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⇒ You tube → Splines

Spline of degree k,

$$S(x) = S_i(x) \quad [t_i, t_{i+1}].$$

$$t_0 < t_1 < t_2 \ldots < t_n \rightarrow \text{knots}.$$

Conditions:

i) $S_i \in P_k$ (each piecewise ~~polynomial~~ function is a polynomial with degree $\leq k$)

ii) Continuity conditions ⇒

[for inner nodes)

$$S_{i-1}(t_i) = S_i(t_i), \quad S'_{i-1}(t_i) = S'_i(t_i),$$

upto $\quad S^{(k-1)}_{(i-1)}(t_i) = S^{k-1}_i(t_i)$

• k will be ≤ 3

if $k = 1$, linear spline

$k = 2$, quadratic spline (due to some reason not much used)

$k = 3$, cubic spline → Much used

Q: check if it is a linear spline →

$$S(x) = \begin{cases} x, & -1 \leq x \leq 0 \\ 1-x, & 0 < x < 1 \\ 2x-2, & 1 \leq x \leq 2 \end{cases}$$

$\Rightarrow$ Cond-2 $S_i \in P_1$
each piecewise func is linear hence this condition is satisfied.

Cond-3
Continuity at $x = 0, 1$ (inner nodes)

$x = 0$, $S(0^-) = 0$
$S(0^+) = 1$
$\therefore$ It is discontinuous, not a linear spline

Q) Check if its quadratic?

$$S(x) = \begin{cases} x^2 & , -10 \le x \le 0 \\ -x^2 & , 0 < x < 1 \\ 1-2x & , 1 \le x \le 2 \end{cases}$$

Let, ~~$S_0(x^2)$~~ $S_0(x) = x^2$, $S_1(x) = -x^2$, $S_2(x) = 1-2x$

Cond-1
$S_0(x)$ & $S_1(x) \to$ quad $\to$ degree 2
$S_2(x) \to$ degree 1
Condition is satisfied.

Cond-3
Continuity of $S(x)$ & $S'(x)$

$S(x)$ for $x = 0, 1$ (inner nodes)

$x = 0$, $S_0(0) = 0$
$S_1(0) = 0$

at $x=1$,

$$S_1(1) = -1$$

$$S_2(1) = -1$$

$S(x)$ is continuous at $x=0$ and $x=1$.

Now, check for $S'(x) \rightarrow$ first derivative

$$S'(x) = \begin{cases} 2x & , -\infty < x \leq 0 \\ -2x & 0 < x < 1 \\ -2 & , 1 \leq x \leq 2 \end{cases}$$

$S'(x)$ at $x=0$

$$S'_0(0) = 0$$

$$S'_1(0) = 0$$

$S'(x)$ at $x=1$

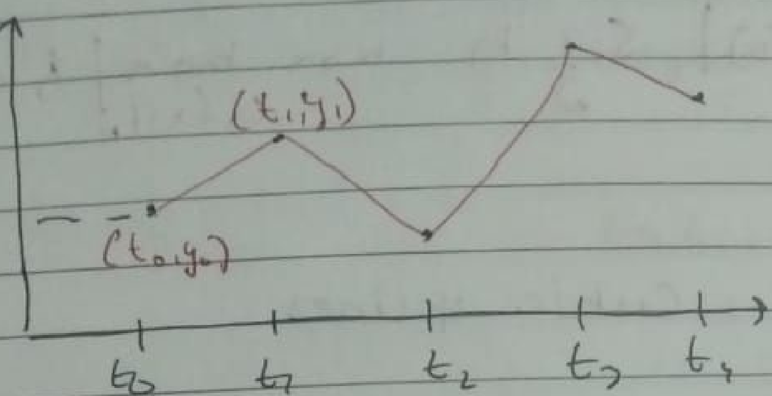
$$S'_1(1) = -2$$

$$S'_2(1) = -2$$

$S'(x)$ is continuous at $x=0,1$

$\therefore$ $\boxed{S(x) \text{ is quadratic spline}}$

⇒ Linear splines



~~each~~ each S_i we can find $S_i(x)$, since we have two points and its easy to get eq of line from two points.

⇒ Accuracy ⇒

let $h = \max_i (t_{i+1} - t_i)$ { max distance b/w two adjacent nodes }

let $f(x)$ ~~be true~~ functn
$S(x)$ is a linear sub spline.
such that. $S(t_i) = f(t_i)$

For $x \in [t_0, t_n]$, we have

i) If $f \in C^2 \Rightarrow f''$ exists.

$$|f(x) - S(x)| \leq \frac{1}{8} h^2 \cdot \max_{t_0 \leq x \leq t_n} |f''(x)|$$

error term

max of second derivative

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2) $f \in C^1 \Rightarrow f'$ exist

$|f(x) - S(x)| \le \frac{1}{2} h \max \max_{t_0 \le x \le t_n} |f'(x)|$

⇒ Natural Cubic splines

$S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \quad \forall \; i = 0 \to 1$

unknown $4 \cdot n$

Cubic Spline →

- It follows the two conditions discussed earlier.

- Natural splines follows an additional constraint that $S''(t_0) = S''(t_n) = 0$

⇒ How to compute $S(x)$

1) Start with $S_i''(x)$, linear spline
2.) Integrate twice

→ Define

$\underbrace{z_i = S''(t_i)}_{\text{unknowns}}$, → for inner nodes

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Let $h_i = t_{i+1} - t_i$

Find $S''(x)$ using Lagrange form (as it is easy to integrate).

for interval ~~t~~t_i to ~~t~~t_{i+1}

$$S_i''(x) = \frac{z_{i+1}}{h_i}(x - t_i) - \frac{z_i}{h_i}(x_j - t_{i+1})$$

Integrate once to get $S_i'(x)$ integrate again to get $S_i(x)$.

Integrating

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x-t_i)^2 - \frac{z_i}{2h_i}(x-t_{i+1})^2 + C_i - D_i$$

~~Int~~ again

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x-t_i)^3 - \frac{z_i}{6h_i}(x-t_{i+1})^3 + C_i(x-t_i) - D_i(x-t_{i+1}).$$

⇒ Interpolating properties :

1) $S_i(t_i) = y_i$
$S_i(t_{i+1}) = y_{i+1}$ } $S(x)$ is continuous

~~∴ property can be proved by putting the values of $x = t_i$ & $y =$ $S(x) = y_i$ & $x = t_{i+1}$ and $S(x) = y_{i+1}$~~

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• From the properties we can get the value of C_i and D_i

Now to get ~~z~~ value of z_i

We will compute $S_i'(x)$ ~~for~~ from $S_i(x)$ and use continuity condition for $S'(x)$

i.e.

i~~ntegral~~ diff. $S_i(x)$ to get $S_i'(x)$

then use condition

$$S_{i-1}'(t_i) - S_i'(t_i) = 0$$