Notes on Multi-product dynamic pricing. — Guillermo Gallego, Garrett ryzin (1994)
Operations research.

§ 1. Problem Formulation — Description

- Yield management problem:
- Joint Pricing / allocation decision, multiple-market. multiple resource.
- product j requires a_{ij} units of resources
- $A = [a_{ij}]$. bill of materials matrix.
- demand intensity $\lambda = (\lambda^1, \cdots, \lambda^n)$. $p = (p^1, \cdots, p^n)$.
- $[0, t]$.

Eg 1. Flight industries: Multiple flight with multiple sets.

- product: path from an Origin to a Destination.
- Resource: A seat on a particular flight leg.

Eg. 2. hotel rooms. Multiple rooms with multiple days to stay.

- Product: particular Subsets of days.
- Resources: room capacities on each day of the planning period.

§ 1.1. Model.

- $\lambda = (\lambda^1, \cdots, \lambda^n)$.
- $p = (p^1, \cdots, p^n)$.
- $\underset{\text{intensities}}{\underline{\lambda(p, s)}}$ | demand in Controlled poisson process.
- n products, m types of resources.
- $(x^1, \cdots, x^n) \to$ resource capacity.

- product j needs a_{ij} of type i resource.
- Counting process. N_s^j. type j products sold up to time s.
- $dN_s^j = 1$ if there's an arrival ~~of~~ the demand for product j.
- $\int_0^t A\, dN_s \le x$ constraints.
- $J^u(x,t) \doteq E_u\left[\int_0^t p_s' dN_s\right]$. p_s' transpose
- $J^*(x,t) \doteq \sup J^u(x,t)$

§ 1.2. Optimality Conditions.

$$\frac{\partial}{\partial t} J^*(x,t) = \sup_{\lambda \in \Lambda(s)} \left\{ r(\lambda, s) - \sum_{j=1}^{n} \lambda^j \left[J^*(x,t) - J^*(x - A^j, t) \right] \right\}.$$

~~Multiperiod~~ Multi-product.

$$\frac{\partial J^*(n,t)}{\partial t} = \sup_{\lambda} \left[r(\lambda) - \lambda \left(J^*(n,t) - J^*(n-1,t) \right) \right].$$

Single product.

Derivation of the optimality Condition: $\lambda_j \delta t$, there is an arrival λ_j; $(1-\lambda_j \delta t)$ there is no arrival λ_j).

$$J^*(x,t) = \sup_{\lambda} \left\{ \lambda_j \delta_t \left[p_j(\lambda) + J^*(x - A^j, t - \delta t) \right] + (1 - \lambda_j \delta t) \left[J^*(x, t - \delta t) \right] + o(\delta t) \right\}$$

Similar argument.

§ 1.3. Deterministic Problem.

- $\lambda(s)$ is deterministic.
- $J^d(x,t) = \max \int_0^t r(\lambda(s), s)\, ds$, where, $r(\lambda(s), s) = \lambda' p(\lambda, s)$ Transpose $1 \times n$, $n \times 1$, 1×1

subject: $\int_0^t A \lambda(s)\, ds \le x$

- The deterministic formulation can be reduced to.

$$J^d(x,t) = \sum_{j=1}^{n} \bar{p}^j \alpha^j$$

$$\alpha^j = \int_0^t \lambda_d^j(s)\,ds \quad , \quad \bar{p}^j = \frac{\int_0^t p_d^j(s)\lambda_d^j(s)\,ds}{\int_0^t \lambda_d^j(s)\,ds}$$

§ 1.4. Upper bound -

Adopting the same technique (Lagrangian Duality), we define the argumented value function.

$$J^u(x,t,\mu) = E_\mu\left[\int_0^t c(\lambda_s, s) - \mu' A \lambda_s)\,ds + \mu' x \geq J^u(x,t)\right.$$

The method of constructing this is similar to the single-item model.

Similarly, $J^u(x,t,\mu) \leq J^d(x,t,\mu)$ and $\underline{J^*(x,t) \leq J^d(x,t)}$

§ 1.5. Asymptotically optimal Heuristics.

Make-to-Stock (MTS) Policy.

- $P_d(s)$ optimal deterministic price path.
- $\lambda_d(s)$ optimal deterministic intensity.

$$z^j = \cancel{\left\lfloor \int_0^t \lambda_d^j(s)\,ds \right.} \quad \left\lfloor \int_0^t \lambda_d^j(s)\,ds \right\rfloor = \lfloor \alpha_j \rfloor$$

- Policy: Preassemble z^j units of products $j = 1, \dots, n$.
- Price at $\underline{P_d(s)}$ and sell them until the inventory (product) is exhausted.

- Use the same inequality $\quad E[(D-d)^+] \leq \dfrac{\sqrt{\sigma^2 + (d-\mu)^2} - (d-\mu)}{2}$

They can show that: $$\frac{J^{dnTS}(x,t)}{J^{*}(x,t)} \geq 1 - \frac{\sum_{j=1}^{N} u^{j}\left(\frac{1}{2}\sqrt{\alpha^{j}}+1\right)}{\sum_{j} \bar{p}^{j}\alpha^{j}}.$$

Hence, ~~the~~ asymptotic property is obtained.