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THE BIFURCATIONS OF DUFFING'S EQUATION: AN APPLICATION OF CATASTROPHE THEORY

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The existence and stability of solutions of Duffing's equation are investigated and their characteristic bifurcations are studied. It is shown how catastrophe theory provides a natural synthesis between the practical and theoretical aspects of the problem. In so doing, ways are indicated in which the mathematical methods of differential dynamics might be of use to the vibration engineer, particularly in the treatment of non-linear problems.

1. INTRODUCTION

Catastrophe theory provides a natural synthesis between the practically observable and theoretical aspects of many problems in dynamics. In this paper the theory is applied to forced non-linear vibrations governed by Duffing's equation. Duffing's equation, which describes the behaviour of many electrical and mechanical systems, [1–3] may be written as

$$m\ddot{x} + c\dot{x} + k(x + \alpha x^3) = f\cos\omega t, \tag{1.1}$$

where m, c, k, α, f and ω represent, respectively, the mass, damping factor, stiffness, and non-linearity of the (mechanical) system, and the amplitude and frequency of the excitation force. Engineers generally prefer to treat the dimensionless form

$$\ddot{x} + 2\zeta\dot{x} + x + \alpha x^3 = \phi\cos\Omega t, \tag{1.2}$$

where ζ is the damping ratio $(=c/2\sqrt{mk})$, ω_n is the natural frequency of the undamped, linear system $(=\sqrt{k/m})$, $\phi = F/m\omega_n^2$ and $\Omega = \omega/\omega_n$.

Although Duffing's equation has been treated extensively in engineering texts (see, e.g., references [1–3]), the existence of stable harmonic solutions has generally been assumed. Here a demonstration of the existence of nearly harmonic solutions (at the forcing frequency) is presented, their stability is investigated, and, more significantly, so are the effects of changes of the "control" parameters ζ, α, ϕ and Ω on these solutions: in particular, the manner in which the control parameters affect the characteristic "jumps" from one amplitude to another observed in systems governed by Duffing's equation [2, 3]. The description of this behaviour fits naturally into catastrophe theory. In the process, it is indicated that differential dynamics [4] offers a unifying scheme for the study of vibrating systems and that catastrophe and bifurcation theory may have considerable value as tools in the practical analysis and modelling of more complicated systems.

For those concepts of differential topology not defined herein the reader is referred to references [4–6]; the survey article of Arnold [7] and the notes of Takens [8] provide an excellent introduction to bifurcation theory. Further references to catastrophe theory can be found in references [9, 10].

2. BIFURCATION AND CATASTROPHES

Bifurcation theory is concerned with the description of the topological variation of the orbit structures of dynamical systems which depend upon a parameter. On the other hand elementary catastrophe theory is the theory of the creation and annihilation of critical points of "generic" families V_λ ($\lambda \in \mathbf{R}^k$) of infinity differentiable real-valued functions [9–11]. As such, elementary catastrophes reflect the simplest generic bifurcations of dynamical systems and, moreover, one has a complete list for $k \leqslant 15$. (A word of warning here: when $k > 5$ the number of possible elementary catastrophes is infinite and their geometry is little understood as yet.) The most obvious application of catastrophe theory is to systems whose state is determined by the local minimization of an explicitly given potential. For gradient-like dynamical systems one can choose a Lyapunov function [5], which can then be taken as the generic potential function locally minimized by the flow. It is also known that for Morse-Smale systems [4] possessing a cross-section Y (such as Duffing's equation for almost all small values of the parameters, a cross-section Y being the $(x, \dot{x})$-plane (see equations (2.2)) given by $t = 0$) one can choose a Lyapunov function V for the flow and then $V|_Y$ is a generic potential function on Y that is locally minimized by the flow.

In our problem, unfortunately, we are unable to find V explicitly and consequently the strategy here is to approximate equation (1.2) by an autonomous system so that nearly sinusoidal harmonic solutions of the former correspond to constant solutions of the latter. In this way we approximate the *precise* catastrophe theory of Duffing's equation and appeal to the Theorem of section 3 to justify the qualitative equivalence of these approaches, at least for small non-zero values of the parameter $\lambda = (\zeta, \alpha, \phi, 1 - \Omega^2)$. It is worth noting here that this approximating autonomous system has a very natural interpretation (see section 3) providing a further phenomenological justification for this approach.

Firstly, consider the general problem of bifurcation, mentioned above, for a differential equation of the form

$$\ddot{x} = f(x, \dot{x}, t, \lambda) \qquad (2.1)_\lambda$$

where f is of period $2\pi/\omega$ in t and depends upon some parameter $\lambda \in \mathbf{R}^k$. Duffing's equation is such a system if one lets $\lambda = (\zeta, \alpha, \phi, 1 - \Omega^2) \in \mathbf{R}^4$. Interest here is mainly in the nature and behaviour of periodic solutions of equation $(2.1)_\lambda$. If equation $(2.1)_\lambda$ is considered as an autonomous first order system,

$$\dot{x} = y, \dot{y} = f(x, y, \theta, \lambda), \dot{\theta} = \omega, \qquad (2.2)_\lambda$$

on $\mathbf{T} = \mathbf{R}^2 \times \mathbf{S}^1$ ($\mathbf{S}^1$ is the unit circle) and one assumes, for example, that f is infinitely differentiable, then the system is sufficiently well behaved to give a one-parameter group of diffeomorphisms† $\phi_t^\lambda : \mathbf{T} \to \mathbf{T}$, $t \in \mathbf{R}$ for each $\lambda \in \mathbf{R}^k$. Here $\phi_t^\lambda(x)$ is the point of $\mathbf{T}$ reached by running along the trajectory of the system $(2.2)_\lambda$ for time t. In particular consider the mapping $\Pi_\lambda = \phi_{2\pi/\omega}^\lambda$. It is clear that by definition Π_λ is the identity on the third co-ordinate θ and hence one need only consider the mapping $P_\lambda : \mathbf{R}^2 \to \mathbf{R}^2$, corresponding to the action of Π_λ on the first two co-ordinates. This is then a diffeomorphism and is known as the *Poincaré map* of equation $(2.1)_\lambda$ [12]. Periodic points of P_λ (i.e., points $x \in \mathbf{R}^2$ such that $P_\lambda^m(x) = x$, some m) correspond to periodic solutions of equation $(2.1)_\lambda$ and, in particular, the fixed points of P_λ

† Here a diffeomorphism is an infinitely differentiable map with an infinitely differentiable inverse [6].

correspond to solutions of period $2\pi/\omega$. Moreover, there is an exact correspondence between the stability types of the periodic points of P_λ and the corresponding solutions of equation $(2.1)_\lambda$ [4]. The general problem of bifurcation of equation $(2.1)_\lambda$ is to determine the set of $\lambda \in \mathbf{R}^4$ where the phase portrait changes and to describe the ways in which it changes as λ passes through this set. Thus part of the task is (i) to describe the behaviour of the fixed points of P_λ as λ ranges over $\mathbf{R}^k$ (for example give a qualitative description of the regions of $\mathbf{R}^k$ where there exist 0, 1, 2, . . . stable fixed points of P_λ and describe what happens as one crosses between these regions) and (ii) to do the same for periodic points of P_λ of any given period. In the general case little is known except for the work of Takens [12]. Here, where one is concerned with the harmonic solutions of Duffing's equation, it will be seen that for $\lambda = (\phi, \Omega) \in \mathbf{R}^2$ with $\zeta, \alpha \neq 0$, their bifurcational behaviour is governed by the cusp or Riemann–Hugoniot catastrophe. Subharmonic solutions are not considered in this paper; their behaviour appears to be considerably more complicated.

Secondly, one can approach the problem from a different point of view and consider the actual physical system (i.e., a harmonically forced non-linear mass-spring system with damping). For reasonable values of the damping and non-linearity parameters and the forcing amplitude and frequency one observes the existence of at least one stable, nearly harmonic solution. In this case the phenomenologically important observables are the amplitude, A, and the phase, ψ, of the solution so one uses one of these,† say A, to define the state of the system. Now one can vary the system by smoothly changing the amplitude, ϕ, of the force and its frequency, Ω, and observe the resulting change in A. Usually A moves continuously with (ϕ, Ω) though sometimes it makes "discontinuous" jumps and settles down at a new value. Thus one can plot out the points where such a jump occurs and temporarily call this set, K, the *bifurcation or catastrophe set*.

If the state of a system is determined by the local minimization of a potential (i.e., if it is so highly dissipative that transients can be ignored) then the number of "structurally stable" ways in which a smooth change in the potential function can give rise to discontinuous changes of state is quite limited and described by the elementary catastrophes. For example, suppose the system above is determined by the minimization of the potential $V_{(\phi,\Omega)}: \mathbf{R} \to \mathbf{R}$, where the parameterization is smooth in the sense that the mapping $V: (\phi, \Omega, A) \to V_{(\phi,\Omega)}(A)$ is infinitely differentiable. Then a smooth change in the parameter (ϕ, Ω) can give rise to a discontinuous jump in the state A in essentially only two ways, provided the family V is generic‡. This requirement is, roughly speaking, that it is impossible to alter the discontinuity type by an arbitrarily small perturbation of V—a reasonable assumption in physical systems which by their nature contain a large amount of stochastic noise. In fact, almost all families are structurally stable in this sense. Consider now the following example.

Canonical cusp catastrophe. Let V be the two-parameter family of functions given by $V_{(a_1,a_2)}(x) = (1/4)x^4 + (1/2)a_1x^2 + a_2x$. Varying (a_1, a_2) gives rise to essentially two different types of potential function: if (a_1, a_2) lies outside the cusp $4a_1^3 + 27a_2^2 = 0$ then there is one minimum; if (a_1, a_2) lies inside then two. The different forms of the graph of $V_{(a_1,a_2)}$ are shown in Figure 1, and the situation is expressed by the pleated surface M_V of Figure 2 whose equation is

$$x^3 + a_1x + a_2 = 0. \tag{2.3}$$

The surface M_V is called the *singular locus* of the family and consists of those points $(a_1, a_2; x) \in \mathbf{R}^2 \times \mathbf{R}$ such that x is a critical point of $V_{(a_1,a_2)}$. Clearly such a definition is

† By the theorem below one can expect that one (possibly unknown) parameter will capture the bifurcational behaviour relative to $\lambda = (\phi, \Omega)$ provided $A \neq 0$. The map $(x, \dot{x}) \mapsto A$ is only smooth when $A \neq 0$.

‡ V is defined to be generic if the map $(\phi, \Omega) \mapsto V_{(\phi,\Omega)}$ is transverse to the natural stratification of $C^\infty(\mathbf{R})$. If V is generic then it is a structurally stable family.

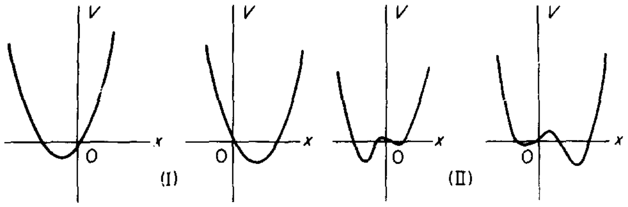


Figure 1. The "potential" function.

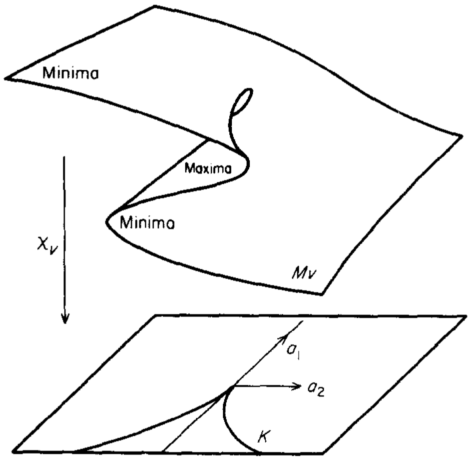


Figure 2. The singular locus and the catastrophe set in a_1, a_2 control space.

extendible to any smoothly parameterized family of functions. The cusp $4a_1^3 + 27a_2^2 = 0$ is called the *catastrophe set*, K, since if $(a_1, a_2) \in K$ then the graph of $V_{(a_1,a_2)}$ is unstable with respect to small perturbations of the parameters. Alternatively, and equivalently, one could define K to be the image of the set of singularities† of the projection χ_V of M_V onto $\mathbf{R}^2$ as indicated in Figure 2—again a definition which is clearly generalizable to arbitrary families of functions. In this way one can obtain rigorously the following result as a corollary to Thom's theorem [11] on the classification of elementary catastrophes. *Theorem*: *Let* $V_{(a_1,a_2)}$: $\mathbf{R}^s \to \mathbf{R}$ *be a smoothly parameterized family of* C^∞ *real-valued functions where the parameter* $(a_1, a_2) \in \mathbf{R}^2$. *Then, if the family is structurally stable,* (i) M_V *is a 2-dimensional submanifold of* $\mathbf{R}^2 \times \mathbf{R}^s$, *and* (ii) *the catastrophe set,* K_V, *and the projection,* χ_V, *of the family has the same local structure as the catastrophe set and projection of either of the following families*: $V_{(a_1,a_2)}(x) = (1/3)\,x^3 + a_1 x$ (*fold catastrophe*); $V_{(a_1,a_2)}(x) = (1/4)\,x^4 + (1/2)\,a_1 x^2 + a_2 x$ (*cusp catastrophe*).

One can now return to practical considerations and look at the local structure of the bifurcation set K as plotted. Does it contain any cusps? The catastrophe set of the fold consists of a straight line. So if K is a curve with cusps one can call the points corresponding to the cusp "tip" *cusp points* and points on a "qualitatively straight" piece of *K fold points*. The theorem above states that if K comes from a structurally stable 2-parameter family of functions then it should consist solely of fold and cusp points. Moreover, it is clear that every such K contains fold points so one is mainly interested in detecting cusp points. One hopes that if one has plotted a cusp point one can identify a pleated surface like M_V sitting above this point. In fact, one *can* demonstrate the existence of cusp points experimentally and the rest of this paper is concerned with a theoretical study of the global bifurcation sets and singular loci associated with Duffing's equation.

† i.e., those points $x \in M_V$ at which χ_V is not a local diffeomorphism.

3. DUFFING'S EQUATION WITH SMALL DAMPING AND SMALL HARMONIC FORCING

One can now return to Duffing's equation

$$\ddot{x} + 2\zeta\dot{x} + x + \alpha x^3 = \phi\cos\Omega t, \tag{3.1}$$

where $|\lambda|$ is assumed small, $\lambda = (\zeta, \alpha, \phi, \rho)$; $\rho = 1 - \Omega^2$. Further, the physically realistic case of non-negative damping is assumed. Analysis for negative damping simply involves a reversal of time: for instance, stable solutions become unstable. Experiment and approximation suggest the existence of nearly periodic solutions at the forcing frequency [1–3]. The analysis below obviates the need to seek specific solutions, which are in any case difficult to make use of since they may contain many harmonics. Here one is able to attach a natural meaning to the "amplitude" and "phase" of such an "impure" response and thus provide a simple method of characterizing it. The approach used is the well-known method of averaging due to Krylov and Bogoliubov [13], the treatment of Hale [14] being followed.

Consider equation (3.1) as the first-order system

$$\dot{x} = y, \dot{y} = -x - 2\zeta y - \alpha x^3 + \phi\cos\Omega t. \tag{3.2}$$

If one makes the transformation $u = x\cos\Omega t - (1/\Omega)y\sin\Omega t$, $v = -x\sin\Omega t - (1/\Omega)y\cos\Omega t$, which has the inverse

$$x = u\cos\Omega t - v\sin\Omega t, y = -\Omega u\sin\Omega t - \Omega v\cos\Omega t, \tag{3.3}$$

then the transformed system is given by

$$\begin{aligned} \dot{u} &= (1/\Omega)[\rho x + \alpha x^3 + 2\zeta y - \phi\cos\Omega t]\sin\Omega t, \\ \dot{v} &= (1/\Omega)[\rho x + \alpha x^3 + 2\zeta y - \phi\cos\Omega t]\cos\Omega t, \end{aligned} \tag{3.4}$$

where $\rho = (1 - \Omega^2)$.

Clearly periodic solutions of equations (3.2) and (3.4) correspond. Notice also that this transformation converts a solution of equation (3.1) of the form $A\cos(\Omega t + \psi)$ to the constant solution $(A\cos\psi, A\sin\psi)$ of equations (3.4). Conversely, any constant solution of equations (3.4) of this form corresponds to a harmonic solution of equation (3.1) with amplitude A and phase ψ.

Of course, constant solutions of equations (3.4) will rarely exist but when $|\lambda|$ is small enough the periodic solutions of equation (3.1) are nearly harmonic whence the corresponding periodic solutions of equations (3.4) are nearly constant and will be well approximated by the constant solutions of the autonomous averaged equations:

$$\begin{aligned} \dot{u} &= (1/2\Omega)\left[-\rho v - \frac{3\alpha}{4}v(u^2+v^2) - 2\zeta\Omega u\right], \\ \dot{v} &= -(1/2\Omega)\left[-\rho u - \frac{3\alpha}{4}u(u^2+v^2) + 2\zeta\Omega v + \phi\right]. \end{aligned} \tag{3.5}$$

To be more precise, one needs the following theorem (cf. reference [14]).

Theorem: If

$$\dot{x} = f(t, x, \lambda), \tag{3.6}$$

where $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^m$ is a parameter, f is infinitely differentiable and of period $T > 0$ in $t \in \mathbf{R}$ and $f(t,x,0) = 0$ for all $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, and the associated autonomous *averaged equation* is defined as

$$\dot{x} = f_0(x, \lambda), \tag{3.7}$$

where $f_0(x,\lambda) = \lim_{T\to\infty}(1/T)\int_0^T f(t,x,\lambda)\mathrm{d}t$, then *by a regular (i.e., diffeomorphic at 0) change of co-ordinates x one can write equation (3.6) in the form* $\dot{x} = f_0(x,\lambda) + O(|\lambda|^2)$, *for* $|\lambda|$ *small.*

Thus provided one works in a small enough neighbourhood of $\lambda = 0$ and provided the family in equation (3.7) is generic one need only be concerned with equation (3.7). Note also that this theorem implies the usual averaging theorem ([14], p. 190).

Corollary: *If* x_0 *is a hyperbolic*† *constant solution of equation* (3.7) *there exists* $\varepsilon > 0$ *such that for all* $\varepsilon > |\lambda| > 0$ *there exists a hyperbolic periodic solution* $x_\lambda(t)$ *of equation* (3.6) *which is unique in some neighbourhood of* x_0, *depends continuously on* λ, *and has the same stability type as* x_0 *(i.e., sink, source or saddle).*

This theorem now can be applied to equations (3.4) and (3.5). Before attempting to calculate the constant solutions of equation (3.5) note that if one writes them in the form $(A\cos\psi, A\sin\psi)$ then by the previous discussion A and ψ approximate the amplitude and phase of the corresponding solution of equation (3.1). To help keep this in mind and make clear what is meant by the amplitude and phase of a non-harmonic periodic solution it is useful to make the following definitions: let (u_0, v_0) be a constant solution of equation (3.5) corresponding to a periodic solution $x(t)$ of equation (3.1); then the *amplitude* of $x(t)$ is $+\sqrt{(u_0^2 + v_0^2)}$ and the *phase* is $\tan^{-1}(v_0/u_0)$.

If the response is not too harmonically impure, these defined quantities would be the amplitude and phase measured by conventional instrumentation with respect to a sinusoidal force reference signal. By the discussion above, if $x(t)$ is sinusoidal, these notions coincide with the usual ones.

An examination of equation (3.5) reveals that for ζ, $\alpha \neq 0$ it can have one, two or three constant solutions. If it has a unique constant solution then this is hyperbolically stable; if it has two, one is hyperbolically stable and the other is degenerate (i.e., it can be destroyed by an arbitrary perturbation of the parameters); and if it has three, two are hyperbolically stable and one is hyperbolic and unstable being, in fact, a saddle point.

The above is qualitatively the situation encountered with the cusp catastrophe, the only difference being that every constant (i.e., critical) solution has one extra stable dimension.‡ Annihilation and creation of constant solutions occurs in qualitatively the same way. By the theorem above corresponding to the hyperbolic constant solutions of equation (3.5) are hyperbolic periodic solutions of equation (3.1) (the unstable hyperbolic periodic solution is a saddle, as reversal of time leaves this solution unstable). One can make the connection with the cusp catastrophe even stronger.

If one transforms to polar co-ordinates (r, θ), equation (3.5) becomes

$$\dot{r} = (1/2\Omega)\,[-2\zeta\Omega r - \phi\sin\theta],\; r\dot{\theta} = (1/2\Omega)\,[\rho r + (3\alpha/4)r^3 - \phi\cos\theta], \tag{3.8}$$

and it is straightforward to check that if (A, ψ) is a constant solution of this then

$$A^2\{\rho + (3/4)\alpha A^2\}^2 + 4\zeta^2\Omega^2 A^2 = \phi^2 \tag{3.9}$$

and

$$\sin\psi = -(2\zeta\Omega/\phi)\,A. \tag{3.10}$$

Equation (3.9) can be rewritten in the form

$$(W+p)^3 + q(W+p) + s = 0 \tag{3.11}$$

where $W = A^2$, $p = 8\rho/9\alpha$, $q = (16/27\alpha^2)(12\zeta^2\Omega^2 - \rho^2)$ and $s = -(16/729\alpha^3)\{8\rho(\rho^2 + 36\zeta^2\Omega^2)$

† A constant solution x_0 is hyperbolic if the eigenvalues of the differential equation linearized at x_0 have non-zero real parts [5].

‡ Here the family V of potentials is being associated with the family $\dot{x} = -\mathrm{grad}\, V$ of differential equations which have the critical points of the functions of V as constant solutions.

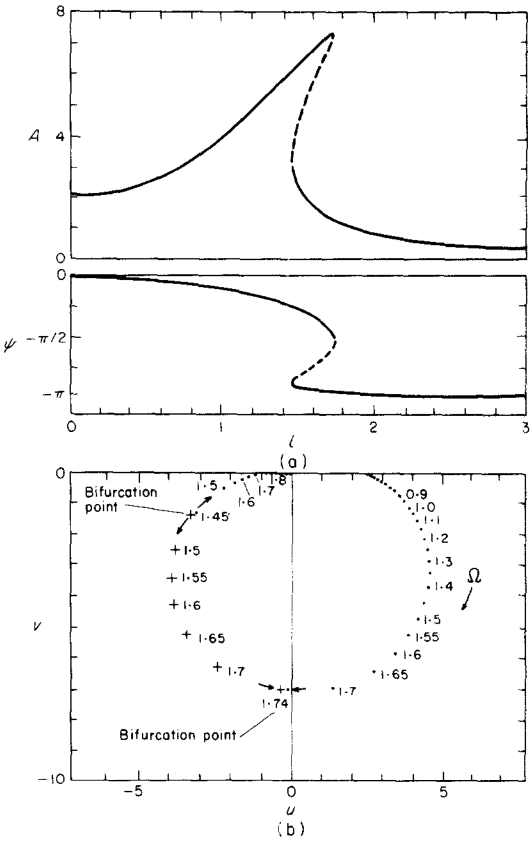


Figure 3. Harmonic solutions of Duffing's equation for $\alpha = 0{\cdot}05$, $\zeta = 0{\cdot}1$, $\phi = 2{\cdot}5$. (a) Frequency response function (A *vs.* Ω and ψ *vs.* Ω); ——, attracting (stable) solutions; -----, saddle solutions; (b) path of fixed points of autonomous equation in u, v space, parameterized by Ω; ●, sinks (attractors); +, saddles.

$+ 81\alpha\phi^2\}$. Compare this to the equation of the singular locus of the canonical cusp catastrophe (2.3).

Solutions to equations (3.9) and (3.10) have been calculated by digital computer. Figure 3(a) shows a typical frequency response function: α, ζ, and ϕ are fixed. For low Ω there is only one stable solution; as Ω is increased a second stable solution and a saddle are born at the bifurcation point. At a yet higher frequency the saddle annihilates the original stable solution again leaving a single stable solution. Corresponding to the frequency response function, one can plot the trajectories of the fixed points of the averaged equation (3.8) in u–v space as Ω is varied. This is clearly an analogue of the familiar Kennedy–Pancu or vector diagram for a linear single degree of freedom system.

4. THE BIFURCATION SET FOR THE HARDENING SPRING ($\alpha > 0$)

Equation (3.9), derived above, gives the amplitude of the response, A, in terms of the parameters ζ, α, ϕ and Ω. One may rewrite the equation as

$$9\alpha^2 A^6 + 24\alpha\rho A^4 + 16(\rho^2 + \sigma^2) A^2 - 16\phi^2 = 0, \tag{4.1}$$

where $\rho = 1 - \Omega^2$ and $\sigma = 2\zeta\Omega$. As has been seen, the equation has, in general, either one or three solutions, representing either one stable attractor or else two attractors, and a saddle type orbit. For some values of the controls, the saddle coalesces with one of the attractors and annihilates it, leaving only a single attractor. One can now proceed to find the *bifurcation set*,† K, for which this occurs. (The phase angle between response and force is derived directly from the amplitude and thus the bifurcation set is common to amplitude and phase. Therefore one need only examine the amplitude characteristics.)

Equation (4.1) may be rewritten as a cubic

$$W^3 + a_2 W^2 + a_1 W + a_0 = 0, \tag{4.2}$$

where $W = A^2$, $a_2 = 8\rho/3\alpha$, $a_1 = 16(\rho^2 + \sigma^2)/9\alpha^2$ and $a_0 = -16\phi^2/9\alpha^2$. It is clear that bifurcation occurs when the equation possesses three real roots, at least two of which are equal, and that away from K one has either one root or three distinct ones. If one defines

$$r = 1/6(a_1 a_2 - 3a_0) - 1/27a_2^3, q = a_1/3 - a_2^2/9, \tag{4.3}$$

the expression

$$q^3 + r^2 = 0 \tag{4.4}$$

gives the condition for two equal roots [15]. Making the necessary substitutions in equation (3.4), one obtains

$$243\alpha^2\phi^4 + 48\alpha\rho(\rho^2 + 9\sigma^2)\phi^2 + 64\sigma^2(\rho^2 + \sigma^2)^2 = 0. \tag{4.5}$$

This gives the bifurcation set $K \subset \mathbf{R}^4$. Since for a given physical system the non-linearity parameter, α, and the damping ratio, ζ, are fixed, it seems reasonable to examine sections of K in (ϕ, Ω)-space for such fixed values.‡ Equation (4.5) is a quadratic in ϕ^2, and thus

$$\phi = \pm\sqrt{(8/81\alpha)(-\rho(\rho^2 + 9\sigma^2) \pm \sqrt{(\rho^2 - 3\sigma^2)^3})}. \tag{4.6}$$

Note that this is identical to the set obtained by the different "stability analysis" of, e.g., reference [1]. Engineering texts generally present stability regions in the A, Ω plane. Our presentation, in the ϕ, Ω plane, shows more clearly the interaction between the two control parameters, which are both necessary here to capture the bifurcational behaviour.

Recalling that $\rho = 1 - \Omega^2$ and $\sigma = 2\zeta\Omega$ one may plot ϕ *vs.* Ω for various values of ζ and α. Such plots have been produced by using a simple digital computer program, and a number of typical ones for the hardening spring ($\alpha > 0$) are shown in Figures 4 and 5. In Figure 4 the non-linearity is varied and in Figure 5 the damping is varied, keeping $\zeta > 0$. The characteristic two cusps, and symmetry about $\phi = 0$ are clear. It should be noted that the cusps do not meet and that for small force amplitudes bifurcations do not occur.

An examination of equation (4.1) and the corresponding equation for phase, derived from equation (3.10)

$$\psi = \sin^{-1}(-\sigma A/\phi) \tag{4.7}$$

indicates that the singular loci above the ϕ, Ω control space are of the forms indicated in Figure 6. The regions of stable (attractor) orbits and unstable (saddle) orbits are indicated. It follows from the Theorem of section 3 that solutions on the underside of the pleated surfaces always represent unstable behaviour.

The case without damping ($\zeta = 0$) is especially interesting. The governing sixth order equation reduces to $9\alpha^2 A^6 + 24\alpha\rho A^4 + 16\rho^2 A^2 - 16\phi^2 = 0$
or

$$((3/4)\alpha|A|^3 + \rho|A|) \pm \phi = 0, \tag{4.8}$$

† Note that strictly this is only the part of the bifurcation set concerned with the harmonic solutions; sub-harmonic bifurcations are not treated here.

‡ Although the remaining control parameters, α and ζ, do not generally affect the structure of these sections, their influence is of considerable interest. See below, Figures 9 and 13 and section 6.

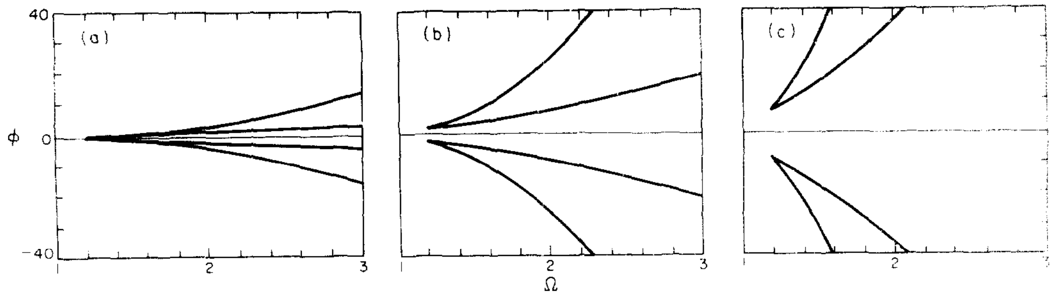


Figure 4. The bifurcation set for the hardening spring in ϕ, Ω space. $\zeta = 0{\cdot}1$. (a) $\alpha = 0{\cdot}5$; (b) $\alpha = 0{\cdot}01$; (c) $\alpha = 0{\cdot}0005$.

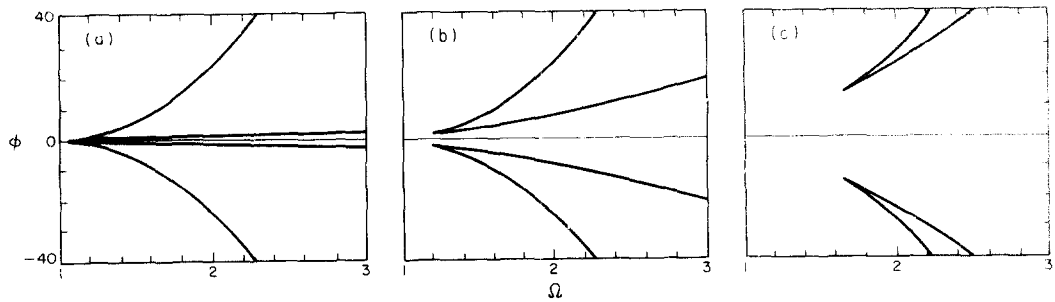


Figure 5. The bifurcation set for the hardening spring in ϕ, Ω space. $\alpha = 0{\cdot}01$. (a) $\zeta = 0{\cdot}01$; (b) $\zeta = 0{\cdot}1$; (c) $\zeta = 0{\cdot}3$.

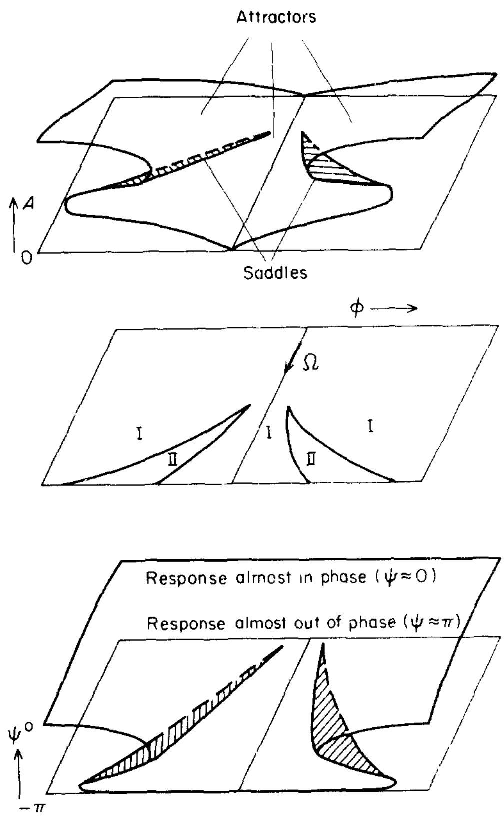


Figure 6. Amplitude and phase surfaces over ϕ, Ω control space for the hardening spring ($\zeta > 0$, $\alpha > 0$). Region I, one attractor; region II, two attractors plus one saddle.

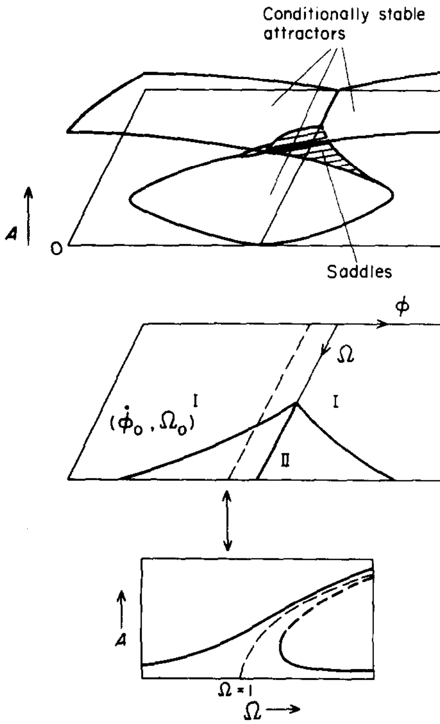


Figure 7. The amplitude surface over ϕ, Ω control space ($\rho = 0$; $\alpha > 0$) with a section of the frequency response function.

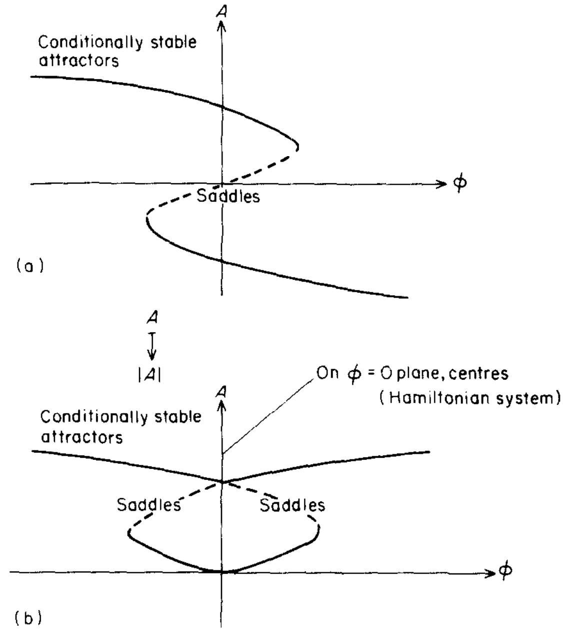
and the corresponding equation for the bifurcation set reduces to

$$\phi = \pm\sqrt{-16\rho^3/81\alpha} \text{ or } 0. \tag{4.9}$$

As $\zeta \to 0$ the two cusps meet, their inner arms becoming coincident on the $\phi = 0$ axis. The folds above these lines "close up" as indicated in Figure 7. In the past the case of zero damping has received more attention than the general case ($|\zeta| > 0$) since the analysis is simpler [1, 2]. However, in this case the jumps from low to high amplitudes alone are predicted, since the two "arms" of the frequency response function, A *vs.* Ω ($\alpha > 0$, $\phi > 0$, fixed) never meet. This is clear from Figure 7; the "A surface" is (topologically) a cone beneath a plane, and solutions, once on the plane, can never leave it.

It was probably this case that misled Zeeman in his original application to memory recall [16] (see also reference [17]). Zeeman predicted the existence of a single cusp in (ϕ, Ω) space with the "canonical" surface lying above it, as in Figure 2. Our equation (4.9) indicates that, there are, in fact, two cusps and that ϕ and Ω act as conflicting factors rather than normal and splitting factors as Zeeman suggests. One can see how, by attributing a meaning to the sign of the amplitude, an incorrect surface was predicted. Replacing A by $|A|$, as in equation (4.8), transforms the canonical cusp form into the correct surface as indicated in Figure 8; also note the interchange of zones of stability. Then one can see that when starting with a large force ((ϕ_0, Ω_0) outside the (double) cusp), variations in the controls lead to no jumps in amplitude, although the phase changes by π as the force moves smoothly through 0.

As Zeeman points out one must be careful in interpreting Figure 8. The single point $\phi = 0$, $A = \sqrt{(-4\rho/3\alpha)}$, and $\Omega > 1$ in Figure 8 (b) represents not one harmonic solution but a whole torus filled with a circle of harmonics. For this point represents the whole circle of harmonic solutions in (u, v)-space: $0 \leqslant \psi \leqslant 2\pi$, $A = \sqrt{(-4\rho/3\alpha)}$. Thus the map $(u, v) \mapsto A$ maps the set of harmonic solutions non-homeomorphically when $\phi = 0$ and $\Omega > 1$, and, in



Sections for fixed $\zeta = 0$, $\alpha > 0$, $\Omega > 1$

Figure 8. A section of the canonical cusp surface compared with a section of the amplitude surface for Duffing's equation with zero damping.

this sense, the picture does not correctly represent the set of harmonic solutions.

In fact, $\zeta = 0$ represents a physically unrealistic case since an arbitrarily small amount of damping is necessary if hyperbolic stable solutions are to exist at all. For zero damping one has a forced, undamped system and the orbits are only conditionally stable.

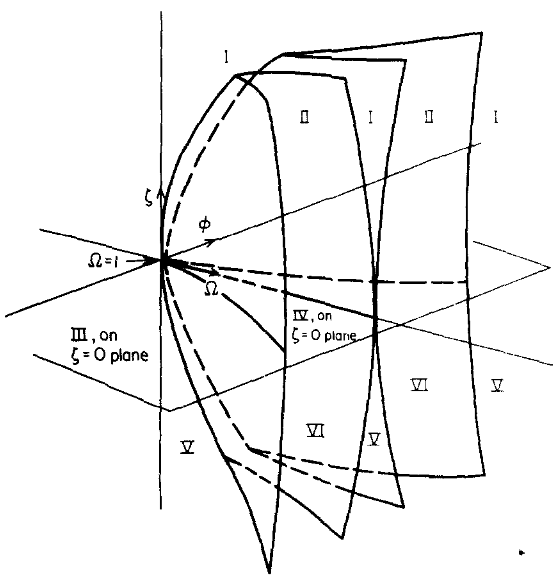


Figure 9. A section of the bifurcation set for the hardening spring ($\alpha > 0$). Region I, one attractor; region II, two attractors plus one saddle; region III, one conditionally stable attractor; region IV, two conditionally stable attractors plus one saddle; region V, one repellor; region VI, two repellors plus one saddle.

It is worth mentioning the case of negative damping ($\zeta < 0$). Since ζ appears in equation (4.6) only as a squared term (σ^2), the bifurcation set is symmetrical about the plane $\zeta = 0$. However, for $\zeta < 0$ all solutions become unstable, as shown in section 3. The attractors become repellors while the saddle remains a saddle. A complete bifurcation set in $\mathbf{R}^3$ for the hardening spring is sketched in Figure 9, with the different regions indicated.

To conclude this section the case of the linear system ($\alpha = 0$) will be briefly mentioned. When the restoring force is completely linear it is well known that jumps do not occur and that the amplitude increases linearly with force [2]. Equation (4.1) reduces to the common equation for a single degree of freedom system:

$$A = \phi/\{(1 - \Omega^2)^2 + 4\zeta^2\Omega^2\}^{1/2}, \tag{4.10}$$

and equation (4.6) indicates that the bifurcation set vanishes, or, more correctly, moves out to infinite force levels.

5. THE BIFURCATION SET FOR THE SOFTENING SPRING ($\alpha < 0$)

Although equation (4.5) governs the bifurcation set for all values of α it will be more convenient to discuss the softening spring separately, since the behaviour differs in some aspects from that of the hardening spring and some qualifications must be made. The most important of these is that there is clearly a limiting amplitude beyond which the spring loses its stiffness, becoming, in theory, of negative stiffness. The treatment given here does not yield this limit since it does not correspond to a bifurcation in the averaged equation. Presumably the parameter values lie outside the region for which this approximation is valid. In this respect our treatment is incomplete; however, a conventional stability investigation of Duffing's equation with negative non-linearity [1] leads to the stability criterion

$$A \leqslant \sqrt{-2/3\alpha} \tag{5.1}$$

This corresponds to $|\lambda|$ small (section 3) (physically we are restricting ourselves to a "sensible" amplitude range in which the spring offers resistance to forces on it). One must, however, bear the limitation of equation (5.1) in mind when defining areas of stability and instability on the amplitude surface above the control space.

Figures 10 and 11 show sections of the bifurcation set in (ϕ, Ω) space for varying α and ζ values ($\alpha < 0$, $\zeta > 0$). The manner in which K "comes in" from infinity as $|\alpha|$ increases is clear. It can also be seen that the tangent to K is parallel to the Ω axis for some Ω. The form of the amplitude surface above it is particularly interesting. The surface is sketched in Figure 12 and sections (1-5) in (A, Ω) space, for constant ϕ, ζ and α, are also shown. These are, of course, frequency response functions. Note that the frequency response function for large forces is not obtained as a result of the folded section (*e–b–c–d* in section 3 of Figure 12) "growing" until it reaches the A axis, but by a meeting of *two* folds. Previous treatments [2, 3] have not predicted the existence of the first fold (in sections 1–4 of Figure 12).† Thus the behaviour is radically different from that of the hardening spring, as will be seen.

It will be noted that practical (experimental) observations are limited to a relatively small area of these curves (or of the surface) since the saddles cannot be observed and the amplitude limitation of equation (5.1) makes observation of the upper surface for low frequency (Ω) or high force (ϕ) levels impossible. In effect, these limitations prevent observation of the fold in sections 1 to 4 and make observations of the upper surface in section 5 very difficult. Thus the interesting meeting of the two folds in section 4 cannot be demonstrated clearly. It is, however, possible to demonstrate its effects indirectly and in a striking manner, as follows.

† Note added in revision: compare with the treatment by Timoshenko, Young & Weaver 1974 *Vibration Problems in Engineering* (4th Edn.), p. 184; New York: John Wiley.

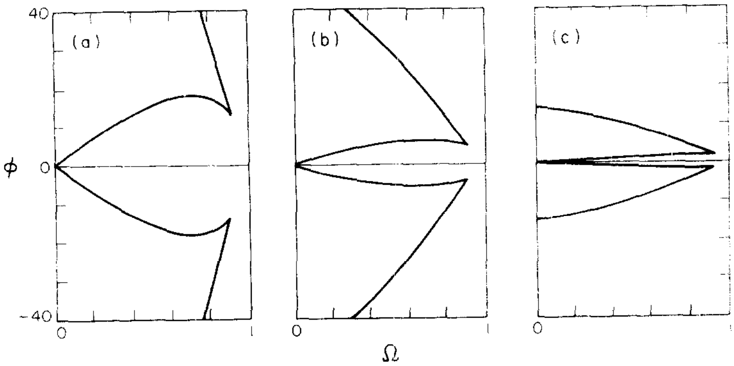


Figure 10. The bifurcation set for the softening spring in ϕ, Ω space. $\zeta = 0{\cdot}05$. (a) $\alpha = -0{\cdot}00001$; (b) $\alpha = -0{\cdot}0001$; (c) $\alpha = -0{\cdot}001$.

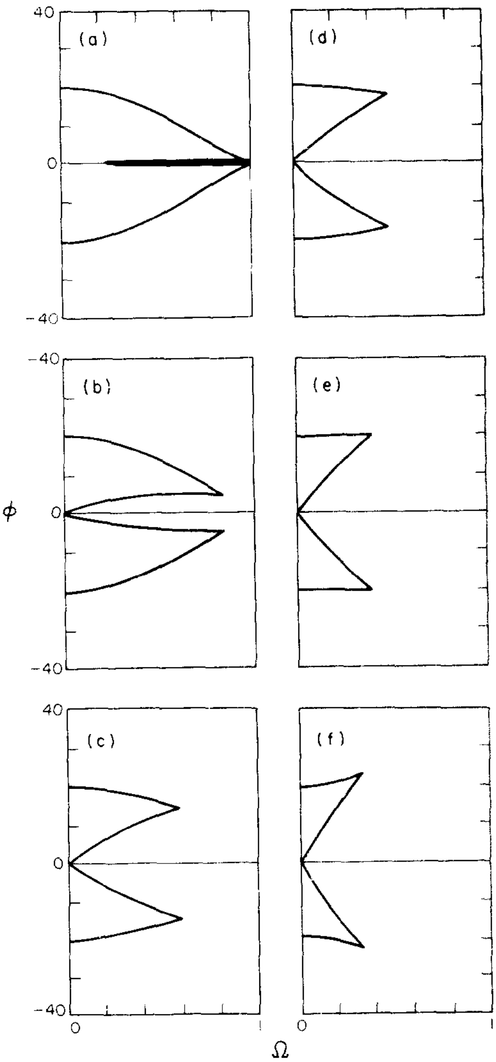


Figure 11. The bifurcation set for the softening spring in ϕ, Ω space. $\alpha = -0{\cdot}0005$. (a) $\zeta = 0{\cdot}01$; (b) $\zeta = 0.1$ (c) $\zeta = 0{\cdot}3$; (d) $\zeta = 0{\cdot}4$; (e) $\zeta = 0{\cdot}55$; (f) $\zeta = 0{\cdot}75$.

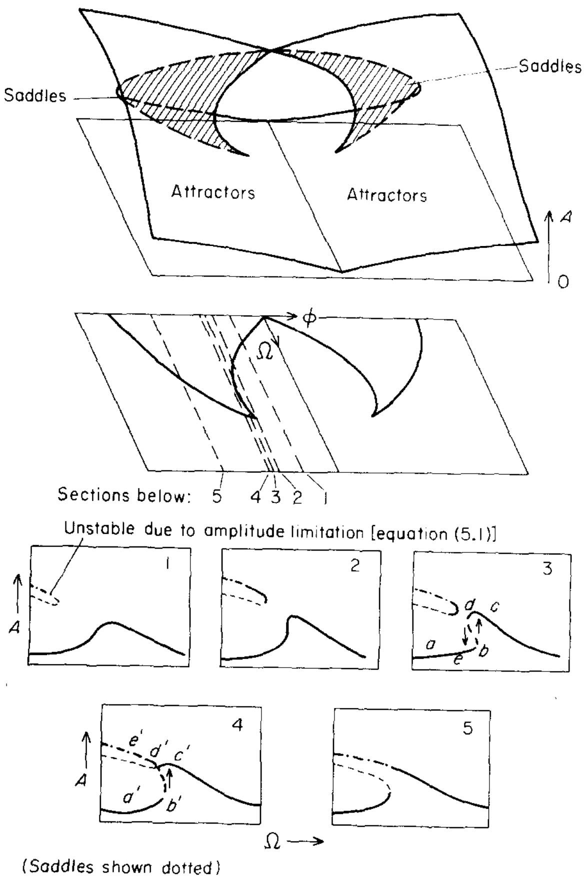


Figure 12. Amplitude surface over ϕ, Ω control space and sections of the frequency response function (A *vs.* Ω) for the softening spring ($\zeta > 0$; $\alpha < 0$).

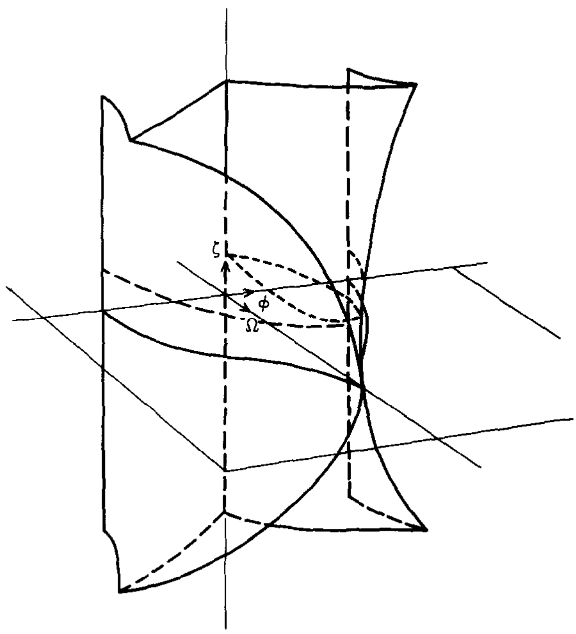


Figure 13. A section of the bifurcation set for the softening spring ($\alpha < 0$).

For a given system ($\zeta > 0$, $\alpha < 0$ fixed) a force level, ϕ, is selected so that a frequency response function of type 3 (Figure 12) applies. The forcing function is applied at a low frequency so that stable oscillations are obtained, the solution corresponding to point a on the lower attractor sheet. As the frequency is increased, the solution moves to point b and a jump occurs to point c as the lower stable solution is annihilated. If the frequency is then reduced a second jump occurs, from d to e, as the upper stable solution is annihilated. So far behaviour is exactly analogous to that of the hardening spring [2, 3]. For slightly larger values of force, however, the qualitative picture changes to type 4.

In this case, although an increase in frequency leads to similar behaviour, $a' \to b' \to c'$, a decrease, $c' \to d' \to e'$, provides access to the upper fold surface, since it now meets the second fold. As has been seen, at the line $A = \sqrt{-2/3\alpha}$ on this surface the stable solution vanishes as the amplitude of oscillations increases and the spring exhibits "negative stiffness". In practice (on an analogue computer model, for example) a slight decrease in frequency beyond point d' causes the spring to lose its stiffness and system to blow up. Thus the region for stable behaviour of the softening spring is effectively bounded by its force level for which the tangent to the bifurcation set is parallel to the frequency axis.

Finally, Figure 13 shows a section of the bifurcation set in $\mathbf{R}^3$ for a fixed non-linearity parameter ($\alpha < 0$).

6. EXPERIMENTAL WORK

In order to investigate the validity of assumptions such as that of a near harmonic response, Duffing's equation was modelled on an analogue computer and a number of measurements made. The problems mentioned in section 5 made quantitative investigation of the softening spring difficult, but tests on the hardening spring showed reasonable agreement with the predicted behaviour. Figure 14 shows a comparison of measured bifurcation sets in ϕ, Ω space with predictions and an isometric drawing of a measured amplitude surface is presented in Figure 15. The only major difference between experimental results and the theory occurs in the bifurcation set for $\alpha = 0{\cdot}01$ (Figure 14). Here the lower "arm" of the measured set diverges considerably (and in a consistent manner) from the theoretical predictions for $\Omega > 1{\cdot}4$. Since the other bifurcation sets (for $\alpha = 0{\cdot}05$ and $0{\cdot}1$) compare so well, this deviation may repay further study.

Observations of orbits in x, $\dot{x}$ space by means of a cathode ray oscilloscope indicated that the response was almost purely harmonic except at very low or high frequencies and high force levels. Thus it can be concluded that the theoretical treatment is satisfactory.

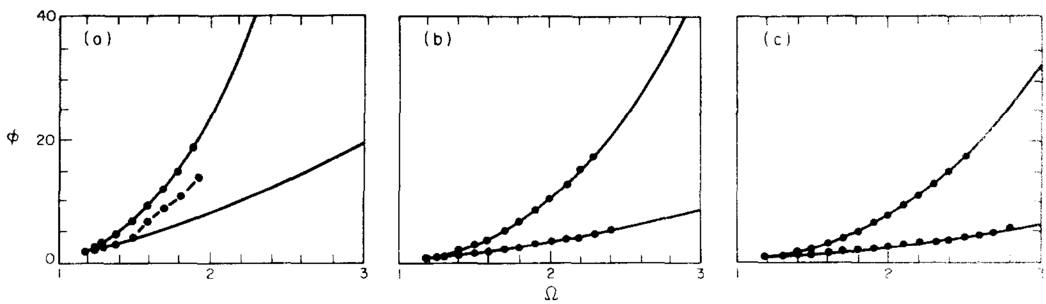


Figure 14. Measured and calculated bifurcation sets for the hardening spring ($\alpha > 0$). The sets are symmetrical about $\phi = 0$ and only one side is shown. (a) $\zeta = 0{\cdot}10$, $\alpha = 0{\cdot}01$; (b) $\zeta = 0{\cdot}10$, $\alpha = 0{\cdot}05$; (c) $\zeta = 0{\cdot}10$, $\alpha = 0{\cdot}10$. ●, Measured bifurcation points; ——, calculated bifurcation set.

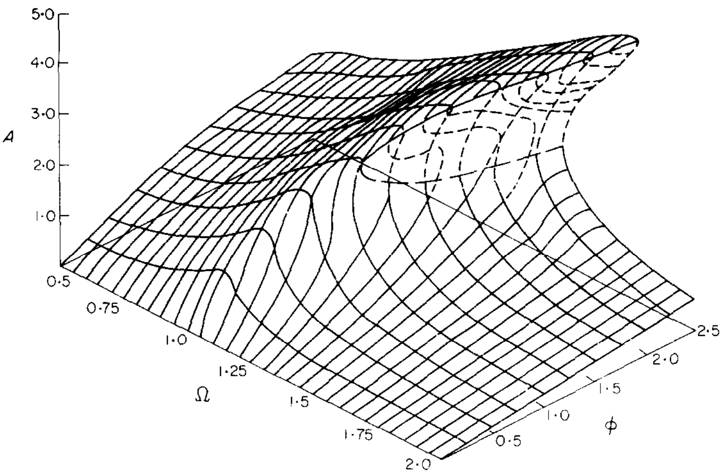


Figure 15. The measured amplitude surface over the ϕ, Ω control space. The surface is symmetrical about $\phi = 0$ plane and only one side is shown. $\zeta = 0{\cdot}1$, $\alpha = 0{\cdot}05$. Unobservable (saddle) region shown dotted.

7. CONCLUSION: MORE BIFURCATIONS

Although the treatment of Duffing's equation has not revealed any radically new facts, we feel that the approach is of general interest to vibration engineers. In many non-linear problems, sudden jumps and discontinuities are observed and as we have indicated, catastrophe theory provides natural descriptions for such behaviour. Moreover, it is helpful in the determination of the general form that mathematical models of such systems may take (*via* the notions of structural stability and genericity). In this respect, the determination of the bifurcation set in $\mathbf{R}^4$ rather than $\mathbf{R}^1$, with ζ, α, ϕ fixed (as in references [2] and [3]) is particularly interesting since it provides a "global" picture of the behaviour and in so doing, explains some strange experimental observations for the softening spring.

An obvious extension of this work is to free the damping and non-linearity parameters and consider Duffing's equation as a three or four parameter family of differential equations. As a two parameter family, parameterized by (ϕ, Ω) with ϕ small, Ω near 1, and fixed $\zeta, \alpha > 0$, Duffing's equation is "generic" but this is no longer true if one introduces an extra parameter such as damping or non-linearity (see Figures 9 and 13). By this we mean than an arbitrarily small perturbation of Duffing's equation in the space of all three or four parameter families could destroy the qualitative nature of the bifurcation set.

This raises an interesting question which is particularly relevant from a physical point of view. Is there a way of "regularizing" Duffing's equation: i.e., is there a simplest, physically meaningful alteration to Duffing's equation which makes the resulting bifurcation set generic? One possibility is that by introducing a non-linearity β into the damping the resulting equation will have generic bifurcations with respect to the parameterization by α, β, ζ, Ω and ϕ. Such a family would include the forced Van der Pol oscillator. This oscillator is not governed by an elementary catastrophe: we should expect to have to use the more general generic theory of bifurcations of vector fields (catastrophe theory) here.

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REFERENCES

1. C. HAYASHI 1964 *Non-linear Oscillations in Physical Systems*. New York: McGraw-Hill Book Company.
2. W. T. THOMPSON 1966 *Vibration Theory and Applications*. London: Allen and Unwin.
3. S. TIMOSHENKO and D. H. YOUNG 1955 *Vibration Problems in Engineering*. Van Nostrand, third edition.
4. S. SMALE 1967 Differentiable dynamical systems. *Bulletin of the American Mathematical Society* **73**, 747–817.
5. M. HIRSCH and S. SMALE 1974 *Differential Equations, Dynamical Systems and Linear Algebra*. New York: Academic Press.
6. J. W. MILNOR 1965 *Topology from the Differentiable Viewpoint*. University Press of Virginia.
7. V. I. ARNOLD 1972 *Russian Maths Surveys* **27**, 54–123. Lectures on bifurcations in versal families.
8. F. TAKENS 1974 *Communication* 2 *of The Mathematical Institute, Rijksuniversiteit, Utrecht*. Introduction to global analysis.
9. E. C. ZEEMAN 1975 *Applications of Catastrophe Theory*. Tokyo: Manifolds.
10. R. THOM and E. C. ZEEMAN 1975 *Warwick Dynamical Systems, Springer Lecture Notes in Mathematics*, 361–401. Catastrophe Theory: its present state and future perspectives.
11. R. THOM 1973 *Stabilité Structurelle et Morphogenésè*. London: Benjamin.
12. F. TAKENS 1974 in *Communication* 3 *of the Mathematical Institute, Rijksuniversiteit, Utrecht: Applications of Global Analysis*. Forced oscillations and bifurcations.
13. N. N. BOGOLIUBOV 1934 *Publication* 8 *of the Ukranian Academy of Science, Kiev*. The application of methods of non-linear mechanics to the theory of stationary oscillations.
14. J. K. HALE 1969 *Ordinary Differential Equations*. New York: Wiley Interscience.
15. M. ABRAMOWITZ and I. A. STEGUN 1965 *Handbook of Mathematical Functions*. Dover.
16. E. C. ZEEMAN 1972 *I.C.T.P. Conference on Neural Networks, Trieste, July* 1972. Catastrophe theory in brain modelling.
17. R. ABRAHAM 1972 (March) *Quatrième Rencontre entre Mathematiciens et Physiciens*, Vol. **4**, *Fasc.* 1, *Université de Lyon*. Introduction to morphology.