Characterizing the Uncertainty of Jointly Distributed Poses in the Lie Algebra

Response to Reviewers

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We would like to begin by thanking the editors and reviewers for their careful review of our paper. The provided comments were incredibly helpful and insightful. Based on the recommendations of the reviewers and editors, we have made several modification to the paper, which are summarized in detail below. In addition, a highlighted manuscript is attached. In the modified manuscript, new text to address comments are highlighted in green and deleted text is crossed out and highlighted in ~~red~~.

Editor

Comment 1 :

The reviews are mixed for this manuscript, although overall, the editors finds the paper well written and the contribution on the stronger side, and very relevant to the community. Reviewers R3 and R10 raise some valid concerns that the editors would recommend addressing, such as the reference to prior work and the validation of this work with an additional experiment, in the spirit of complementing the theoretical argumentation with practical demonstrations of the relevance of this work.

Thank you for carefully reviewing our paper. We have addressed these concerns.

More specifically, we have made the following major changes to the paper:

1. We have rewritten the introduction to further clarify our work.
2. We have added a related works section to more clearly place our work in the literature and have added the citations and references suggested by reviewer 10.
3. We have added an additional experiment on a real-world dataset to practically demonstrate the relevence of the work as suggested by reviewer 3.

#E.1

Reviewer 2

Comment 1 :

This paper develops a framework for modeling the uncertainty of jointly distributed poses and describes how to perform the equivalent of the pose uncertainty characterization methods proposed by Smith, Self, and Cheeseman while characterizing uncertainty in the Lie Algebra. This paper also propones an evaluation on simulated and open-source datasets that shows that the proposed methods result in more accurate uncertainty estimates. An accompanying C++ library implementation is also released.

All in all, I must say that I really enjoyed reading this manuscript. It is very well structured and written, and it definitively has a high value as a review of works related to uncertainty measuring and SE(3) pose representations.

The topic is of the maximum interest to the reader of this journal, hence I believe it is worth publishing. Sections IV, V, VI and VII are of special interest for their didactic and survey value. Experimental validation is good enough.

Thank you for your kind comments and we are glad that you enjoyed reading our paper. #R2.1

Reviewer 3

Comment 1 :

This article introduces a framework for expressing pose uncertainty in the Lie Algebra. Therefore, the authors describe how uncertainty can be characterized in the Lie Algebra, and introduce methods for pose operations using this formulation, such as pose composition or inversion, that are commonly used in pose-graph SLAM. All operations take into account correlations between poses instead of assuming independent poses, which is usually not the case in pose-graph SLAM. The authors quantify the improvement of the uncertainty estimation experimentally using simulated cases and real-world data sets.

The main contribution of this article is the formulation of pose uncertainty, including the common pose operations pose composition, pose inversion and relative pose extraction, for Lie groups. Compared to existing works in the literature, the authors thereby do not assume independence between the poses, which is usually not the case in SLAM, but model the uncertainty framework for jointly correlated poses. The improvements in terms of accuracy of the estimated uncertainty is evaluated in simulation on pose-pairs as well as on a real-world SLAM data set. Furthermore, the authors describe how to convert to the presented uncertainty parametrization from other representations and release an accompanying C++ library implementation.

This article is generally well written and presented, the explanations are mostly clear and transparent to follow, and the described method is theoretically well-founded. The contributions of the article are clearly stated and supported by the description of the framework and the experiments.

Thank you for your careful summary of our paper and kind comments. #R3.1

Comment 2 :

The main criticism of this reviewer refers to the presented experimental results on real-word data, which could, to this reviewer's opinion, be better described and also extended to strengthen the contribution of this article.

Thank you for your very helpful critique. We agree and have revised Sec. -B as follows to improve the experimental description:

To evaluate uncertainty propagation through the relative pose ~~extraction~~operation, we used ~~iSAM [?] to find a solution to~~ the Manhattan 3500 dataset [?] to generate a set of pose pairs on which to perform the relative pose operation. After using iSAM [?] to find a solution to the SLAM problem, we ~~We then~~ extracted joint mean and covariance as described in Section IX-B for pairs of poses at offsets varying from 5 to 50 in increments of 5 as well as for offsets of 100, 200, and 500 nodes. A visualization of the extracted pose pairs and the correlation between them for offsets of 50 are shown in Fig. 10 (a-c). For each pose pair, we plot a line between the two poses colored according to the degree of correlation between the two poses with respect to the x, y, and θ dimensions. Equivalent visualizations for offsets of 10, 100, 200, and 500 are shown in Fig. 11, Fig. 12, Fig. 13, and Fig. 14.

For each extracted pose pair, we performed a Monte Carlo simulation to determine a ground truth relative pose distribution against which we could compare the first-order propagation methods being investigated. First, the mean and joint marginal covariance for each pose pair was extracted from the iSAM solution. Then, for a given pair, a large number of sample pose pairs were generated from the extracted pose pair distribution and the relative pose between each of those pairs was evaluated. This gave us a sample distribution of the relative transformation between the two poses in the original pair which was used as groundtruth. We then repeated the process for each joint pose pair distribution extracted from the dataset.

Using this Monte Carlo derived relative pose covariance as a reference, we can use the following covariance error metric to evaluate the predicted covariance for the various methods we compare:

$$\epsilon \triangleq \sqrt{\mathrm{tr}\left((\mathbf{\Sigma} - \mathbf{\Sigma}_{mc})^\top (\mathbf{\Sigma} - \mathbf{\Sigma}_{mc})\right)}, \quad (1)$$

where $\mathbf{\Sigma}_{mc}$ is the sample relative pose covariance matrix generated via Monte Carlo for the given pose.

~~To investigate the importance of taking into account correlation when estimating relative pose, we performed a Monte Carlo simulation to estimate the true relative pose covariance as described in (??) - (??), where $\boldsymbol{\xi}^m_{g1}$ and $\boldsymbol{\xi}^m_{g2}$ are the perturbation variables sampled from the extracted joint covariance and $\bar{\mathbf{T}}_{g1}$ and $\bar{\mathbf{T}}_{g2}$ are the mean values extracted from the iSAM solution. We then used the metric defined in (1) to~~To evaluate ~~the covariance error of~~ our method proposed in (??) when taking into account correlation (using all four terms in (??)) or ignoring correlation (using only the first two terms in (??)), we used both methods to propagate pose uncertainty through the relative pose transformation for each pair and evaluated the error of the predicted covariance according to (1). Summary statistics are shown in Table I and visualizations of the error for offsets of 50, 10, 100, 200, and 500 are shown in (d-e) of Fig. 10, Fig. 11, Fig. 12, Fig. 13, and Fig. 14.

#R3.2

Comment 3 :

Regarding the evaluation of the proposed framework, this reviewer especially appreciates the fact that the authors evaluate the effect of taking into account correlations vs. the assumption of independent poses, before moving on to a 'full SLAM example'. However, regarding the experiment on SLAM data (Sec. IX-B), this reviewer finds it quite hard to follow the explanations of the authors on the design as well as on the evaluation metrics of the conducted experiment. This reviewer would suggest revising this section to improve transparency.

Thank you for your suggestion.

We have revised the description of Sec. IX-B as described in the response to the previous comment.

#R3.3

Comment 4 :

Furthermore, why was the solution of iSAM used for the poses, while there should be ground truth available *for benchmarking data sets?*

Thank you for your question.

It is true that a groundtruth pose could be obtained from the simulated data set, however the goal of the proposed method is not to find the final pose from the SLAM solution, but to correctly characterize the uncertainty of the relative pose for a given pair of poses extracted from a given SLAM solution. Thus, the mean of two poses as well as the joint covariance of the two poses was extracted from the iSAM solution and Monte-Carlo simulation was used to provide a "ground-truth" estimate of the relative pose distribution. We then compared the relative pose distribution predicted by our method and that proposed by Smith, Self, & Cheeseman with the pose predicted by Monte-Carlo. #R3.4

Comment 5 :

While the experiments authentically substantiate the claim of the authors that their framework results in more accurate uncertainty evaluation, it would be interesting to see how this finally affects the accuracy of the SLAM estimate, since from this reviewer's point of view, the final benefit from better uncertainty estimation should be increased accuracy of the estimate. Therefore, this reviewer would ask the authors to quantify also RMSE accuracy for the SLAM solution experiment.

Thank you for your comment.

While the factor graph framework used to formulate the Pose-Graph SLAM problem is affected by the modeled uncertainty of the individual factors, the primary focus of the relative pose operation presented in this paper is not on improving the accuracy of the SLAM solution, but on accurately characterizing uncertainty when performing operations on the correlated pose estimates derived from the SLAM solution. For example, after solving the SLAM problem, it is often necessary to determine the relative pose between globally referenced poses estimated by the SLAM solution in order to evaluate the likelihood of a potential loop-closure.

To make this more clear, we have added the following paragraph to the introduction:

> *In addition to accurately characterizing the uncertainty of a single pose, it is often necessary to propagate the uncertainty of one or more poses through non-linear operations such as pose composition, pose inversion, and relative pose extraction. For example, given an odometry sequence consisting of multiple uncertain relative pose transformations, it may be necessary to compose the entire sequence of transformations together to obtain a distribution describing the relative pose of the robot over an entire sequence. Alternatively, after estimating the pose of the robot at multiple time steps via maximum likelihood estimation (MLE) or SLAM, it can be very helpful to extract the relative pose of the robot between two time steps to evaluate whether a potential loop-closure is consistent with the estimated trajectory [**?**, **?**].*

#R3.5

Comment 6 :

Furthermore, additional experiments on a second real-world data set would, to this reviewer's opinion, strengthen the contribution of the article in terms of confirming that the benefits from this framework not only apply in simulation, but also in real-world examples. However, the evaluation does not have to be as extensive as for the first experiments, the authors could focus only on the most important aspects here.

We thank you for your suggestion and have extended the evaluation by performing the following experiment as decribed in Section X-C:

The results of the previous experiment show that our proposed method results in more accurate characterizations of pose uncertainty when used to evaluate the relative pose between pairs of poses extracted from an estimated SLAM solution. This increased accuracy is important when evaluating potential loop-closures. To emphasise this, we performed an additional experiment on the CSAIL real-world dataset collected at MIT [?].

We first divided the factors in the graph into a group of verified accurate loop-closures and odometry. We then used iSAM [?] to estimate a solution to the graph using odometry only and extracted the joint block covariance for the pairs of nodes associated with each loop-closure. As before, we extracted this joint covariance twice using both the proposed parameterization and that used by SSC. We then estimated the relative pose between the two base poses using the method proposed in Section Fix This and via the SSC tail-to-tail operation and evaluated the proposed loopclosures by thresholding the squared Mahalanobis distance.

For a given loopclosure $\mathbf{T}_{jk}$, the SSC Mahalanobis distance was evaluated by applying the SSC tail-to-tail operation to the mean and associated joint covariance for the two poses $\mathbf{x}_{gj}$ and $\mathbf{x}_{gk}$ extracted from iSAM to obtain $\hat{\mathbf{x}}_{jk}^{odom}$ and $\mathbf{\Sigma}_{jk}^{odom}$ and then calculating the distance as follows:

$$\delta_{SSC} = (\hat{\mathbf{x}}_{jk}^{odom} - \mathbf{x}_{jk})^\top {\mathbf{\Sigma}_{jk}^{odom}}^{-1} (\hat{\mathbf{x}}_{jk}^{odom} - \mathbf{x}_{jk}), \tag{2}$$

where $\mathbf{x}_{jk}$ is $\mathbf{T}_{jk}$ expressed as a parameter vector in $\mathbb{R}^3$ as defined in [?] for $\mathrm{SO}(2)$.

Under the proposed parameterization, the Mahalanobis distance calculation was preformed by using the relative pose operation defined in Section FIX THIS to estimate $\bar{\mathbf{T}}_{jk}^{odom}$ and $\mathbf{\Sigma}_{jk}^{odom}$ from the poses $\mathbf{T}_{gj}$ and $\mathbf{T}_{gk}$ and their joint covariance matrix extracted from iSAM and then calculating the distance as follows:

$$\delta_{Lie} = \log(\mathbf{T}_{jk} {\bar{\mathbf{T}}_{jk}^{odom}}^{-1})^\top {\mathbf{\Sigma}_{jk}^{odom}}^{-1} \log(\mathbf{T}_{jk} {\bar{\mathbf{T}}_{jk}^{odom}}^{-1}). \tag{3}$$

Finally, both distances where compared against the chi-squared threshold value 7.814 corresponding to a 95 percent likelihood for three degrees of freedom. The results are shown in Fig. 15, Fig.16, and Table III.

When using the proposed Lie algebra based method, 100 percent of the verified loop-closures fall within the expected chi-squared bounds, while when using SSC more than ten percent fall outside it (see Fig. 16 and Table III). #R3.6

Comment 7 :

To this authors experience from other articles, '§ VII' is not a common abbreviation for 'Section VI'. Please consider using e.g. 'Sec. IV'.

Throughout the paper, all section references were changed from ~~§~~ → Section. #R3.7

Comment 8 :

P2: typo: non-linearm

~~non-linearm~~ → non-linear #R3.8

Comment 9 :

Eq. 35: 'E' is not introduced

We thank the reviewer for pointing out this ommission. We made the following changes to the manuscript to resolve this issue:

Computing the covariance $\mathbf{\Sigma}$ amounts to evaluating $E[\boldsymbol{\xi}_{ik}\boldsymbol{\xi}_{ik}^{\top}]$, where $E[\cdot]$ signifies the expected value of a given random variable. #R3.9

Comment 10 :

Fig. 4: Color scheme is not optimal (color for ellipse for independent solution does not match legend, green ellipse on green ellipse). Furthermore, it took this reviewer a moment to understand the plot - it is not directly clear that the black lines are trajectories from the same starting point. Maybe the caption could be revised to improve clarity.

We thank the reviewer for their comment. We have changed the caption to Fig. 4 as follows to clarify the figure both in terms of general understanding and color scheme:

Plots of 10000 sample trajectories (shown in black), each made up of a sequence of 10 noisy pose transformations starting from the same point. The 95% likely uncertainty ellipse predicted by first order uncertainty propagation through the SSC head-to-tail operation is shown in red overlayed on top of the sample trajectories, while a representation of the flattened 95% likely uncertainty position ellipsoids predicted by the Lie algebra pose composition methods when correlation is and is not taken into account are ~~shown in~~ overlayed in transparent green and cyan, respectively. #R3.10

Reviewer 10

We thank you for your very helpful comments and suggestions. We have completely rewritten several portions of the paper based on your suggestions and feel that the suggestions provided have significantly increased the clarity of the paper.

Comments 1 – 9 :

Unfortunately, the paper was actually quite frustrating to read and requires lot of effort on the part of the reader and often at odds with existing literature.

I found the contributions to be limited, and the current presentation is not sufficiently clear.

Claims seem excessive. previous methods are not as poor as is claimed.

Much of the confusion on the use of "variables are not correlated".

*misleading or incorrect: Abstract, Section IV, Conclusion: "existing approaches assume that individual poses are independent" – this is claim highly ambiguous. All (mean/covariance) SLAM techniques that are MLE model the correlation between variables with f(x), h(x) etc. The whole point of building a pose/factor graph is to model the statistical dependence structure between all variables. I had a lot of trouble figuring out precisely what the authors are trying to say. Independent measurements vs independent variables are two different things. arfoot [1] specifically state that *noise* terms are assumed uncorrelated (like most SLAM literature). Work needs to be done to clear this significant ambiguity.*

fig. 4 "Indep. Lie Algebra Comp ([1])" you show a bean shaped but projecting covariance should be elliptical?

What is the point behind "compose odometry measurements that are potentially correlated"? Composition of two normal distributions is just the composition of two normal distributions

Are you talking about estimating joint marginal covariance amongst multiple variables (poses) AFTER MLE based mean estimation has completed?

Thank you for your comments.

It seems that the majority of your problems with the paper stem from a lack of clarity or misunderstanding of the goal of the work.

The primary goal of the work is describe how to propagate uncertainty through the pose composition, pose inverse, and relative pose operations upto first order when using a similar pose uncertainty parameterization to that proposed by Barfoot et al. [1], and while taking into account the potential correlation of the input poses.

While a variety of methods have been proposed to characterize pose uncertainty, many methods use a pose parameterization, such as that proposed by Smith, Self, and Cheeseman et al. [6, 7], that represents pose uncertainty by treating a vector of parameters as a Gaussian random variable even though it does not reside within a Euclidean vector space. Modeling uncertainty in this way means that such representations are unable to represent the true distribution of pose uncertainty which is by definition non-eliptical.

Recent work by Chirikjian et al. [8,9] and Barfoot et al. [1, 17] show that by using the Lie algebra to represent this uncertainty, the bean or banana shapes of uncertain poses can be accurately represented. The major difference between these two approaches is that Chirikjian et al. define a distribution in the group space and then use "exponential coordinates" to represent it. Barfoot on the other hand represents pose uncertainty by using a mean group element that is perturbed by multiplying it by the exponential of a zero mean multi-variate Gaussian random variable defined in the Lie algebra. This leads to a simpler method for propagating uncertainty through the composition operation. It is important to note that the distribution defined by Barfoot is not Gaussian, but rather a new distribution parameterized by a mean Lie group element and the covariance of the perturbation in the Lie algebra.

However, although Barfoot et al. [1] clearly describes how to propagate pose uncertainty through the pose composition operation, they make the assumption that the poses being composed are independent. While, this is often true, it is not always the case. For example, in many cases, odometry may be correlated over time depending on the specifics of the environment and method of motion. In addition, after solving simultaneous localization and mapping (SLAM) or after estimating multiple poses via maximum likelihood estimation (MLE), the resulting poses are correlated with one another.

In addition, Barfoot et al. [1] focuses primarily on the pose composition operation. We describe here how to both take into account pose correlation when operating on multiple poses and how to perform the pose composition, pose inversion, and relative pose operations that are commonly used during robotic state estimation tasks.

We are sorry if our paper was unclear. We have tried to improve the clarity of the paper by:

1. Rewriting a substantial portion of the introduction.
2. Adding a related work section to better clarify where our work fits into the literature.
3. Reworking the descriptions of several of the experiments.

In addition, to address your comment about the ambiguity of claiming that "existing approaches assume that individual poses are independent", we have changed the quoted phrase in the abstract as follows:

> *However, existing Lie group based uncertainty propogtion techniques ~~approaches~~ assume that individual poses are independent.*

Similarly, the conclusion has been changed as follows:

> *However, recent ~~work~~ methods that represent uncertainty via a perterbation defined in the Lie algebra assume~~s~~ that individual poses are independent from one another and have primarily focused on pose composition, ignoring the equally important inverse and relative pose operations.*

Comments 10 – 15 : #R10.9

There are known expressions for propagating covariance through non-linear functions which use higher order terms than commonly used in the jacobian approach in common use today – i.e. J P J' + Q. This is precisely why methods like Kalman filtering introduce Q, because it is a way "make up" for inaccuracies in the J P J' term. This paper 'characterizes covariance propagation with expressions using up to fourth degree (you used order) terms. Although this 'characterization should be carefully compared to other methods in a systematic way, the presentation here is rather—or just seems—haphazard.

The paper does a poor job relative performance (in absolute reproducible terms) to existing methods. For example Laplace's approximation is not even mentioned once. Reference to the Laplace approximation method is missing.

Other references such as Kaess & Delleart Covariance recovery is cited, however, the results from those covariance estimates (e.g. covariances generated by iSAM) are omitted from the results. If the intended contribution here is that something like iSAM or Ceres do not provide a way to recover the joint marginals of multiple variables, but this paper describes a method/implementation to do that, then it should clearly be stated as such – although I don't think that is true (see EKF-SLAM argument below).

There isn't a clear "Related Work" section interpreting and placing this paper properly in context The citations count yes, but important interpretations dont transpire. One of the most important aspects to literature survey (the covariance recovery used in iSAM) is late on pg.10 without any real context or comparative placement. What about Google Ceres?

EKF-SLAM estimates mean and covariances measurement jacobians (Euler/Lie/Spin), basically eqs. (16), (18), (19), (26), (28), and (30) and produces a dense covariance matrix, because it encodes the correlations between each of the variables in the state vector.

*Laplace approximations is known Bayesian inference method to find covariances for nonlinear problems, how does your method compare – specifically relative to fig 8 and 9 If you truly did a Monte Carlo based solution to accurately find the *true* joint marginal covariances, then Laplace approximation might have been less work?*

Thank you for your comments.

We agree that the paper could have been more clear in describing how our work relates to previous work. To correct this, we have rewritten the introduction and added a related work section (Section II) that covers the following topics:

1. The importance of consistent uncertainty characterization.
2. Early methods via first-order derivatives on Euler angle parameterizations.
3. Higher order methods that enable increased accuracy at increased computational cost.
4. The use of Lie group theory to enable accurate representation of non-Gaussian pose uncertainty distributions.
5. Joint marginal extraction from existing estimation algorithms.

The requested citations are included.

To specifically address your questions, the related work section includes the following paragraphs:

Other methods try to improve approximation accuracy by incorporating non-linear information into the approximation process. The unscented transform (UT) [?] propagates a set of deterministic sample points through the non-linear operation and then uses a weighted average to approximate the final distribution as a Gaussian. Laplace approximation [?] follows a similar derivation to first-order approximation, but also takes into account second order derivative information. These methods result in more accurate Gaussian approximations then first-order methods when the true resulting distribution is heavily non-Gaussian at the expense of greater computational cost.

There have also been a variety of methods proposed for extracting estimates of uncertainty from the solutions to estimation algorithms. Kalman filter-based methods such as EKF-SLAM [?, ?, ?, ?] explicitly track the covariance matrix of the values they estimate, although the parameterization of that covariance is somewhat dependent on the specific algorithm or implementation [?]. It is also possible to extract estimates of uncertainty from maximum likelihood estimation (MLE)-based estimation methods. Kaess et al. [?] propose a method that is able to extract a Gaussian uncertainty approximation for the estimated variables from the information matrix used to find the MLE Pose-Graph SLAM solution. Google Ceres [?] also has the capability to extract covariance matrices using similar methods to those proposed in [?]. While the extraction of pose uncertainty from existing estimation solutions is not the primary contribution of this article, it is important to point out that when multiple poses are estimated jointly whether via Kalman filter based methods or MLE, the resulting poses are often highly correlated and thus propagation methods that operate on those estimated poses need to be able to handle such correlation.

We agree that it is important to acknowledge that there are a variety of methods for propogating uncertainty through non-linear functions and that each such method varies in its degree of accuracy and computational expense. However, we also want to note that this is not the primary focus of the paper. The primary focus is on describing how to perform first order propagation through the pose composition, pose inversion, and relative pose operations when characterizing uncertainty in the Lie algebra following the method described in [1] and when taking into account correlation.

We also agree with you that methods such as EKF-SLAM, iSAM, and Ceres all provide methods for extracting joint marginal covariances. Our goal is not to extract those covariances, but to derive how to perform operations on them after they have been extracted. Part of the rewritten introduction was specifically written to clarify this:

For example, it is well known that the estimated poses obtained via a maximum likelihood estimation (MLE) algorithm such as Pose-Graph SLAM, tend to be heavily correlated [?]. ... [A]fter estimating the pose of the robot at multiple time steps via MLE or SLAM, it can be very helpful to extract the relative pose of the robot between two time steps to evaluate whether a potential loop-closure is consistent with the estimated trajectory [?, ?].

Comment 16 : #R10.15

Further issue with "[Euler parameterization of rotations] are not correlated": [cross]-correlation/covariance vs. manifold parameterization are two very different things!

Thank you for your comment.

We also could not find the direct quote "not correlated" in the manuscript. However, we agree with you that correlation and parameterization are two very different things. Additionally, the SSC first-order jacobian based methods are able to handle correlation, the primary problem with that method is the choice of parameterization. We have tried to make this clear throughout the text.

Comment 17 : #R10.16

The paper can be shortened.
- Section II is textbook theory .. as appendix.
- Section III, Smith paper can also be condensed, reader can find original.
- eqs. (19), (23), (26), and in fig.3 are same appart from which jacobian used. maybe just once and discuss how the Jacobians are generated
- simplify "first-order-jacobian" vs. adjoint (higher degree) method, fig. 3 and eqs. (39), (45), (51).

Comment 18 : #R10.17

True that?
- paper compares two parameterizations:
— Euler
— Lie
- four methods of non-linear cov. propagation:
— first-order-jacobian,
— adjoint transformation,
— Sigma point (unscented) transform
— Direct sampling as Monte Carlo
- important fifth is Laplace approximation.
- how much of this "table" is populated by presentation of the paper?

Thanks for your questions.

The answer to your first question is true. The paper is primarily concerned with comparing two different parameterizations Euler and Lie.

The answer to your second question is partially true. The paper compares two differen't first-order covariance propogation methods, one for each parameterization. First-order jacobian propogation is used for Euler, while a first-order method based on the Baker-Campbell-Hausdorff formula is used for Lie. Comparing against higher order methods is outside the scope of this paper as the primary focus is on the choice of parameterization, however higher order methods can always be applied at the cost of higher computational cost if increased accuracy is needed.

We assume the table you are referring to is Fig. 3? This figure provides a quick reference guide to show how uncertainty is modeled and how uncertainty propagation is defined for each operation (or not implemented) for SSC [**?**], [**?**], and our proposed method. Our work is listed under the last row.

Comment 19 : #R10.18

succinctly indicate in figs. 10-14 – comparing left and right columns

Thank you for your suggestion.

The following was added to the caption for Fig. 10:

Thus, the left and right columns should be compared and the darker the color of the plotted relative pose transformations, the larger the error. #R10.19

Comment 20 :

How do you substantiate left is good right is bad? Section VII. B., Fig 8, 9 would suggest that actually both left and right columns are wrong but left is just a little less wrong (inconsistent)?

Thanks for your question.

The figures you are refering to actually correspond to two different experiments. Figures 8 and 9 pertain to -A that is looking at the composition of a sequence of poses, derived from correlated odometry measurements. Figures 10-14 pertain to -B and are refering to the estimated relative pose transformations for a large set of pose pairs that were estimated by iSAM.

The covariance error used in -B can be treated as a measure of consistency since a large covariance error corresponds to over/under approximating the covariance estimate. The degree of consistency/inconsistency is dependent on the how correlated the underlying poses are with one another when comparing the Lie based method while taking into account/ignoring correlation. While the degree of consistency/inconsistency is dependent on the noise level of the true underlying distribution when comparing the Lie and Euler parameterizations since the Euler based method is unable to accuratelly characterize uncertain poses with high rotation noise.

We also further substantiate this by performing an additional experiment. See response to comment number #R3.6. #R10.20

Comment 21 :

Where does ground truth comparison data come from?

Thanks for your question.

The groundtruth comparison data for each datapoint in Figures 8 and 9 were generated via Monte-Carlo as described in Section -B. However, to generate Figures 8 and 9, we performed parameter sweeps over the number of poses in the trajectory, the translation noise scale, and the rotation noise scale.

The groundtruth comparison data for Section -B is also generated via Monte-Carlo simulation, but the individual pose pairs and their associated joint marginals are generated by applying iSAM to the Manhattan3500 dataset. After solving the SLAM problem, a pair of poses and their associated joint marginal covariance is extracted from the solution. Then, a large number of sample pose pairs are generated from that pose pair distribution and the relative pose between each of those pairs is evaluated. This gives us a sample distribution of the relative transformation between the two poses in the original pair. We use this sampled distribution as the ground truth relative pose distibution for the given pair. This process is then repeated for a large subset of the possible pose pairs in the dataset. We have rewritten the description of this experiment in the manuscript. See response to comment #R3.2. #R10.21

Comment 22 :

pg. 10 "The exact accuracy of our proposed uncertainty characterization method is dependent on a variety of parameters including the number of poses compounded end-to-end and the rotation and translation noise.", undermine figs. 10-14 and are somewhat rhetorical.

Thank you for your comment.

It is important to point out that for any method that is not exact there are conditions for which its accuracy will drop-off. The experiments described in Section -A were designed to directly explore these conditions. They also serve to show that the accuracy of the common method of using Euler angles and first-order jacobian propagation drops off much more quickly than was previously supposed. #R10.22

Comment 23 :

I'm assuming you are visualizing joint (Gaussian) posterior marginal covariance elements between different variables in the problem, x1 vs x50; showing some of them(?) by color? P(X1, X50 — Z:) = MultiVariateNormal, where Z: indicates all measurements (factors, f(x), h(x),...). Are drawing the diagonal or off-diagonal elements of joint posterior covariance?

Thanks for your question.

In the top row of Figs 10-14, we are drawing the off diagonal elements of the joint marginal covariance for a given pair of poses as estimated by iSAM. Specifically we are showing the correlation between the x coordinates of the two poses, the y coordinates of the two poses, and the heading of the two poses. #R10.23

Comment 24 :

Figs 10-14, are you comparing against a method of your own choosing as a drop in replacement for what is in use today?

Thank you for your question.

There are two primary comparisons occuring in Figs. 10-14.

The first is a comparison between our proposed methods and the equivelent if correlation were to be ignored.

The second is a comparison between our proposed methods and SSC (Euler) jacobian based methods, since these methods are often used in practice and since we want to point out the error that occurs by using those techniques. #R10.24

Comment 25 :

It's hard to determine the relative order of error between known and proposed methods.

Thanks for your comment.

The relative order of error is shown in Table I and Table II. #R10.25

Comment 26 :

Figs. 8,9 say both methods are inconsistent, but Figs. 10-14 suggests that one method is right and the other is worse (wrong) – that doesn't make sense.

Thanks for your comment.

Figures 8 and 9 show that the consistency of first order uncertainty propagation drops off as the number of poses being composed or rotation noise increases. However, they also show that the proposed Lie based method consistently results in more consistent uncertainty estimates than the Euler based method or the Lie based method when correlation is ignored.

This drop-off in consistency is expected since higher-order terms are being dropped in all first-order methods and as the number of poses or rotation noise increases, this effect become more pronounced. However, for small numbers of poses or low rotation noise, the proposed Lie based method is relativelly consistent.

Each relative pose plotted in Figures 10-14 show the error for a single relative pose computation. Thus, Figures 10-14 show that there is a significant difference in error for the compared methods, even when a single operation is being performed and when the noise (and correlation) is consistent with what is observed when operating on poses derived from a SLAM solution. #R10.26

Comment 27 :

What is your method of finding reference covariances, pg. 12: "... we formed a Monte Carlo simulation to estimate the true relative pose covariance as described in..." – how did you do this on Manhattan3500? Doing "Monte Carlo" is insufficient detail. How did you get accurate marginal covariance estimates, how did you deal with cycles? Clarify how you got reference data for figs. 10-14. If by Monte Carlo you mean you forward propagated a set of samples, as shown in fig. 4, then you didn't solve the SLAM problem and the results are even more suspicious. That enters realm of non-parametric methods on graphs.

Thanks for your question.

See the response to comment #R10.21. #R10.27

Comment 28 :

Similarly, pg.11 "...the joint composition method in the Lie algebra is consistently more accurate than when correlation is ignored" – that doesnt say anything?

Thanks for your comment.

This comment was included since current state-of-the-art techniques such as [1] make the assumption that the poses being composed are independent. However, it is important that this correlation is not ignored since ignoring it when it is present leads to inconsistency.

To improve clarity, it has been changed as follows:

In addition, the joint composition method in the Lie algebra is consistently more ~~accurate~~consistent than when correlation is ignored. #R10.28

Comment 29 :

Top rows of fig. 10-14 refer to self correlation between variables? top rows and right columns supposed to be similar? Top rows coming from iSAM?

Thanks for your questions.

The top row of Figures 10-14 refer to the cross correlation between pairs of poses as estimated by iSAM. These correspond to the off diagonal elements of the joint marginal covariances for different pairs of poses.

Monte-Carlo is then used to evaluate a "ground-truth" distribution for the relative pose between each pose pair. See the response to comment #R3.2

This Monte-Carlo derived distribution is then used for evaluating the methods compared in the bottom two rows.

There is definitely a relationship between which pairs of poses are correlated in the top row and which poses have increased error on the right column the middle row. However, a relationship with the bottom row is less clear. #R10.29

Comment 30 :

On pg. 12 you say, about figs. 10-14 and the discussion in IX.A., "we then extracted joint mean and covariance ... VIII-B for pairs ... a visualization ... pairs and the correlation ... are shown in Fig. 10 (a-c) ... Equivalent ... Fig. 11-14": are you now just using iSAM's covariance recovery just for the top rows in figs 10-14, or are you also using iSAM mean results (but not marginal covariances) and then doing your own calculation of left right columns?

Thanks for your question.

We are using iSAM's covariance recovery to extract the joint marginal covariance for various pose pairs and then performing our own calculation of the left and right columns based on that extracted joint covariance. See the response to comment #R3.2.

We also ensure that the extracted covariance is in the correct parameterization for the method being used. #R10.30

Comment 31 :

Fig. 6 is probably not that useful?

Thanks for the comment.

We agree. It will be removed in the final paper. #R10.31

Comment 32 :

"preforms" → "performs"

~~preforms~~ → performs #R10.32

Comment 33 :

What do you mean with the very last sentence: "...correlated ... such as ... presence of wheel slip". I do not see the connection between wheel slip and correlation

Thanks for your comment.

We often make the assumption that odometery measurements are indepedent of one another. This is often not the case.

For example, if wheel slip is due to irregularities in one of the wheels or if it is due to patches of loose gravel or mud in the environment, then it makes sense that noise in consecutive odometry measurements is likely to be correlated.

We added the following line to the introduction to address this:

> *Alternatively, wheel slip due to an irregular wheel or loose gravel may lead to sequence of short term odometry measurements being correlated over time.*

We also modified the quoted line in the conclusion as follows:

> *It can also be used to compose odometry measurements that are potentially correlated with one another such as can be the case in the presence of wheel slip due to wheel irregularity or loose gravel in the environment.*

#R10.33

Comment 34 :

pg. 8, "As far as we can tell, this is the first time ..." please clarify what you mean by "first" and "this" in context of Chirkjian?

Thanks for your comment.

We have changed this statement in the paper to the following:

> *As far as we can tell, this is the first time this operation has been published while using the Lie algebra to characterize uncertainty as defined in [1].*

#R10.34

Comment 35 :

There might be a mistake with eq. 43., it should be $(T'_{ij})^{-1} = (T_{ij}T'_{jj})^{-1} = exp(-\xi'_{jj})T_{ji} = exp(\xi'_{jj})T_{ji}$

Thanks for pointing this out.

This is a matter of right handed or left handed convention. As defined in (14), in this paper we multiply the perturbation on the left side of the group mean. However, all of the same results can be derived based on the other convention as well.

We don't feel that there was a mistake in the derivation, however, we have added additional details to the derivation to potentially clarify the problem:

$$
\begin{aligned}
\mathbf{T}_{ji} &= \mathbf{T}_{ij}^{-1} \\
&= (\exp(\boldsymbol{\xi}_{ij}^{\wedge})\bar{\mathbf{T}}_{ij})^{-1} \\
&= \bar{\mathbf{T}}_{ij}^{-1}\exp(-\boldsymbol{\xi}_{ij}^{\wedge}) \\
&= \exp((-\mathrm{Ad}_{\bar{\mathbf{T}}_{ij}^{-1}}\boldsymbol{\xi}_{ij})^{\wedge})\bar{\mathbf{T}}_{ij}^{-1} \\
&= \exp(\boldsymbol{\xi}'_{ij})\bar{\mathbf{T}}_{ij}^{-1}
\end{aligned} \tag{4}
$$

We have also added the following footnote to equation (14):

> *In this paper we follow the convention of using the exponential map to perturb a mean group element on the left side. A valid alternative would be to multiply on the right. The primary difference between these two forms is the frame in which the perturbation is applied. All of the results in the paper have equivellent derivations for right handed perturbation as well, however, in this article we perform multiplication on the left to follow the convention used in [1].*

#R10.35

Comment 36 :

Section X not perhaps also be in an Appendix?

#R10.36