# Mathematical formulas for QM(02/16)

In order to do QIP(quantum information process), we need to know how to represent

1. Quantum state
2. Transformation of a quantum state
3. Measurement of a quantum state
4. Composition systems: 2 qubits (quantum entanglement)
5. $1-4$ in an imperfect world

Quantum state

The most complete characterization of a quantum variable at a given time is "Quantum states are states of (lack of my)knowledge".

| Bit | | Qubit |
|---|---|---|
| state 0 | $\rightarrow$ | state $\|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ |
| state 1 | $\rightarrow$ | state $\|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ |
| no knowledge of classical computing | $\rightarrow$ | $\|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\|0\rangle + \beta\|1\rangle$ <br> $\langle\psi\| = [\ \alpha^* \ \ \beta^*\ ] = \alpha^*\langle 0\| + \beta^*\langle 1\|$ |

**Table 1.** Classical state vs quantum state

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ where $|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Qubit can be also represented as a density matrix

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} [\ \alpha^* \ \ \beta^*\ ] \\ &= \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix} \end{aligned}$$

A diagonal density matrix implies a classical system.

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The state of a quantum state is described by ket $|\psi\rangle \Rightarrow$ All kets are elements of abstract vector space $\mathcal{H}$, called Hilbert space,

Properties

1. If $|\psi\rangle$ and $|\phi\rangle$ are in $\mathcal{H}$ then "superposition" $a|\psi\rangle + b|\phi\rangle \in \mathcal{H}$.

2. Inner product $\langle \phi | \psi \rangle$

For a real physical system, $\langle \phi | \phi \rangle = 1$; if two vectors are orthogonal then $\langle \phi | \psi \rangle = 0$.

Example

$$\begin{aligned} \langle 0|0 \rangle &= \langle 1|1 \rangle \\ &= 1 \end{aligned}$$

$$\begin{aligned} \langle 0|1 \rangle &= \langle 1|0 \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle \psi | \psi \rangle &= \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \alpha\alpha^* + \beta\beta^* \\ &= 1 \end{aligned}$$

$|0\rangle$ and $|1\rangle$ form an orthogonal basis set.

Bloch sphere

Relation to measurements

Results of a ‘basic’ measurement is associated with a basic vector. For a qubit, possible outcomes are $|0\rangle$,$|1\rangle$, which are called post measurement qubit state. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ denote the qubit before measurement.

The probability to get result 0 is $\mathrm{Prob}(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2$.

The probability to get result 1 is $\mathrm{Prob}(1) = |\langle 1|\psi\rangle| = |\beta|^2$.