Notes on Deterministic Control.

§1. Problem Formulation. — A basic Version.

This note focuses on deterministic Control in operations management problems. The main reference is based on three papers. (i). Optimal Dynamic Pricing of inventories with Stochastic Demand over Finite Horizons. by Guillermo Gallego & Garrett van Ryzin (ii). A multi product Dynamic Pricing problem and its applications to network yield management by the Same Authors. (iii). Reoptimization and Self-Adjusting Price Control for Network Revenue management. by Jasin. Speaking of the formulation, a common framework, the intensity control, is adopted. We will start by a more general and easy model adopted in (i). The models in (ii) (iii) are basically some extensions.

§1.1. Demand.

usually, a firm operates in a market with imperfect competition. i.e., the firm can influence demand by varying its price p. The market demand is denoted by a function $\lambda(p)$.

Then realized demand is formulated as a Poisson process with intensity $\lambda(p)$.

~~That the firm~~ An important characteristic is used frequently, that is, in a very short time interval, say, δt, the probability that there is an arrival is $\delta t \cdot \lambda(p) + O(\delta t)$ while no items are sold the probability is essentially $1 - \lambda(p)\delta t - O(\delta t)$.

This is due to the characteristic of Poisson Process.

Remark 1. The derivation of Poisson distribution.

Consider the time interval t. broken into small sub-intervals of length δt.

$P(1, \delta t) = \lambda \delta t$; sufficient small δt, at most 1 success.

$P(0, \delta t) = 1 - \lambda \delta t$.

λ: arrival rate. Arrival rate $\cdot$ time = Arrivals. when time is sufficiently small. it turns to probability.

$$P(0, t+\delta t) = P(0,t)(1-\lambda \delta t).$$

$$\Rightarrow \frac{P(0, t+\delta t) - P(0,t)}{\delta t} = -\lambda P(0,t) \Rightarrow \frac{dP(0;t)}{dt} = -\lambda P(0,t)$$

Solving ODE.

$$\Rightarrow P(0,t) = C \cdot e^{-\lambda t}. \xrightarrow{P(0,0)=1} P(0,t) = e^{-\lambda t}.$$

Similarly. $P(n, t+\delta t) = P(n;t)(1-\lambda \delta t) + P(n-1;t)\lambda \delta t$

$$\frac{dP(n,t)}{dt} + \lambda P(n;t) = \lambda P(n-1,t). \quad *$$

it's a recursive differential function. A technique is adopted. we aim to find $\mu(t)$ such that:

$$\mu(t) \cdot \left[\frac{dP(n;t)}{dt} + \lambda P(n;t)\right] = \frac{d}{dt}(\mu(t) P(n;t)).$$

integrating factor $\mu(t) \Rightarrow e^{\lambda t}$ hence the equation (*) becomes:

$$\frac{d(e^{\lambda t} \cdot P(n;t))}{dt} = \lambda e^{\lambda t} P(n-1,t)$$

$$\Rightarrow n=1 \Rightarrow \frac{d(e^{\lambda t} \cdot P(1;t))}{dt} = \lambda e^{\lambda t} \cdot P(0,t) = \lambda.$$

$$\Rightarrow e^{\lambda t} \cdot P(1,t) = \int \lambda dt = \lambda t + C$$

$p(1;0)=0 \Rightarrow C=0.$ hence $p(1;t)=\lambda t.e^{-\lambda t}$.

By induction. We have: $P(n;t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$.

§ 1.2. Revenue Rate.

A very important assumption in the context of revenue mangement in the demand model is that $\lambda(p)$ has an inverse function: $\underline{p(\lambda)}$. i.e. $\lambda(p)$ is one-to-one correspondence. Different regularity constraints are made about $p(\lambda)$ in different papers[1].

the revenue rate is then: $\underline{\underline{r(\lambda) \doteq \lambda \cdot p(\lambda)}}$

§ 1.3. Formulation.

[context: One Firm. One product, n stock, decide price to maximize revenue]

- N_s. the number of items sold up to time s., a counting process.
- n. stock.
- $\lambda(\cdot)$. regular demand function

$$\int_0^t dN_s \le n.$$

- zero salvage value.
- $J_u(n,t) \doteq \mathbb{E}_u\left[\int_0^t p_s dN_s\right]$. $J_u(n,0) \doteq 0$. $J_u(0,t) \doteq 0$.

$$J^*(n,t) = \sup_{u\in\mathcal{U}} J_u(n,t).$$ ~~Hence this problem is~~

§ 1.4. Optimality Conditions and Structural Results.

$$J^*(n,t) = \sup_\lambda \left[\lambda\delta t(p(\lambda) + J^*(n-1,t-\delta t)) + (1-\lambda\delta t)(J^*(n,t-\delta t)) + o(\delta t)\right].$$

$r(\lambda)=\lambda p(\lambda)$, $\delta t \to 0$. Interchanging limit and Supreme Needs regularity check.

$$\frac{\partial J^*(n,t)}{\partial t} = \sup_\lambda \left[r(\lambda) - \lambda(J^*(n,t) - J^*(n-1,t))\right].$$

1: We Omit the assumption upon regularity here, as we focus more on the model and Basic introduction of the deterministic Control.

§1.5. Key Results:

(a). Existence of unique solutions.

As long as $\lambda(p)$ is regular demand function, then the solution exists and is unique.

(b). Monotonic Properties.

$J^*(n,t)$ is strictly increasing and strictly concave in both n and t.

$\lambda^*(n,t)$ .. strictly increasing in n and strictly decreasing in t.

$p^*(n,t)$ .... decreasing ... n .... increasing in t.

(c). Optimal Solution of $\lambda(p) = \alpha e^{-\alpha p}$.

$$J^*(n,t) = \log\left(\sum_{i=0}^{n} (\lambda^* t)^i \frac{1}{i!}\right).$$

$$p^*(n,t) = J^*(n,t) - J^*(n-1,t) + 1$$

§2. Deterministic Control. §2.1 Formulation

Usually, the demand function might not fit $\lambda(p) = ae^{-\alpha p}$. The dynamic programming suffers the curse of dimensionality. For example. if either n or t increases, the computational efforts might explodes. Then the deterministic problem is involved. It would create an upper bound for the optimal revenue. A key difference from the stochastic control is that the demand rate is "deterministic".

- Stock x. a continuous amount.
- finite time $t > 0$.
- $p(s)$. price.
- $\lambda(p(s)) \to$ demand rate.

$$J^D(x,t) = \max_{\{\lambda(s)\}} \int_0^t r(\lambda(s))\, ds$$

subject to: $\int_0^t \lambda(s)\, ds \le x$

One Formula directly distinguishes deterministic control with stochastic control is,

$\underbrace{E_u \int_0^t \lambda_s \, ds}_{\text{stochastic}}$ v.s. $\underbrace{\int_0^t \lambda(s) \, ds}_{\text{deterministic}}$.

- Given price policy u, the total demand is still unknown and it's random.
- Given price policy $p(s)$, the total demand is deterministic $\underline{\int_0^t \lambda(p(s)) ds}$.

§ 2.2 Optimal Solution of the deterministic problem

"run-out" rate. $\lambda^0 \doteq x/t$.

under such rate, all the stock can be sold.

Let $\lambda^* \in \arg\max_{\lambda(s)} r(\lambda(s))$ $\Rightarrow$ λ^* represents the maximizer of the revenue function.

PROPOSITION 2. The optimal solution to the deterministic problem is $\lambda(s) = \lambda^D \doteq \min\{\lambda^*, \lambda^0\}$.

$0 \le s \le t$. $p(s) = p^D \doteq \max\{p^*, p^0\}$.

$J^D(x,t) = t \min\{r^*, r^0\}$.

Proof Sketch: ① if $\lambda^* \le \lambda^0$, i.e. $\lambda^* t \le x$ then $\lambda(s) = \lambda^*$ is optimal solution.

Since it maximize $r(\lambda)$ pointwise

② if $\lambda^* > \lambda^0$, i.e., $\lambda^* t > x$. the $r(\lambda(s))$ is increasing

in $[0, x]$. optimal $\lambda = \frac{x}{t} = \lambda^0$

Hence $\lambda(s) = \min\{\lambda^*, \lambda^0\}$

~~For example $r(\lambda)$ if $\lambda^* \le \lambda^0 \Rightarrow$ then, r^0 cannot be reached since~~

~~$r(\lambda)$~~ r^*

② Price $P(\lambda)$ is a inverse function of $\lambda(p)$.

③. $J^D(x,t) = t \cdot \min\{r^*, r^0\}$??? why?

§2.3. Upper Bound.

Theorem 2. If $\lambda(p)$ is a regular demand function, then for all $0 \le n < +\infty$ and $0 \le t < \infty$, we have.

$$J^*(n,t) \le J^D(n,t)$$

Proof Sketch. Step 1: $E_u[\int_0^t \lambda_s ds] = n$.

Step 2: $J_u(n,t,\mu) = E_u[\int_0^t (r(\lambda(s)) - \mu\lambda_s) ds] + n\mu \ge J_u(n,t)$.

Step 3 $J_\mu(n,t,\mu) \le \int_0^t \max_{\lambda(s)} r(\lambda(s)) - \mu\lambda(s)\, ds + n\mu = J^D(n,t,\mu)$.

Expectation $\le$ Maximizing the integrand inside pointwise.

Step 4 Strong duality, $\inf_\mu J^D(n,t,\mu) = J^D(n,t)$.

$\Rightarrow$. $J^*(n,t) \le J^D(n,t)$.

§2.4 Heuristics.

Deterministic control suggests a fixed price heuristic. if we let $p^D = \max\{p^0, p^*\}$ for the entire horizon under the stochastic demand. how the performance's?

Remark :

Note that, in ~~the~~ last section, we show that the deterministic control leads to an upper bound. Here, the optimal-fixed-pricing policy is only a feasible control under the stochastic demand setting and hence any heuristics definitely is smaller than $J^*(n,t)$. But, we are gonna use deterministic control as a medium to explore the property of such heuristics.

As the deterministic control suggests a fixed price, then it is ~~better~~ nice to find a better fixed price, for example, the one maximizing $p\,E[\min\{n, N_{\lambda(p)t}\}]$.

Intuitively, the best matches the uncertainty.

Theorem 3. $\dfrac{J^{OFP}(n,t)}{J^*(n,t)} \geq \dfrac{J^{FP}(n,t)}{J^*(n,t)} \geq 1 - \dfrac{1}{2\sqrt{\min\{n, \lambda^* t\}}}$.

Proof Sketch:

- Fixed price: $p\,E[N_{\lambda(p)t} - (N_{\lambda(p)t} - n)^+]$.
- Gallego C(1992): $E[(N-n)^+] \leq \dfrac{\sqrt{\sigma^2 + (n-\mu)^2} - (n-\mu)}{2}$, $\quad x^+ \doteq \max(x, 0)$.
- $\lambda^* t > n$, run-out policy is optimal. $\underline{p^0}$

  If run out, then n would be sold out in expectation

  The arrival rate of the poisson process is $\boxed{n/t}$.

  Hence $E[\underline{N_{\lambda(p)t}}] = n$. The standard deviation: $\sigma^2 = n$.

  $$J^{FP}(n,t) \geq n p^0 \left(1 - \frac{1}{2\sqrt{n}}\right) = r^0 t \left(1 - \frac{1}{2\sqrt{n}}\right).$$
- $\lambda^* t < n$, p^* is optimal.

  then, the rate is λ^*. $E[N_{\lambda(p)t}] = \lambda^* t$.

  $$J^{FP}(n,t) \geq p^* \left(\lambda^* t - \frac{\sqrt{\lambda^* t + (n - \lambda^* t)^2} - (n - \lambda^* t)}{2}\right) \geq p^* \lambda^* t \left(1 - \frac{1}{2\sqrt{\lambda^* t}}\right) = r^* t \left(1 - \frac{1}{2\sqrt{\lambda^* t}}\right).$$
- $J^*(n,t) \leq J^D(n,t)$

  $$\frac{J^{FP}(n,t)}{J^*(n,t)} \geq \frac{J^{FP}(n,t)}{J^D(n,t)} \Rightarrow$$

  $\lambda^* t > n$: $\dfrac{J^{FP}(n,t)}{J^D(n,t)} \geq \dfrac{n p^0 \left(1 - \frac{1}{2\sqrt{n}}\right)}{n \cdot p^0} = 1 - \dfrac{1}{2\sqrt{n}}$

  $\lambda^* t < n$: $\dfrac{J^{FP}(n,t)}{J^D(n,t)} \geq 1 - \dfrac{1}{2\sqrt{\lambda^* t}}$
- proof is completed.