Theorem: Worst case running time using Shell's increments is $\Theta(N^2)$

Proof:

lower bound, Ω ✓ O ✓

(A) Show there is some input that actually takes $\Omega(N^2)$ to run.

Is there such an example?

$h_t = \lfloor N/2 \rfloor$ ✓

$h_k = \lfloor \frac{h_{k+1}}{2} \rfloor$

h_k {5, 3, 1

{50, 25, 12, 6, 3, 1.

→ N is a power of 2.

→ N/2 largest at even positions ✓

→ N/2 small at odd positions ✓

Position: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Array: 1 9 2 10 3 11 4 12 5 13 6 14 7 15 8 16

List: 1 9 2 10 3 11 4 12 5 13 6 14 7 15 8 16

8 sort: 1 9 2 10 3 11 4 12 5 13 6 14 7 15 8 16

4 sort: 1 ... 4 12 5 13 6 14 7 15.

2 sort: 1 9 2 10 3 11

1 sort: 1 2 3 4 5 6 7 8 9 . . . . . .

$\Omega(N^2)$.

(B) Big O related:

Upper bound on the worst case running time is $O(N^2)$.

5 sort, 3 sort, k sort ← h_k. A pass with increment h_k consists of h_k insertion sorts of about $\frac{N}{h_k}$ elements.

Insertion sort is quadratic. $\underline{O(n^2)}$ ✓

h_k sort: $h_k \cdot O\left(\left(\frac{N}{h_k}\right)^2\right) = O\left(\cancel{h_k} \cdot \left(\frac{N}{\cancel{h_k}}\right)^2\right)$

↳ Eg if $h_k = 1$ → $1 \cdot O\left(\left(\frac{N}{1}\right)^2\right) = O(N^2)$ known.

For one pass: $= \boxed{O\left(\frac{N^2}{h_k}\right) \text{ per pass}}$

Total cost for all passes:

$$\sum_{i=1}^{t} O\left(\frac{N^2}{h_i}\right) = O\left(\sum_{i=1}^{t}\left(\frac{N^2}{h_i}\right)\right)$$

$$= O\left(N^2 \sum_{i=1}^{t} \frac{1}{h_i}\right)$$

Because $h_i = 1$, $\sum_{i=1}^{t} \frac{1}{h_i} = 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots}$

$= 1 + {<1}$

$= < 2 \cdot$

Total cost $= O(N^2 \cdot) \cdot < 2 \cdot$ $\quad < 2$

$O(N^2)$