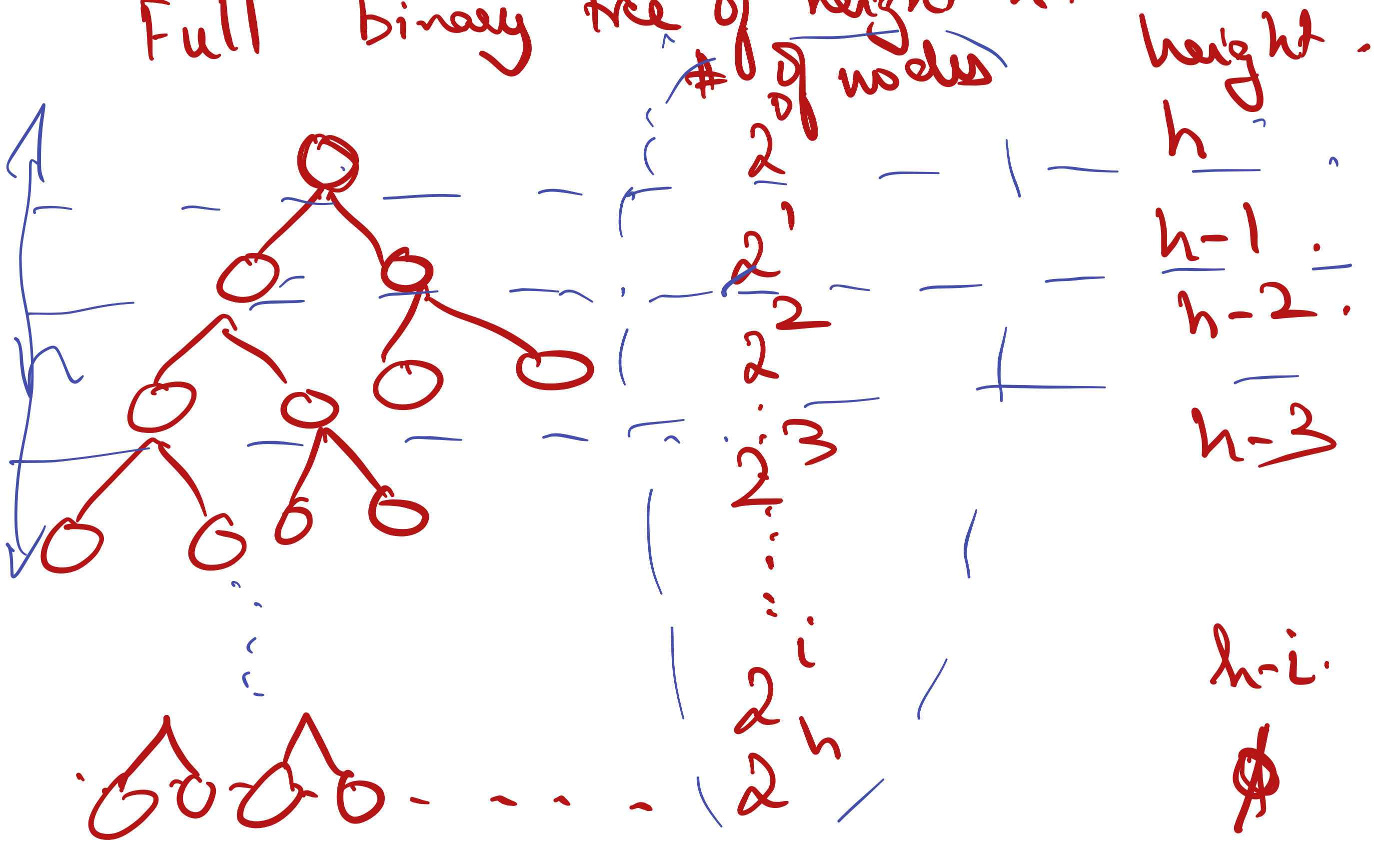
Theorem : For a perfect binary tree of ht h containing $2^{h+1}-1$ nodes, sum of the heights of the nodes is

$$\underline{2^{h+1}-1+(h+1)}$$

Full binary tree of height h.



1 node at hight h.
2 nodes at height $h-1$
2^i nodes : $h-i$.

$$S = 2^0 \cdot h + \underbrace{2^1 \cdot (h-1)}_{i=1} + 2^2 \cdot (h-2) + \cdots 2^i(h-i) + \cdots$$

$$S = \sum_{i=0}^{h} 2^i (h-i).$$

$$S = h + 2(h-1) + 4(h-2) + \dots 2^{h-1}(h-(h-1)) \quad \text{--- } 1$$

$$2S = 2h + 4(h-1) + 8(h-2) + \dots 2^h \cdot 1 \quad \text{--- } 2$$

$$2 - 1$$

$$S = -h + 2h - (2(h-1)) + 4(h-1) - (4(h-2))$$

$$+8(n-2)-(8(n-3))$$

$\vdots$

$$= -n+\cancel{2n}-\cancel{2n}+2$$
$$+\cancel{4n}-4-\cancel{4n}+8+$$

$$\cdots$$

$$= -n+(2+4+\cdots$$
$$+2^{n-1}+2^{n})$$

$$S = -n+\underbrace{2^{n+1}-1}-1.$$

$$S = \left(2^{n\cancel{+1}} - 1\right) - (n+1)$$