Advancements in Fair Allocation and Social Welfare Optimization

YOU YI TXUAN, CEN ZHENG ALEXANDRA XIAO YIN
(Computer science and technology, School of computer science,
Beijing University of Technology)

Abstract: This paper delves into the complex problem of fair allocation of indivisible goods among agents and the optimization of social welfare in various computational and economic contexts. We explore the latest advancements in algorithms for approximate maximin share (MMS), Nash social welfare (NSW), and the integration of fairness with efficiency in allocation mechanisms contain approximating nash social welfare by matching and local search.
Keywords: fair allocation, approximate maximin share, Nash social welfare, allocation mechanisms, local search.

1. Introduction

The fair allocation of resources is a fundamental issue in computer science, economics, and social choice theory. It involves the equitable distribution of indivisible goods while ensuring efficiency and fairness. This paper reviews recent breakthroughs in algorithmic solutions for MMS allocations, which guarantee that each agent receives a bundle of goods worth at least their MMS value. We consider the problem of allocating a set G of m indivisible items among a set A of n agents, where each agent i ∈ A has a valuation function vi : 2G → R≥0 and weight (entitlement) wi > 0 such that P i ∈A wi = 1. The Nash social welfare (NSW) problem asks for an allocation S = (Si)i ∈A that maximizes the weighted geometric mean of the agents' valuations,

$$\mathrm{NSW}(S) = \prod_{i \in A} (v_i(S_i))^{w_i}$$

We refer to the special case when all agents have equal weight (i.e., wi = 1/n) as the symmetric NSW problem, and call the general case the asymmetric NSW problem. Throughout, we let wmax B maxi ∈A wi . For α > 1, an α-approximate solution to the NSW problem is an allocation S with NSW(S) ≥ OPT/α, where OPT denotes the optimum value of the NSW-maximization problem.

2. Approximate Maximin Share Allocations

2.1 Introduction to maximin share allocations

The problem of allocating a set of indivisible goods among agents with additive valuations has been extensively studied in the context of fair division. The MMS criterion ensures that each agent receives a bundle of goods worth at least their guaranteed minimum value. However, achieving exact MMS allocations is often infeasible when the number of agents exceeds two, leading to the exploration of approximate MMS allocations.The study by Hannaneh Akrami and Jugal Garg presents a novel algorithm that breaks the 3/4 barrier for approximate MMS allocations. By introducing new reduction rules and analysis techniques, the authors demonstrate the existence of (3/4 + 3/3836)-MMS allocations, a significant improvement over previous results.

2.2 Approximation Algorithm for Nash Social Welfare

2.1.1 Algorithm 1: Approximating the submodular NSW problem.

Input: Valutions $(v_i)_{i\in A}$ Over G. weights $w \in \mathbb{R}^{A}_{>0}$ such that $\Sigma_{i\in A} w_i = 1$, and $\varepsilon > 0$

Output: Allocation $S = (S_i)_{i\in A}$

Find a matching $\tau: A \longrightarrow G$ maximizing $\prod_{i\in A} v_i(\tau(i))^{w_i}$ and set $H := \tau([n]), J := G\backslash H$

$R = (R_i)_{i\in A} := LocalSearch(J, (v_i)_{i\in A})$

Find a matching $\sigma: A \longrightarrow H$ maximizing $\prod_{i=1}^{n} v_i(R_i + \sigma(i))^{w_i}$

Return $S = (R_i + \sigma(i))_{i\in A}$

2.1.2 Algorithm 2: $\boldsymbol{LocalSearch(J, (v_i)_{i\in A})}$

$\overline{A} \leftarrow \{i \in A: v_i(J) > 0\}$

$\ell(i) \leftarrow argmax\{v_i(\ell): \ell \in J\} for\ i \in \overline{A}$

Define $\overline{v_i}(S) := v_i(\ell(i)) + v_i(S)$

$R_k \leftarrow J$ for some $k \in \overline{A}$ and $R_i \leftarrow \phi$ for $i \in A - k$

While $\exists i, k \in \overline{A}$ and $j \in R_i$ such that $(\frac{\overline{v_i}(R_i - j)}{\overline{v_i}(R_i)})^{w_i} \cdot (\frac{\overline{v_k}(R_k + j)}{\overline{v_k}(R_k)})^{w_k} > 1 + \overline{\varepsilon}$ do

$R_i \leftarrow R_i - j$ and $R_k \leftarrow R_k + j$

Return $\mathcal{R} := (R_i)_{i\in A}$.

3. Fair Division Beyond Additivity

3.1 Introduction to Non-Additive Valuations

The move beyond additive valuations is motivated by the limitations of the additive model in capturing the nuanced preferences of agents, especially when the value of additional goods diminishes (decreasing marginal utility). This section introduces the classes of valuations that extend beyond additivity, focusing on SPLC and submodular valuations, which are more expressive and applicable to a broader range of scenarios.

Submodular valuations further relax the assumptions of additivity by allowing the value of a good to depend on the entire set of goods an agent already possesses. This section discusses the challenges in achieving MMS and APS allocations under submodular valuations, where the problem of finding such allocations is known to be NP-hard.

3.2 Approximation Algorithms for MMS and APS

The paper introduces a relax-and-round paradigm that is applied to both SPLC and submodular valuations. For SPLC valuations, the algorithm involves a linear programming relaxation that ensures each agent receives at least half of their MMS value. For submodular valuations, a greedy algorithm is proposed that utilizes concave extensions of submodular functions to achieve a 1/3-APS allocation.

A key conceptual contribution of the paper is the development of novel upper bounds for MMS and APS values using market equilibrium and concave extensions. The authors demonstrate how to leverage these bounds to approximate APS and MMS values for monotone valuation functions, which is crucial for the design of the algorithms.

4. Innovations in Reduction Rules and Techniques for Fair Division

4.1 Introduction to Reduction Rules

Reduction rules are procedural tools used to simplify the allocation problem by allocating subsets of goods to agents under certain conditions, thereby reducing the complexity of the problem. The introduction of novel reduction rules has been crucial in advancing the state-of-the-art in fair division algorithms. While traditional reduction rules have been effective for additive valuations, they often reach their limits when applied to more complex valuation structures.

This section discusses the limitations of traditional rules and the necessity for new rules that can handle non-additive valuations, such as submodular and SPLC valuations.

4.2 Application of Reduction Rules in Algorithms

The practical application of these reduction rules is demonstrated through their integration into fair division algorithms. The paper discusses how these rules are applied in the algorithmic process to progressively simplify the allocation problem while preserving the fairness criteria. A detailed examination of the computational challenges faced when applying new reduction rules is provided. The paper discusses how these challenges are addressed through clever initialization of bags, counterintuitive allocation strategies, and the use of polynomial-time approximation schemes (PTAS).

The theoretical implications of these new reduction rules and analysis techniques are explored, highlighting their potential impact on the development of fair division theory. Practical applications in various domains, such as economics, computer science, and social choice theory, are also discussed.

5. Conclusion

The exploration of fair allocation and social welfare optimization has been a journey through the complex interplay of computational theory, economic principles, and social justice. The collective research presented in the documents has not only expanded the horizons of our understanding but also provided tangible advancements in the algorithms and techniques that underpin equitable resource distribution.

The breakthrough in surpassing the long-standing 3/4 barrier for approximate MMS allocations marks a significant milestone. It is a testament to the power of innovative thinking in algorithm design, where new reduction rules and analysis techniques have dismantled previously insurmountable obstacles. The development of polynomial-time algorithms for 1/2-MMS allocation under SPLC valuations and the establishment of constant-factor approximations for NSW with subadditive valuations have set new precedents for efficiency and fairness in resource allocation.

The research has also highlighted the importance of extending our models beyond additivity to capture the nuances of real-world valuations. The incorporation of SPLC and submodular valuations into the fair division framework has opened up new avenues for research and application. These models better represent the decreasing marginal utility observed in various economic and social contexts, thus making the allocations more reflective of reality.

The pseudopolynomial time algorithm for finding EF1 and Pareto efficient allocations represents a practical stride forward, offering a more accessible solution to the age-old problem of fair division. This work has also laid the groundwork for approximation algorithms that can provide near-optimal solutions in polynomial time, significantly advancing our capability to tackle complex allocation problems.

Moreover, the theoretical underpinnings of these advancements have deep implications for computational social choice and algorithmic game theory. The research has reinforced the connection between market equilibrium, efficiency, and fairness, offering new perspectives on how these concepts can be harmoniously integrated.

In conclusion, the body of work reviewed in this paper represents a comprehensive push forward in the field of fair allocation and social welfare optimization. It has not only provided us with new tools and algorithms but also with a deeper understanding of the underlying principles that govern fair and efficient resource distribution. As we stand on the precipice of new discoveries, the future of this field looks promising, with potential applications ranging from economic markets to the equitable distribution of public goods. The advancements made are not just academic achievements; they are steps toward a more equitable and just society, where the allocation of resources is guided by fairness, efficiency, and the enlightened application of computational intelligence.

References

[1] Akrami, Hannaneh, and Jugal Garg. "Breaking the 3/4 barrier for approximate maximin share." In Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 74-91. Society for Industrial and Applied Mathematics, 2024.

[2] Chekuri, Chandra, Pooja Kulkarni, Rucha Kulkarni, and Ruta Mehta. "1/2-Approximate MMS Allocation for Separable Piecewise Linear Concave Valuations." In Proceedings of the AAAI Conference on Artificial Intelligence, vol. 38, no. 9, pp. 9590-9597. 2024.

[3] Garg, Jugal, Edin Husić, Wenzheng Li, László A. Végh, and Jan Vondrák. "Approximating nash social welfare by matching and local search." In Proceedings of the 55th Annual ACM Symposium on Theory of Computing, pp. 1298-1310. 2023.

[4] Chaudhury, Bhaskar Ray, Jugal Garg, Peter McGlaughlin, and Ruta Mehta. "Competitive equilibrium with chores: Combinatorial algorithm and hardness." In Proceedings of the 23rd ACM Conference on Economics and Computation, pp. 1106-1107. 2022.

[5] Khot, Subhash, Richard J. Lipton, Evangelos Markakis, and Aranyak Mehta. "Inapproximability results for combinatorial auctions with submodular utility functions." Algorithmica 52 (2008): 3-18.

[6] Dobzinski, Shahar, Wenzheng Li, Aviad Rubinstein, and Jan Vondrák. "A constant factor approximation for Nash social welfare with subadditive valuations." arXiv preprint arXiv:2309.04656 (2023).

[7] Barman, Siddharth, Sanath Kumar Krishnamurthy, and Rohit Vaish. "Finding fair and efficient allocations." In Proceedings of the 2018 ACM Conference on Economics and Computation, pp. 557-574. 2018.

[8] Charikar, Moses, Prasanna Ramakrishnan, Kangning Wang, and Hongxun Wu. "Breaking the metric voting distortion barrier." In Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 1621-1640. Society for Industrial and Applied Mathematics, 2024.