Convex Optimization

BY YALIN LIAO

The following problem

$$\begin{aligned}&\min_{x\in\mathbb{R}^d} f(x)\\&h_i(x)=0\\&g_i(x)\leq 0\end{aligned}$$

is a convex optimization if $f(x), g_i(x)$ are convex and $h_i(x)$ are affine functions.

Assume that $f(x)$ is a convex function. Let $A=\{x|f(x)\leq 0\}$ and $B=\{x|f(x)=0\}$. Then A is convex but B may not be convex.

For any $x, y\in A$ and $\lambda\in[0,1]$,

$$\begin{aligned}f(\lambda x+(1-\lambda)y) &\leq \lambda f(x)+(1-\lambda)f(y)\\&\leq \lambda\cdot 0+(1-\lambda)\cdot 0\\&\leq 0\end{aligned}$$

This implies $\lambda x+(1-\lambda)y\in A$ so A is a convex set.

Consider $f(x)=x^2-4x+3$. Then $B=\{1,3\}$ is obviously not a convex set.

KKT condition

$$\begin{cases}\nabla f(x)+\mu\nabla g(x)=0\\ g(x)\leq 0\\ \mu\geq 0\\ \mu g(x)=0\end{cases}$$

In general case, KKT condition is not solvable. Instead, it's easier to solve

$$\begin{cases}\nabla f(x)+\mu\nabla g(x)=0\\ g(x)\leq 0\\ \mu\geq 0\\ \mu g(x)=-t\\ t>0\,\mathrm{but\,small}\end{cases}$$

which is a relaxion of KKT condition.