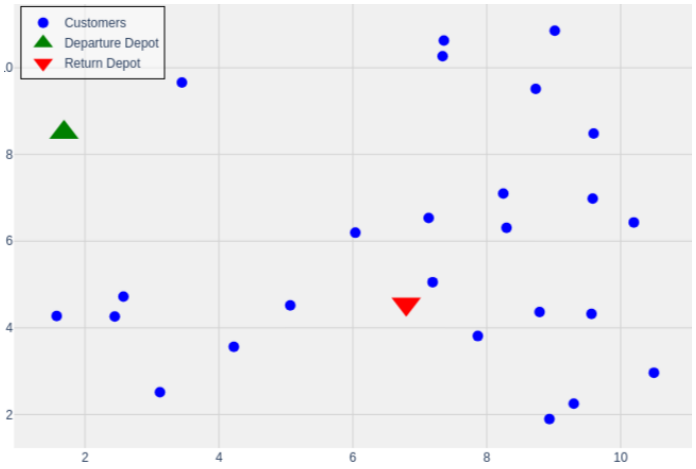
List of Figures

1 Example of a VRP problem 3

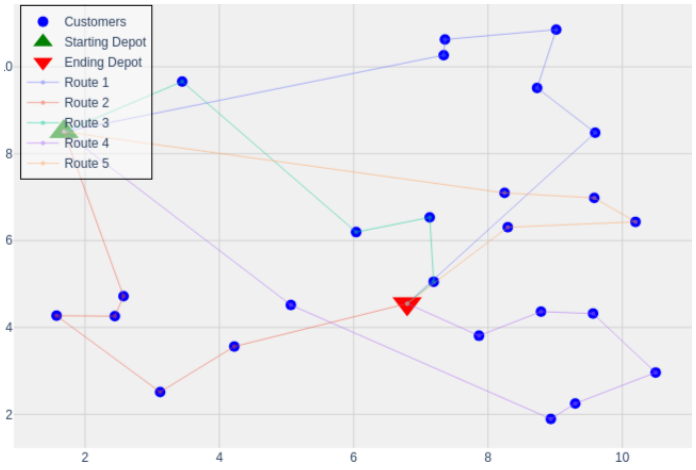
2 Comparison of Runtime(log-scaled) VS Propotion of Negative Arcs 15

1 Introduction

The Vehicle Routing Problem (VRP) is a logistics challenge that involves combinatorial optimization and mixed-integer programming. It is an extension of the traditional Traveling Salesman Problem (TSP). While the TSP aims to identify the most efficient single route for one individual to visit a set of customers by minimizing a specific cost metric (e.g., travel time or distance), the VRP seeks to develop multiple routes for a fleet of vehicles that collectively serve all customers in the most cost-effective manner(Dantzig and Ramser, 1959). The necessity for multiple vehicles arises primarily from vehicle capacity constraints(Dantzig and Ramser, 1959) and other operational restrictions, such as delivery time windows(Kohl, 1995) and route length limitations(Contardo and Martinelli, 2014). Although introducing these specialized constraints makes the optimization process more complex, they are essential for ensuring the model's applicability in real-world logistics and distribution systems.



(a) Random generated graph instance



(b) Solved VRP problem

Figure 1: Example of a VRP problem

Figure 1 shows a random generated VRP instance. Graph (a) of this figure illustrates a variation of the VRP where each vehicle starts from a designated starting depot (green triangle), visits some

customers (blue points), and ends at a different depot (red triangle). Graph (b) shows a feasible solution, where each customer is visited exactly once, and the routing adheres to the specified constraints such as vehicle capacity.

There are numerous algorithms designed to solve the Vehicle Routing Problem (VRP) and its variations, including but not limited to dynamic programming (Novoa and Storer, 2009), tabu search algorithm (Brandão, 2004), mathematical programming (Kara, 2011), and novel methods that utilize AI models (Czuba and Pierzchała, 2021). Most of these algorithms are heuristic, as the VRP is strictly NP-hard(Laporte and Nobert, 1987), and empirically, heuristic methods provide sufficiently cost-effective solutions(Arnold and Sörensen, 2019). In this paper, I focus specifically on a heuristic, mixed-integer programming (MIP) based algorithm: the column generation method (Choi and Tcha, 2007). This approach typically employs a set-covering formulation of the VRP, where the solution space comprises all feasible routes. Since the number of these routes grows exponentially with the number of customers, a direct enumeration is often infeasible. Column generation addresses this challenge by starting from a small subset of routes (variables) and then iteratively adding new routes (columns) to the model. Through this process, we are guaranteed to find the optimal solution of the linear relaxation of the VRP, where route variables may assume fractional values between 0 and 1. Once the relaxed problem is solved, I will use all the generated route variables to solve a 0-1 integer programming version of the same problem. While optimality cannot be assured, the objective value from the binary programming model usually differs by less than five percent compared to its linear relaxed counterpart.

In the context of column generation, the VRP is decomposed into a restricted master problem(RMP), which optimizes the objective function over a limited set of variables, and subproblems, whose role is to generate additional columns (routes) to improve the current solution (Desaulniers, Desrosiers, and (Eds.), 2006). While the continuous-relaxed master problem is a large-scale linear program, addressing its complexity is not my primary focus here. Instead, I concentrate on improving the efficiency when solving the subproblem, usually presented as a resource-constrained shortest path problem(Chabrier, 2006). In this subproblem, our goal is to find elementary routes that have a negative cost; such routes are capable of improving the current solution (Choi and Tcha, 2007). The label-setting algorithm is a popular choice for solving this problem (Boland et al., 2006). This algorithm represents routes as unique labels and, when run, maintains a priority queue of labels. It attempts to extend each label iteratively in a feasible manner until it is dominated or reaches the finish point. Although this is an exact algorithm that solves the problem to optimal, it empirically suffers significantly when the graph contains a very high proportion of negative edges or has a large number of vertices, which drastically increases its runtime. To resolve this, I proposed two heuristic variations of this algorithm.

The remainder of this paper is organized as follows. In Section 2, I present a brief literature review. Section 3 outlines the problem and its formulation. In Section 4, I describe the label-setting algorithms used to solve the subproblem, including the two heuristic variations that I propose. Section 5 presents the testing results of these algorithms on generated graph instances, detailing runtime, the number of columns generated, and iterations performed. Finally, Section 6 provides the conclusions and discusses potential directions for future work.

2 Related work

The Vehicle Routing Problem (VRP) was first introduced by Dantzig and Ramser (1959), who framed it as the "Truck Dispatching Problem." They extended the traditional Traveling Salesman Problem (TSP) by incorporating delivery quantities for each customer and capacity limits for the trucks. Since then, a multitude of VRP variants have emerged to better reflect practical constraints. For example, Solomon (1987) proposed the Vehicle Routing Problem with Time Windows (VRPTW), where certain customers must be served within specified time intervals. In their study, an insertion-type heuristic was shown to consistently yield strong performance. Subsequently, Dror and Trudeau (1990) introduced the Split Delivery Vehicle Routing Problem (SDVRP), permitting multiple deliveries to a single customer. Later, Laporte et al. (2002) developed the Stochastic Vehicle Routing Problem (SVRP), introducing uncertainty in customer demands and the potential for delivery failure if a vehicle arrives without sufficient capacity. They proposed an integer L-shaped method to minimize both expected failure and travel costs.

Dantzig and Ramser (1959) also provided the first mathematical formulation for the VRP, using a two-index, arc-based representation in which decision variables indicate whether an arc is included in the solution. This formulation has since been extended (e.g., Gavish and Graves, 1978) to include three-index versions that specify which vehicle travels each arc (e.g., Pham, 2021). In this work, however, I adopt the set-partitioning formulation, as described by Balinski and Quandt (1964). This formulation can be derived via Dantzig-Wolfe decomposition (G. B. Dantzig and Wolfe, 1960) applied to the three-index VRP formulation (Petersen and Jepsen, 2009). By using this decomposition, the master problem is stated such that subtour elimination constraints, capacity constraints, and any other route-specific restrictions are inherently satisfied. Common subtour elimination strategies include the MTZ constraints (Miller et al., 1960) and the DFJ constraints (G. Dantzig et al., 1954). Consequently, the variables in the master problem represent feasible elementary routes, and the only additional requirement is that every customer is covered exactly once. New variables (columns) are generated by solving a pricing subproblem, which often takes the form of a shortest path problem (Chabrier, 2006).

The literature on VRP algorithms can be broadly divided into two categories: exact and heuristic methods. Among exact algorithms, Fischetti et al. (1994) presented a branch-and-bound approach based on an additive procedure for the Capacitated VRP (CVRP), while Bard et al. (2002) proposed a branch-and-cut scheme for the VRPTW. Another notable exact method is the branch-and-cut algorithm with a network flow formulation for the CVRP by Baldacci et al. (2004). On the heuristic side, Prins (2004) developed a simple yet effective hybrid genetic algorithm, and Altınel and Öncan (2005) introduced an enhanced Clarke-Wright savings algorithm for the CVRP. More recently, Ma et al. (2024) proposed Neural k-Opt (NeuOpt), a learning-to-search solver for routing problems. Additionally, column generation has been an influential technique, many works are dedicated in this area including recent ones like the work of Yuan et al. (2021).

Column generation is designed for linear or mixed-integer programs with potentially exponentially many variables. It begins with a restricted set of columns and solves the resulting master problem. Based on the dual information obtained, one or more pricing subproblems are then

solved to identify new columns that may improve the objective function (Desaulniers, Desrosiers, and Solomon, 2006). To the best of my knowledge, Gilmore and Gomory (1961) were the first to apply this methodology to a Mixed-Integer Programming (MIP) problem, specifically the cutting stock problem. Since then, column generation has been successfully applied to numerous mathematical programming challenges, including aircraft scheduling (Desaulniers et al., 1997), graph coloring (Nemhauser and Park, 1991), and traffic assignment (Ribeiro et al., 1989). When applied to the VRP, the subproblem typically takes the form of a Resource-Constrained Shortest Path Problem (RCSPP) (Chabrier, 2006), often involving negative-cost edges that pose additional computational difficulties.

3 Problem setup

Typically, VRP problems involve a single depot where routes commence and conclude. However, in my approach, I duplicate the depot into a "starting depot" and an "ending depot" for modeling convenience, without altering the underlying setup.

3.1 Master problem: set-covering formulation of a vehicle routing problem

Suppose there are n customers indexed by $\{1,\ldots,n\}$ in a directed graph G that I aim to serve. Additionally, the graph G includes a starting depot with index 0 and an ending depot with index $n+1$.

The problem entails assigning a certain number of vehicles to serve all customers. Each vehicle must depart from the starting depot and arrive at the ending depot. Although there is no strict constraint on the number of vehicles that can be used, a "soft upper bound" is set to n, implying that each customer could potentially be served by a different vehicle. Each customer has a strictly positive service demand, and each vehicle is subject to a capacity constraint regarding this demand. All vehicles possess identical service capabilities, and by default, I set the capacity as:

$$\textbf{max}\left\{\max(\text{All customers' demand}), \frac{6n}{4}\right\}$$

This capacity setting is an arbitrary choice to ensure an appropriate setup of the VRP instance generated.

Each edge in the graph has an associated time cost, which is computed based on the Euclidean distance from the generated graph. The objective is to determine a set of routes that minimizes the total time cost while ensuring that the service capacity of each vehicle is not exceeded.

In summary, the restricted master problem(RMP) can be expressed as follows:

Parameters

r: Index for routes. A feasible route must satisfy three conditions:

1. Start from the starting depot and end at the ending depot;
2. Its cost must not exceed the capacity constraint;
3. It is elementary, which means that no vertex is visited twice.

c_r: cost of route r

R: The set of all routes r

V: The set of all customer vertices.$|V| = n$.

$R_i : \{r \in R : \text{r visits vertex } i\}$

$R_i' : \{\text{subsets of } R_i\}$

Formulation

$$\begin{aligned} \min \quad & \sum_{r \in R'} c_r x_r \\ \text{s.t.} \quad & \sum_{r \in R_i'} x_r \geq 1 \qquad \forall i \in V \quad (1) \\ & x_r \geq 0 \qquad\quad \forall r \in R \quad (2) \end{aligned}$$

Variable Definition

$$x_r = \begin{cases} 1 & \text{if we use route } r \\ 0 & \text{otherwise} \end{cases}$$

The formulation can be interpreted as follows: the objective function minimizes the sum of costs associated with the active routes. There are two sets of constraints. The first set, known as the *covering constraints*, ensures that all customers are visited at least once. The number of these constraints is equal to the number of customers, with a clear one-to-one correspondence between each customer and its associated covering constraint. The second set consists of the standard non-negativity constraints for the variables x_r. It is unnecessary to include the constraint $\sum_{r \in R_i'} x_r = 1$, as visiting a customer more than once would never be optimal.

Solving the integer version of this problem can be challenging, but we can instead solve a continuous relaxed version of this RMP, which simply extends the domain of x_r to the non-negative real line. However, the number of routes (which corresponds to the number of variables) grows exponentially with the number of customers, so I employ a column generation algorithm to include only the relevant routes.

Nonetheless, the algorithm needs to be initialized in a feasible manner to commence its operation. Denote the initial set of routes as R_{start}. I use the following algorithm to initialize our column pool:

Algorithm 1 Initializing routes

Input: Number of customers $n > 0$
Output: A set of routes R_{start}
$R_{\text{start}} \leftarrow \emptyset$
for $i \leftarrow 1$ **to** n **do**
  Create new route r_i by linking starting depot $\rightarrow$ customer $i \rightarrow$ ending depot
  $R_{\text{start}} \leftarrow R_{\text{start}} \cup r_i$
end
return R_{start}

There are at least three compelling reasons for utilizing this algorithm. First, the number of routes in the initial column pool is equal to the number of customers, ensuring a manageable number of selected routes. Second, the routes are guaranteed to be feasible; if a route r_i for customer i is infeasible, it indicates that there is no possible way to serve this customer without exceeding the capacity constraint, thereby rendering the problem itself infeasible. Third, all customers are guaranteed to be served, ensuring a complete and viable initial solution.

After replacing R' with R_{start}, we can initiate the first iteration of the algorithm. To generate additional routes (columns) that will enhance the solution, we proceed to the subproblem, which is modeled as a resource-constrained shortest path problem.

3.2 Subproblem: resource-constrained shortest path problem

We now have our master problem initialized and solved for the first time (which solution should be $x_r = 1 \forall r \in \{1,..,n\}$ with an "optimal" objective value of $\sum_{r\in\{1,..,n\}} c_r$), we need to find new variables(columns) to further improve our solution. To do this, we will need to start from analysiing the dual problem of the master problem.

Let $\pi_i \forall i \in V$ denote the dual variables associated with the covering constraints. Following the intuition of Lahaie (2008), we derive the dual problem as the following:

Formulation

$$\begin{aligned} \max \sum_{i\in V} \pi_i \\ \text{s.t.} \quad c_r - \sum_{i\in V} \delta_{ir}\pi_i \geq 0 \quad \forall r \in R \\ \pi_i \geq 0 \quad \forall i \in V \end{aligned}$$

Variable definition

$$\delta_{ir} = \begin{cases} 1 & \text{if route } r \text{ passes customer i} \\ 0 & \text{otherwise} \end{cases}$$

Now we are looking for routes r such that $c_r - \sum_{i \in V} \delta_{ir} \pi_i < 0$ because we know that following the intuition of the Dantzig-Wolfe decomposition (Ralphs and Galati, 2010) these routes can improve the solution.

To find such route, the problem can be remodeled as the following:

Formulation

$$\begin{aligned} \min_{r \in R} \quad & c_r - \sum_{i \in V} \delta_{ir} \pi_i \\ \text{s.t.} \quad & \delta_{ir} \in \{1, 0\} \end{aligned}$$

And to solve this problem, we consider this as a resource-constrained shortest path problem(RCSPP) which goal is to find the shortest path from the starting depot to the ending depot after modifying the original graph with the following algorithm:

Algorithm 2 Modify graph

Input: A directed graph G with edge set E & A set of dual multipliers $\{\pi_i : i \in \{1, ..., n\}\}$
Output: An updated graph $G^{'}$

```
G' ← G
for i ← 1 to n do
    E'_i ← set of arcs in E' that ends at vertex(customer) i
    for e'_i ∈E'_i do
        c'_i ← cost of edge e'_i
        c'_i ← c'_i − π_i
    end
end
return G'
```

After this modification, we can solve the RCSPP and check if the path find has a negative cost. If yes, then we have found a new route that can reduce the total cost, hence we add this route back to the master problem and resolve it.

We will repeat this master problem-subproblem loop until no columns with negative cost can be found in the subproblem, which indicates that we have solved the continuous-relaxed VRP to optimal, then, we can use the columns generated to solve a binary integer programming problem to get a primal bound.

4 Algorithms to solve the subproblem

To solve the RCSPP problem, we use a label-setting algorithm based on dynamic programming, where paths are represented by labels that contain sets of resources. Irnich and Desaulniers (2005) provides an excellent survey on this type of algorithm. For a detailed explanation, refer to their work; here, we will provide a brief summary tailored to our specific case.

In the document of the boost graph library in C++, Drexl (2006) described the algorithm as the following:

> "The (RCSPP)functions are an implementation of a priority-queue-based label-setting algorithm. At each iteration, the algorithm picks a label l from a priority queue (the set of unprocessed labels) and checks it for dominance against the current set of labels at the vertex i where l resides. If l is dominated by some other label residing at i, l is deleted from the set of labels residing at i. If l is undominated, it is extended along all out-edges of i. Each resulting new label is checked for feasibility, and if it is feasible, it is added to the set of unprocessed labels and to the set of labels residing at the respective successor vertex of i. If a new label is not feasible, it is discarded. The algorithm stops when there are no more unprocessed labels. It then checks whether the destination vertex could be reached (which may not be the case even for a strongly connected graph because of the resource constraints), constructs one or all Pareto-optimal (i.e., undominated) s-t-path(s) and returns. "

The concept of *dominance*, commonly used in the context of labeling algorithms (Boland et al., 2006), provides a way to compare solution labels. More formally, a label l_a *dominates* another label l_b if it satisfies one of the following conditions:

1. l_a serves at least as many customers as l_b while having a strictly lower service cost.
2. l_a serves more customers than l_b at the same service cost.

At the onset of the algorithm, a single label l is placed on the depot. In the exact version of the algorithm, the resource vector associated with l is initialized as a $(n+2)$-dimensional vector of 0's. This vector includes: a floating-point resource to track capacity usage, an integer resource to record the last visited vertex (preventing two-vertex cycles), and n binary resources to indicate which customers have been visited (Beasley and Christofides, 1989). In the heuristic variants, only the floating-point and integer resources are initialized, and binary resources are added selectively as needed. When extending a label from one vertex to another, each arc e_{ij} in G' incurs an associated cost. To represent this, we define a resource cost vector for each arc e_{ij}, with length $n+2$ (or fewer, depending on the heuristic), structured as follows:

$$\{d_j, j, 0, 0, 0, ..., 1, ...0, 0\}$$

where:

d_j : service demand of customer j

i : this resource is recording the index of the last vertex we visited. Once we extend the label l awayfrom vertex j, this index will replace the original one in label l This way, we can prevent loops with only two vertices

$\{0,0,0,..,1,..0,0\}$: This is a binary vector in which only the jth component is 1, other components are all zeros. This is an indicator of "if customer j has already been visited". Consequently, we can only visit j if the vertex j has never been visited before.

As per the above explanation, all non-dominated routes are retained. Therefore, we can consider all non-dominated routes with negative costs at the ending depot, rather than just selecting the single "shortest path".

I will consider the basic elementary labeling algorithm and its four variations, which are described below. The first 2 algorithms are taken directly out of past literature (Beasley and Christofides, 1989), and the last 3 haven been modified in someways. These variations are closely related, and the development of each one is often based on the results observed from the previous version.

Algorithm 3 Elementary labeling algorithm(ELA)

Input: A graph G
Output: A set of columns and related solution
Initialize routes by using Algorithm 1.
while *new columns can still be found* **do**
    Solve the master problem with Gurobi.
    Adjust the edge costs with Algorithm 2.
    Solve the subproblem with the basic elementary labeling algorithm.
    Take all non-dominated paths with negative cost as new columns.
end
With the columns generated, solve an integer version of the problem to get a primal bound.

Algorithm 4 State-Space relaxed ELA(SSRELA)

Input: A graph G
Output: A set of columns and related results
Initialize routes by using Algorithm 1.
while *new columns can still be generated* **do**
  Solve the master problem with Gurobi.
  Adjust the edge cost with Algorithm 2.
  Solve the state-space relaxation RCSPP.
  while *there exists non-elementary paths* **do**
    Add resources for duplicated vertices.
    Resolve the RCSPP.
  end
  Take all paths with negative cost.
end
With the columns generated, solve an integer version of the problem to get a primal bound.

The "state-space relaxed ELA" refering to the idea of Kohl (1995), in which we start from adding no resource for vertices and incrementally add node resources until all paths with negative costs are elementary. The node resources added in one iteration will not be carry over to the next iteration.

Algorithm 5 K-best SSRELA

Input: A graph G
Output: A set of columns and related results
Note: This algorithm is the same as Algorithm 4, but when adding new columns, only the k routes with the least cost are taken.
Initialize routes by using Algorithm 1.
while *new columns can still be generated* **do**
  Solve the master problem with Gurobi.
  Adjust the edge cost with Algorithm 2.
  Solve the state-space relaxation RCSPP.
  while *there exists non-elementary paths* **do**
    Add resources for duplicated vertices.
    Resolve the RCSPP.
  end
  Take the k routes with the least cost as new columns.
end
With the columns generated, solve an integer version of the problem to get a primal bound.

Algorithm 6 "Ignoring" SSRELA(I-SSRELA)

Input: A graph G
Output: A set of columns and related results
Initialize routes by using Algorithm 1.
while *true* **do**
  Solve the master problem with Gurobi.
  Adjust the edge cost with Algorithm 2.
  Solve the state-space relaxation RCSPP.
  if *we found elementary routes with negative cost* **then**
    Take these routes as new columns.
    continue
  end
  else
    Add resources for duplicated vertices.
    if *no duplicated vertices found* **then**
      break
    end
    Resolve the RCSPP.
    Go back to the start of the first *if* statement.
  end
end
With the columns generated, solve an integer version of the problem to get a primal bound.

I call algorithm 6 the "ignoring" algorithm because instead of solving the subproblem to optimal at each iteration, we simply "ignore" any non-elementary paths at the ending depot as long as we can find elementary paths with negative cost. The idea is to generate new columns as fast as possible, instead of exploiting the solution space at each iteration. This modification can drastically improve the efficiency.

Algorithm 7 "Ignoring" SSRELA with multiplicity(I-M-SSRELA)

Input: A graph G
Output: A set of columns and related results
Note: This algorithm has the same setup as Algorithm 6, but when adding resources for duplicated vertices, only add the resource for the vertex that has the highest multiplicity.

In Algorithm 7 the concept of multiplicity is inspired by a similar concept in Boland et al. (2006). Here we define the multiplicity as: "the number of times a vertex i appears as a duplicated vertex in all non-dominated routes with negative cost".

5 Algorithm efficiency testing

I have conducted computational experiments on the algorithms to evaluate their performance. The test graph instances are generated using the following method: for each different number of cus-

tomers, I will use a set of fixed seeds to generate 10 graphs. My focus is on testing the execution efficiency of the different algorithm variations. For each algorithm, I will report relevant statistics and provide interpretations of these results. The programming language used for the tests is C++, and the following subsections will be named based on the subproblem-solving method.

5.1 Results for the elementary labeling algorithm(ELA)

This algorithm is the baseline algorithm I am considering. For this algorithm, the statistics I want to report are run time, number of iterations run, and number of columns generated. The reulsts are presented in table 1 and table 2.

From the summary statistics reported in Table 2, we can reach these conclusions: first, the runtime of the algorithm seems to grow exponentially with the number of customer vertices in the graph. This might be related to the fact that increasing the number of customers increases the dimension of the vector of resources considered in the labeling algorithm, as well as the number of potential routes, which also grows exponentially. Second, the same thing is happening for the number of columns generated, and we can infer that this is mainly due to the fact that having more vertices in the graph increases the complexity of the graph itself since we are generating arcs between all possible pairs of nodes. Third, in all 10 instances and across all numbers of customers reported, the algorithm terminates after two iterations. My explanation for this phenomenon is that the way I am initializing the column pool is so bad such that even a slight change in the routes can result in a better solution, hence our algorithm is exploring the solution space in a maybe unnecessarily excessive way. With the huge number of columns found, the restricted master problem easily arrives at the optimal in the second iteration.

5.2 Results for the State-space relaxed ELA(SSRELA)

This algorithm is an extension of the baseline algorithm and aims to improve performance by adding node resources only for those customers that are repeatedly visited in negative-cost paths, as described in Kohl (1995). For this algorithm, we will report the previously considered statistics, as well as the average number of node resources added across all iterations and the maximum number of resources added in a single iteration. Since the runtime difference between 7 and 12 customers is quite marginal, we will now only report results for graphs with more than 12 customers. The results are presented in Tables 3 and 4.

From Table 4, we observe a marginal improvement in the runtime for graph instances with 12 customers. This improvement is due to the fact that the maximum number of resources added to the algorithm is, on average, close to 8, indicating a successful reduction in the size of the state space. However, for instances with 16 customers, the efficiency actually decreases. Although this result might seem counterintuitive, the recorded data provides an explanation: in all 10 instances with 16 customers, node resources were added for all 16 customers during the first iteration. This causes our state-space relaxation of the ELA to converge to the non-relaxed form. Consequently, the actual time used includes both the ”time to solve the baseline ELA” and the ”time to repeatedly

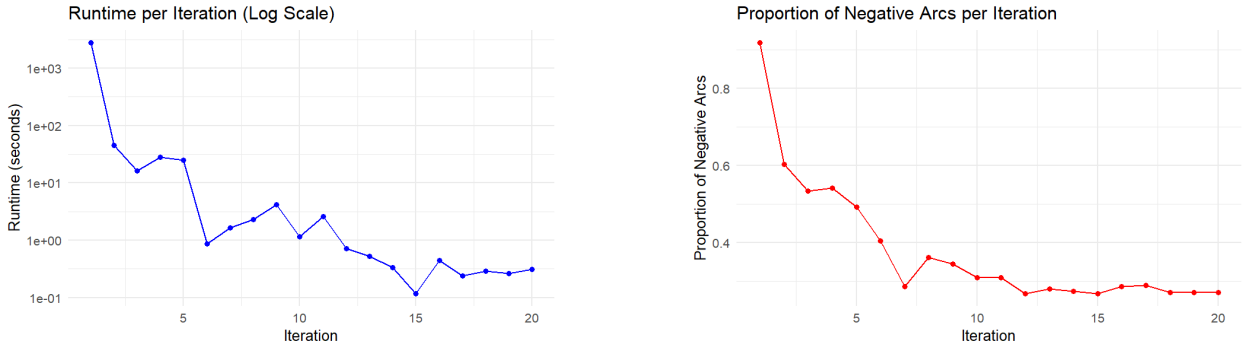
solve the state-space relaxed ELA to add new node resources", which is significantly longer than solving the problem once with the baseline ELA.

5.3 Results for the K-best SSRELA

In the first algorithm, we observe that the number of columns generated, even for relatively small graphs, can be enormous, exploring a vast chunk of the solution space. Although GUROBI's outstanding solving power allows it to handle LPs with tens of thousands of variables effectively, we still need to consider whether such a large number of columns is truly necessary.

The results for the second algorithm also raise some interesting questions. For small graph sizes, the SSRELA appears to effectively reduce the state space. However, as the number of customers increases, the graph's complexity also grows. In such cases, we need to add node resources for all customers to avoid non-elementary paths, causing the algorithm to terminate in just 2 iterations, just like what happened in the baseline ELA. This prevents us from observing the dynamic process of the algorithm.

To address these two problems, we propose a new variation of the algorithm that does the following: while maintaining the same setup as the previous algorithm, it returns at most k "best" routes in each iteration, selecting the elementary routes with the lowest associated costs. We do not expect an improvement in efficiency, as the basic setup remains unchanged. Instead, the purpose of this algorithm is to shed light on the evolution process of the algorithm. To achieve this goal, I will present two plots: one showing the runtime of each iteration on a logarithmic scale, and the other showing the proportion of negative edges in the updated graph G'. The results are displayed in Figure 1.



(a) KBest Runtime

(b) KBest Negative Arcs

Figure 2: Comparison of Runtime(log-scaled) VS Propotion of Negative Arcs

The plots indicate that both statistics decrease exponentially as the iterations progress. This phenomenon suggests a positive correlation between runtime and the proportion of negative-cost edges in the graph.

There are two potential ways to further improve the efficiency of the algorithms. The first approach is to use a different variation of the labeling algorithm to enhance runtime performance

given the same number of negative-weight edges in the graph. The second approach is to identify new routes more efficiently by using a heuristic approach since finding the route with the most reduced cost is not required for every iteartionMehrotra and Trick (2007). This way, we can improve the solution even without modifying the labeling algorithm itself. In this paper, we will adopt the second approach.

5.4 Results for the "ignoring" algorithm(I-SSRELA)

As we observed earlier, adding new improving routes (columns) to the master problem can significantly reduce the time spent on solving the subproblem. This is the key reason for using this variation. We call this variation the "ignoring" algorithm because it ignores the existence of non-elementary paths with negative costs as long as there are non-dominated elementary paths with negative costs at the ending depot. Instead of solving the subproblem to optimality at each iteration, we proceed to the next iteration as soon as we find paths that can improve the solution. We will only add node resources when no elementary negative-cost edges can be found with the "ignoring" method to ensure that the master problem is solved to optimal in our algorithm.

This time, because the algorithm performs so well empirically, I will start by reporting the test results for graph instances with 16 customers. The statistics reported for each graph instance include runtime, number of iterations, number of columns generated, and average number of node resources added. The results are shown in Tables 5 and 6.

From these two tables, we can observe the following changes: First, the efficiency in terms of runtime is drastically improved compared to all the algorithms we considered before. Solving a graph instance with 16 customers now takes, on average, less than 2 seconds, compared to the nearly one hundred seconds observed with the ELA. Second, the number of columns generated is also reduced, indicating that we are exploring a smaller portion of the solution space. Third, the algorithm no longer terminates after 2 iterations; it requires more iterations to solve the problem. However, this is not problematic as long as each iteration can be solved in a relatively short time. Nonetheless, when the number of customer vertices increases to 20, we observe that more node resources are added on average in each iteration, which again slows down the algorithm. How can we improve this?

5.5 Result for the "Ignoring" algorithm with multiplicity(I-M-SSRELA)

As demonstrated previously, when the number of customers increases to 20, the complexity of the graph further increases, and consequently, the average number of resources added also rises.

To further improve our algorithm's efficiency, we will introduce a concept inspired by Boland et al. (2006) called "multiplicity." In their paper, they define this concept as the number of times a node V appears in a certain path p. However, in our context, we define the multiplicity of a node V as the number of times it appears as a repeatedly visited vertex in all non-elementary paths with negative costs at the ending depot. The results are shown in Tables 7 and 8. By adding node resources only for this "most represented customer," we aim to avoid adding too many resources in a single iteration, thereby making the algorithm more stable and efficient. We will compare this

algorithm to the previous one to determine if there is any improvement.

First, let us take a look at the graph instances that have 20 customers. We can observe from the result presented that the runtime seems to receive a marginal improvement on average, and the standard deviation is also smaller since we are using this conservative way of adding resources for nodes. In terms of the columns generated, this algorithm is quite similar to the previous one, indicating that they might be exploring a very similar decision space. Another interesting finding is that iterations run are on average similar, but the avereage numebr of nodes added seems to be halved. This suggests that we have successfully avoided adding too much node resources in this case.

Now let us examine the graph instances with 24 customers. The efficiency drastically decreases, with longer runtimes and more iterations required. However, the maximum number of node resources added remains similar, indicating that our method has successfully stabilized the dimension of the state space even when the number of customers in the graph grows. The extra number of iterations may be the ”price” we are paying for this stability. This interpretation aligns with the results for the ELA algorithms, where we include node resources for all customers, and the algorithm consistently terminates after the second iteration.

6 Conclusions and Future Work

This paper has examined the computational complexity associated with solving the Vehicle Routing Problem (VRP) via column generation, placing particular emphasis on the efficiency of the subproblem, formulated as a resource-constrained shortest path problem (RCSPP). Traditional exact labeling algorithms, such as the Elementary Labeling Algorithm (ELA) and its state-space relaxed variants, can handle the RCSPP optimally but often struggle when faced with graphs containing a high proportion of negative-cost edges. This difficulty substantially increases runtime, as these methods must explore large portions of the solution space and frequently add node resources to eliminate non-elementary routes.

To address this challenge, I introduced two heuristic “ignoring” variations: I-SSRELA and I-M-SSRELA. Rather than insisting on identifying all non-elementary paths at every iteration, these heuristics prioritize quickly finding and incorporating improving routes (columns), thus accelerating the column generation process. By “ignoring” non-elementary paths as long as elementary paths with negative cost are available, the algorithm more rapidly converges to improving solutions. Additionally, the I-M-SSRELA approach employs multiplicity-based resource addition, which selectively targets only the most frequently duplicated vertices. This measured approach prevents an excessive explosion of the state space hence stabilizing the algorithm’s performance.

Testing results on generated graph instances underscore the effectiveness of these heuristics. Compared to classical methods, the proposed heuristics significantly reduced computation times, total column counts, and the complexity of state-space expansions. While they require more iterations, each iteration runs efficiently enough that overall performance is substantially improved.

In conclusion, the heuristic variations presented in this work offer a promising avenue for addressing the computational bottlenecks inherent in solving the VRP subproblem within a column generation framework. Future research may focus on refining these heuristics, integrating learning-based methods to guide the heuristic selection of node resources, or exploring hybrid strategies that combine exact and heuristic approaches. Such developments have the potential to further streamline VRP solution methods and expand their applicability to larger, more complex real-world instances.

Table 1: Test Results for the ELA

Number of Customers	Runtime (seconds)									
	1	2	3	4	5	6	7	8	9	10
7	0.015	0.015	0.019	0.024	0.057	0.018	0.045	0.088	0.011	0.047
12	2.263	2.443	0.695	0.804	0.482	3.465	1.750	1.323	0.330	0.761
16	18.485	39.356	20.635	114.606	754.298	328.890	12.860	558.779	80.296	174.006
Number of Customers	Number of Columns Generated									
	1	2	3	4	5	6	7	8	9	10
7	14	14	26	27	69	25	63	128	12	70
12	2215	2822	1014	1040	694	3534	1953	1379	409	1085
16	9032	13387	10756	21577	34870	26902	8138	33821	17308	25860
Number of Customers	Number of Total Iterations									
	1	2	3	4	5	6	7	8	9	10
7	2	2	2	2	2	2	2	2	2	2
12	2	2	2	2	2	2	2	2	2	2
16	2	2	2	2	2	2	2	2	2	2

Table 2: Summary Statistics for the Test Results of ELA

Number of Customers	Metric	Median	Mean	SD
7	Runtime (seconds)	0.0215	0.0339	0.02487
	Columns Generated	26.5	44.8	37.2821
	Iterations	2	2	0
12	Runtime (seconds)	1.0635	1.4316	1.0254
	Columns Generated	1232	1614.5	996.9258
	Iterations	2	2	0
16	Runtime (seconds)	97.451	210.2211	258.1517
	Columns Generated	19442.5	20165.1	9965.2882
	Iterations	2	2	0

Table 3: Results for the SSRELA

Number of Customers	Runtime (seconds)									
	1	2	3	4	5	6	7	8	9	10
12	0.885	1.663	0.09	0.167	0.118	2.741	1.003	0.373	0.06	0.145
16	37.286	128.194	46.927	272.088	2017.06	808.648	24.695	993.18	185.762	556.049
Number of Customers	Number of Columns Generated									
	1	2	3	4	5	6	7	8	9	10
12	1503	2588	763	712	447	2730	1640	1254	329	744
16	9032	13387	10756	21577	34870	26902	8138	33821	17308	25860
Number of Customers	Number of Iterations									
	1	2	3	4	5	6	7	8	9	10
12	2	2	3	3	4	2	3	3	4	4
16	2	2	2	2	2	2	2	2	2	2
Number of Customers	Average Number of Node Resources Added									
	1	2	3	4	5	6	7	8	9	10
12	4.5	6	10/3	8/3	7/4	5	3	14/3	6/4	7/4
16	9	8	11	19/2	9	17/2	11	17/2	8	19/2
Number of Customers	Max number of node resources									
	1	2	3	4	5	6	7	8	9	10
12	9	10	7	8	7	9	9	9	6	7
16	16	16	16	16	16	16	16	16	16	16

Table 4: Summary Statistics for the results of SSRELA

Number of Customers	Metric	Median	Mean	SD
12	Runtime (seconds)	0.27	0.7245	0.8839
	Columns Generated	1008.5	1271	847.7459
	Iterations	3	3	0.8165
	Avg. Node Resources	3.1667	3.4167	1.5595
	Max Resources	8.5	8.1	1.2867
16	Runtime (seconds)	228.925	506.9889	630.4176
	Columns Generated	19442.5	20165.1	9965.2883
	Iterations	2	2	0
	Avg. Node Resources Added	9	9.2	1.0852
	Max Resources	16	16	0

Table 5: Results for the ”Ignoring” Algorithm(I-SSRELA)

Number of Customers	Runtime (seconds)									
	1	2	3	4	5	6	7	8	9	10
16	0.834	0.729	1.577	1.178	4.754	2.948	1.102	2.041	1.179	1.812
20	54.653	84.769	107.098	83.89	9.393	5.277	30.409	102.281	59.457	5.169
Number of Customers	Number of Columns Generated									
	1	2	3	4	5	6	7	8	9	10
16	406	529	438	539	724	644	385	649	556	514
20	1090	1509	1842	1401	1016	895	1335	1156	1143	695
Number of Customers	Number of Iterations									
	1	2	3	4	5	6	7	8	9	10
16	13	11	15	9	18	12	14	14	12	12
20	18	20	22	23	19	12	20	25	17	15
Number of Customers	Average Number of Node Resources Added									
	1	2	3	4	5	6	7	8	9	10
16	5/13	0	1	5/9	5/18	1/12	16/14	2/14	0	6/12
20	32/18	33/20	33/22	1	11/19	0	5/20	19/25	44/17	17/15

Table 6: Summary Statistics for the results of the ”Ignoring” Algorithm(I-SSRELA)

Number of Customers	Metric	Median	Mean	SD
16	Runtime (seconds)	1.378	1.8154	1.2243
	Columns Generated	534	538.4	110.5372
	Iterations	12.5	13	2.4495
	Avg. Node Resources Added	0.3312	0.4087	0.4010
20	Runtime (seconds)	57.055	54.2396	39.8956
	Columns Generated	1149.5	1208.2	327.4123
	Iterations	19.5	19.1	3.8427
	Avg. Node Resources Added	1.0667	1.1238	0.7790

Table 7: Result of the ”Ignoring” algorithm with multiplicity(I-M-SSRELA)

Number of Customers	Runtime (seconds)									
	1	2	3	4	5	6	7	8	9	10
20	43.661	37.408	70.536	85.922	9.918	5.353	31.253	102.858	77.174	8.351
24	1635.71	63.777	170.607	449.068	346.311	435.662	3858.53	769.326	1068.3	245.196
Number of Customers	Number of Columns Generated									
	1	2	3	4	5	6	7	8	9	10
20	1078	1509	1389	1364	1016	895	1334	1153	1090	693
24	2482	1780	1441	2263	1841	2265	1891	2666	2460	1857
Number of Customers	Number of Iterations									
	1	2	3	4	5	6	7	8	9	10
20	24	20	23	24	19	12	20	29	27	18
24	41	17	39	25	52	38	77	32	63	30
Number of Customers	Average Number of Node Resources Added									
	1	2	3	4	5	6	7	8	9	10
20	20/24	13/20	10/23	7/24	6/19	0	4/20	17/20	40/17	11/18
24	39/41	4/17	22/39	3/25	26/52	18/38	157/77	12/32	124/63	13/30
Number of Customers	Maximum Number of Node Resources Added in One Iteration									
	1	2	3	4	5	6	7	8	9	10
20	6	7	7	2	4	0	3	5	9	8
24	5	4	4	2	5	6	10	4	10	6

Table 8: Summary Statistics for the results of the ”Ignoring” algorithm with multiplicity(I-M-SSRELA)

Number of Customers	Metric	Median	Mean	SD
20	Runtime (seconds)	40.5345	47.2434	35.0406
	Columns Generated	1121.5	1152.1	250.8287
	Iterations	21.5	21.6	4.881
	Avg. Node Resources Added	0.522947	0.653962	0.657017
	Max Node Resources Added	5.5	5.1	2.846
24	Runtime (seconds)	442.365	904.2487	1140.5243
	Columns Generated	2077.0	2094.6	388.1111
	Iterations	38.5	41.4	18.094
	Avg. Node Resources Added	0.4868	0.7660	0.6879
	Max Node Resources Added	5.0	5.6	2.5905

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