Reinforcement Learning

- Blue slides: Mitchell
- Green slides: Alpaydin

Reinforcement Learning (RL)

- How an **autonomous agent** that **sense** and **act** in the environment can **learn to choose optimal actions** to achieve its **goals**.
- Examples: mobile robot, optimization in process control, board games, etc.
- Ingredients: **reward/penalty** for each action, where the reinforcement signal can be significantly **delayed**.
- One approach: Q **learning**

Introduction: Agent

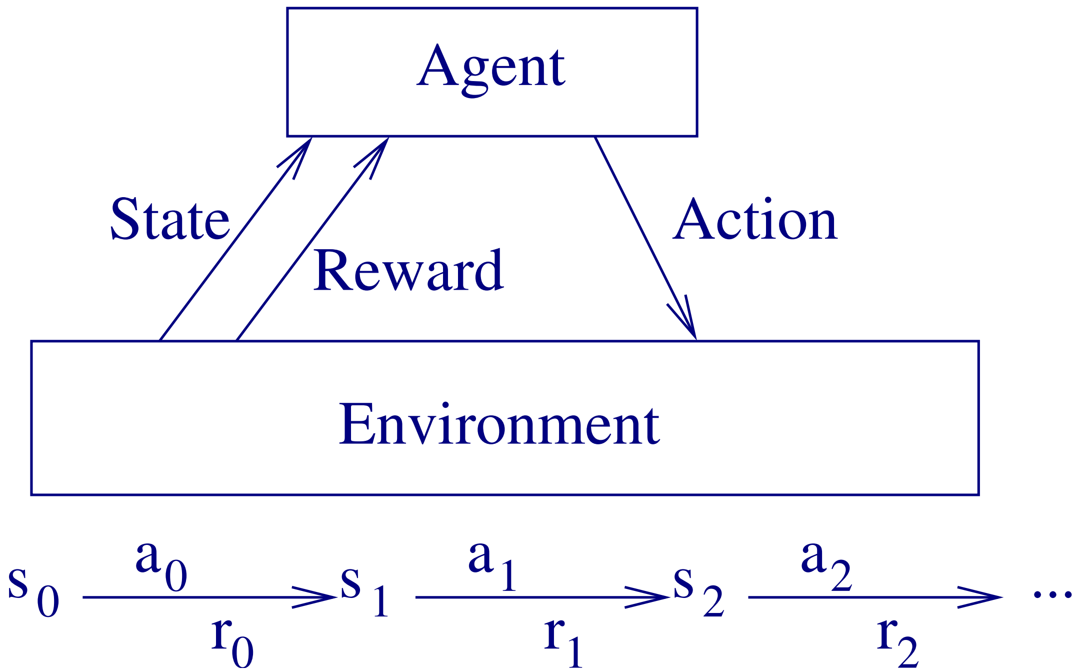
Terminology:

- **State**: state of the environment, obtained through sensors
- **Action**: alter the state
- **Policy**: choosing actions that achieve a particular goal, based on the current state.
- **Goal**: desired configuration (or state).

Desired policy:

- From any initial state, choose actions that **maximize the reward accumulated over time** by the agent.

RL Task



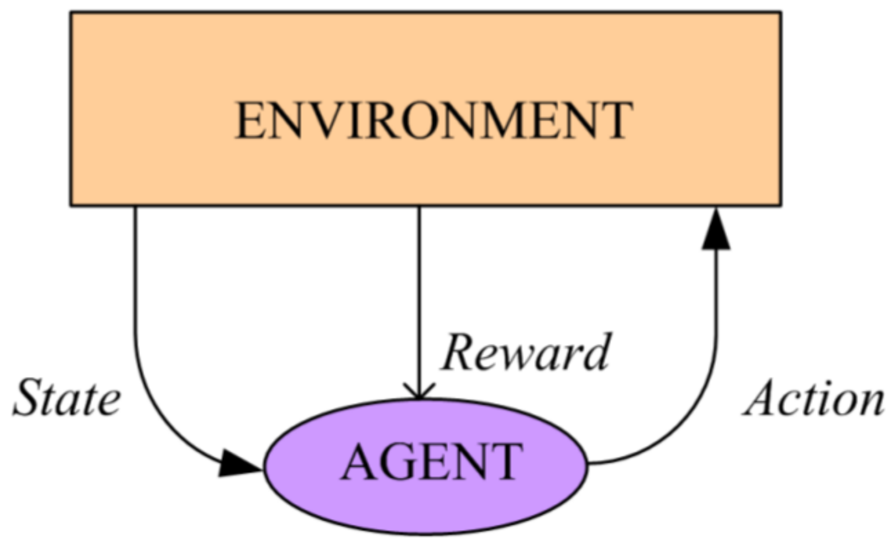
- Goal: learn to choose actions that maximize **discounted, cumulative award**:

$$r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots, \text{ where } 0 \leq \gamma < 1.$$

- That is, we want to learn a policy $\pi : S \rightarrow A$ that maximizes the above, where S is the set of states, and A that of actions.

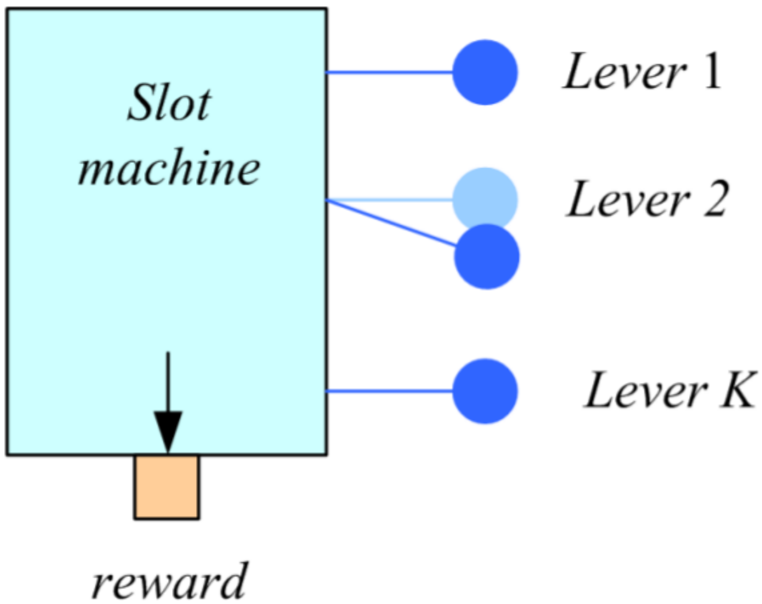
[Alpaydin] Introduction

- Game-playing: Sequence of moves to win a game
- Robot in a maze: Sequence of actions to find a goal
- Agent has a state in an environment, takes an action and sometimes receives reward and the state changes
- Credit-assignment
- Learn a policy



[Alpaydin] Single State: K-Armed Bandit

- Among K levers, choose the one that pays best
  - $Q(a)$: value of action a
  - Reward is r_a
  - Set $Q(a) = r_a$
  - Choose a^* if
    - $Q(a^*)=\max_a Q(a)$



- Rewards stochastic (keep an *expected* reward):

$$Q_{t+1}(a) \leftarrow Q_t(a) + \eta\left[r_{t+1}(a) - Q_t(a)\right]$$

Variations of RL Tasks

- Deterministic vs. nondeterministic action outcomes.
- With or without prior knowledge about the effect of action on environmental state.
- Partially or fully known environmental state (e.g., Partially Observable Markov Decision Process [POMDP]).

RL Compared to Other Learning Algorithms

- Planning (in AI)
- Function approximation: $\pi : S \rightarrow A$.
- Differences:
  - Delayed reward
  - Exploration vs. exploitation
  - Partially observable states
  - Life-long learning: leveraging on existing knowledge, to make learning of a new complex task easier.

The Learning Task

Markov Decision Process: only immediate state matters.

- State s_t, action a_t at time step t.
- Reward from environment: $r_t = r(s_t, a_t)$
- State transition by environment: $s_{t+1} = \delta(s_t, a_t)$
- $r(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ may be **unknown** to the agent!
- Task: learn $\pi : S \rightarrow A$ to select $a_t = \pi(s_t)$.
- Question: how to specify which π to learn?

[Alpaydin] Elements of RL (Markov Decision Process)

- s_t : State of agent at time t
- a_t: Action taken at time t
- In s_t, action a_t is taken, clock ticks and reward r_{t+1} is received and state changes to s_{t+1}
- Next state prob: $P\ (s_{t+1} \mid s_t\ ,\ a_t\ )$
- Reward prob: $p\ (r_{t+1} \mid s_t\ ,\ a_t\ )$
- Initial state(s), goal state(s)
- Episode (trial) of actions from initial state to goal
- (Sutton and Barto, 1998; Kaelbling et al., 1996)

Discounted Cumulative Reward: $V^{\pi}(s_t)$

- Obvious approach is to find π that maximizes the cumulative reward when π is executed:

$$\begin{aligned} V^{\pi}(s_t) &\equiv r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots \\ &\equiv \sum_{i=0}^{\infty} \gamma^i r_{t+i}, \end{aligned}$$

  where $0 \leq \gamma < 1$ is the discount rate.

- π is repeatedly executed: $a_t = \pi(s_t), a_{t+1} = \pi(s_{t+1}), \ldots$
- When $\gamma = 0$, only the current reward is used.
- When $\gamma \to 1$, future rewards become more important.

Choosing a Policy

- Optimal policy π^*

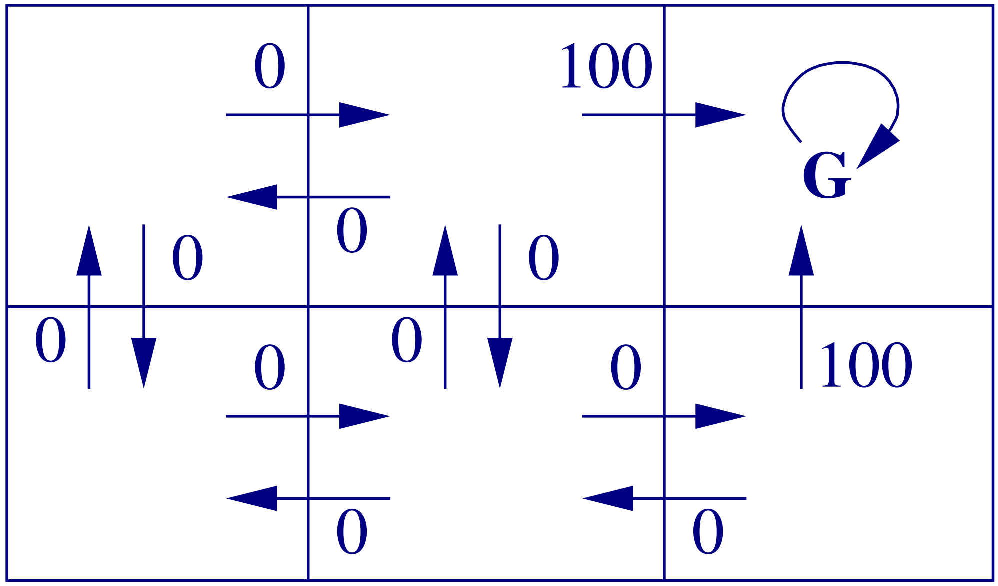
$$\pi^* = \underset{\pi}{\operatorname{argmax}}\, V^\pi(s), \forall s$$

- Want a policy that does its best for all states.
- Cumulative reward under optimal policy π^*:

$$V^*(s) \equiv V^{\pi^*}(s),$$

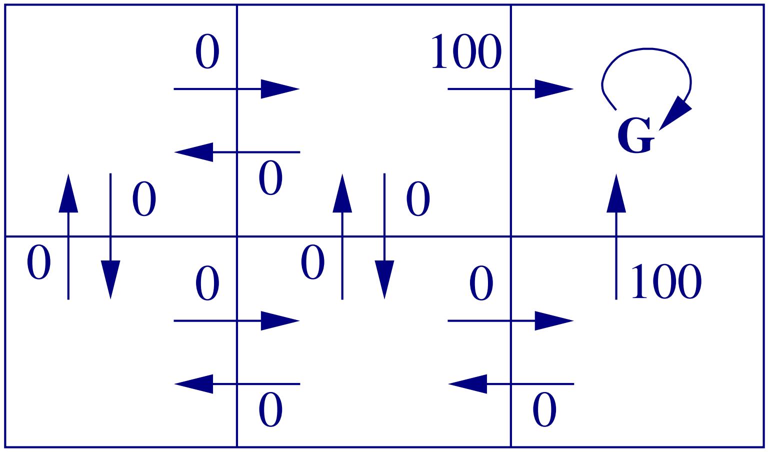
for short.

Example: Grid World

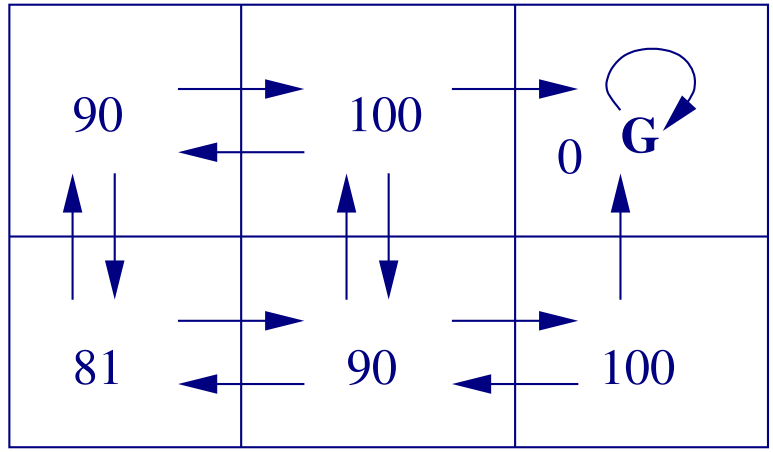


- Immediate reward given only when entering the goal state G.
- Given any initial state, we want to generate an action sequence to maximize V.

Grid World: $V^*(s)$ Values



(a) $r(s,a)$ values



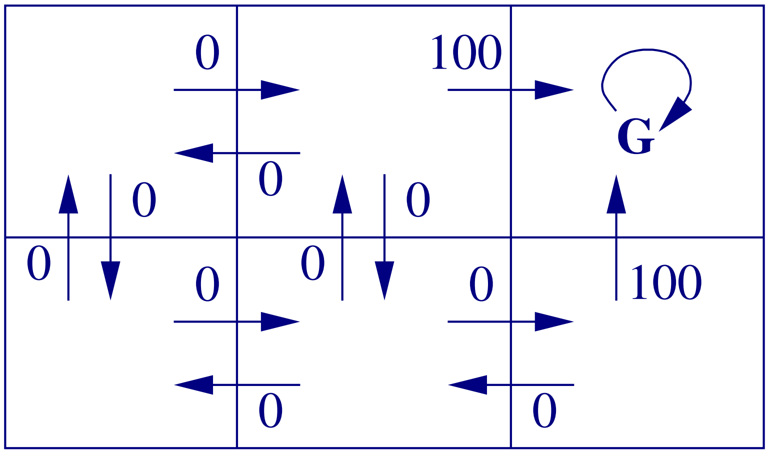
(b) $V^*(s)$ values

- Discount rate: $\gamma = 0.9$.
- Top middle: $100 + \gamma 0 + \gamma^2 0 + ... = 100$
- Top left: $0 + \gamma 100 + \gamma^2 0 + ... = 90$
- Bottom left: $0 + \gamma 0 + \gamma^2 100 + ... = 81$
- Note that these values are supposed to be **obtained using the optimal policy** π^*.

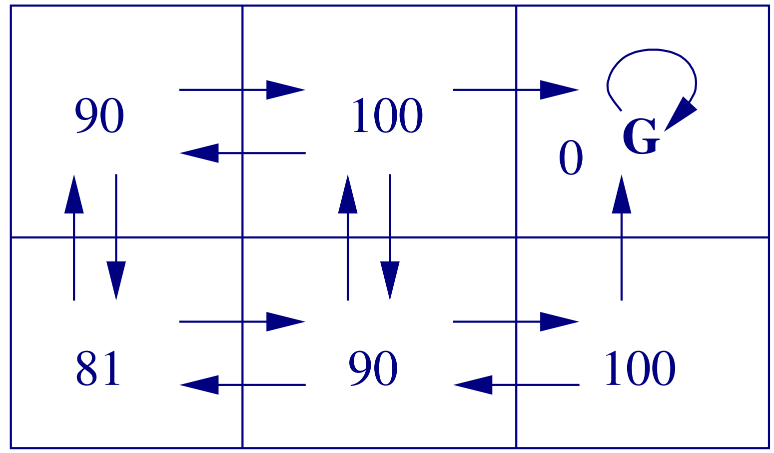
Q Learning

- Policy is hard to learn directly, because training experience does not provide $< s, a >$ pairs.
- Only available info: sequence of immediate rewards $r(s_i, a_i)$ for $i = 0, 1, 2, ...$
- In this case, it is easier to learn an **evaluation function** and construct a policy based on that.

Optimal Policy using $V^*(s)$



(a) $r(s, a)$ values



(b) $V^*(s)$ values

- If reward $r(s, a)$, state transition $\delta(s)$, and evaluation function $V^*(s)$ are known the following gives an optimal policy:

$$\pi^*(s) = \underset{a}{\operatorname{argmax}} \left[r(s, a) + \gamma V^*(\delta(s, a)) \right]$$

- For example, top middle state: move right = $100 + \gamma 0 = 100$, move left = $0 + \gamma 90 = 81$, move down = $0 + \gamma 90 = 81$.

[Alpaydin] Model-Based Learning

- Environment, $P(s_{t+1} \mid s_t, a_t)$, $p(r_{t+1} \mid s_t, a_t)$ known
- There is no need for exploration
- Can be solved using dynamic programming
- Solve for

$$V^*(s_t) = \max_{a_t}\left(E[r_{t+1}] + \gamma \sum_{s_{t+1}} P(s_{t+1} \mid s_t, a_t) V^*(s_{t+1}) \right)$$

- Optimal policy

$$\pi^*(s_t) = \arg\max_{a_t}\left(E[r_{t+1} \mid s_t, a_t] + \gamma \sum_{s_{t+1}} P(s_{t+1} \mid s_t, a_t) V^*(s_{t+1}) \right)$$

Problems with Policy Based on $V^*(s)$

- Requires perfect knowledge of $r(s, a)$ and $\delta(s, a)$, to exactly predict the outcome and reward of a particular action.
- In practice, the above is impossible.
- Thus, even when $V^*(s)$ is known, $\pi^*(s)$ cannot be found. Refer to:

$$\pi^*(s) = \operatorname*{argmax}_a \left[r(s, a) + \gamma V^*(\delta(s, a)) \right]$$

- Solution: use a surrogate – the Q function.

The Q Function

Can we get by without explicit knowledge of $r(s,a)$ and $\delta(s,a)$?

- $Q(s,a)$: evaluation function whose value is the **maximum discounted cumulative reward** obtainable when action a is taken in state s:

$$Q(s,a) \equiv r(s,a) + \gamma V^*(\delta(s,a))$$

- The derived policy is then:

$$\pi^*(s) = \operatorname*{argmax}_a Q(s,a)$$

  Note that if $Q(s,a)$ can be learned without any reference to $r(s,a)$ and $\delta(s,a)$, we have solved our problem.

- Further problem: how to **estimate** $Q(s,a)$?

Learning the Q Function: Getting Rid of $V^*(\delta(s, a))$

- $Q(s, a)$ is defined over all possible actions a from state s. But note that one of these actions is optimal for state s, and thus:

$$V^*(s) = \max_{a'} Q(s, a')$$

- With the above,

$$Q(s, a) \equiv r(s, a) + \gamma V^*(\delta(s, a))$$

can be rewritten as:

$$Q(s, a) \equiv r(s, a) + \gamma \max_{a'} Q(\delta(s, a), a'),$$

thus getting rid of $V^*(\delta(s, a))$.

Learning the Q Function: Getting Rid of r and δ

In state s, execute action a, and observe immediate reward r and resulting state s'. Then, simply use those r and s' you got without worrying about $r(s,a)$ or $\delta(s,a)$.

- Initialize the estimate $\hat{Q}(s,a)$ to zero.
- Iteratively update, with estimated function $\hat{Q}(s,a)$:

$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a').$$

The Q Learning Algorithm

1. For each s, a, initialize the table entry $\hat{Q}(s,a)$ to zero.
2. Observe the current state s.
3. Do forever:
   - Select action a and execute.
   - Receive immediate reward r.
   - Observe resulting state s'.
   - Update table entry for $\hat{Q}(s,a)$ as:

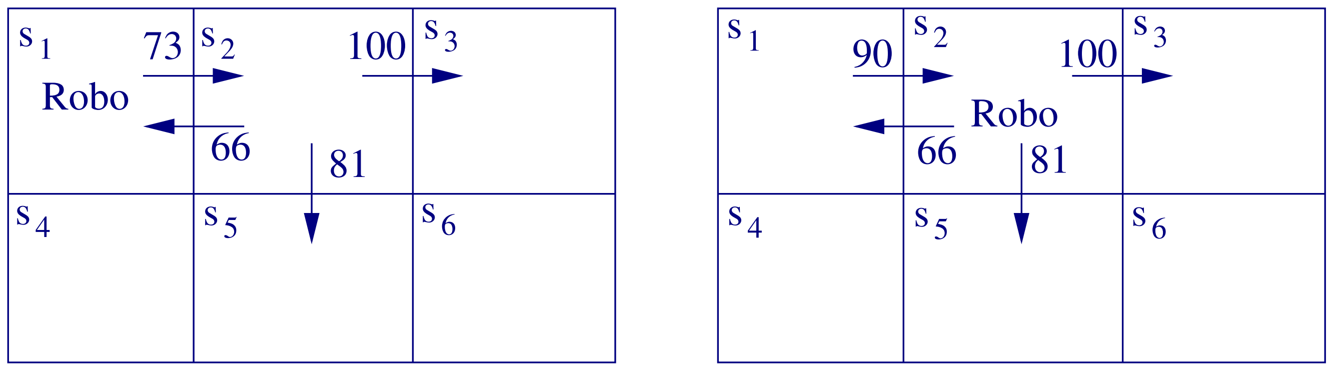
$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a').$$

   - $s \leftarrow s'$

Q Learning Properties

- For deterministic Markov decision processes
- $\hat{Q}$ converges to Q, when
  - process is deterministic **MDP**,
  - r **is bounded** (and non-negative), and
  - actions are chosen so that every state-action pair is visited **infinitely often**.

Example



(a) Initial state, in s_1 (b) Next state, in s_2

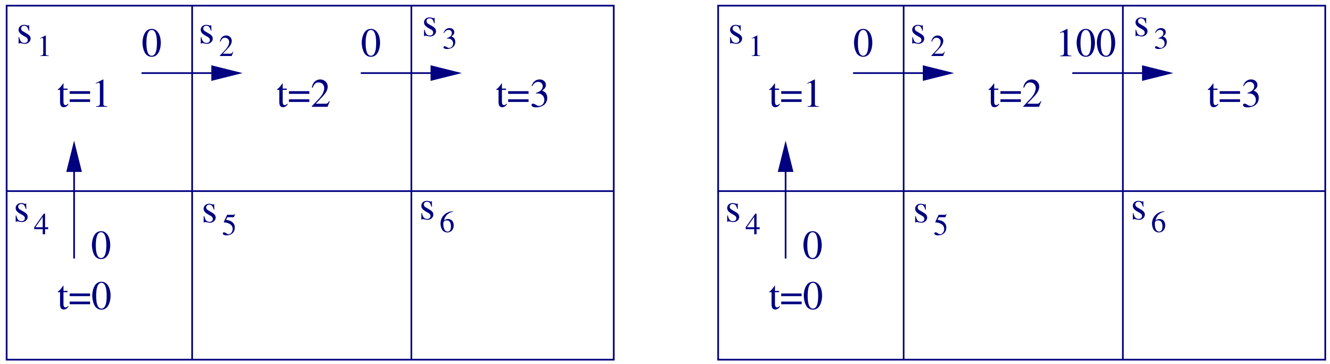
Arrows represent the $\hat{Q}$ values.

- Move right ($a = a_{right}$) and get immediate reward $r = 0$, with discount rate $\gamma = 0.9$:

$$\begin{aligned} \hat{Q}(s_1, a_{right}) &\leftarrow r + \gamma \max_{a'} \hat{Q}(s_2, a') \\ &\leftarrow 0 + 0.9 \max\{66, 81, 100\} \\ &\leftarrow 90 \end{aligned}$$

- Note that in (b), the $\hat{Q}(s_1, a_{right})$ value is updated from 73 to 90.

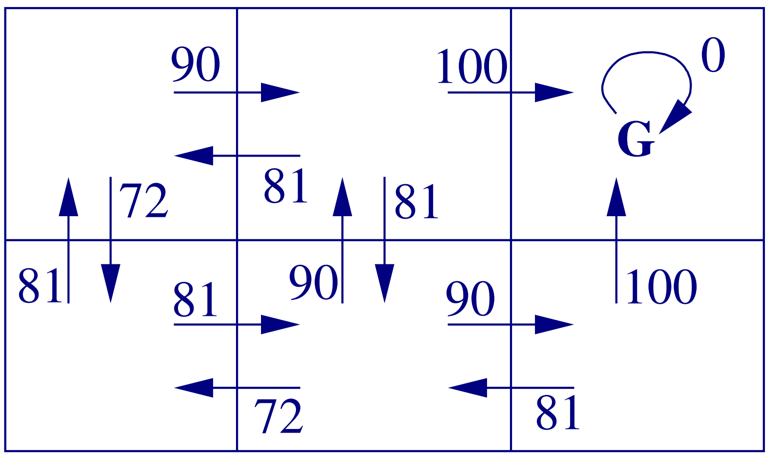
Exercise, from scratch



(a) Initial state $Q(s,a) = 0$ (b) After one iteration

- Robot moved from $s_4 \to s_1 \to s_2 \to s_3$.
- How do the various $Q(s,a)$ values get updated?
  - For the first iteration?
  - For the next iteration of $s_4 \to s_1 \to s_2 \to s_3$?

Final learned $\hat{Q}$



- For this domain, following actions that have max $Q(s, a)$ will lead you to the goal through an optimal path.

Convergence of $\hat{Q}$ to Q

- Properties (for non-negative rewards):

$$\forall s, a, n : \hat{Q}_{n+1}(s, a) \geq \hat{Q}_n(s, a)$$

$$\forall s, a, n : 0 \leq \hat{Q}_n(s, a) \leq Q_n(s, a)$$

- In general, convergence is guaranteed under three conditions:
  1. The system is a deterministic MDP.
  2. The reward is bounded $(\forall s, a) \;\; |r(s, a)| < c$ for a fixed constant c.
  3. All (s, a) pairs are visited infinitely often.

Proof of Convergence: Sketch

- The table entry $\hat{Q}(s, a)$ with the largest error must have its error reduced by a factor of γ whenever it is updated.
- The updated $\hat{Q}(s, a)$ will be based on the error-prone $\hat{Q}(s, a)$ only partially. The accurate immediate reward r used in the Q update rule will help reduce the error.
- *Proof*: Define a full interval to be an interval during which each table entry $\langle s, a \rangle$ is visited. During each full interval the largest error in $\hat{Q}$ table is reduced by factor of γ.

Convergence of Q

Let $\hat{Q}_n$ be table after n updates, and Δ_n be the maximum error in $\hat{Q}_n$; that is

$$\Delta_n = \max_{s,a} |\hat{Q}_n(s,a) - Q(s,a)|$$

For any table entry $\hat{Q}_n(s,a)$ updated on iteration $n+1$, the error in the revised estimate $\hat{Q}_{n+1}(s,a)$ is

$$\begin{aligned}
|\hat{Q}_{n+1}(s,a) - Q(s,a)| &= |(r + \gamma \max_{a'} \hat{Q}_n(s',a')) \\
&\quad -(r + \gamma \max_{a'} Q(s',a'))| \\
&= \gamma |\max_{a'} \hat{Q}_n(s',a') - \max_{a'} Q(s',a')| \\
&\leq \gamma \max_{a'} |\hat{Q}_n(s',a') - Q(s',a')| \\
&\leq \gamma \max_{s'',a'} |\hat{Q}_n(s'',a') - Q(s'',a')| \\
|\hat{Q}_{n+1}(s,a) - Q(s,a)| &\leq \gamma \Delta_n
\end{aligned}$$

Convergence in Q

- Main result:

$$|\hat{Q}_{n+1}(s,a) - Q(s,a)| \leq \gamma \Delta_n$$

- That is, error in the updated $\hat{Q}(s,a)$ is less than γ times the max error in the table before the update.

- Note that $\gamma < 1.0$.

- Given initial Δ_0, after k visits to $\langle s,a \rangle$, the error will be at most $\gamma^k \Delta_0$, and as $k \to \infty$, $\Delta_k \to 0$.

Constructing the Policy from the Learned Q

1. Greedy: given state s, pick $\operatorname{argmax}_a Q(s, a)$.
   - May cause the agent to **exploit** early successes and ignore interesting possibilities.
   - This would prevent the agent from visiting all (s, a) pairs infinitely often.
2. Probabilistic: pick action a_i with probability:

$$P(a_i|s) = \frac{k^{\hat{Q}(s,a_i)}}{\sum_j k^{\hat{Q}(s,a_j)}}$$

where $k > 0$ controls **exploration (low k)** vs. **exploitation (high k, greedy)**.

Updating Sequence

No specific order of (s, a) visit is necessary for convergence.

However, this can be inefficient.

1. Perform update in reverse order, once the goal has been reached.

2. Store past state-action transitions.

Nondeterministic Case

What if reward and next state are non-deterministic?

We redefine V, Q by taking expected values

$$\begin{aligned} V^{\pi}(s) &\equiv E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots] \\ &\equiv E\left[\sum_{i=0}^{\infty} \gamma^i r_{t+i}\right] \end{aligned}$$

$$Q(s,a) \equiv E[r(s,a) + \gamma V^*(\delta(s,a))]$$

Nondeterministic Case

$Q(s,a)$ can be redefined as follows:

$$\begin{aligned} Q(s,a) &\equiv E[r(s,a) + \gamma V^*(\delta(s,a))] \\ &= E[r(s,a)] + \gamma E[V^*(\delta(s,a))] \\ &= E[r(s,a)] + \gamma \sum_{s'} P(s'|s,a) V^*(s') \end{aligned}$$

Finally, rewriting it recursively, we get:

$$Q(s,a) = E[r(s,a)] + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q(s',a')$$

Nondeterministic Case: Learning

Using the original learning rule can result in oscillation in $\hat{Q}(s,a)$, and thus no convergence. Taking a decaying weighted average can solve the problem:

$$\hat{Q}_n(s,a) \leftarrow (1-\alpha_n)\hat{Q}_{n-1}(s,a)+\alpha_n \left[r + \gamma \max_{a'} \hat{Q}_{n-1}(s',a')\right]$$

where

$$\alpha_n = \frac{1}{1 + visits_s(s,a)}$$

and α determines how much the old and new $\hat{Q}$ values will be used. The α_n formula above is known to allow convergence (there can be other formulas).

Temporal Difference Learning

Q learning reduces the difference between $\hat{Q}$ of a state and its immediate successor (one-step look ahead). This can be generalized to include more distant successors.

Q learning reduces the difference between $\hat{Q}$ of a state

- $\hat{Q}(s_t, a_t)$ is estimated based $\hat{Q}(s_{t+1}, \cdot)$, where $s_{t+1} = \delta(s_t, a_t)$.

- One-step look ahead:

$$Q^{(1)}(s_t, a_t) \equiv r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

- Two-step look ahead:

$$Q^{(2)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \gamma^2 \max_a \hat{Q}(s_{t+2}, a)$$

- n-step look ahead:

$$Q^{(n)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \ldots + \gamma^{(n-1)} r_{t+n-1} + \gamma^n \max_a \hat{Q}(s_{t+n}, a)$$

Learning in TD

TD(λ) for learning Q using various lookaheads ($0 \leq \lambda \leq 1$):

$$Q^{\lambda}(s_t, a_t) \equiv (1-\lambda) \left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \ldots \right]$$

which can be rewritten recursively:

$$\begin{aligned}
&Q^{\lambda}(s_t, a_t) \\
&= (1 - \lambda) \left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \ldots \right] \\
&= \ldots \\
&= r_t + \gamma(1 - \lambda) \max_a \hat{Q}(s_{t+1}, a) + \gamma\lambda \left[r_{t+1} + \gamma(1 - \lambda) \max_a \hat{Q}(s_{t+2}, a) + \ldots \right] \\
&= r_t + \gamma \left[(1 - \lambda) \max_a \hat{Q}(s_{t+1}, a) + \lambda Q^{\lambda}(s_{t+1}, a_{t+1}) \right]
\end{aligned}$$

Note: there's a typo in Mitchell's book.

$$r_t + \gamma \left[(1 - \lambda) \max_a \hat{Q}(\underbrace{s_t}_{\text{typo}}, a) + \lambda Q^{\lambda}(s_{t+1}, a_{t+1}) \right]$$

TD(λ) Properties

$$Q^{\lambda}(s_t, a_t) = r_t + \gamma \left[(1-\lambda) \max_{a} \hat{Q}(s_{t+1}, a_t) + \lambda Q^{\lambda}(s_{t+1}, a_{t+1}) \right]$$

- TD(0): same as $Q^{(1)}$.
- TD(1): only observed r_{t+i} values are considered.
- When $Q = \hat{Q}$, Q^{λ} values are the same for any $0 \leq \lambda \leq 1$.

Curious Properties of TD(λ)

Why is TD(λ) not 0 when $\lambda = 1$? Note that TD(0)$= Q^{(1)}$.

$$Q^{\lambda}(s_t, a_t) = (1-\lambda)\left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \ldots\right]$$

It's because of the infinite sum that involve λ:

$$\begin{aligned}
Q^{\lambda} &= (1-\lambda)Q^{(1)} + (1-\lambda)\lambda Q^{(2)} + (1-\lambda)\lambda^2 Q^{(3)} + \ldots \\
&= (1-\lambda)(r_t + ..) + (1-\lambda)\lambda(r_t + \gamma r_{t+1}..) + (1-\lambda)\lambda^2(r_t + \gamma r_{t+1} \\
&= (1-\lambda)r_t + (1-\lambda)\lambda r_t + (1-\lambda)\lambda^2 r_t + \ldots \\
&= (1-\lambda)\sum_{n=0}^{\infty} \lambda^n r_t + \ldots \\
&= (1-\lambda)\frac{1}{1-\lambda} r_t + \ldots \\
&= r_t + \ldots
\end{aligned}$$

TD(λ) Properties

- Sometimes converges faster than Q learning
- Converges for learning V^* for any $0 \leq \lambda \leq 1$ (Dayan, 1992)
- Tesauro's TD-Gammon uses this algorithm

[Alpaydin] Q-Learning

Initialize all $Q(s, a)$ arbitrarily
For all episodes
  Initalize s
  Repeat
    Choose a using policy derived from Q, e.g., ϵ-greedy
    Take action a, observe r and s'
    Update $Q(s, a)$:
$$Q(s, a) \leftarrow Q(s, a) + \eta(r + \gamma \boxed{\max_{a'} Q(s', a')} - Q(s, a))$$
    $s \leftarrow s'$
  Until s is terminal state

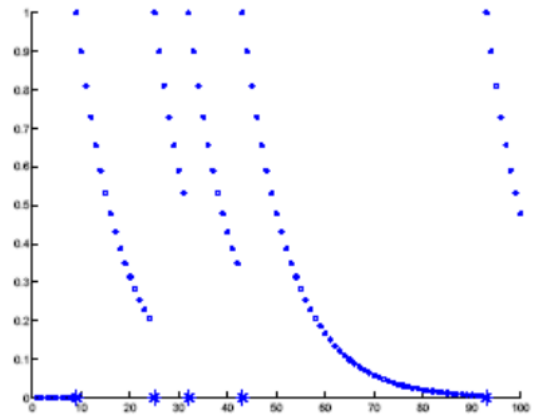
[Alpaydin] SARSA

Initialize all $Q(s, a)$ arbitrarily
For all episodes
    Initalize s
    Choose a using policy derived from Q, e.g., ϵ-greedy
    Repeat
        Take action a, observe r and s'
        Choose a' using policy derived from Q, e.g., ϵ-greedy
        Update $Q(s, a)$:
            $Q(s, a) \leftarrow Q(s, a) + \eta(r + \gamma \boxed{Q(s', a')} - Q(s, a))$
        $s \leftarrow s', \ a \leftarrow a'$
    Until s is terminal state

[Alpaydin] Eligibility Trace

Keep a record of previously visited states (actions)

$$e_t(s,a) = \begin{cases} 1 & \text{if } s = s_t \text{ and } a = a_t \\ \gamma\lambda e_{t-1}(s,a) & \text{otherwise} \end{cases}$$



$$\delta_t = r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)$$
$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \eta \delta_t e_t(s,a), \forall s, a$$

[Alpaydin] SARSA(λ)

Initialize all $Q(s,a)$ arbitrarily, $e(s,a) \leftarrow 0, \forall s,a$

For all episodes

  Initalize s

  Choose a using policy derived from Q, e.g., ϵ-greedy

  Repeat

    Take action a, observe r and s'

    Choose a' using policy derived from Q, e.g., ϵ-greedy

    $\delta \leftarrow r + \gamma Q(s',a') - Q(s,a)$

    $e(s,a) \leftarrow 1$

    For all s,a:

      $Q(s,a) \leftarrow Q(s,a) + \eta \delta e(s,a)$

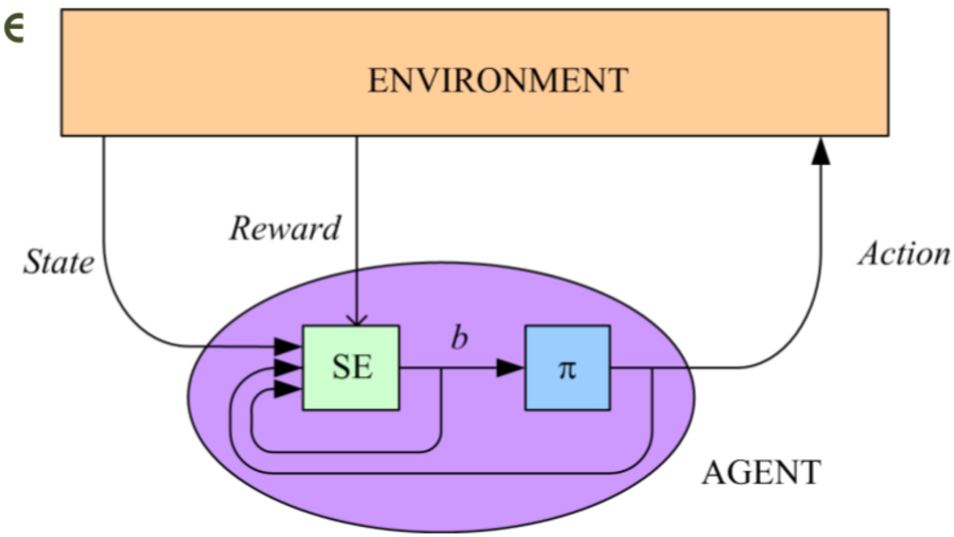
      $e(s,a) \leftarrow \gamma \lambda e(s,a)$

    $s \leftarrow s', \;\; a \leftarrow a'$

  Until s is terminal state

[Alpaydin] Partially Observable States

- The agent does not know its state but receives an observation $p(o_{t+1} \mid s_t, a_t)$ which can be used to infer a belief about states
- Partially observable MDP



Subtleties and Ongoing Research

- Replace $\hat{Q}$ table with neural net or other generalizer.
- Handle case where state is only partially observable (partially observable MDP, or POMDP).
- Design optimal exploration strategies.
- Extend to continuous action, state.
- Learn and use $\hat{\delta} : S \times A \rightarrow S$.
- Relationship to dynamic programming.