Note on simplifying the objective function

Example $f(x,y) = A\sin x\cos y + B\cos x\sin y + C\cos^2 x + D\sin^2 x$

$$\begin{aligned}
f(x,y) &= A\cdot\frac{\sin(x+y)+\sin(x-y)}{2} + B\cdot\frac{\sin(x+y)-\sin(x-y)}{2} \\
&\quad + C\cdot\frac{1+\cos(2x)}{2} + D\cdot\frac{1-\cos(2x)}{2} \\
&= \frac{1}{2}[(A+B)\sin(x+y) + (A-B)\sin(x-y)] + \frac{1}{2}(C-D)\cos(2x) + \frac{1}{2}(C+D) \\
&\quad \text{Let } K_1 = \sqrt{(A+B)^2+(A-B)^2} \\
&= \frac{1}{2}K_1\left[\sin(x+y)\frac{A+B}{K_1} + \frac{A-B}{K_1}\sin(x-y)\right] + \frac{1}{2}(C-D)\cos(2x) + \frac{1}{2}(C+D) \\
&\quad \text{Since angles } x+y \text{ and } x-y \text{ are different, we cannot simplify the first term.}
\end{aligned}$$

Try

$$\begin{aligned}
R_Y(x)|0\rangle &= \begin{bmatrix} \cos\frac{x}{2} & -\sin\frac{x}{2} \\ \sin\frac{x}{2} & \cos\frac{x}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{x}{2} \\ \sin\frac{x}{2} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
R_Y(y)|0\rangle &= \begin{bmatrix} \cos\frac{y}{2} & -\sin\frac{y}{2} \\ \sin\frac{y}{2} & \cos\frac{y}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{y}{2} \\ \sin\frac{y}{2} \end{bmatrix}
\end{aligned}$$

Then we see

$$\begin{aligned}
R_Y(x)\otimes R_Y(y)|00\rangle &= R_Y(x)|0\rangle \otimes R_Y(y)|0\rangle \\
&= \begin{bmatrix} \cos\frac{x}{2} \\ \sin\frac{x}{2} \end{bmatrix} \otimes \begin{bmatrix} \cos\frac{y}{2} \\ \sin\frac{y}{2} \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{x}{2}\cos\frac{y}{2} \\ \cos\frac{x}{2}\sin\frac{y}{2} \\ \sin\frac{x}{2}\cos\frac{y}{2} \\ \sin\frac{x}{2}\sin\frac{y}{2} \end{bmatrix} \\
&= \cos\frac{x}{2}\cos\frac{y}{2}|00\rangle + \cos\frac{x}{2}\sin\frac{y}{2}|01\rangle + \sin\frac{x}{2}\cos\frac{y}{2}|10\rangle + \sin\frac{x}{2}\sin\frac{y}{2}|11\rangle \\
&= |0\rangle\left(\cos\frac{x}{2}\cos\frac{y}{2}|0\rangle + \cos\frac{x}{2}\sin\frac{y}{2}|1\rangle\right) + |1\rangle\left(\sin\frac{x}{2}\cos\frac{y}{2}|0\rangle + \sin\frac{x}{2}\sin\frac{y}{2}|1\rangle\right)
\end{aligned}$$

Without other gates, use PauliZ to measure the first qubit

$$\begin{aligned}
\mathbb{E}_{Z[0]} &= \left[\left(\cos\frac{x}{2}\cos\frac{y}{2}\right)^2 + \left(\cos\frac{x}{2}\sin\frac{y}{2}\right)^2\right]\cdot 1 + \left[\left(\sin\frac{x}{2}\cos\frac{y}{2}\right)^2 + \left(\sin\frac{x}{2}\sin\frac{y}{2}\right)^2\right]\cdot(-1) \\
&= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \\
&= \cos x
\end{aligned}$$

Let U be an unitary matrix. Then

$$
\begin{aligned}
UR_Y(x) \otimes R_Y(y)|00\rangle &= UR_Y(x)|0\rangle \otimes R_Y(y)|0\rangle \\
&= U\begin{bmatrix} \cos\frac{x}{2} \\ \sin\frac{x}{2} \end{bmatrix} \otimes \begin{bmatrix} \cos\frac{y}{2} \\ \sin\frac{y}{2} \end{bmatrix} \\
&= \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \begin{bmatrix} \cos\frac{x}{2}\cos\frac{y}{2} \\ \cos\frac{x}{2}\sin\frac{y}{2} \\ \sin\frac{x}{2}\cos\frac{y}{2} \\ \sin\frac{x}{2}\sin\frac{y}{2} \end{bmatrix} \\
&= \begin{bmatrix} u_{11}\cos\frac{x}{2}\cos\frac{y}{2} + u_{12}\cos\frac{x}{2}\sin\frac{y}{2} + u_{13}\sin\frac{x}{2}\cos\frac{y}{2} + u_{14}\sin\frac{x}{2}\sin\frac{y}{2} \\ u_{21}\cos\frac{x}{2}\cos\frac{y}{2} + u_{22}\cos\frac{x}{2}\sin\frac{y}{2} + u_{23}\sin\frac{x}{2}\cos\frac{y}{2} + u_{24}\sin\frac{x}{2}\sin\frac{y}{2} \\ u_{31}\cos\frac{x}{2}\cos\frac{y}{2} + u_{32}\cos\frac{x}{2}\sin\frac{y}{2} + u_{33}\sin\frac{x}{2}\cos\frac{y}{2} + u_{34}\sin\frac{x}{2}\sin\frac{y}{2} \\ u_{41}\cos\frac{x}{2}\cos\frac{y}{2} + u_{42}\cos\frac{x}{2}\sin\frac{y}{2} + u_{43}\sin\frac{x}{2}\cos\frac{y}{2} + u_{44}\sin\frac{x}{2}\sin\frac{y}{2} \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_{Z[0]} = & \Big[\Big(u_{11}\cos\frac{x}{2}\cos\frac{y}{2} + u_{12}\cos\frac{x}{2}\sin\frac{y}{2} + u_{13}\sin\frac{x}{2}\cos\frac{y}{2} + u_{14}\sin\frac{x}{2}\sin\frac{y}{2}\Big)^2 \\
& + \Big(u_{21}\cos\frac{x}{2}\cos\frac{y}{2} + u_{22}\cos\frac{x}{2}\sin\frac{y}{2} + u_{23}\sin\frac{x}{2}\cos\frac{y}{2} + u_{24}\sin\frac{x}{2}\sin\frac{y}{2}\Big)^2\Big] \cdot \\
& + \Big[\Big(u_{31}\cos\frac{x}{2}\cos\frac{y}{2} + u_{32}\cos\frac{x}{2}\sin\frac{y}{2} + u_{33}\sin\frac{x}{2}\cos\frac{y}{2} + u_{34}\sin\frac{x}{2}\sin\frac{y}{2}\Big)^2 \\
& + \Big(u_{41}\cos\frac{x}{2}\cos\frac{y}{2} + u_{42}\cos\frac{x}{2}\sin\frac{y}{2} + u_{43}\sin\frac{x}{2}\cos\frac{y}{2} + u_{44}\sin\frac{x}{2}\sin\frac{y}{2}\Big)^2\Big] \cdot (-1)
\end{aligned}
$$

Expand the first term,

$$
\begin{aligned}
T_1 = & u_{11}^2\cos^2\frac{x}{2}\cos^2\frac{y}{2} + u_{12}^2\cos^2\frac{x}{2}\sin^2\frac{y}{2} + u_{13}^2\sin^2\frac{x}{2}\cos^2\frac{y}{2} + u_{14}^2\sin^2\frac{x}{2}\sin^2\frac{y}{2} \\
& + 2u_{11}u_{12}\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{x}{2}\sin\frac{y}{2} + 2u_{11}u_{13}\cos\frac{x}{2}\cos\frac{y}{2}\sin\frac{x}{2}\cos\frac{y}{2} \\
& + 2u_{11}u_{14}\cos\frac{x}{2}\cos\frac{y}{2}\sin\frac{x}{2}\sin\frac{y}{2} + 2u_{12}u_{13}\cos\frac{x}{2}\sin\frac{y}{2}\sin\frac{x}{2}\cos\frac{y}{2} \\
& + 2u_{12}u_{14}\cos\frac{x}{2}\sin\frac{y}{2}\sin\frac{x}{2}\sin\frac{y}{2} + 2u_{13}u_{14}\sin\frac{x}{2}\cos\frac{y}{2}\sin\frac{x}{2}\sin\frac{y}{2} \\
= & u_{11}^2\frac{1+\cos x}{2} \cdot \frac{1+\cos y}{2} + u_{12}^2\frac{1+\cos x}{2} \cdot \frac{1-\cos y}{2} + u_{13}^2\frac{1-\cos x}{2} \cdot \frac{1+\cos y}{2} \\
& + u_{14}\frac{1-\cos x}{2} \cdot \frac{1-\cos y}{2} + u_{11}u_{12}\cos^2\frac{x}{2}\sin y + u_{11}u_{13}\sin x\cos^2\frac{y}{2} \\
& + \frac{1}{2}u_{11}u_{14}\sin x\sin y + \frac{1}{2}u_{12}u_{13}\sin x\sin y + u_{12}u_{14}\sin x\sin^2\frac{y}{2} + u_{13}u_{14}\sin^2\frac{x}{2}\sin y \\
= & \frac{1}{4}u_{11}^2\cos x\cos y + \frac{1}{2}u_{11}^2(\cos x + \cos y) + \frac{1}{4}u_{11}^2 + \frac{1}{4}u_{12}^2 + \frac{1}{2}u_{12}^2(\cos x - \cos y) - \frac{1}{4}u_{12}^2\cos x\cos y \\
& + \frac{1}{4}u_{13}^2 + \frac{1}{2}(\cos y - \cos x) + \frac{1}{4}u_{14}\cos x\cos y + u_{11}u_{12}\frac{1+\cos x}{2}\sin y + u_{11}u_{13}\sin x\frac{1-\cos y}{2} \\
& + \frac{1}{2}(u_{11}u_{14} + u_{12}u_{13})\sin x\sin y + u_{12}u_{14}\sin x\,\frac{1-\cos y}{2} + u_{13}u_{14}\frac{1-\cos x}{2}\sin y \\
= & \frac{1}{4}(u_{11}^2 - u_{12}^2 + u_{14})\cos x\cos y + \frac{1}{2}(u_{11}u_{14} + u_{12}u_{13})\sin x\sin y - \frac{1}{2}(u_{11}u_{13} + u_{12}u_{14})\sin x\cos y \\
& + \frac{1}{2}(u_{11}u_{12} - u_{13}u_{14})\cos x\sin y + \frac{1}{2}(u_{11}^2 + u_{12}^2 - 1)\cos x + \frac{1}{2}(u_{11}^2 - u_{12}^2 + 1)\cos y \\
& + \frac{1}{2}(u_{11}u_{13} + u_{12}u_{14})\sin x + \frac{1}{2}(u_{11}u_{12} + u_{13}u_{14})\sin y + \frac{1}{4}(u_{11}^2 + u_{12}^2 + u_{13}^2)
\end{aligned}
$$

It seems $\sin^2 x, \sin^2 y$ can not be generated.

Remark.

When using 2 qubits, embedding angle is $\frac{x}{2}$.

When using $4 = 2^2$ qubits, embedding angle should be $\frac{x}{4}$?

Consider the 4 qubit case,

Can we consider the phase embedding?