Notes on Reoptimization and Self-Adjusting Pricing Control for Network Revenue Management.

§1. Model.

- Discrete time. T-periods dynamic pricing with m resources and n products.
- Product is defined as a combination of resources.
- $A = [A_{ij}]$. "the bill of material" or "capacity consumption matrix" $A = [A_{ij}]$.
- $C = (C_i)$. Initial capacity of resources.
- $D_t(P_t) = (D_{tj}(P_t))$. demand. at most one customer arrives during each period and that demands across different periods are i.i.d.
- $\lambda(P_t) = \mathbb{E}[D_t(P_t)]$. denote the expected demand rate.
- $r(P_t) = P_t \lambda(P_t) = \sum_j P_{t,j} \lambda_j(P_t)$. revenue rate.
- ~~$\sum_j \lambda_j(P_t) \cdot P_{tj}$~~ $\sum_j \lambda_j(P_t) \leq 1$

§2. Stochastic & Deterministic Formulations.

$$J_{opt} = \max_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=1}^{T} P_t^{\pi} \cdot D_t^{\pi}\right].$$

$$\text{Subject to. } \sum_{t=1}^{T} A D_t^{\pi} \leq C$$

$$V_t(C_t) = \max_{P_t} \mathbb{E}\left[P_t D_t + V_{t+1}(C_t - AD_t)\right].$$

Bellman Equation.

stochastic.

$$J_D = \max_{p \in \Omega^p} \sum_{t=1}^{T} r(p_t).$$

$$\text{Subject to } \sum_{t=1}^{T} A\lambda(P_t) \leq C$$

$$\Longleftrightarrow$$

$$J_D = \max_{\lambda \in \Omega^\lambda} \sum_{t=1}^{T} r(\lambda_t).$$

$$\text{Subject to. } \sum_{t=1}^{T} A\lambda_t \leq C$$

$$J_{opt}^{\theta} \leq J_D^{\theta}.$$

§3. Reoptimized Static Control.

- Static Control has $O(\sqrt{\theta})$ revenue loss.

- Reoptimized Static Control (RSC).

1. At time 1, $p_1 = p(\lambda^D)$.
2. At time $t \geq 1$, do the following.
   a. $\{\lambda_s\}_{s=t}^{T} = \arg\max_{x \in \Omega^x} \{\sum_{s=t}^{T} r(x_s) \mid \text{s.t. } \sum_{s=t}^{T} A x_s \leq C_t\}$.
   b. Compute $\hat{p}_t = p(\lambda_t)$.
   c. If $C_t = 0$, set $p_t = \bar{p}$. Otherwise, set $p_t = \hat{p}_t$

turnoff price.

- Compared to the static Control, the ~~Reoptimized control~~ can update the info of the current capacity.

☆ Theorem 1. If $\mu^D > 0$, there exists a positive constant ψ independent of $\theta \geq 1$.

Such that

$$J_D^\theta - \mathbb{E}[R_{RSC}^\theta] \leq \psi(1 + \log\theta).$$

Proof Sketch: It takes a long Journey to get the final result.

Step 1. Construction and Optimality.

- $\hat{\lambda}_t = \lambda^D - [\nabla^2 r(\xi_t)]^{-1} A' (A [\nabla^2 r(\xi_t)]^{-1} A')^{-1} \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}$,
- $\hat{\mu}_t = \mu^D - (A [\nabla^2 r(\xi_t)]^{-1} A')^{-1} \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}$
- $\hat{C}_t^\theta = (\theta - t + 1)[C - \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}]$.

Aims to show that under some certain condition $\left[\hat{C}_t^\theta > 0 \ \& \ \vec{\Omega} \left| \sum_{s=1}^{t-1} \frac{\Delta_s}{\theta - s} \right| < \bar{\varphi}\right]$.

$\hat{\lambda}_t, \hat{\mu}_t, \hat{C}_t^\theta$ is the optimality under the reoptimized Control.

Step 2 Bound, construction &

- As the optimality in step 1 requires the condition $\{\hat{C}_t^{\theta} > 0 \; \& \; \vec{Q} \left| \sum_{s=1}^{t-1} \frac{\Delta s}{\theta - s} \right| < \varphi\}$, comparing J_0^{θ} and $\mathbb{E}[R_{RSC}^{\theta}]$ needs to discuss by cases.
- τ^{θ} is a random variable, where if $t \le \tau^{\theta}$, the condition definitely holds.
- By the Good property of a Martingale $\{S_t = \frac{\Delta_{t-1}}{\theta - t + 1} + \frac{\Delta_{t-2}}{\theta - t + 2} + \dots + \frac{\Delta_1}{\theta - 1}\}_{t \ge 0}$,

and $\sum_{k=1}^{\infty} \frac{1}{k} \sim (\log k)$.

It can be shown that $\mathbb{E}(\theta - \tau^{\theta}) \le 1 + M \log \theta$.

Step 3 divide the case and prove.

Some key points which are vital to the proof, however, ~~which~~ are not ~~listed~~ included in the original proof, are well noted in the following pages.

$C_1^\theta = \hat{C}_1^\theta = \theta c$. (Induction about $\hat{C}_t^\theta$).

suppose it is true for some $s = t-1$, then:

$$C_{t-1}^\theta - AD_{t-1} = \hat{C}_{t-1}^\theta - AD_{t-1}$$

$$= (\theta - t + 2)\left[c - \sum_{s=1}^{t-2} \frac{\delta_s}{\theta - s}\right] - AD_{t-1}$$

$$= (\theta - t + 2)c - (\theta - t + 2)\left[\frac{\delta_1}{\theta - 1} + \frac{\delta_2}{\theta - 2} + \dots + \frac{\delta_{t-3}}{\theta - t + 3} + \delta_{t-2}\right] - \underline{A D_{t-1}}$$

$\underline{D_{t-1}}$,

$$\delta_{t-1} = D_{t-1} - E[D_t(p_t)] = \underline{D_{t-1} - \lambda_{t-1}} \Rightarrow D_{t-1} = \delta_{t-1} + \lambda_{t-1} = \delta_{t-1} + \hat{\lambda}_{t-1} = \delta_{t-1} + \lambda^D - [\nabla^2 r(\xi_{t-1})]^{-1} A' (A \nabla^2 r(\xi_{t-1})^{-1} A')^{-1} \dots$$

~~Thus~~, $\hat{\lambda}_{t-1} = \lambda^D - \underline{[\nabla^2 r(\xi_{t-1})]^{-1} A' (A [\nabla^2 r(\xi_{t-1})]^{-1} A')^{-1} \sum_{s=1}^{t-2} \frac{\delta_s}{\theta - s}}$

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$$A\hat{\lambda}_{t-1} = A\lambda^D - \sum_{s=1}^{t-2} \frac{\delta_s}{\theta - s}$$

Hence: $C_{t-1}^\theta - AD_{t-1} = (\theta - t + 2)c - \left[\delta_{t-2} + \frac{\theta - t + 2}{\theta - t + 3}\delta_{t-3} + \dots + \frac{\theta - t + 2}{\theta - 1}\delta_1\right] - \left[A\lambda^D - \sum_{s=1}^{t-2}\frac{\delta_s}{\theta - s} + \delta_{t-1}\right]$.

$A\lambda^D = c.$

$$C_{t-1}^{\theta} - AD_{t-1} = (\theta - t + 1)c - \left[\delta_{t-2} + \frac{\theta - t + 2}{\theta - t + 3}\delta_{t-3} + \cdots + \frac{\theta - t + 2}{\theta - 1}\delta_1 - \frac{\delta_1}{\theta - 1} - \frac{\delta_2}{\theta - 2} - \cdots - \frac{\delta_{t-2}}{\theta - t + 2} + \delta_{t-1} \right].$$

$$= (\theta - t + 1)c - \left[\delta_{t-1} + \frac{\theta - t + 1}{\theta - t + 2}\delta_{t-2} + \frac{\theta - t + 1}{\theta - t + 3}\delta_{t-3} + \cdots + \frac{\theta - t + 1}{\theta - 1}\delta_1 \right].$$

$$= \hat{C}_t^{\theta}$$

(Induction about $\hat{\lambda}_t$ and $\hat{\mu}_t$).

the lagrangian KKT condition Verification.

$$\nabla r(\hat{\lambda}_t) = \nabla r(\lambda^D) + \nabla^2 r(\xi_t)(\hat{\lambda}_t - \lambda^D)$$

$$= A'\mu^D - \underline{\underline{\nabla^2 r(\xi_t) \cdot [\nabla^2 r(\xi_t)]^{-1}}} A'(A[\nabla^2 r(\xi_t)]^{-1}A')^{-1} \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}$$

$$= A'\mu^D - A'(A[\nabla^2 r(\xi_t)]^{-1}A')^{-1} \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}.$$

$$= A'\hat{\mu}_t$$

$$(\theta - t + 1)A\hat{\lambda}_t = (\theta - t + 1)\left(A\lambda^D - A[\nabla^2 r(\xi_t)]^{-1}A'(A[\nabla^2 r(\xi_t)]^{-1}A')^{-1} \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}\right)$$

$$= (\theta - t + 1)\left(c - \sum_{s=1}^{t-1} \frac{\delta_s}{\theta - s}\right).$$

$$= \cancel{(\theta - t + 1)} \cdot C_t^{\theta}$$

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$$\sum_{t=2}^{\theta-1} E[S_{t,i}^2] = \sum_{t=2}^{\theta-1}\left[\frac{E[\Delta_{t-1,i}^2]}{(\theta-t+1)^2} + \frac{E[\Delta_{t-2,i}^2]}{(\theta-t+2)^2} + \dots + \frac{E[\Delta_{1,i}^2]}{(\theta-1)^2}\right] = O(\log\theta)$$

Proof:

First let's simplify the inside:

$$\frac{E[\Delta_{t-1,i}^2]}{(\theta-t+1)^2} \le \frac{E[\Delta_{t-1,i}^2]}{\theta-t} - \frac{E[\Delta_{t-1,i}^2]}{\theta-t+1}$$

and let $k = \max_{t=1,\dots,\theta} E[\Delta_{t,i}^2]$. then:

$$\sum_{t=2}^{\theta-1} E[S_{t,i}^2] \le k \cdot \sum_{t=2}^{\theta-1}\left[\left[\frac{1}{\theta-t} - \frac{1}{\theta-t+1}\right] + \left(\frac{1}{\theta-t+1} - \frac{1}{\theta-t+2}\right) + \dots + \left(\frac{1}{\theta-1} - \frac{1}{\theta}\right)\right].$$

$$= k \cdot \sum_{t=2}^{\theta-1}\left[\frac{1}{\theta-t} - \frac{1}{\theta}\right].$$

$$= k \cdot \sum_{t=2}^{\theta-1} \frac{1}{\theta-t} - k \cdot \frac{(\theta-2)}{\theta}$$

$$= k \cdot \sum_{t=2}^{\theta-1} \frac{1}{\theta-t} + k\frac{2}{\theta} - k.$$

$$= k \cdot \left[\frac{1}{\theta-2} + \frac{1}{\theta-3} + \dots + 1\right] + k \cdot \frac{2}{\theta} - k.$$

$$\le k\left[\frac{1}{\theta} + \frac{1}{\theta-1} + \dots + 1\right] - k$$

$$\le \underbrace{k\left[\frac{1}{\theta} + \frac{1}{\theta-1} + \dots + 1\right]}_{\downarrow} = O(\log\theta)$$

KKT condition of deterministic Control. [Explicit Version].

$$J_D = \max_{\lambda \in \Omega^\lambda} \sum_{t=1}^{T} r(\lambda_t).$$

Subject to. $\sum_{t=1}^{T} A\lambda_t \leq C$

Let $\mu^D = [\mu_i^D]_{m\times 1}$ denote the Lagrangian multiplier.

The Lagrangian function can be denoted by:

$$L(\lambda_t, \mu^D) = \sum_{t=1}^{T} r(\lambda_t) - \mu^{D'}\left(\sum_{t=1}^{T} A\lambda_t - C\right).$$

where $\mu^{D'}$ denote the transpose of μ^D.

A key info we might use is that, in deterministic control. the demand realization is deterministic and the $\lambda_t^D = \lambda^D$. the optimal demand rate is identical for any period. We can further simplify the Lagrangian function as follows.

$$L(\lambda^D, \mu^D) = \sum_{t=1}^{T} r(\lambda^D) - \mu^{D'}\left(\sum_{t=1}^{T} A\lambda^D - c\right) = T\, r(\lambda^D) - \mu^{D'}(TA\lambda^D - c)$$

KKT condition.

$$\frac{\partial L(\lambda^D, \mu^D)}{\partial \lambda^D}:$$

Stationarity:

$$\nabla r(\lambda^D) - A'\mu^D = 0$$

Primal Feasibility:

$$(TA\lambda^D - c) \leq 0.$$

Dual feasibility:

$$\mu^D \geq 0$$

Complementary ~~Completeness~~ slackness:

$$\mu^{D'}(TA\lambda^D - c) = 0.$$