COURSEWORK: SYSTEMS IDENTIFICATION
THE FINAL REPORT SHOULD BE SUBMITTED ON BLACKBOARD BY THE 20TH OF JANUARY 2020

Objectives.

Consider a stationary stochastic process $y(\cdot)$ generated as shown in Fig. 0.1.

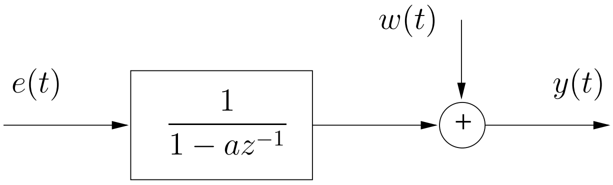

Figure 0.1 Dynamic system generating the stationary process $y(\cdot)$.

where the processes $e(\cdot)$ and $w(\cdot)$ are independent, $e(\cdot) \sim WN(0, \lambda^2)$, $w(\cdot) \sim WN(0, \mu^2)$, and $a \in \mathbb{R}$, $|a| < 1$.

The purpose of this coursework assignment is to:

- analyze a stationary process by a static representation using expected value, variance and covariance functions;
- show theoretically the identification capabilities of AR models of increasing order;
- compare through simulations the theoretical findings with the empirical estimates obtained by Least Squares (LS) techniques;
- show the effects of coloured noise on the LS estimates;
- implement a recursive LS algorithm and evaluate its effectiveness compared with the performance of the batch LS algorithm.

A. Analysis

A1. For generic values of a, λ^2, μ^2, determine the expected value of the process $\overline{y} = E[y(t)]$ (constant, *why?*), the variance $\mathrm{var}[y(t)] = E\{[y(t) - \overline{y}]^2\}$, and the covariance function $\gamma_y(\tau) = E\{[y(t) - \overline{y}][y(t-\tau) - \overline{y}]\}$.

A2. For generic values of a, λ^2, μ^2, determine by a PEM identification algorithm a model of the process $y(\cdot)$ by using the family AR(1) of dynamic process models

$$\mathscr{M}_1(\theta_1): \ y(t) = a_1 y(t-1) + \xi(t), \quad \theta_1 := a_1$$

and the family AR(2) of dynamic process models:

$$\mathscr{M}_2(\theta_2): \ y(t) = a_1 y(t-1) + a_2 y(t-2) + \xi(t), \quad \theta_2 := [a_1\,, a_2]^\top$$

where $\xi(\cdot)$ is a suitable white process.

Determine the optimal (in the PEM sense) values θ_1° and θ_2° of the parameters θ_1 and θ_2, respectively.

Moreover, determine the variances $\text{var}[\varepsilon_{\theta_1^\circ}(t)]$ and $\text{var}[\varepsilon_{\theta_2^\circ}(t)]$ of the prediction errors associated with the models $\mathscr{M}_1(\theta_1^\circ)$ and $\mathscr{M}_2(\theta_2^\circ)$, respectively.

A3. Now, assume that $e(\cdot)$ and $w(\cdot)$ are replaced by $\tilde{e}(\cdot)$ and $\tilde{w}(\cdot)$ that are still independent processes, but *they are not zero-mean*. Specifically: $\tilde{e}(\cdot) = 1 + e(\cdot)$ (with $e(\cdot) \sim WN(0,\lambda^2)$), and $\tilde{w}(\cdot) = 4 + w(\cdot)$ (with $w(\cdot) \sim WN(0,\mu^2)$), all the other conditions being the same as before. Answer again Questions A1 and A2 by also commenting on the differences compared to your answers under the initial assumptions on the characteristics of $e(\cdot)$ and $w(\cdot)$.

A4. Assume that the true parameters a, λ^2, μ^2 take on the following values: $a = 1/2$, $\lambda^2 = 9$, $\mu^2 = 1$. Compute the numerical values of all quantities determined in Points *A1* and *A2* above. Compare $\text{var}[\varepsilon_{\theta_1^\circ}(t)]$ and $\text{var}[\varepsilon_{\theta_2^\circ}(t)]$. Comment on your findings.

B. Simulations

B1. Consider the initial setting (hence not the one mentioned in Point A3 above) and the true values set at Point *A4* ($a = 1/2$, $\lambda^2 = 9$, $\mu^2 = 1$) and set $N = 1000$. By running the recursive equation

$$y(t) = ay(t-1) + e(t) + w(t) - aw(t-1)\,, \quad t = 1,\ldots,N \tag{0.1}$$

and by making use of a suitable MATLAB random number generator library function for generating a sequence of independent, zero mean, variables of suitable variance, obtain by Eq. (0.1) a simulated data set $B = \{y(1),\ldots,y(N)\}$. (Set the initial conditions $y(0)$, $w(0)$ to zero, to initiate the recursion.)

Apply the Least Squares Algorithm to the data set B you have generated to obtain the estimate $\hat{\theta}_1$ of the parameter θ_1 for the AR(1) case and the estimate $\hat{\theta}_2$ of the vector of parameters θ_2 for the AR(2) case.

Compare the estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ with the respective optimal (in the PEM sense) values θ_1° and θ_2° determined at Point *A2* (and evaluated by using the true parameters above). Comment on your findings.

Set $N = 2000$ and repeat the overall simulation. Comment on your findings.

Now, construct 100 independent batches of simulated data $B_i = \{y_i(1),\ldots,y_i(N)\}, i = 1,\ldots,100$, with $N = 1000$, each obtained as above. Apply the Least Squares Algorithm to each batch of data B_i that you have generated to obtain an estimate $\hat{\theta}_1^{(i)}$ of the parameter θ_1 for the AR(1) case and an estimate $\hat{\theta}_2^{(i)}$ of the vector of parameters θ_2 for the AR(2) case. Compute the empirical mean and variances of $\hat{\theta}_1^{(i)}$ and $\hat{\theta}_2^{(i)}$:

$$\overline{\theta}_1 = \frac{1}{100}\sum_{i=1}^{100}\hat{\theta}_1^{(i)}; \quad \sigma^2_{\theta_1} = \frac{1}{100}\sum_{i=1}^{100}\left(\hat{\theta}_1^{(i)} - \overline{\theta}_1\right)^2;$$

$$\overline{\theta}_2 = \frac{1}{100}\sum_{i=1}^{100}\hat{\theta}_2^{(i)}; \quad \sigma^2_{\theta_2} = \frac{1}{100}\sum_{i=1}^{100}\begin{bmatrix} \left(\hat{\theta}_{2,1}^{(i)} - \overline{\theta}_{2,1}\right)^2 & \left(\hat{\theta}_{2,1}^{(i)} - \overline{\theta}_{2,1}\right)\left(\hat{\theta}_{2,2}^{(i)} - \overline{\theta}_{2,2}\right) \\ \left(\hat{\theta}_{2,1}^{(i)} - \overline{\theta}_{2,1}\right)\left(\hat{\theta}_{2,2}^{(i)} - \overline{\theta}_{2,2}\right) & \left(\hat{\theta}_{2,2}^{(i)} - \overline{\theta}_{2,2}\right)^2 \end{bmatrix}$$

where the notation $\hat{\theta}_{2,1}^{(i)}$ stands for the first component of vector $\hat{\theta}_2^{(i)}$, etc.

Compare the mean values of the estimates $\overline{\theta}_1$ and $\overline{\theta}_2$ with the respective optimal values θ_1° and θ_2° determined at Point *A2* (and evaluated by using the true parameters above). Comment on your findings.

Finally, construct 100 independent larger batches of simulated data $\tilde{B}_i = \{y_i(1), \ldots, y_i(N)\}, i = 1, \ldots, 100$, with $N = 2000$, each obtained as above and repeat all the above simulation campaign. Compare the results with the ones obtained on batches B_i and comment on your findings.

B2. Assume that $e(\cdot)$ is a *coloured noise* generated according to the recursive equation:

$$e(t) = -\frac{1}{2}e(t-1) + \xi(t)$$

where $\xi(\cdot) \sim WN(0,1)$, $\xi(\cdot)$ independent from $w(\cdot)$.

Without modifying the Least Squares Algorithm because of the fact that the noise $e(\cdot)$ is now coloured, repeat the whole simulation campaign above (question B1) again applying the Least Squares Algorithm for both AR(1) and AR(2) case. What can be observed? Comment on your findings.

B3. Take the data set B with $N = 1000$ considered in Point B above and implement a recursive version of the LS algorithm implemented in the context of Point B. Apply this recursive algorithm to obtain the estimate $\hat{\theta}_1^{(\mathrm{rec})}$ of the parameter θ_1 for the AR(1) case and the estimate $\hat{\theta}_2^{(\mathrm{rec})}$ of the vector of parameters θ_2 for the AR(2) case, where the superscript "(rec)" stands for "recursive". Comment on your findings.