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On the number of partitions of n into k different parts ☆

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ABSTRACT

We study the number of partitions of n into k different parts by constructing a generating function. As an application, we will prove mysterious identities involving convolution of divisor functions and a sum over partitions. By using a congruence property of the overpartition function, we investigate values of a certain convolution sum of two divisor functions modulo 8.

1. Introduction

A partition of a positive integer n is a non-increasing sequence whose sum equals n. We define $p(n)$ as the number of partitions of n and for convenience, we define $p(0) = 1$. Various properties of $p(n)$ have been studied in many ways. For the literature, consult [1]. In this note, we will investigate $p(k, n)$ which counts the number of partitions of n into k different parts. For example, there are 7 partitions of 5 as follows:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Therefore, $p(1, 5) = 2$, $p(2, 5) = 5$ and $p(k, 5) = 0$ for all $k > 2$.

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Since many important partition functions such as the ordinary partitions and overpartitions can be expressed in terms of $p(k,n)$, it is interesting to find how to find $p(k,n)$. We will express $p(k,n)$ in terms of the number of divisor function. The form of its generating function looks very interesting in the sense that the generating function looks reminiscent of the Jacobi–Trudi identity for Schur polynomials and the algorithm to find coefficients in the generating function is reminiscent of the Pieri rule. For the reference on symmetric functions, consult [6].

Before stating the generating series for $p(k,n)$, we need to introduce some notation. We denote the partition λ of n either as $1^{\ell_1}2^{\ell_2}\cdots$ or as a sum of a non-increasing sequence. Here, ℓ_j in the first representation is the multiplicity of the part j. Thus, for the partition λ of n, $n=\sum j\ell_j$. If λ is a partition of n, then we denote $\lambda\vdash n$ and we define the weight of a partition $|\lambda|$ as the number being partitioned. In other words, if $\lambda\vdash n$, then $|\lambda|=n$. The following Lambert series $D(j,q)$ will play a key role in this article as it will serve as a building block for the generating function for $p(k,n)$:

$$D(j,q)=\sum_{i=1}^{\infty}\frac{q^{ji}}{(1-q^i)^j}.$$

Then, for a given partition $\lambda=1^{\ell_1}2^{\ell_2}\cdots j^{\ell_j}\cdots$, we define $D(\lambda,q)$ as

$$D(\lambda,q)=\prod_{j=1}^{\infty}D(j,q)^{\ell_j}.$$

We define $P(k,q)=\sum_{n=1}^{\infty}p(k,n)q^n$ as a generating function for $p(k,n)$. Now we are ready to state our first theorem.

Theorem 1. *For all positive integers k,*

$$P(k,q)=\sum_{\lambda\vdash k}d_\lambda D(\lambda,q), \tag{1.1}$$

where the sum runs all partitions of k and d_λ is a constant defined by

$$\frac{(-1)^{|\lambda|-\#(\lambda)}}{\prod j^{\ell_j}\ell_j!}.$$

Here, $\#(\lambda)$ denotes the number of parts in the partition λ.

Remark. If λ is a partition of n and λ has k different parts, then n should be larger than or equal to $1+2+\cdots+k=\frac{k(k+1)}{2}$. Therefore, $p(k,n)=0$ if $n<\frac{k(k+1)}{2}$. Therefore, the coefficient of q^k in $P(k,q)$ is 0 if $k\geqslant 2$. Since the coefficient of q^k of $D(\lambda,q)$ is 1 if $\lambda\vdash k$, we see that

$$\sum_{\lambda\vdash k}d_\lambda=0,$$

for all $k\geqslant 2$. Moreover, we see that $k!|d_\lambda|$ is the number of permutations of $|\lambda|$ with cycle type λ. Therefore, we obtain

$$\sum_{\lambda\vdash k}|d_\lambda|=1,$$

for all $k\geqslant 1$.

For example, here are some $P(k,q)$ for small values of k:

$$P(1,q) = D\bigl(1^1,q\bigr),$$
$$P(2,q) = \frac{1}{2}D\bigl(1^2,q\bigr) - \frac{1}{2}D\bigl(2^1,q\bigr),$$
$$P(3,q) = \frac{1}{6}D\bigl(1^3,q\bigr) - \frac{1}{2}D\bigl(2^1 1^1,q\bigr) + \frac{1}{3}D\bigl(3^1,q\bigr).$$

Before stating our next proposition which gives the coefficients of $D(\lambda,q)$, we need to introduce a notation. For a fixed positive integer j, we define

$$d^{(j)}(n) = \sum_{d|n}(n-1)(n-2)\cdots(n-j+1), \tag{1.2}$$

for $j \geqslant 2$ and $d^{(1)} = d(n) = \sum_{d|n} 1$.

Remark. We can express $d^{(j)}(n)$ as a linear combination of $\sigma_j(n)$'s. Here, $\sigma_j(n) = \sum_{d|n} d^j$.

Proposition 2. *Let $\lambda = j_1 + j_2 + \cdots + j_\ell$ be a partition of k. If we denote the n-th coefficient of power series expansion for $D(\lambda,q)$ as $\tau(\lambda,n)$, then*

$$\tau(\lambda,n) = \sum_{n_1+n_2+\cdots+n_\ell=n} d^{(j_1)}(n_1)d^{(j_2)}(n_2)\cdots d^{(n_\ell)}(n_\ell),$$

where the n_i's are positive integers and the empty sum is defined as 0.

By combining Theorem 1 and Proposition 2, we obtain an expression for $p(k,n)$.

Theorem 3. *For a fixed positive integer k, we can express $p(k,n)$ in terms of $\sigma_j(n)$.*

Here are some examples for the first few k's:

$$p(1,n) = d(n), \tag{1.3}$$

$$p(2,n) = \frac{1}{2}\sum_{k=1}^{n-1} d(k)d(n-k) + \frac{1}{2}\bigl(\sigma_1(n) - \sigma_0(n)\bigr), \tag{1.4}$$

$$p(3,n) = \frac{1}{6}\sum_{\substack{n_1+n_2+n_3=n\\ n_i>0}} d(n_1)d(n_2)d(n_3) - \frac{1}{2}\sum_{k=1}^{n-1}\sigma_1(k)d(n-k) + \frac{1}{2}\sum_{k=1}^{n-1} d(k)d(n-k)$$
$$+ \frac{1}{6}\sigma_2(n) - \frac{1}{2}\sigma_1(n) + \frac{1}{3}d(n). \tag{1.5}$$

In particular, we obtain that

$$\sum_{k=1}^{n-1} d(k)d(n-k) = \sigma_1(n) - \sigma_0(n) + 2p(2,n). \tag{1.6}$$

The left side of (1.6) is an additive convolution of a multiplicative function $d(n)$. Unlike the Dirichlet convolution of a multiplicative function (in this case, we can use L-series to investigate the convolution), an exact formula for such a convolution is not well known. Moreover, since there is no known closed formula for $\sum_{k=1}^{n-1} d(k)d(n-k)$, (1.6) is very interesting.

Since $p(n)=\sum_{k(k+1)/2\leqslant n} p(k,n)$ for all $n\geqslant 1$, the following corollary is clear.

Corollary 4. *We can express $p(n)$ in terms of $\sigma_j(n)$.*

On the other hand, Fine [3, Section 22] showed that

$$p(1,n)=\sum_{\lambda\vdash n}\ell_1-2\ell_1\ell_2+3\ell_1\ell_2\ell_3-\cdots.$$

By employing Fine's argument, we can see the following proposition.

Proposition 5. *For a fixed positive integer k,*

$$p(k,n)=\sum_{\lambda\vdash n}\binom{k}{k}\ell_1\ell_2\cdots\ell_k-\binom{k+1}{k}\ell_1\ell_2\cdots\ell_{k+1}+\binom{k+2}{k}\ell_1\ell_2\cdots\ell_{k+2}-\cdots.$$

Combining these two representations for $p(k,n)$ (Theorem 3 and Proposition 5), we can derive many mysterious formulas relating the sum involving power of divisor functions and the sum over partitions. For example,

$$d(n)=\sum_{\lambda\vdash n}\ell_1-2\ell_1\ell_2+3\ell_1\ell_2\ell_3-\cdots, \tag{1.7}$$

$$\sum_{k=1}^{n-1}d(k)d(n-k)-\sigma_1(n)+d(n)=2\sum_{\lambda\vdash n}\binom{2}{2}\ell_1\ell_2-\binom{3}{2}\ell_1\ell_2\ell_3+\binom{4}{2}\ell_1\ell_2\ell_3\ell_4-\cdots \tag{1.8}$$

$$\begin{aligned}&\sum_{\substack{n_1+n_2+n_3=n\\ n_i>0}}d(n_1)d(n_2)d(n_3)-3\sum_{k=1}^{n-1}\bigl(\sigma_1(k)-d(k)\bigr)d(n-k)+\sigma_2(n)-3\sigma_1(n)+2d(n)\\ &\quad=6\sum_{\lambda\vdash n}\binom{3}{3}\ell_1\ell_2\ell_3-\binom{4}{3}\ell_1\ell_2\ell_3\ell_4+\binom{5}{3}\ell_1\ell_2\ell_3\ell_4\ell_5-\cdots.\end{aligned}$$

Remark. (1.7) is given in [3, Section 22].

For example, there are 5 partitions of 5 into 2 different parts. In the left side of (1.8), $\sum_{k=1}^{4}d(k)d(5-k)=14$, $\sigma_1(5)=6$, and $d(5)=2$. Therefore, the left side of (1.8) equals 10. The right side of (1.8) is 10 since the summand in right side of (1.8) is 4 if the partition λ is $1^1 2^2$, and 6 if the partition λ is $1^3 2^1$, and 0, otherwise. Therefore, we can see that each side of (1.8) gives twice of $p(2,5)$ as we expected.

As we have already seen, we can express $p(n)$ in terms of $p(k,n)$. Beside of the ordinary partition function $p(n)$, the overpartition function $\overline{p}(n)$ can also be expressed in terms of $p(k,n)$. By using this connection between $\overline{p}(n)$ and $p(k,n)$, we will investigate values of the sum

$$A(n)=\sum_{k=1}^{n-1}d(k)d(n-k) \tag{1.9}$$

modulo 8. An overpartition of n is a partition of n in which we may overline the first occurrence of the part. For example, there are 8 overpartitions of 3:

$$3,\ \bar{3},\ 2+1.\ \bar{2}+1.\ 2+\bar{1}.\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1.$$

Overpartitions have played an important role in basic hypergeometric series and related fields. For more information and references on overpartitions, see the seminal paper of S. Corteel and J. Lovejoy [2] and the references therein. The behavior of the overpartition function modulo 16 implies the following theorem.

Theorem 6. *The arithmetic density of the set* $\{n \in \mathbb{N} \mid A(n) \equiv 0 \pmod 8\}$ *is* 1, *where* $A(n)$ *is a convolution of two divisor functions defined in* (1.9).

2. Proofs of the results

We start this section with a simple observation. If d divides n, we can write n as the sum of n/d copies d. Therefore, each divisor of n corresponds to the partitions of n into 1 different part. Hence, we see that $p(1,n) = d(n)$. Before proceeding further, we note that $D(k^1, q)$ generates the number of representations of n as $n = (m_1 + m_2 + \cdots + m_k)\ell$, where m_i and ℓ are positive integers, and the different orders of m_j's are counted as different.

Proof of Theorem 1. For a generating function for $p(2,n)$, since $P(1,q)$ generates the partitions into 1 different part, by multiplying $D(1,q)$ by $P(1,q)$, we obtain a generating series for the number of representations for n as $n = m_1n_1 + m_2n_2$ such that m_i and n_j are positive integers and that n_1 and n_2 might be equal. Since we want the case when n_1 and n_2 are different from each other, we need to subtract the case $n_1 = n_2$. This case is generated by $D(2,q)$. Hence, $P(1,q)D(1,q) - D(2,q)$ generates the number of representations for n as $n = m_1n_1 + m_2n_2$, where n_1 and n_2 are different positive integers at this time. Because we regard different orders of representations as the same partition, we find that

$$P(2,q) = \frac{1}{2}\big(P(1,q)D\big(1^1,q\big) - D\big(2^1,q\big)\big).$$

Similarly, we see that $P(2,q)D(1^1,q)$ generates the number of representations of n as $n = m_1n_1 + m_2n_2 + m_3n_3$, where n_i and m_j are positive integers such that n_1 and n_2 are different and different places of n_3 in the summation are counted as different. The case when n_3 is the same as either n_1 or n_2 is generated by $P(1,q)D(2^1,q)$. However, $P(1,q)D(2^1,q)$ generates the case $n_1 = n_2 = n_3$. Therefore, we need to subtract $D(3^1,q)$ to cancel this case. In summary, $P(2,q)D(1^1,q) - P(1,q)D(2^1,q) + D(3^1,q)$ generates the number of representations of n as $n = m_1n_1 + m_2n_2 + m_3n_3$, where n_i and m_j are positive integers such that n_1, n_2 and n_3 are different and different places of n_3 in the summations are counted as different. Thus, we arrive at

$$P(3,q) = \frac{1}{3}\big(P(2,q)D\big(1^1,q\big) - P(1,q)D\big(2^1,q\big) + D\big(3^1,q\big)\big).$$

By the same argument, we obtain the following recurrence relation for $P(k,q)$:

$$P(k,q) = \frac{1}{k}\big(P(k-1,q)D\big(1^1,q\big) - P(k-2,q)D\big(2^1,q\big) + \cdots + (-1)^{k-1}D\big(k^1,q\big)\big). \tag{2.1}$$

Before continuing, we need to define an operation for the partitions. If j is a part of the partition λ, we define $\lambda - \{j\}$ as the partition which has all parts of λ except one j. Then, for d_λ, we find the following recurrence relation from (2.1).

Proposition 7. *For each positive integer k,* $d_{k^1} = \frac{(-1)^{k-1}}{k}$*. Moreover, for a given partition* λ*,*

$$d_\lambda = \sum_j \frac{(-1)^{j-1}}{|\lambda|} d_{\lambda-\{j\}},$$

where the sum runs over all different parts of λ*.*

By using the recurrence relation given in Proposition 7, we can calculate d_λ for a given partition λ by using induction, which concludes Theorem 1. □

A proof of Proposition 2 follows easily by expanding $D(j,q)$ as a geometric series.

Proof of Proposition 2. To prove Proposition 2, we need to determine the n-th coefficient of $D(j,q)$. By expanding the geometric sum, we find that

$$D(j,q) = \sum_{i=1}^{\infty} \frac{q^{ji}}{(1-q^i)^j} = \frac{1}{(j-1)!} \sum_{n=1}^{\infty} d^{(j)}(n) q^n,$$

where $d^{(k)}(n)$ is defined as (1.2). Hence, from the definition of $D(\lambda, q)$, Proposition 2 follows. □

Now, we turn to the values of $A(n) = \sum_{k=1}^{n-1} d(k)d(n-k)$. From now on, a property holds almost always means that the set which satisfies the property has arithmetic density 1. As we already mentioned in the introduction, we will investigate $A(n)$ via the connection between $p(k,n)$ and $\overline{p}(n)$. To this end, we relate $\overline{p}(n)$ with $p(k,n)$ as follows. Since there are two choices (we may overline or may not overline) for each different part, we observe that

$$\overline{p}(n) = \sum_{k=1}^{\infty} 2^k p(k,n),$$

for all $n \geqslant 1$. Therefore, we find that

$$\overline{p}(n) \equiv 2p(1,n) + 4p(2,n) + 8p(3,n) \pmod{16}, \tag{2.2}$$

for all $n \geqslant 1$. Before going further, we introduce the following lemma which provides divisible properties we need. Though the following lemma is a summary of the results proven in K. Mahlburg [5] and the author [4], we give a brief sketch for the completeness.

Lemma 8. *Let* $r_k(n)$ *be the number of representations of n as the sum of k squares of integers and* $r_{1,2}(n)$ *be the number of representations of n as* $x^2 + 2y^2$*, where x and y are integers. Then,* $r_1(n)$*,* $r_2(n)$ *and* $r_{1,2}(n)$ *are almost always* 0*. Moreover, for any fixed integer j,* $r_3(n)$*,* $d(n)$*,* $\sigma_1(n)$*, and* $\sigma_2(n)$ *are almost always divisible by* 2^j*. Finally,* $\overline{p}(n)$ *is almost always divisible by* 16*.*

Sketch of proof. It is clear that $r_1(n)$ is almost always 0. By the famous result of Landau, $r_2(n)$ is almost always 0. An analogous proof for the fact $r_{1,2}(n)$ is almost always 0 is given in [4, Lemma 3]. Regarding $r_3(n)$, by the famous result of Gauss and by the divisibility property of Hurwitz class number, $r_3(n)$ is divisible by 2^j provided there are at least j distinct odd primes dividing the square-free part of n. Moreover, it is clear that $d(n)$, $\sigma_1(n)$, and $\sigma_2(n)$ are also divisible by 2^j if n has at least j distinct odd primes in its square-free part. Let $B_j(x)$ be the number of integers $n \leqslant x$ having at most j odd primes in its square-free part. Then, it is well known that $B_j(x)/x$ tends to 0 as x goes to infinity (for example, consult [4, Lemma 2]). Therefore, $r_3(n)$, $d(n)$, $\sigma_1(n)$ and $\sigma_2(n)$ are almost always divisible by 2^j. Concerning $\overline{p}(n)$, it is now proven that $\overline{p}(n)$ is almost always divisible by 128 [4, Theorem 1]. □

Now we are ready to give a proof of Theorem 6.

Proof of Theorem 6. From (1.4), we see that

$$A(n) = 2p(2,n) - \sigma_1(n) + d(n).$$

Since $\sigma_1(n)$ and $d(n)$ are almost always divisible by 8, proving $p(2,n)$ is almost always divisible by 4 is enough for the proof for Theorem 6. Since $p(1,n) = d(n)$ and $\overline{p}(n)$ are almost always divisible by 16 by Lemma 8, we can conclude that $p(2,n)$ is almost always divisible by 4 if $p(3,n)$ is almost always divisible by 2. From (1.5), we obtain that

$$\begin{aligned} 6p(3,n) = & \sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3) \\ & - 3\sum_{k=1}^{n-1}\big(\sigma_1(k) - d(k)\big)d(n-k) + \sigma_2(n) - 3\sigma_1(n) + 2d(n). \end{aligned} \tag{2.3}$$

We are going to show that the right side of (2.3) is almost always divisible by 4. First, we investigate the sum

$$\sum_{\substack{x+y+z=n \\ x,y,z>0}} d(x)d(y)d(z)$$

modulo 4. Note that $d(x)$ is odd if and only if x is a square of a positive integer and $\sigma_1(n)$ is odd if and only if $n = m^2$, or $2m^2$. When $x + y + z = n$, we have three cases: (1) $x = y = z$, (2) two of them are the same and the other is different from the other two, and (3) all of them are different. When $x = y = z$, $d(x)^3$ is not divisible by 4 only if x is a square. In other words, unless $n = 3m^2$, $d(x)^3$ is divisible by 4. Note that the set $\{n \in \mathbb{N} \mid n = 3m^2\}$ has arithmetic density 0, so

$$\sum_{\substack{x+y+z=n \\ x=y=z>0}} d(x)d(y)d(z)$$

is almost always divisible by 4. For the second case, there are three possible choices of pairs which are the same and for convenience, say $x = y$ and $x \neq z$. Then, $d(x)^2 d(z)$ might not be divisible by 4 if x is a square. For the last case, by considering permutations of x, y, z, we see that $\sum_{\substack{x+y+z=n \\ x>y>z>0}} d(x)d(y)d(z)$ is not divisible by 4 only if $n = x^2 + y^2 + z^2$. In summary, we arrive at

$$\sum_{\substack{x+y+z=n \\ x,y,z>0}} d(x)d(y)d(z) \equiv 3 \sum_{\substack{2m^2+z=n \\ m,z>0}} d\big(m^2\big)^2 d(z) + 2 \sum_{\substack{x^2+y^2+z^2=n \\ x>y>z}} 1 \pmod 4, \tag{2.4}$$

for almost all n. By Lemma 8, we see that $\sum_{\substack{x^2+y^2+z^2=n \\ x>y>z}} 1$ is almost always divisible by 2. Moreover, by employing a similar argument, we observe that

$$\sum_{\substack{x+y=n \\ x,y>0}} \bigl(\sigma_1(x) - d(x)\bigr)d(y) \equiv \sum_{\substack{2m^2+z=n \\ m,z>0}} \sigma_1\bigl(2m^2\bigr)d(z) \pmod 4, \tag{2.5}$$

for almost all n. By adding (2.4) and (2.5), we find that

$$\sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3) - 3\sum_{k=1}^{n-1}\bigl(\sigma_1(k) - d(k)\bigr)d(n-k)$$

is almost always divisible by 4 since the number of representations of n as $2x^2 + y^2$ is almost always 0 from Lemma 8. Since $\sigma_2(n)$, $\sigma_1(n)$ and $d(n)$ are almost always divisible by 4, we have seen that $p(3,n)$ is almost always even, which finishes the proof of Theorem 6. □

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