Bayesian Learning

- Olive slides marked [Alp]: Alpaydin
- Blue slides: Mitchell.

Bayesian Learning

- Probabilistic approach to inference.
- Quantities of interest are governed by prob. dist. and optimal decisions can be made by reasoning about these prob.
- Learning algorithms that directly deal with probabilities.
- Analysis framework for non-probabilistic methods.

Two Roles for Bayesian Methods

Provides practical learning algorithms:

- Naive Bayes learning
- Bayesian belief network learning
- Combine prior knowledge (prior probabilities) with observed data
- Requires prior probabilities

Provides useful conceptual framework

- Provides "gold standard" for evaluating other learning algorithms
- Additional insight into Occam's razor

Basic Probability Formulas

- *Product Rule*: probability $P(A \wedge B)$ of a conjunction of two events A and B:

$$P(A, B) = P(B, A) = P(A \wedge B) = P(A|B)P(B) = P(B|A)P(A)$$

- *Sum Rule*: probability of a disjunction of two events A and B:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

- *Theorem of total probability*: if events $A_1, \ldots, A_n$ are mutually exclusive with $\sum_{i=1}^{n} P(A_i) = 1$, then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Bayes Theorem

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

- $P(h)$ = prior probability that h holds, before seeing the training data
- $P(D)$ = prior probability of observing training data D
- $P(D|h)$ = probability of observing D in a world where h holds
- $P(h|D)$ = probability of h holding given observed data D
- Some useful tricks:
  - $P(h, D) = P(D, h)$
  - $P(h|D) = \frac{P(h,D)}{P(D)}$
  - $P(D, h) = P(D|h)P(h)$, from $P(D|h) = \frac{P(D,h)}{P(h)}$

Bayes Theorem: Example

Does patient have cancer or not?

> A patient takes a lab test and the result comes back positive. The test returns a correct positive result in only 98% of the cases in which the disease is actually present, and a correct negative result in only 97% of the cases in which the disease is not present. Furthermore, $.001$ of the entire population have this cancer.

$$P(cancer) = \qquad P(\neg cancer) =$$

$$P(\oplus | cancer) = \qquad P(\ominus | cancer) =$$

$$P(\oplus | \neg cancer) = \qquad P(\ominus | \neg cancer) =$$

How does $P(cancer|\oplus)$ compare to $P(\neg cancer|\oplus)$?

Bayes Theorem: Example

$$P(cancer) = 0.001 \qquad P(\neg cancer) = 1 - 0.001 = 0.999$$

$$P(\oplus|cancer) = 0.98 \qquad P(\ominus|cancer) = 1 - 0.98 = 0.02$$

$$P(\oplus|\neg cancer) = 1 - P(\ominus|\neg cancer) \qquad P(\ominus|\neg cancer) = 0.97$$

$$= 1 - 0.97 = 0.03$$

How does $P(cancer|\oplus)$ compare to $P(\neg cancer|\oplus)$?

$$\begin{aligned}
P(cancer|\oplus) &= \frac{P(\oplus|cancer)P(cancer)}{P(\oplus)} \\
&= \frac{0.98 \times 0.001}{P(\oplus)} = \frac{0.00098}{P(\oplus, cancer) + P(\oplus, \neg cancer)} \\
&= \frac{0.00098}{P(\oplus|cancer)P(cancer) + P(\oplus|\neg cancer)P(\neg cancer)} \\
&= \frac{0.00098}{0.98 \times 0.001 + 0.03 \times 0.999} = 0.031664
\end{aligned} \tag{1}$$

$P(\neg cancer|\oplus) = 1 - P(cancer|\oplus) = 1 - 0.031664 = 0.96834.$

Conditional Independence

Definition: X is *conditionally independent* of Y given Z if the probability distribution governing X is independent of the value of Y given the value of Z; that is, if

$$(\forall x_i, y_j, z_k)\, P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

more compactly, we write

$$P(X|Y,Z) = P(X|Z)$$

Example: $Thunder$ is conditionally independent of $Rain$, given $Lightning$

$$P(Thunder|Rain, Lightning) = P(Thunder|Lightning)$$

Choosing Hypotheses

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

Generally want the most probable hypothesis given the training data

Maximum a posteriori hypothesis h_{MAP}:

$$\begin{aligned} h_{MAP} &= \arg\max_{h \in H} P(h|D) \\ &= \arg\max_{h \in H} \frac{P(D|h)P(h)}{P(D)} \\ &= \arg\max_{h \in H} P(D|h)P(h) \end{aligned}$$

Choosing Hypotheses

- If all hypotheses are equally probable a priori:

$$P(h_i) = P(h_j), \forall h_i, h_j,$$

then, h_{MAP} reduces to:

$$h_{ML} \equiv \operatorname*{argmax}_{h \in H} P(D|h).$$

$\longrightarrow$ Maximum Likelihood hypothesis.

Brute Force MAP Hypothesis Learner

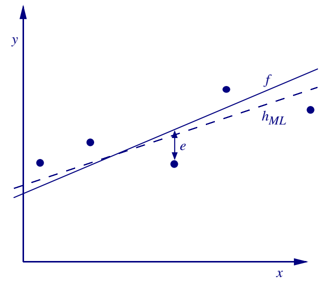
1. For each hypothesis h in H, calculate the posterior probability

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

2. Output the hypothesis h_{MAP} with the highest posterior probability

$$h_{MAP} = \operatorname*{argmax}_{h \in H} P(h|D)$$

Learning A Real Valued Function



Consider any real-valued target function f

Training examples $\langle x_i, d_i \rangle$, where d_i is noisy training value

- $d_i = f(x_i) + e_i$
- e_i is random variable (noise) drawn independently for each x_i according to some Gaussian distribution with mean=0

Then the maximum likelihood hypothesis h_{ML} is the one that minimizes the sum of squared errors:

$$h_{ML} = \arg\min_{h \in H} \sum_{i=1}^{m} (d_i - h(x_i))^2$$

Setting up the Stage

- Probability density function:

$$p(x_0) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} P(x_0 \leq x < x_0 + \epsilon)$$

- ML hypothesis

$$h_{ML} = \underset{h \in H}{\operatorname{argmax}}\, p(D|h)$$

- Training instances $\langle x_1, ..., x_m \rangle$ and target values $\langle d_1, ..., d_m \rangle$, where $d_i = f(x_i) + e_i$.

- Assume training examples are mutually independent given h,

$$h_{ML} = \underset{h \in H}{\operatorname{argmax}} \prod_{i=1}^{m} p(d_i|h)$$

Note: $p(a, b|c) = p(a|b, c) \cdot p(b|c) = p(a|c) \cdot p(b|c)$

Derivation of ML for Func. Approx.

From $h_{ML} = \text{argmax}_{h \in H} \prod_{i=1}^{m} p(d_i|h)$:

- Since $d_i = f(x_i) + e_i$ and $e_i \sim \mathcal{N}(0, \sigma^2)$, it must be:

$$d_i \sim \mathcal{N}(f(x_i), \sigma^2).$$

  – $x \sim \mathcal{N}(\mu, \sigma^2)$ means random variable x is normally distributed with mean μ and variance σ^2.

- Using pdf of $\mathcal{N}$:

$$h_{ML} = \underset{h \in H}{\text{argmax}} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(d_i - \mu)^2}{2\sigma^2}}.$$

$$h_{ML} = \underset{h \in H}{\text{argmax}} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(d_i - h(x_i))^2}{2\sigma^2}}.$$

Derivation of ML

$$h_{ML} = \underset{h\in H}{\operatorname{argmax}} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(d_i - h(x_i))^2}{2\sigma^2}}.$$

- Get rid of constant factor $\frac{1}{\sqrt{2\pi\sigma^2}}$, and put on log:

$$\begin{aligned}
h_{ML} &= \underset{h\in H}{\operatorname{argmax}} \ln \prod_{i=1}^{m} e^{-\frac{(d_i - h(x_i))^2}{2\sigma^2}} \\
&= \underset{h\in H}{\operatorname{argmax}} \sum_{i=1}^{m} \ln e^{-\frac{(d_i - h(x_i))^2}{2\sigma^2}} \\
&= \underset{h\in H}{\operatorname{argmax}} \sum_{i=1}^{m} -\frac{(d_i - h(x_i))^2}{2\sigma^2} \\
&= \underset{h\in H}{\operatorname{argmin}} \sum_{i=1}^{m} (d_i - h(x_i))^2 \qquad (2)
\end{aligned}$$

Least Square as ML

Assumptions

- Observed training values d_i generated by adding random noise to true target value, where noise has a normal distribution with zero mean.
- All hypotheses are equally probable (uniform prior).
  - Note: it is possible that $MAP \neq ML$!

Limitations

- Possible noise in x_i not accounted for.

Minimum Description Length

Occam's razor: prefer the shortest hypothesis.

$$
\begin{aligned}
h_{MAP} &= \operatorname*{argmax}_{h \in H} P(D|h)P(h) \\
h_{MAP} &= \operatorname*{argmax}_{h \in H} \log_2 P(D|h) + \log_2 P(h) \\
h_{MAP} &= \operatorname*{argmin}_{h \in H} -\log_2 P(D|h) - \log_2 P(h)
\end{aligned}
$$

Surprisingly, the above can be interpreted as h_{MAP} preferring shorter hypotheses, assuming a particular encoding scheme is used for the hypothesis and the data.

According to information theory, the shortest code length for a message occurring with probability p_i is $-\log_2 p_i$ bits.

MDL

$$h_{MAP} = \underset{h \in H}{\operatorname{argmin}} - \log_2 P(D|h) - \log_2 P(h)$$

- $L_C(i)$: description length of message i with respect to code C.
- $-\log_2 P(h)$: description length of h under optimal coding C_H for the hypothesis space H.

$$L_{C_H}(h) = -\log_2 P(h)$$

- $-\log_2 P(D|h)$: description length of training data D given hypothesis h, under optimal encoding $C_{D|H}$.

$$L_{C_{D|H}}(D|h) = -\log_2 P(D|h)$$

- Finally, we get:

$$h_{MAP} = \underset{h \in H}{\operatorname{argmin}} L_{C_{D|H}}(D|h) + L_{C_H}(h)$$

MDL

- MAP:

$$h_{MAP} = \underset{h \in H}{\operatorname{argmin}} L_{C_{D|H}}(D|h) + L_{C_H}(h)$$

- MDL: Choose h_{MDL} such that:

$$h_{MDL} = \underset{h \in H}{\operatorname{argmin}} L_{C_1}(h) + L_{C_2}(D|h)$$

  which is the hypothesis that minimizes the **combined length** of the hypotheis itself, and the data described by the hypothesis.

- $h_{MDL} = h_{MAP}$ if $C_1 = C_H$ and $C_2 = C_{D|H}$.

Bayes Optimal Classifier

- What is the most probable hypothesis given the training data, **vs.** What is the most probable classification?
- Example:
  - $P(h_1|D) = 0.4$, $P(h_2|D) = 0.3$, $P(h_3|D) = 0.3$.
  - Given a new instance x, $h_1(x) = 1$, $h_2(x) = 0$, $h_3(x) = 0$.
  - In this case, probability of x being positive is only 0.4.

Bayes Optimal Classification

If a new instance can take classification $v_j \in V$, then the probability $P(v_j|D)$ of correct classification of new instance being v_j is:

$$P(v_j|D) = \sum_{h_i \in H} \underbrace{P(v_j|h_i)}_{(A)} \underbrace{P(h_i|D)}_{(B)}$$

Thus, the optimal classification is

$$\operatorname*{argmax}_{v_j \in V} \sum_{h_i \in H} P(v_j|h_i)P(h_i|D).$$

Bayes Optimal Classifier

What is the assumption for the following to work?

$$P(v_j|D) = \sum_{h_i \in H} P(v_j|h_i)P(h_i|D)$$

Let's consider $H = \{h, \neg h\}$:

$$\begin{aligned}
P(v|D) &= P(v,h|D) + P(v,\neg h|D) \\
&= \frac{P(v,h,D)}{P(D)} + \frac{P(v,\neg h,D)}{P(D)} \\
&= \frac{P(v|h,D)P(h|D)P(D)}{P(D)} \\
&\quad + \frac{P(v|\neg h,D)P(\neg h|D)P(D)}{P(D)} \\
&\quad \{\text{if } P(v|h,D) = P(v|h), \text{ etc.}\} \\
&= P(v|h)P(h|D) + P(v|\neg h)P(\neg h|D)
\end{aligned}$$

Bayes Optimal Classifier: Example

- $P(h_1|D) = 0.4$, $P(h_2|D) = 0.3$, $P(h_3|D) = 0.3$.
- Given a new instance x, $h_1(x) = 1$, $h_2(x) = 0$, $h_1(x) = 0$.
  - $P(\ominus|h_1) = 0$, $P(\oplus|h_1) = 1$, etc.
  - $P(\oplus|D) = 0.4 + 0 + 0$,
    $P(\ominus|D) = 0 + 0.3 + 0.3 = 0.6$
  - Thus, $\text{argmax}_{v \in O\{\oplus,\ominus\}} P(v|D) = \ominus$.
- Bayes optimal classifiers maximize the probability that a new instance is correctly classified, given the available data, hypothesis space H, and prior probabilities over H.
- Some oddities: The resulting hypotheis can be outside of the hypothesis space.

Gibbs Sampling

Finding $\mathrm{argmax}_{v \in V} P(v|D)$ by considering every hypothesis $h \in H$ can be infeasible. A less optimal, but error-bounded version is **Gibbs sampling**:

1. Randomly pick $h \in H$ with probability $P(h|D)$.

2. Use h to classify the new instance x.

The result is that missclassification rate is at most $2\times$ that of BOC.

Naive Bayes Classifier

Given attribute values $\langle a_1, a_2, ..., a_n \rangle$, give the classification $v \in V$:

$$v_{MAP} = \underset{v_j \in V}{\operatorname{argmax}} P(v_j | a_1, a_2, ..., a_n)$$

$$\begin{aligned} v_{MAP} &= \underset{v_j \in V}{\operatorname{argmax}} \frac{P(a_1, a_2, ..., a_n | v_j) P(v_j)}{P(a_1, a_2, ..., a_n)} \\ &= \underset{v_j \in V}{\operatorname{argmax}} P(a_1, a_2, ..., a_n | v_j) P(v_j) \end{aligned}$$

- Want to estimate $P(a_1, a_2, ..., a_n | v_j)$ and $P(v_j)$ from training data.

Naive Bayes

- $P(v_j)$ is easy to calculate: Just count the frequency.
- $P(a_1, a_2, ..., a_n|v_j)$ takes the number of posible instances $\times$ number of possible target values.
- $P(a_1, a_2, ..., a_n|v_j)$ can be approximated as

$$P(a_1, a_2, ..., a_n|v_j) = \prod_i P(a_i|v_j).$$

- From this naive Bayes classifier is defined as:

$$v_{NB} = \operatorname*{argmax}_{v_j \in V} P(v_j) \prod_i P(a_i|v_j)$$

- Naive Bayes only takes number of distinct attribute values $\times$ number of distinct target values.

Naive Bayes uses cond. indep. to justify

$$\begin{aligned} P(X, Y|Z) &= P(X|Y, Z)P(Y|Z) \\ &= P(X|Z)P(Y|Z) \end{aligned}$$

Naive Bayes Algorithm

Naive_Bayes_Learn($examples$)

For each target value v_j

$\hat{P}(v_j) \leftarrow$ estimate $P(v_j)$

For each attribute value a_i of each attribute a

$\hat{P}(a_i|v_j) \leftarrow$ estimate $P(a_i|v_j)$

Classify_New_Instance(x)

$$v_{NB} = \underset{v_j \in V}{\operatorname{argmax}} \hat{P}(v_j) \prod_i \hat{P}(x_i|v_j)$$

Naive Bayes: Example

Consider *PlayTennis* again, and new instance:

$$x = \langle Outlk = sun, Temp = cool, Humid = high, Wind = strong \rangle$$

$$V = \{Yes, No\}$$

Want to compute:

$$v_{NB} = \operatorname*{argmax}_{v_j \in V} P(v_j) \prod_i P(x_i | v_j)$$

$$P(Y)\, P(sun|Y)\, P(cool|Y)\, P(high|Y)\, P(strong|Y) = .005$$

$$P(N)\, P(sun|N)\, P(cool|N)\, P(high|N)\, P(strong|N) = .021$$

Thus, $v_{NB} = No$

Naive Bayes: Subtleties

1. Conditional independence assumption is often violated

$$P(a_1, a_2 \ldots a_n | v_j) = \prod_i P(a_i | v_j)$$

- ...but it works surprisingly well anyway. Note don't need estimated posteriors $\hat{P}(v_j|x)$ to be correct; need only that

$$\operatorname*{argmax}_{v_j \in V} \hat{P}(v_j) \prod_i \hat{P}(a_i | v_j) = \operatorname*{argmax}_{v_j \in V} P(v_j) P(a_1 \ldots, a_n | v_j)$$

- Naive Bayes posteriors often unrealistically close to 1 or 0.

Naive Bayes: Subtleties

What if none of the training instances with target value v_j have attribute value a_i? Then

$$\hat{P}(a_i|v_j) = 0, \text{ and...}$$

$$\hat{P}(v_j) \prod_i \hat{P}(a_i|v_j) = 0$$

Typical solution is Bayesian estimate for $\hat{P}(a_i|v_j)$

$$\hat{P}(a_i|v_j) \leftarrow \frac{n_c + mp}{n + m}$$

where

- n is number of training examples for which $v = v_j$,
- n_c number of examples for which $v = v_j$ and $a = a_i$
- p is prior estimate for $\hat{P}(a_i|v_j)$
- m is weight given to prior (i.e. number of “virtual” examples)

Extra Slides: Will be covered, time permitting

Expectation Maximization (EM)

When to use:

- Data is only partially observable
- Unsupervised clustering (target value unobservable)
- Supervised learning (some instance attributes unobservable)

Some uses:

- Train Bayesian Belief Networks
- Unsupervised clustering (AUTOCLASS)
- Learning Hidden Markov Models

EM for Estimating k Means

Given:

- Instances from X generated by mixture of k Gaussian distributions
- Unknown means $\langle \mu_1, \ldots, \mu_k \rangle$ of the k Gaussians
- Don't know which instance x_i was generated by which Gaussian

Determine:

- Maximum likelihood estimates of $\langle \mu_1, \ldots, \mu_k \rangle$

Think of full description of each instance as $y_i = \langle x_i, z_{i1}, z_{i2} \rangle$, where

- z_{ij} is 1 if x_i generated by jth Gaussian
- x_i observable
- z_{ij} unobservable

EM for Estimating k Means

EM Algorithm: Pick random initial $h = \langle \mu_1, \mu_2 \rangle$, then iterate

E step: Calculate the expected value $E[z_{ij}]$ of each hidden variable z_{ij}, assuming the current hypothesis $h = \langle \mu_1, \mu_2 \rangle$ holds.

$$E[z_{ij}] = \frac{p(x = x_i | \mu = \mu_j)}{\sum_{n=1}^{2} p(x = x_i | \mu = \mu_n)} = \frac{e^{-\frac{1}{2\sigma^2}(x_i - \mu_j)^2}}{\sum_{n=1}^{2} e^{-\frac{1}{2\sigma^2}(x_i - \mu_n)^2}}$$

M step: Calculate a new maximum likelihood hypothesis $h' = \langle \mu'_1, \mu'_2 \rangle$, assuming the value taken on by each hidden variable z_{ij} is its expected value $E[z_{ij}]$ calculated above. Replace $h = \langle \mu_1, \mu_2 \rangle$ by $h' = \langle \mu'_1, \mu'_2 \rangle$.

$$\mu_j \leftarrow \frac{\sum_{i=1}^{m} E[z_{ij}]\ x_i}{\sum_{i=1}^{m} E[z_{ij}]}$$

EM Algorithm

Converges to local maximum likelihood h

and provides estimates of hidden variables z_{ij}

In fact, local maximum in $E[\ln P(Y|h)]$

- Y is complete (observable plus unobservable variables) data
- Expected value is taken over possible values of unobserved variables in Y

General EM Problem

Given:

- Observed data $X = \{x_1, \ldots, x_m\}$
- Unobserved data $Z = \{z_1, \ldots, z_m\}$
- Parameterized probability distribution $P(Y|h)$, where
  - $Y = \{y_1, \ldots, y_m\}$ is the full data $y_i = x_i \cup z_i$
  - h are the parameters

Determine:

- h that (locally) maximizes $E[\ln P(Y|h)]$

General EM Method

Define likelihood function $Q(h'|h)$ which calculates $Y = X \cup Z$ using observed X and current parameters h to estimate Z

$$Q(h'|h) \leftarrow E[\ln P(Y|h')|h, X]$$

EM Algorithm:

Estimation (E) step: Calculate $Q(h'|h)$ using the current hypothesis h and the observed data X to estimate the probability distribution over Y.

$$Q(h'|h) \leftarrow E[\ln P(Y|h')|h, X]$$

Maximization (M) step: Replace hypothesis h by the hypothesis h' that maximizes this Q function.

$$h \leftarrow \underset{h'}{\operatorname{argmax}}\, Q(h'|h)$$

Derivation of k-Means

- Hypothesis h is parameterized by $\theta = \langle \mu_1 ... \mu_k \rangle$.
- Observed data $X = \{\langle x_i \rangle\}$
- Hidden variables $Z = \{\langle z_{i1}, ..., z_{ik} \rangle\}$:
  - $z_{ik} = 1$ if input x_i is generated by th k-th normal dist.
  - For each input, k entries.
- First, start with defining $\ln p(Y|h)$.

Deriving $\ln P(Y|h)$

$$p(y_i|h') = p(x_i, z_{i1}, z_{i2}, \ldots, z_{ik}|h') = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{k} z_{ij}(x_i - \mu'_j)^2}$$

Note that the vector $\langle z_{i1}, \ldots, z_{ik} \rangle$ contains only a single 1 and all the rest are 0.

$$\begin{aligned}
\ln P(Y|h') &= \ln \prod_{i=1}^{m} p(y_i|h') \\
&= \sum_{i=1}^{m} \ln p(y_i|h') \\
&= \sum_{i=1}^{m} \left(\ln \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2\sigma^2} \sum_{j=1}^{k} z_{ij}(x_i - \mu'_j)^2 \right)
\end{aligned}$$

Deriving $E[\ln P(Y|h)]$

Since $P(Y|h')$ is a linear function of z_{ij}, and since $E[f(z)] = f(E[z])$,

$$\begin{aligned} E[\ln P(Y|h')] &= E\left[\sum_{i=1}^{m}\left(\ln\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2\sigma^2}\sum_{j=1}^{k} z_{ij}(x_i - \mu_j')^2\right)\right] \\ &= \sum_{i=1}^{m}\left(\ln\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2\sigma^2}\sum_{j=1}^{k} E[z_{ij}](x_i - \mu_j')^2\right) \end{aligned}$$

Thus,

$$\begin{aligned} Q(h'|h) &= Q(\langle \mu_1', \ldots, \mu_k' \rangle | h) \\ &= \sum_{i=1}^{m}\left(\ln\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2\sigma^2}\sum_{j=1}^{k} E[z_{ij}](x_i - \mu_j')^2\right) \end{aligned}$$

Finding $\operatorname{argmax}_{h'} Q(h'|h)$

With

$$E[z_{ij}] = \frac{e^{-\frac{1}{2\sigma^2}(x_i-\mu_j)^2}}{\sum_{n=1}^{2} e^{-\frac{1}{2\sigma^2}(x_i-\mu_n)^2}}$$

we want to find h' such that

$$\begin{aligned} \operatorname*{argmax}_{h'} Q(h'|h) &= \operatorname*{argmax}_{h'} \sum_{i=1}^{m} \left(\ln \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2\sigma^2} \sum_{j=1}^{k} E[z_{ij}](x_i - \mu_j')^2 \right) \\ &= \operatorname*{argmin}_{h'} \sum_{i=1}^{m} \sum_{j=1}^{k} E[z_{ij}](x_i - \mu_j')^2, \end{aligned}$$

which is minimized by

$$\mu_j \leftarrow \frac{\sum_{i=1}^{m} E[z_{ij}] x_i}{\sum_{i=1}^{m} E[z_{ij}]}.$$

Deriving the Update Rule

Set the derivative of the quantity to be minimized to be zero:

$$
\begin{aligned}
& \frac{\partial}{\partial \mu'_j} \sum_{i=1}^{m} \sum_{j=1}^{k} E[z_{ij}](x_i - \mu'_j)^2 \\
= \quad & \frac{\partial}{\partial \mu'_j} \sum_{i=1}^{m} E[z_{ij}](x_i - \mu'_j)^2 \\
= \quad & 2 \sum_{i=1}^{m} E[z_{ij}](x_i - \mu'_j) = 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{m} E[z_{ij}]x_i - \sum_{i=1}^{m} E[z_{ij}]\mu'_j &= 0 \\
\sum_{i=1}^{m} E[z_{ij}]x_i &= \mu'_j \sum_{i=1}^{m} E[z_{ij}] \\
\mu'_j &= \frac{\sum_{i=1}^{m} E[z_{ij}]x_i}{\sum_{i=1}^{m} E[z_{ij}]}
\end{aligned}
$$

See Bishop (1995) *Neural Networks for Pattern Recognition*, Oxford U Press. pp. 63–64.

[Alp] Losses and Risks

- Actions: α_i
- Loss of α_i when the state is C_k : λ_{ik}
- Expected risk (Duda and Hart, 1973)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x})$$

$$\text{choose } \alpha_i \text{ if } R(\alpha_i \mid \mathbf{x}) = \min_k R(\alpha_k \mid \mathbf{x})$$

[Alp] Losses and Risks; 0/1 Loss

$$\lambda_{ik} = \begin{cases} 0 \text{ if } i = k \\ 1 \text{ if } i \neq k \end{cases}$$

$$\begin{aligned} R(\alpha_i \mid \mathbf{x}) &= \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x}) \\ &= \sum_{k \neq i} P(C_k \mid \mathbf{x}) \\ &= 1 - P(C_i \mid \mathbf{x}) \end{aligned}$$

For minimum risk, choose the most probable class

[Alp] Losses and Risks: Reject

$$\lambda_{ik} = \begin{cases} 0 & \text{if } i = k \\ \lambda & \text{if } i = K+1, \quad 0 < \lambda < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$R(\alpha_{K+1} \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda P(C_k \mid \mathbf{x}) = \lambda$$

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k \neq i} P(C_k \mid \mathbf{x}) = 1 - P(C_i \mid \mathbf{x})$$

choose C_i if $P(C_i \mid \mathbf{x}) > P(C_k \mid \mathbf{x}) \; \forall k \neq i$ and $P(C_i \mid \mathbf{x}) > 1 - \lambda$

reject otherwise

[Alp] Discriminant Functions

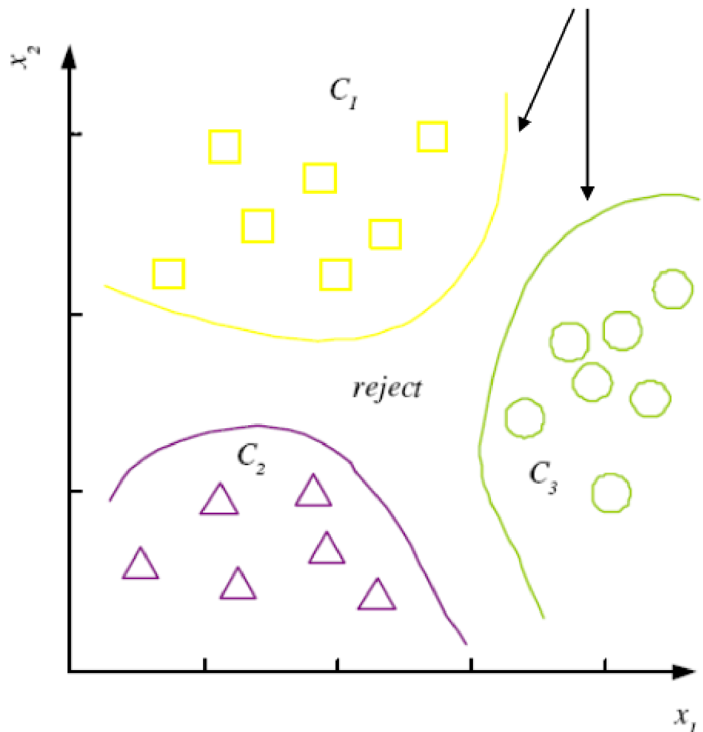
choose C_i if $g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})$

$$g_i(\mathbf{x}) = \begin{cases} -R(\alpha_i \mid \mathbf{x}) \\ P(C_i \mid \mathbf{x}) \\ p(\mathbf{x} \mid C_i)P(C_i) \end{cases}$$

K decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_K$

$$\mathcal{R}_i = \{\mathbf{x} \mid g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})\}$$

$g_i(\mathbf{x}), i = 1, \ldots, K$



[Alp] $K = 2$ Classes

- Dichotomizer (K=2) vs Polychotomizer (K>2)
- $g(\boldsymbol{x}) = g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})$

$$\text{choose} \begin{cases} C_1 \text{ if } g(\mathbf{x}) > 0 \\ C_2 \text{ otherwise} \end{cases}$$

- *Log odds:* $\log \frac{P(C_1 \mid \mathbf{x})}{P(C_2 \mid \mathbf{x})}$

[Alp] Utility Theory

- Prob of state k given exidence $\mathbf{x}$: $P\ (S_k \mid \mathbf{x})$
- Utility of α_i when state is k: U_{ik}
- Expected utility:

$$EU(\alpha_i \mid \mathbf{x}) = \sum_k U_{ik} P(S_k \mid \mathbf{x})$$

Choose α_i if $EU(\alpha_i \mid \mathbf{x}) = \max_j EU(\alpha_j \mid \mathbf{x})$

[Alp] Association Rules

- Association rule: $X \rightarrow Y$
- *People who buy/click/visit/enjoy X are also likely to buy/click/visit/enjoy Y.*
- A rule implies association, not necessarily causation.

[Alp] Association Measures

- Support $(X \rightarrow Y)$:

$$P(X,Y) = \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers}\}}$$

- Confidence $(X \rightarrow Y)$:

$$P(Y \mid X) = \frac{P(X,Y)}{P(X)}$$

$$= \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers who bought } X\}}$$

- Lift $(X \rightarrow Y)$:

$$= \frac{P(X,Y)}{P(X)P(Y)} = \frac{P(Y \mid X)}{P(Y)}$$