DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

COURSEWORK

The final report should be submitted electronically on Blackboard by the end of the 18th of March 2020
Include your code in an appendix at the end of the report. Submit a single pdf file.
Standard plagiarism and late submission policies apply.

- The coursework is complementary to the lectures: you will work on discrete-time systems from a different point of view. The aim of the coursework is to encourage your own understanding of the problems arising when interfacing a discrete-time controller with a real device (i.e. a nonlinear continuous-time system which is not totally captured by the linearized models).
- The report is an individual work.
- The coursework is divided in part A and part B. For part B, it is important that you understand that a "result" is just the bare minimum. What makes a report a good coursework is the comments which go with the result.
- No help will be provided on the coursework (apart for clarifications on the text of this document). The reason is that your ability to work independently is part of the assessment. Google is your friend! All functions that are needed to solve this coursework are standard in MATLAB and are documented on-line. Questions such as "How do I place the eigenvalues of the closed-loop system?" will not be answered (see MATLAB function *place*).
- All the elements of the coursework can be completed from the beginning. Possibly, the only exception is point B10. Point B10 asks to solve a linear quadratic regulator problem, which is a topic covered in lecture L8. However, note that the solution of that point relies on just one MATLAB function, so it is debatable if you really need L8 to complete the point...
The state-space representation is a prerequisite to the M.Sc. (topic covered in "Control Engineering" ELEC96009 in the autumn term). If in doubt, note that a revision of state-space will be covered in lecture L7.

Consider an inverted pendulum on a cart. The model is described by the equations

$$M\ddot{s}(t) + F\dot{s}(t) - \mu(t) = 0, \qquad \ddot{\phi}(t) - \frac{g}{L}\sin\phi(t) + \frac{1}{L}\ddot{s}(t)\cos\phi(t) = 0, \tag{1}$$

where $s(t)$ is the displacement of the pivot; $\phi(t)$ is the angular rotation of the pendulum; $\mu(t)$ is the external force exerted on the carriage; $M = 1\,kg$ is the mass of the carriage, $L = 0.842\,m$ is the effective pendulum length, $F = 1\,kg\,s^{-1}$ is a friction coefficient, and $g = 9.8093\,m\,s^{-1}$ is the gravitational acceleration.

Points A1 to A6 have to be completed analytically.

A1) Write the equations of the system in the standard format

$$\begin{aligned} \dot{x} &= f(x,u)\,, \\ y &= g(x,u)\,, \end{aligned}$$

where

$$x(t) := \begin{bmatrix} s(t) \\ \dot{s}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix}, \qquad u(t) = \mu(t), \qquad y(t) = \begin{bmatrix} s(t) \\ \phi(t) \end{bmatrix}.$$

A2) Compute all the equilibrium points of the system for $\mu(t) = 0$.

A3) Write the equations describing the linearised system around the equilibrium point $x(t) = 0$.

A4) Express the obtained linear dynamics in the standard state space format

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{2}$$

A5) Show that the pair (A, B) is reachable.

A6) Show that the pair (A, C) is observable.

Points B1 to B11 have to be completed using some simulation software.
The software code has to be included in the report.
Note that if you use MATLAB, there are already plenty of built-in functions to complete this coursework. Check the manual.

In the subsequent design use the following constraints as guidelines:

- $|\phi(t)| < \frac{\pi}{4}$ for all $t \geq 0$.
- Avoid excessively large $|\dot{\phi}(t)|$ and $|\dot{s}(t)|$ for all $t \geq 0$.

B1) Design a state feedback controller $u(t) = Kx(t)$ which stabilizes the linearized system (2) in $x(t) = 0$. Comment your design.

B2) Using the controller in point B1, display plots (no more than four) of $y(t)$ and $u(t)$ for various initial states $x(0)$. You are free to choose the initial conditions, however your choice should be adequate to show the performance/limitation of your design. Comment on these plots. **Use the same initial conditions in all the following points.** (Hint: since you will have to use the same initial states in the rest of the coursework, you may realize that your initial selection of initial states is not adequate to show differences in performance. Thus, you may want to come back to point B2 and pick other initial states.)

B3) Display plots of $y(t)$ for the nonlinear system (1), from the same initial states $x(0)$ and using the same controller $u(t)=Kx(t)$ of point B2. Comment on these plots.

B4) Suppose that the control law is implemented with a discrete-time controller connected to the nonlinear system (1) via an impulsive sampler (sampling the continuous-time state $x(t)$) and a hold (generating the continuous-time input $u(t)$). Find a value of the sampling time T for which the closed-loop system is asymptotically stable. Find a value of T for which the closed-loop system is unstable. Plot the corresponding $y(t)$ and comment on the plots.

B5) Similarly to point B4, apply the discrete-time control law to the nonlinear system (1). This time, discuss about the possible degrading of the performance for different values of T. Display plots of $y(t)$ and comment on the plots.

B6) The relations between the matrices describing the continuous-time state-space description (A, B, C, D) and the discrete-time one obtained by step response invariance (A_d, B_d, C_d, D_d) are:

$$A_d = e^{AT}\,, \qquad B_d = \int_0^T e^{A(T-\tau)} B d\tau\,, \quad C_d = C\,, \quad D_d = D\,.$$

Compute A_d and B_d with respect to a suitable value of T (Hint: choose T as the value associated in point B4 with the asymptotically stable closed-loop).

B7) Design a discrete-time state feedback control law $u(k) = K_d x(k)$ such that the closed-loop associated to the discretized linearized system computed in point B6 is asymptotically stable (Hint: place the poles of $A_d + K_d B_d$ mapping the eigenvalues you chose in point B1 to the z-plane). Display plots of $y(t)$ and comment on the plots.

B8) Apply the discrete-time state feedback control law $u(k) = K_d x(k)$ to the nonlinear system (1). Display plots of $y(t)$, comment on the plots and compare with the results of point B5.

B9) Design a discrete-time state feedback control law $u(k) = K_d^* x(k)$ for the discretized linearized system computed in point B6 which minimizes the cost

$$J = \sum_{k=0}^{\infty} x^\top(k) Q x(k) + u^\top(k) u(k).$$

Select Q in such a way that the displacement $s(t)$ and the angular rotation $\phi(t)$ are penalized 100 times more than their respective velocities. Display plots of $y(t)$ and comment on the plots.

B10) Apply the discrete-time optimal control law $u(k) = K_d^* x(k)$ to the nonlinear system (1). Display plots of $y(t)$, comment on the plots and compare with the results of point B5 and B8. (Hint: in point B5 you applied a controller designed for continuous-time as a discrete-time controller. In point B8 you applied a controller designed for discrete-time as a discrete-time controller. In point B10 you applied an optimal controller designed for discrete-time as a discrete-time controller. Thus, the performances should be different).

B11) Draw some final general conclusions on the importance of the choice of the sampling time T.