IB MATHEMATICS EXTENDED ESSAY

MAP PROJECTIONS

How do we quantify geometric distortions of different map projections using differential geometry?

June 6, 2025

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Contents

1 Introduction

The problem of projecting Earth's spherical surface onto a two-dimensional plane stands as one of the most persistent intellectual challenges in human history (Snyder, 1987). Since antiquity, this fundamental limitation of cartography has compelled scholars to develop increasingly complex mathematical solutions while confronting unavoidable geometric compromises. Far from being merely technical instruments for navigation, maps are powerful cultural artifacts that reveal how societies conceptualise space, territory and their prominence in the world (Harley, 1989).

One of the earliest systematic treatments of cartography was established in the 2nd century CE by Claudius Ptolemy, whose *Geographia* introduced the revolutionary concept of a coordinate grid system (Claudius et al., 2002). This pioneering work not only shaped medieval and Renaissance cartographic development for centuries but also demonstrated the inherent mathematical nature embedded in this problem (Thrower, 2008).

During the Islamic Golden Age, (8th to 13th century), scholars such as al-Khwārizmī and al-Idrīsī advanced mathematical geography by synthesising Ptolemaic theory with new empirical data from Arab navigators and astronomers (Edson et al., 2011). Parallel developments occurred in East Asia, where Chinese cartographers created sophisticated grid-based maps for administrative and military purposes, demonstrating the cross-cultural importance of spatial representation (Yee, 1994).

A major breakthrough in projections came during the Age of Exploration (15th to 17th century), as European empires ventured across the vast oceans and the demand for more precise navigation tools intensified (Brotton, 2014). This period witnessed the development of several landmark projections, most notably Gerardus Mercator's 1569 cylindrical projection. While its conformal properties proved invaluable for marine navigation, its severe areal distortion at high latitudes later became a subject of considerable criticism (Battersby et al., 2014). Contemporary cartographers including Johannes Stabius, Johannes Werner, and Oronce Finé developed alternative projections such as azimuthal and cordiform maps, reflecting the growing interest and recognition of the trade-offs involved in flattening a sphere (Bugayevskiy & Snyder, 2013).

While cartographers grappled with practical representations of Earth, mathematicians were developing the theoretical foundations that would later revolutionise map projections. The origins of differential geometry can be traced to the 17th and 18th centuries, when Christiaan Huygens and Leonhard Euler began systematically studying the curvature of surfaces (Struik, 1988). Their work on osculating circles and developable surfaces laid the groundwork for understanding how three-dimensional forms could be mathematically described.

The two field of mathematics and cartography collided most dramatically in the 20th century, when mathematicians began applying differential geometry to optimise map projections (Snyder, 1987). Today, modern computational maps continue to build upon this mathematical foundation, powering navigation systems worldwide.

Against this historical backdrop, this Extended Essay aims to investigate the Research Question:

How do we quantify geometric distortions of map projections using the tools from differential geometry?

I will explore how tools such as the First Fundamental Form, Gaussian curvature, and Tissot's Indicatrix allow us to analyse and visualise distortions in length, area, and angle on maps. By modelling Earth as a perfect unit sphere, I will examine three widely used projections - the Mercator, Mollweide, and Lambert Conformal Conic - and use my mathematical framework to calculate the geometric distortions each introduces.

2 Mathematical Frameworks

2.1 Come up with title

We know that every position on a map corresponds to a unique point on Earth. In mathematical language, this is a bijection (one-to-one and onto mapping) - when a function is both an injection and surjection.

Definition 1. Bijection (Libretexts, 2021)

Let $f : A \mapsto B$

Injection (one-to-one) - for all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$
Surjection (onto) - for every $y \in B$, there exists $x \in A$ such that $f(x) = y$

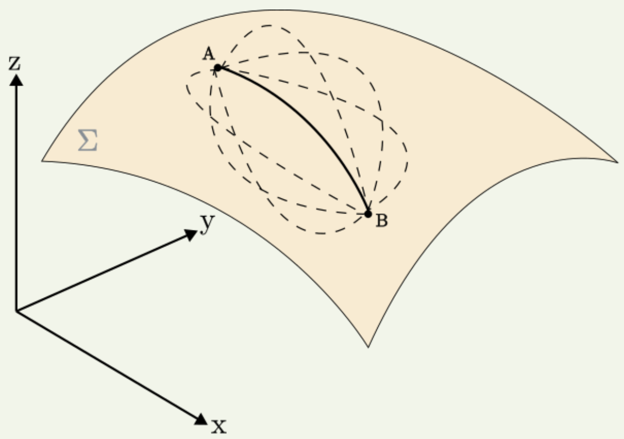
A bijection function holds both of these properties.

It helps to think of the map and the Earth as two sets containing points, where the function f maps each point on the Earth (set A) to a unique point on the map (set B) because it formalizes the intuitive concept of a perfect map. Injection ensures two distinct places on Earth cannot merge into one point on the map while surjection guarentees every map location corresponds to a real place on Earth.

It is also necessary to define the notion of *distance*, as ambiguity may arise without a clear specification. For instance, distance could refer to the shortest path through ambient space (the chord passing through the Earth's interior), the straight-line or "as-the-crow-flies" distance, or the actual travel distance along roads between two locations. The distance which cartographers will deal with is called the intrinsic or geodesic distance (Ozuch, 2024).

Definition 2. Geodesic distances

Given a surface Σ embedded in ambient space, if A and B are points of S, we can then consider the infinite amount of the lengths of curves included in Σ joining these two points (dashed lines).



Out of these curves, the one with the minimum length (solid line) is the intrinsic or geodesic distance on the surface. This will be denoted as $d(A, B)$.

Having established the necessary definitions, we can now mathematically describe what a faithful map is: a map that preserves all distances between any pair of points, up to a scale change. This property is called *isometry* - a challenge that many cartographers try to achieve for centuries (Wolfram, 2025).

2.2 Gaussian Curvature

Gauss curvature, commonly denoted as k, is the product of the minimal and maximal curvatures of geodesics starting from a point.

<insert diagram>

The precise mathematical definition of curvature is too complicated beyond the scope of this Extended Essay, so I will describe the intuitive understanding of it instead. Construct a tangent plane at a point on the surface; if:

- the surface is contained on one side of its tangent plane will have positive curvature.
- the surface is intersected with its tangent plane will have negative curvature.
- the surface with one of its segments contained in the tangent plane will have 0 curvature.

Another way to understand curvature is by comparing the properties of objects on that surface. One such comparison tool is the Toponogov's Theorem.

2.3 Parametric Surfaces

A surface in three-dimensional space is often most effectively described using a parametric representation. In this framework, a surface is defined by a vector-valued function

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \tag{1}$$

where u and v are independent parameters that vary over a two-dimensional region D in the uv-plane. The set of all such position vectors forms the parametric surface S (Dawkins, 2024).

Unlike implicit surfaces defined by equations of the form $F(x,y,z) = 0$, parametric surfaces allow for the direct generation of points and the computation of geometric properties such as tangent vectors through partial derivatives, which will later on prove itself useful in the context of cartography.

If we express the two parameters u and v as a function of another variable t, then by applying the chain rule to $\vec{r}(u(t), v(t))$ with respect to t, we get:

$$\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{dt} \tag{2}$$

This derivative represents the tangent to the curve formed by $u(t)$ and $v(t)$ (Patrikalakis et al., 2009).

Another important property of surfaces, particularly relevant in cartography, is the notion of smoothness. Depending on the desired level of accuracy, the Earth's surface may be modeled in various ways: as a perfect sphere, an oblate spheroid, or a more complex geoid that represents mean sea level variations. Regardless of the chosen model, a key geometric feature of such surfaces is that, when observed at an infinitesimally small scale, they resemble a flat plane. In other words, the surface is locally planar, a property that characterizes it as smooth (Ozuch, 2024).

As mentioned in the introduction, we will model the Earth as a perfect unit sphere due to the limited scope of this Extended Essay. Let ϕ and λ be the longitude and latitude of the Earth, acting as parameters of the surface, and let $M(\phi, \lambda)$ be a point on the Earth's surface.

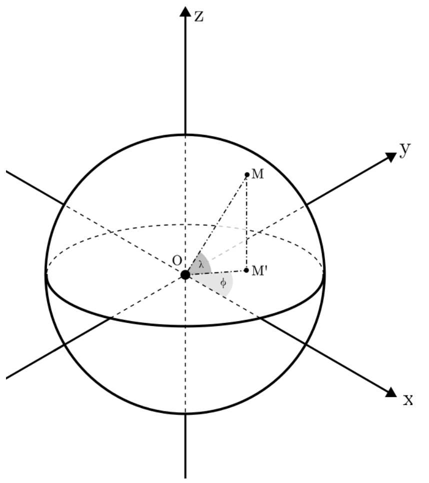


Figure 1: Earth as a sphere centered at $(0,0,0)$

Fix the center of the spherical Earth at the origin of the Cartesian coordinate $(0,0,0)$, with the poles aligning with the z-axis. Let M' be the projection of point M onto the xy-plane in 3D Cartesian space. We can see that the z-coordinate of M would be

$$z = MM' = \sin\lambda$$

To find the x and y-coordinate of point M, we first find the coordinate in terms of OM' using trigonometry and then substitute $OM' = \cos\lambda$, obtaining

$$x = OM'\cos\phi = \cos\lambda\cos\phi$$
$$y = OM'\sin\phi = \cos\lambda\sin\phi$$

Thus, the parametric equation of the model Earth is

$$\vec{r}(\phi,\lambda) = \cos\lambda\cos\phi\vec{i} + \cos\lambda\sin\phi\vec{j} + \sin\lambda\vec{k} \tag{3}$$