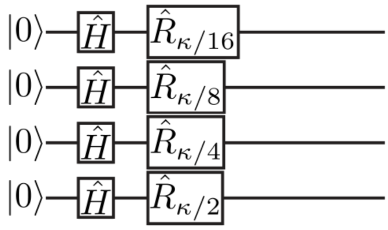
Variational quantum algorithms for nonlinear problems

Figure 6 (a)



This quantum circuit realizes the wave function $|\psi\rangle = (1/\sqrt{2^n})\sum_{q_1, q_2, \ldots, q_n} \exp(i\kappa\sum_{j=1}^n q_j 2^{-j})|q_1, q_2, \ldots, q_n\rangle$ whicn represents a plane wave of wave vector κ. $R_\varphi = \mathrm{diag}(1, \exp(i\varphi))$ is a phase shift gate of phase shift φ.

Remark. We can see he phase shift gate $R_\theta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{smallmatrix}\right]$ is unitary via the calculation

$$R_\theta R_\theta^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Case $n = 1$

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}(e^{i\cdot\kappa\cdot 0\cdot 2^{-1}}|0\rangle + e^{i\cdot\kappa\cdot 1\cdot 2^{-1}}|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + e^{i2^{-1}\kappa}|1\rangle) \\ &= \frac{1}{\sqrt{2}}[|0\rangle + (\cos(2^{-1}\kappa) + i\sin(2^{-1}\kappa))|1\rangle] \end{aligned}$$

Case $n = 2$

$$\begin{aligned} |\psi\rangle &= \frac{1}{2}(e^{i\kappa(0\cdot 2^{-1}+0\cdot 2^{-2})}|00\rangle + e^{i\kappa(1\cdot 2^{-1}+0\cdot 2^{-2})}|01\rangle + e^{i\kappa(0\cdot 2^{-1}+1\cdot 2^{-2})}|10\rangle + e^{i\kappa(1\cdot 2^{-1}+1\cdot 2^{-2})}|11\rangle) \\ &= \frac{1}{2}(|00\rangle + e^{i2^{-1}\kappa}|01\rangle + e^{i2^{-2}\kappa}|10\rangle + e^{i(2^{-1}+2^{-2})\kappa}|11\rangle) \\ & \frac{1}{2}[|00\rangle + (\cos(2^{-1}\kappa) + i\sin(2^{-1}\kappa))|01\rangle + (\cos(2^{-2}\kappa) + i\sin(2^{-2}\kappa))|01\rangle \\ & +(\cos((2^{-1}+2^{-2})\kappa) + i\sin((2^{-1}+2^{-2})\kappa))|11\rangle] \end{aligned}$$

The plan

Consider the case $\delta = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix}$ and the objective function $f(\delta_1, \delta_2) = c_1\sin^2\delta_1 + c_2\sin^2\delta_2 + c_3\sin\delta_1\sin\delta_2 + c_4\cos\delta_1\cos\delta_2 + c_5\sin\delta_1\cos\delta_2 + c_6\cos\delta_1\sin\delta_2 + c_7\sin\delta_1 + c_8\sin\delta_2 + c_8\cos\delta_1 + c_9\cos\delta_2 + c_{10}$.

Using the following identities,

(1) $\cos^2\delta_1 = 1 - \sin^2\delta_1, \cos^2\delta_2 = 1 - \sin^2\delta_2$

(2) $\sin^2\delta_1 = \frac{1-\cos 2\delta_1}{2}, \sin^2\delta_2 = \frac{1-\cos 2\delta_2}{2}$

(3) $\sin\delta_1\sin\delta_2 = \frac{1}{2}[\cos(\delta_1-\delta_2) - \cos(\delta_1+\delta_2)], \cos\delta_1\cos\delta_2 = \frac{1}{2}[\cos(\delta_1-\delta_2) + \cos(\delta_1+\delta_2)]$

(4) $\sin\delta_1\cos\delta_2 = \frac{1}{2}[\sin(\delta_1+\delta_2) + \sin(\delta_1-\delta_2)], \cos\delta_1\sin\delta_2 = \frac{1}{2}[\sin(\delta_1+\delta_2) - \sin(\delta_1-\delta_2)]$

we can express $f(\delta_1, \delta_2)$ in the following form

$$\begin{aligned} f(\delta_1, \delta_2) &= c_1\text{cos}2\delta_1 + c_2\text{cos}2\delta_2 + c_3\text{sin}(\delta_1 + \delta_2) + c_4\text{sin}(\delta_1 - \delta_2) + \\ &\quad c_5\text{cos}(\delta_1 + \delta_2) + c_6\text{cos}(\delta_1 - \delta_2) + c_7\text{sin}\delta_1 + c_8\text{sin}\delta_2 + c_8\text{cos}\delta_1 + c_9\text{cos}\delta_2 + c_{10} \end{aligned}$$

We can think the objective function is linear in terms of $\sin x, \cos x, \text{sin}^2x, \text{sin}^2x, \sin x \cos x$.