1. **Prove that Rock-Paper-Scissors game has a unique Nash equilibrium, which is uniform over these three actions.**

For rock-paper-scissors, we can find a Nash equilibrium through mixed strategies. A mixed strategy means that a player chooses different actions with certain probabilities. Let p be the probability of choosing rock, q the probability of choosing scissors, and r the probability of choosing paper. Since each player can only choose one action, we have:

$$p+q+r=1$$

We want to find a mixed strategy where no player is tempted to change their strategy to get a better expected outcome, regardless of the opponent's strategy. Let P, Q, and R be the probabilities that the opponent chooses rock, scissors, and paper, respectively. We can calculate the expected payoffs to find the optimal strategy.

If the player chooses rock, the expected payoff is: $Erock=P\cdot 1+Q\cdot 0+R\cdot(-1)$

If the player chooses scissors, the expected payoff is: $Escissors=P\cdot 0+Q\cdot 1+R\cdot(-1)$

If the player chooses paper, the expected payoff is: $Epaper=P\cdot(-1)+Q\cdot(-1)+R\cdot 1$

To make the player indifferent to changing their strategy, the expected payoffs for these three choices must be equal, that is: $Erock=Escissors=Epaper$

By combining the above equations, we get: $P-R=0$ $Q-R=0$ $P-Q=0$

Since $P+Q+R=1$, we can solve to find $P=Q=R=1/3$. This means that when each player chooses rock, scissors, or paper with a probability of 1/3 at random, no player can obtain a higher expected payoff by changing their strategy, and thus this is a Nash equilibrium.

This Nash equilibrium is unique because it is the only one where all players use the same probability for each action. At this equilibrium, each player's expected payoff is zero, because in the long run, the probability of each action being chosen is the same, and the probability of each action being defeated and defeating the other is also the same."

2. Prove that if a finite game admits a potential function then it has at least one pure equilibrium.

- Let $(G = (N, (S_i)i \in N, (u_i)i \in N))$ be a finite game where:
- (N) is the set of players.
- (S_i) is the set of pure strategies for player (i), with $(S = \prod_{i \in N} \quad S_i)$ being the set of all pure strategy profiles.
- $(u_i: S \to \mathbb{R})$ is the payoff function for player (i).

Suppose there exists a potential function $(\Phi: S \to \mathbb{R})$ such that for all players $(i \in N)$ and all strategy profiles $(s \in S)$, if player (i) changes their strategy from (s_i) to (s_i') while the others keep their strategies fixed at (s_{-i}), the following holds: $[u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i})]$

The potential function (Φ) captures the incentives of the players such that any increase in the potential corresponds to an increase in the payoff for the deviating player. This implies that each player's incentive to change their strategy can be measured directly by the change in (Φ).

Since (S) is finite, the potential function (Φ) has a finite range. Therefore, there exists at least one strategy profile $(s^* \in S)$ where (Φ) attains its maximum value, say $(\Phi(s^*))$.

Consider the strategy profile (s^*) where (Φ) is maximized. Suppose (s^*) is not a Nash equilibrium. Then, there exists a player (i) and a deviation (s_i') such that: $[u_i(s_i', s_{-i}) > u_i(s)]$

By the definition of the potential function, this inequality implies: $[\Phi(s_i', s_{-i}) > \Phi(s)]$

However, this contradicts the assumption that $(\Phi(s))$ is the maximum value of (Φ). Hence, no such deviation (s_i') exists, and (s^) must be a Nash equilibrium.

3. Please give the statement of Myerson's Lemma and prove it.

Myerson's Lemma is a fundamental result in auction theory and mechanism design. It provides a characterization of the expected revenue that can be obtained from an auction or mechanism. Specifically, the lemma states that the expected revenue from a truthful (incentive-compatible) mechanism is equal to the expected "virtual surplus" generated by the mechanism.

Proof: The expected payment ($\mathbb{E}[P_i(v)]$) made by each bidder (i) can be expressed as the expected value of the product of the allocation probability and the virtual value. We use integration by parts to transform the expectation over the payments into an expectation over the virtual values: [$\mathbb{E}[P_i(v)] = \mathbb{E}[\int_0^{v_i} X_i(t, v_{-i}), dt]$] where ($v_{-i}$) represents the bids of all bidders except (i).

Substitute the virtual value ($\phi_i(v_i)$) into the payment expression: [$\mathbb{E}[P_i(v)] = \mathbb{E}[v_i X_i(v) - \int_0^{v_i} (1 - F_i(t)), dt]$]

Summing over all bidders, the expected revenue (R) is given by: [$R = \mathbb{E}[\sum_{i=1}^{n} P_i(v)] = \mathbb{E}[\sum_{i=1}^{n} (v_i X_i(v) - \int_0^{v_i} (1 - F_i(t)), dt)]$]

Simplifying the integral term using the definition of the virtual value: [$\int_0^{v_i} (1 - F_i(t)), dt = \frac{1-F_i(v_i)}{f_i(v_i)}$] which gives: [$R = \mathbb{E}[\sum_{i=1}^{n} \phi_i(v_i) X_i(v)]$]

4. Please give a complete proof of the $1/\sqrt{M}$-approximate mechanism of the single-minded auction.

- Let (N) be the set of bidders.
- Let (M) be the set of items.
- Each bidder ($i \in N$) wants a specific bundle ($S_i \subseteq M$) and has a valuation (v_i) for this bundle.
- Let (x_i) be an indicator variable that is 1 if bidder (i) is allocated their bundle (S_i) and 0 otherwise.

The objective is to maximize the social welfare: [Maximize $\sum_{i \in N} v_i x_i$].Subject to the constraint that no item is allocated more than once.

The greedy algorithm sorts bidders in non-increasing order of their valuations (v_i).It allocates items to each bidder (i) in this order if the items in (S_i) have not been allocated yet.Let (OPT) be the optimal social welfare and (GREEDY) be the social welfare achieved by the greedy algorithm.

Consider any optimal solution (OPT).Let (N_{OPT}) be the set of bidders selected in (OPT).Define (N_k) as the first (k) bidders selected by the greedy algorithm.

Divide the bidders into groups of size ($\sqrt{M}$).Each group contains ($\sqrt{M}$) bidders, and there are ($\frac{n}{\sqrt{M}}$) such groups if (n) is the number of bidders.

Consider the group that contributes the most to the optimal social welfare (OPT).Let (OPT_{max}) be the maximum contribution of any group to (OPT).Therefore, we have: [$OPT \leq \frac{n}{\sqrt{M}} OPT_{max}$]The greedy algorithm ensures that for the first ($\sqrt{M}$) bidders, the total valuation is at least (OPT_{max}).

5. Please give a complete proof of Arrow's Impossibility Theorem. (See 7a.arrow.pdf).

Proof of Arrow's Impossibility Theorem

- Let ($N = 1,2,\dots,n$) be the set of individuals.
- Let ($A = a, b, c, \dots$) be the set of alternatives.
- Each individual (i) has a preference order (P_i), which is a complete, transitive ordering of (A).

A social welfare function (F) maps the set of individual preferences (($P_1, P_2, \dots, P_n$)) to a single collective preference order (P).

Assume that there is no single individual (i) whose preferences always dictate the social preference order.

If every individual prefers alternative (a) to alternative (b) (i.e., (a P_i b) for all (i)), then the social preference should reflect (a) being preferred to (b) (i.e., (a P b)).

The social preference between any two alternatives (a) and (b) should depend only on the individual preferences between (a) and (b), not on preferences involving other alternatives.

Proof by Contradiction

Assume that there exists a social welfare function (F) that satisfies all three conditions.

Consider three alternatives (a, b,) and (c).

Suppose we have a situation where the preference profiles of individuals are as follows:

[

Individual 1: $aP_1b \quad bP_1c$

Individual 2: $bP_2c \quad cP_2a$

Individual 3: $cP_3a \quad aP_3b$

]

According to Pareto efficiency, if all individuals prefer (a) to (c), then the social preference should also prefer (a) to (c).

By the IIA criterion, the preference between (a) and (b) should be independent of (c).

Consider the possible collective preference orders. Because of Pareto efficiency and the given individual preferences, we must have:

(a P b)

(b P c)

(c P a)

This forms a cycle: (a P b), (b P c), (c P a).

The cycle (a P b), (b P c), (c P a) violates the transitivity requirement of preference order, where transitivity means if (a) is preferred to (b), and (b) is preferred to (c), then (a) must be preferred to (c).

6. Prove the correctness of the LMM Algorithm. (See NE.pdf).

- Let (x,y) be any NE of our instance.

Take K i.i.d. samples (actions) $(r_1, \ldots, r_k)$ from the distribution x.

Let $\tilde{x}$ be the "empirical" strategy which plays r_i uniformly at random. Similarly with $\tilde{y}$.

We will show, when k is large enough, below could happen:

$$|e_i^T Ry - e_i^T R\tilde{y}| \leq \epsilon/2$$

$$\text{and}$$

$$|x^T Ce_j - \tilde{x}^T Ce_j| \leq \epsilon/2$$

$$\text{where}$$

$$i, j \in [n]$$

If so, we have

$$e_i^T R\tilde{y} \leq e_i^T Ry + \epsilon/2 \leq \frac{1}{k}\sum_{j=1} \quad e_{r_j}^T Ry + \epsilon/2 \leq \frac{1}{k}\sum_{j=1} \quad e_{r_j}^T R\tilde{y} + \epsilon = \tilde{x}^T R\tilde{y} + \epsilon$$

We focus on a bad case that $|e_i^T Ry - e_i^T R\tilde{y}| \geq \epsilon/2$ for fixed i.

By Chernoff bound, we have (by setting $X_j = e_i^T Re_{r_j}$)

$\Pr[|e_i^T Ry - e_i^T R\tilde{y}| > \epsilon/2] \leq 2\exp(-k\epsilon^2/2).$

By the union bound, we have 2n bad cases, so the probability that any of the bad cases happens is at most $4n \exp(-k\epsilon^2/2)$

For $k > 2\log(4n)/\epsilon^2$, the probability above is less than 1

7. (Optional) Prove the correctness of Lemke-Howson algorithm. (See LH-algo.pdf).

First, note that the label set of (0, 0) and any Nash equilibrium is exactly M ∪ N, so (0, 0) and all Nash equilibrium points are in Uk for any k. Furthermore, let v = (v1, v2) be (0, 0) or any Nash equilibrium point. Without loss of generality, suppose k ∈ L(v1), where v1 is a corner point of the polytope P. Among all edges in G1 that v1 is incident to, there is only one direction leading to a vertex v 0 1 without label k (i.e. loosing the binding constraint corresponding to label k). It is easy to see that (v 0 1 , v2) ∈ Uk, therefore there is only one neighbor of v in Uk. For part (2), let v = (v1, v2) be any other point in Uk. Then there must be a duplicated label in L(v1) and L(v2), denoted by l. Similarly to (2), there is exactly one direction of v1's edges in P to drop the label l, and the new vertex v 0 1 has all labels v1 has except l, so (v 0 1 , v2) ∈ Uk. It is symmetric for v2. Hence there are two neighbors of v in Uk. In other words, in a non-degenerate bimatrix game (A, B) the set of k-almost completely labeled vertices in G and their induced edges consist of disjoint paths and cycles. The endpoints of the paths are the artificial equilibrium (0,0) and the equilibria of the game.